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# Overpartition rank differences modulo 7 by Maass forms



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## ABSTRACT

Using that the overpartition rank function is the holomorphic part of a harmonic Maass form, we deduce formulas for the rank differences modulo 7. To do so we make improvements on the current state of the overpartition rank function in terms of harmonic Maass forms by giving simple formulas for the transformations under  $SL_2(\mathbb{Z})$  as well as formulas for orders at cusps.

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## 1. Introduction

A partition of an integer  $n$  is a non-increasing sequence of positive integers that sum to  $n$ . For example the partitions of 4 are 4,  $3 + 1$ ,  $2 + 2$ ,  $2 + 1 + 1$ , and  $1 + 1 + 1 + 1$ . An overpartition of  $n$  is a partition of  $n$  in which the first appearance of a part may be overlined. For example the overpartitions of 4 are 4,  $\overline{4}$ ,  $3 + 1$ ,  $3 + \overline{1}$ ,  $\overline{3} + 1$ ,  $\overline{3} + \overline{1}$ ,  $2 + 2$ ,  $\overline{2} + 2$ ,  $2 + 1 + 1$ ,  $2 + \overline{1} + 1$ ,  $\overline{2} + 1 + 1$ ,  $\overline{2} + \overline{1} + 1$ ,  $1 + 1 + 1 + 1$ , and  $\overline{1} + 1 + 1 + 1$ . The Dyson rank of a partition or an overpartition is defined as the largest part minus the number of parts, in particular the rank does not depend on whether or not a part is overlined.

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For the partitions 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1 the respective ranks are 3, 1, 0, −1, and −3.

The first point of interest with the rank is that it explains certain congruences for the partition function. With  $p(n)$  denoting the number of partitions of  $n$ , we have  $p(5n+4) \equiv 0 \pmod{5}$  and  $p(7n+5) \equiv 0 \pmod{7}$ . The rank explains  $p(5n+4) \equiv 0 \pmod{5}$  in that if we group the partitions of  $5n+4$  according to the value of their rank modulo 5, then we have five groups of equal size. The rank similarly explains the congruence modulo 7. While the rank of overpartitions does not so simply explain congruences for overpartitions, it does yield refinements of congruences for overpartitions [2, Theorem 1.2] and it does play a part in explaining congruences for the number of appearances of the smallest part in the overpartitions of  $n$  [7].

We let  $\overline{N}(m, n)$  denote the number of overpartitions of  $n$  with rank  $m$  and let  $\overline{N}(k, t, n)$  denote the number of overpartitions of  $n$  with rank congruent to  $k$  modulo  $t$ . The generating function for  $\overline{N}(m, n)$  is given by

$$\mathcal{O}(z, \tau) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{\frac{n(n+1)}{2}}}{(zq, z^{-1}q; q)_n},$$

where here and throughout  $q = \exp(2\pi i\tau)$  for  $\tau \in \mathcal{H}$ , that is  $\text{Im}(\tau) > 0$ , and we are using the standard product notation

$$\begin{aligned} (z; q)_n &= \prod_{j=0}^{n-1} (1 - zq^j), & (z; q)_{\infty} &= \prod_{j=0}^{\infty} (1 - zq^j), \\ (z_1, \dots, z_k; q)_n &= (z_1; q)_n \cdots (z_k; q)_n, & (z_1, \dots, z_k; q)_{\infty} &= (z_1; q)_{\infty} \cdots (z_k; q)_{\infty}, \\ [z; q]_{\infty} &= (z, q/z; q)_{\infty}. \end{aligned}$$

Both the rank of partitions and overpartitions have been studied extensively. In [2] Bringmann and Lovejoy established for  $z$  a root of unity that  $\mathcal{O}(z; \tau)$  is the holomorphic part of a harmonic weak Maass form of weight  $\frac{1}{2}$  and Dewar made certain refinements in [5]. The work of Bringmann and Lovejoy in [2] was done in a fashion similar to the work of Bringmann and Ono in [3] for the rank of partitions, upon which Garvan [6] has recently made impressive improvements. In [12] Lovejoy and Osburn gave formulas for the rank differences  $\overline{N}(r, t, n) - \overline{N}(s, t, n)$ , for  $t = 3$  and  $t = 5$ , in terms of infinite products and generalized Lambert series. Determining these difference formulas is equivalent to determining the 3 dissection of  $\mathcal{O}(\exp(2\pi i/3); \tau)$  and the 5 dissection of  $\mathcal{O}(\exp(2\pi i/5); \tau)$ . We will give similar formulas for  $t = 7$  and for this we revisit  $\mathcal{O}(z; q)$  as a harmonic weak Maass form. We could also derive these results from identities between generalized Lambert series as in [12]. The Lambert series method has the advantage that one does not need to work out modular transformation formulas or introduce harmonic Maass forms, but one must derive identities for all of the rank differences. Using harmonic Maass forms has the advantage that one could prove identities for the rank differences

individually. Both methods require that we must first guess the identities to prove them and the proofs usually reduce to verifying an identity between modular forms.

With  $\zeta_c = \exp\left(\frac{2\pi i}{c}\right)$ , we complete  $\mathcal{O}(\zeta_c^a; \tau)$  to a harmonic Maass form  $\mathcal{M}(a, c)$  in Section 2. It turns out this harmonic Maass form is the sum of an easily understood modular form and a harmonic Maass form of Zwegers. With this we give explicit and compact transformation formulas under  $\mathrm{SL}_2(\mathbb{Z})$  in Sections 3 and 4. These transformation formulas allow us to not only reprove that  $\mathcal{O}(\zeta_c^a; \tau)$  is the holomorphic part of a harmonic Maass form of weight  $\frac{1}{2}$  on  $\Gamma_1(16c^2)$ , but also determine a larger subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  on which  $\mathcal{M}(a, c)$  has a rather simple multiplier. The contents of this are in Corollaries 2.2, 3.3, and 4.1. Additionally the transformation formulas for  $\mathrm{SL}_2(\mathbb{Z})$  allow us to give formulas and bounds on the orders of our functions at the cusps of  $\mathrm{SL}_2(\mathbb{Z})$ . As an application of the transformation and order formulas, along with the Valence formula for modular functions, we are able to prove the following theorem which gives the 7-dissection of  $\mathcal{O}(\zeta_7; \tau)$ . It is by determining a large subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  on which  $\mathcal{M}(a, c)$  is not a harmonic Maass form, but has a simple multiplier system, that we can prove this identity by checking a small number of coefficients. In particular the proof will be to check 110 coefficients of an equivalent identity. As such we see it is possible to use Maass forms to find and prove new concrete identities related to overpartitions.

**Theorem 1.1.** *Let  $\zeta_7$  be a primitive seventh root of unity,  $J_a = (q^a, q^{14-a}; q^{14})_\infty$  for  $1 \leq a \leq 7$ ,  $J_0 = (q^{14}; q^{14})_\infty$ , and  $A(x, y, z) = x\zeta_7 + y\zeta_7^2 + z\zeta_7^3 + z\zeta_7^4 + y\zeta_7^5 + x\zeta_7^6$ . Then*

$$\mathcal{O}(\zeta_7; \tau) = R_0(q^7) + qR_1(q^7) + q^2R_2(q^7) + q^3R_3(q^7) + q^4R_4(q^7) + q^5R_5(q^7) + q^6R_6(q^7),$$

where,

$$\begin{aligned} R_0(q) &= \frac{A(-16, -8, -8)J_0}{J_1^3 J_2^3 J_3^3 J_4 J_5 J_6 J_7^2} + \frac{A(-1, 1, -1)J_0}{J_1^3 J_2^3 J_3^3 J_5^3 J_6^2} + \frac{A(1, -1, -1)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_6^2} + \frac{A(-5, -1, -3)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_5^3} \\ &\quad + \frac{A(20, 8, 12)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^3 J_7} + \frac{A(-3, -5, -1)qJ_0 J_4}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^3 J_7^2} + \frac{A(4, 4, 0)qJ_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_5 J_6^3 J_7} + \frac{A(3, 5, 1)q^2 J_0 J_3^2}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^3 J_6^3 J_7^2}, \\ R_1(q) &= \frac{A(2, -4, 2)J_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7} + \frac{A(18, 6, 10)J_0 J_5}{J_1^3 J_2^3 J_3^3 J_4^3 J_6^3 J_7} + \frac{A(-12, -2, -8)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_7^2} + \frac{A(18, 6, 10)qJ_0 J_4}{J_1^3 J_2^3 J_3^3 J_5 J_6^3 J_7^2} \\ &\quad + \frac{A(-18, -6, -10)q^2 J_0 J_3^2}{J_1^3 J_2^3 J_4^3 J_5 J_6^3 J_7^2} + \frac{A(-10, -2, -6)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_5^2 J_7} + \frac{A(-12, -2, -8)qJ_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_5^2 J_7^2}, \\ R_2(q) &= \frac{A(12, 10, 6)J_0}{J_1^3 J_2^3 J_3^3 J_5 J_6^2 J_7^2} + \frac{A(-30, -6, -18)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5 J_7^2} + \frac{A(-14, -6, -10)J_0}{J_1^3 J_2^3 J_3^3 J_4 J_5^3 J_6 J_7} + \frac{A(16, 2, 10)J_0 J_6}{J_1^3 J_2^3 J_3^3 J_4^3 J_5^3 J_7^2} \\ &\quad + \frac{A(1, -1, 1)qJ_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(37, -1, 25)J_0}{2J_1^3 J_2^3 J_3^3 J_4^3 J_5 J_7^2} + \frac{A(-1, -3, -1)qJ_0}{J_1^3 J_2^3 J_3^3 J_4 J_5^2 J_6^2 J_7^2} + \frac{A(-2, 2, -2)qJ_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_6 J_7^2} \\ &\quad + \frac{A(1, -1, 1)J_0 J_3^3 J_4^2 J_5}{2J_1^3 J_6^3 J_7^2} + \frac{A(1, -1, 1)J_0 J_2 J_3 J_4^3}{J_1^3 J_6 J_7^2} + \frac{A(-2, 2, -2)J_0 J_2^2 J_4^2 J_5^2}{J_1^3 J_3 J_6^2 J_7^2}, \end{aligned}$$

$$\begin{aligned}
R_3(q) &= \frac{A(8, -8, 8)J_0}{J_1^3 J_2^3 J_3^3 J_6^2 J_7^3} + \frac{A(-8, 8, -8)J_0 J_5^3}{J_1^3 J_2^3 J_3^3 J_4^2 J_6^3 J_7^3} + \frac{A(-40, 8, -32)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^2} + \frac{A(28, -2, 20)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_5^3 J_6^3} \\
&+ \frac{A(20, 6, 12)qJ_0 J_3}{J_1^3 J_2^3 J_4^2 J_5^2 J_6^3 J_7^3} + \frac{A(8, -8, 8)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_7^3} + \frac{A(-8, -6, -4)qJ_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^3 J_7^3} \\
&+ \frac{A(-8, 0 - 6)q}{J_0 J_7} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{7n(n+1)}}{1 - q^{7n+2}}, \\
R_4(q) &= \frac{A(-4, -4, -2)J_0 J_4}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(2, -2, 2)J_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(-6, 2, -6)J_0}{J_1^3 J_2^3 J_3^3 J_4^3 J_6^3 J_7^3} \\
&+ \frac{A(-10, -2, -8)qJ_0 J_4}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^3 J_7^3} + \frac{A(4, 4, 2)qJ_0 J_3}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^2} + \frac{A(10, 2, 8)q^2 J_0 J_3^3}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^3 J_7^3} \\
&+ \frac{A(10, 2, 6)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_7^2}, \\
R_5(q) &= \frac{A(24, 12, 16)J_0 J_4}{J_1^3 J_2^3 J_3^3 J_6^2 J_7^3} + \frac{A(8, 8, 6)J_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(-8, -8, -6)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_7^3} \\
&+ \frac{A(-24, -12, -16)qJ_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(-24, -8, -18)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^2} + \frac{A(0, -4, 0)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_7^2} \\
&+ \frac{A(16, 4, 10)qJ_0}{J_1^3 J_2^3 J_4^2 J_5^2 J_7^3} + \frac{A(6, 4, 4)q}{J_0 J_7} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{7n(n+1)}}{1 - q^{7n+3}}, \\
R_6(q) &= \frac{A(-40, -8, -24)J_0}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^3} + \frac{A(2, 4, 0)J_0 J_4}{J_1^3 J_2^3 J_3^3 J_5^2 J_6^2 J_7^2} + \frac{A(18, 4, 12)J_0 J_5^2}{J_1^3 J_2^3 J_3^3 J_4^2 J_6^2 J_7^3} + \frac{A(40, 8, 24)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^2} \\
&+ \frac{A(-18, -4, -12)J_0}{J_1^3 J_2^3 J_4^2 J_5^2 J_6^3} + \frac{A(6, -4, 4)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^2} + \frac{A(-18, -4, -12)J_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_7^3} \\
&+ \frac{A(22, 4, 12)qJ_0}{J_1^3 J_2^3 J_3^3 J_4^2 J_5^2 J_6^2 J_7^3} + \frac{A(8, 2, 4)}{J_0 J_7} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{7n(n+1)}}{1 - q^{7n+1}}.
\end{aligned}$$

One advantage to writing the identity in this form is that we can also read off formulas for the rank differences,

$$R_{r,s}(d; q) = \sum_{n=0}^{\infty} (\overline{N}(r, 7, 7n + d) - \overline{N}(s, 7, 7n + d)) q^n.$$

For this, we note

$$\begin{aligned}
\mathcal{O}(\zeta_7; \tau) &= \sum_{n=0}^{\infty} \sum_{r=0}^6 N(r, 7, n) \zeta_7^r q^n \\
&= \sum_{n=0}^{\infty} \sum_{r=1}^6 (N(r, 7, n) - N(0, 7, n)) \zeta_7^r q^n
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r=1}^3 (\zeta_7^r + \zeta_7^{7-r}) \sum_{d=0}^6 \sum_{n=0}^{\infty} (N(r, 7, 7n+d) - N(0, 7, 7n+d)) q^{7n+d} \\
&= \sum_{r=1}^3 (\zeta_7^r + \zeta_7^{7-r}) \sum_{d=0}^6 q^d R_{r,0}(d; q^7),
\end{aligned}$$

where we have used that  $1 + \zeta_7 + \zeta_7^2 + \cdots + \zeta_7^6 = 0$  and  $N(r, 7, n) = N(7-r, 7, n)$ . However  $\zeta_7, \zeta_7^2, \dots, \zeta_7^6$  are linearly independent over  $\mathbb{Q}$ , so if

$$\mathcal{O}(\zeta_7; \tau) = \sum_{r=1}^3 (\zeta_7^r + \zeta_7^{7-r}) \sum_{d=0}^6 q^d S(d; q^7),$$

and each  $S(d; q)$  is a series in  $q$  with rational coefficients, then  $S(d; q) = R_{r,0}(d; q)$ . As an example, from the formula for  $R_0(q)$ , we have that

$$\begin{aligned}
R_{1,0}(0; q) &= -\frac{16J_0}{J_1^3 J_2^3 J_3^3 J_4 J_5 J_6 J_7^2} - \frac{J_0}{J_1^3 J_2^3 J_3^3 J_5^3 J_6^2} + \frac{J_0}{J_1^3 J_2^3 J_3^2 J_4^3 J_6^3} - \frac{5J_0}{J_1^3 J_2^2 J_3^3 J_4^3 J_5^3} \\
&\quad + \frac{20J_0}{J_1^3 J_2^3 J_3^2 J_4^2 J_5^3 J_7} - \frac{3qJ_0 J_4}{J_1^3 J_2^3 J_3^2 J_5^2 J_6^3 J_7^2} + \frac{4qJ_0}{J_1^3 J_2^3 J_4^3 J_5 J_6^3 J_7} + \frac{3q^2 J_0 J_3^2}{J_1^3 J_2^3 J_4^3 J_5^2 J_6^3 J_7^2}.
\end{aligned}$$

## 2. Preliminaries

We begin by defining the functions needed for the Maass forms. For  $u, v, z \in \mathbb{C}$ ,  $\tau \in \mathcal{H}$ , and  $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$  we have

$$\begin{aligned}
\vartheta(z; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(\pi i n^2 \tau + 2\pi i n \left(z + \frac{1}{2}\right)\right), \\
\mu(u, v; \tau) &= \frac{\exp(\pi i u)}{\vartheta(v; \tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \exp(\pi i n(n+1)\tau + 2\pi i n v)}{1 - \exp(2\pi i n \tau + 2\pi i u)}.
\end{aligned}$$

Next for  $u, z \in \mathbb{C}$ ,  $y = \text{Im}(\tau)$ , and  $a = \text{Im}(u)/\text{Im}(\tau)$  we define

$$\begin{aligned}
E(z) &= 2 \int_0^z \exp(-\pi w^2) dw, \\
R(u; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left( \text{sgn}(n) - E((n+a)\sqrt{2y}) \right) (-1)^{n-\frac{1}{2}} \exp(-\pi i n^2 \tau - 2\pi i n u).
\end{aligned}$$

For  $a, b \in \mathbb{R}$  we set

$$g_{a,b}(\tau) = \sum_{n \in \mathbb{Z} + a} n \exp(\pi i n^2 \tau + 2\pi i n b).$$

Finally for  $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$  we set

$$\tilde{\mu}(u, v; \tau) = \mu(u, v; \tau) + \frac{i}{2}R(u - v; \tau).$$

In his revolutionary PhD thesis [16], Zwegers studied these functions and gave their transformation formulas.

To work with  $\mathcal{O}(z; \tau)$  as a Maass form, we relate it to the functions studied by Zwegers, rather than following the development of Bringmann and Lovejoy. This is similar to Mao's work on the  $M_2$ -rank for overpartitions [14]. Relating functions to  $\mu(u, v; \tau)$  is also how Hickerson and Mortenson studied mock theta functions in [8]. While we should be able to derive our results from Bringmann and Lovejoy's work, we will see using the functions of Zwegers allows for less notation, keeps the transformation formulas simple, and allows us to easily deduce the orders at cusps. The initial step in relating  $\mathcal{O}(z; \tau)$  to  $\tilde{\mu}(u, v; \tau)$  is little more than rearranging fractions.

**Proposition 2.1.** *Let  $z = \exp(\pi i u)$ , then*

$$\mathcal{O}(z, \tau) = \frac{(1-z)}{(1+z)} + 2z \frac{(1-z)(-q; q)_{\infty} (q^2; q^2)_{\infty}^2}{(1+z)(q; q)_{\infty} [z^2; q^2]_{\infty}} - i2zq^{-1/4} \frac{(1-z)}{(1+z)} \mu(u, \tau; 2\tau).$$

**Proof.** This proposition is basically Lemma 2.1 of [13], but in different notation. By [11] we have

$$\begin{aligned} \mathcal{O}(z, \tau) &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \right) \\ &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \\ &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{z(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(z-z^{-1})(1-zq^n)} \\ &\quad - \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{z^{-1}(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(z-z^{-1})(1-z^{-1}q^n)} \\ &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{z(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(z-z^{-1})(1-zq^n)} \\ &\quad + \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2}}{(z-z^{-1})(1-zq^n)} \\ &= \frac{(-q; q)_{\infty} (1-z)}{(q; q)_{\infty} (1+z)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2} (1+2zq^n + z^2 q^{2n})}{(1-z^2 q^{2n})}. \end{aligned} \tag{2.1}$$

First by the  $r = 0$  and  $s = 1$  case of Theorem 2.1 of [4] we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1-z^2 q^{2n})} = \frac{(q^2; q^2)_{\infty}^2}{[z^2; q^2]_{\infty}}. \quad (2.2)$$

Next by Proposition 1.3 of [16] we have

$$\begin{aligned} \vartheta(\tau; 2\tau) &= -iq^{-\frac{1}{4}} (q, q, q^2; q)_{\infty} = -iq^{-\frac{1}{4}} \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}, \\ \vartheta(-\tau; 2\tau) &= iq^{-\frac{1}{4}} \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}. \end{aligned}$$

We find

$$\mu(u, -\tau; 2\tau) = \frac{z}{\vartheta(-\tau; 2\tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2}}{1-z^2 q^{2n}} = -izq^{\frac{1}{4}} \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2}}{1-z^2 q^{2n}}$$

and similarly

$$\mu(u, -\tau; 2\tau) = izq^{\frac{1}{4}} \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1-z^2 q^{2n}}.$$

However, from Proposition 1.4 of [16] we have that

$$\mu(u, -\tau; 2\tau) = -z^2 \mu(u, \tau; 2\tau) - izq^{\frac{1}{4}}.$$

We see (2.1) then becomes

$$\begin{aligned} \mathcal{O}(z; \tau) &= 2z \frac{(1-z)(-q; q)_{\infty} (q^2; q^2)_{\infty}^2}{(1+z)(q; q)_{\infty} [z^2; q^2]_{\infty}} \\ &\quad + \frac{(1-z)}{(1+z)} \left( iz^{-1} q^{-\frac{1}{4}} \mu(u, -\tau; 2\tau) - iz^{-1} q^{-\frac{1}{4}} \mu(u, \tau; 2\tau) \right) \\ &= \frac{(1-z)}{(1+z)} + 2z \frac{(1-z)(-q; q)_{\infty} (q^2; q^2)_{\infty}^2}{(1+z)(q; q)_{\infty} [z^2; q^2]_{\infty}} - i2zq^{-\frac{1}{4}} \frac{(1-z)}{(1+z)} \mu(u, \tau; 2\tau). \quad \square \end{aligned}$$

The term  $\frac{(1-z)}{(1+z)}$  may seem out of place, however we will find it is necessary to accommodate  $R(u-v; \tau)$ . We recall Theorem 1.16 of [16] states

$$R(a\tau - b) = -\exp(\pi i a^2 \tau - 2\pi i a(b + \tfrac{1}{2})) \int_{-\bar{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz,$$

for  $a \in (-\frac{1}{2}, \frac{1}{2})$ . Following the proof we find that for  $a = -\frac{1}{2}$  we instead have

$$R(-\frac{\tau}{2} - b) = \exp(\frac{\pi i \tau}{4} + \pi i b) - \exp(\frac{\pi i \tau}{4} + \pi i(b + \frac{1}{2})) \int_{-\bar{\tau}}^{i\infty} \frac{g_{0, b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz.$$

We then deduce the following.

**Corollary 2.2.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Then*

$$\begin{aligned} \mathcal{O}(\zeta_c^a; \tau) &= \frac{2\zeta_c^a(1-\zeta_c^a)}{(1+\zeta_c^a)} P(a, c; \tau) - \frac{i2\zeta_c^a(1-\zeta_c^a)}{(1+\zeta_c^a)} N(a, c; \tau) \\ &\quad + i\sqrt{2} \frac{(1-\zeta_c^a)}{(1+\zeta_c^a)} \int_{-\bar{\tau}}^{i\infty} \frac{g_{0, \frac{1}{2}-\frac{2a}{c}}(2z)}{\sqrt{-i(z+\tau)}} dz, \end{aligned}$$

where

$$N(a, c; \tau) = q^{-\frac{1}{4}} \tilde{\mu}(\frac{2a}{c}, \tau; 2\tau), \quad P(a, c; \tau) = \frac{(-q; q)_{\infty} (q^2; q^2)_{\infty}}{(q; q)_{\infty} [\zeta_c^{2a}; q^2]_{\infty}}.$$

We set

$$\begin{aligned} \mathcal{M}(a, c; \tau) &= \mathcal{O}(\zeta_c^a; \tau) - i\sqrt{2} \frac{(1-\zeta_c^a)}{(1+\zeta_c^a)} \int_{-\bar{\tau}}^{i\infty} \frac{g_{0, \frac{1}{2}-\frac{2a}{c}}(2z)}{\sqrt{-i(z+\tau)}} dz \\ &= \frac{2\zeta_c^a(1-\zeta_c^a)}{(1+\zeta_c^a)} P(a, c; \tau) - \frac{i2\zeta_c^a(1-\zeta_c^a)}{(1+\zeta_c^a)} N(a, c; \tau). \end{aligned}$$

We do not handle the case when  $z = -1$ , however Bringmann and Lovejoy were able to handle this case in [2].

Next we consider the generalized Lambert series appearing in the  $R_3(q)$ ,  $R_5(q)$ , and  $R_6(q)$  terms of the dissection for  $\mathcal{O}(\zeta_7; \tau)$ . Upon replacing  $q$  by  $q^7$  and multiplying by the appropriate power of  $q$ , these are

$$\begin{aligned} \frac{q^{10} (q^{98}; q^{98})_{\infty}}{(q^{49}; q^{49})_{\infty}^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+14}}, & \quad \frac{q^{12} (q^{98}; q^{98})_{\infty}}{(q^{49}; q^{49})_{\infty}^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+21}}, \\ \frac{q^6 (q^{98}; q^{98})_{\infty}}{(q^{49}; q^{49})_{\infty}^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+7}}. & \end{aligned}$$

We note

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+7k}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)} (1 + q^{49n+7k})}{1 - q^{98n+14k}}$$



$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{98n+14k}} + q^{7k} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)+49n}}{1 - q^{98n+14k}} \\
&= \frac{(q^{98}; q^{98})_{\infty}^2}{[q^{14k}; q^{98}]_{\infty}} + q^{7k} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)+49n}}{1 - q^{98n+14k}},
\end{aligned}$$

but

$$\mu(14k\tau, 49\tau; 98\tau) = \frac{iq^{7k+\frac{49}{4}}(q^{98}; q^{98})_{\infty}}{(q^{49}; q^{49})_{\infty}^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+14k}},$$

and so

$$\begin{aligned}
\frac{q^{7k-k^2}(q^{98}; q^{98})_{\infty}}{(q^{49}; q^{49})_{\infty}^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(n+1)}}{1 - q^{49n+7k}} &= \frac{q^{7k-k^2}(q^{98}; q^{98})_{\infty}^3}{(q^{49}; q^{49})_{\infty}^2 [q^{14k}; q^{98}]_{\infty}} \\
&\quad - iq^{-k^2+7k-\frac{49}{4}} \mu(14k\tau, 49\tau; 98\tau).
\end{aligned}$$

We then define the function  $N_7(k; \tau)$ , for  $7 \nmid k$ , by

$$N_7(k; \tau) = q^{-k^2+7k-\frac{49}{4}} \tilde{\mu}(14k\tau, 49\tau; 98\tau).$$

Working with  $\tilde{\mu}(u, v; \tau)$  is advantageous in that the transformation under the action of the modular group  $\mathrm{SL}_2(\mathbb{Z})$  is known and quite elegant. To work with the overpartition rank in terms of  $\tilde{\mu}$  we have introduced an extra product term, it is also possible to not introduce this product and instead work with the function  $f$  from chapter three of [16]. We reprove that  $\mathcal{O}(z; q)$  is the holomorphic part of a harmonic weak Maass form when  $z \neq \pm 1$  is a root of unity and determine the order of the holomorphic part at cusps. By keeping track of the multipliers of our functions, we will be able work on a fairly large subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ , which has relatively few cusps.

We recall  $\mathrm{SL}_2(\mathbb{Z})$  is the group of  $2 \times 2$  integer matrices with determinant 1. The principal congruence subgroup of level  $N$  is

$$\Gamma(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \gamma \equiv \beta \equiv 0 \pmod{N} \right\}.$$

A subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  is called a congruence subgroup if  $\Gamma \supseteq \Gamma(N)$  for some  $N$ . Two congruence subgroups we will use are

$$\begin{aligned}
\Gamma_0(N) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 0 \pmod{N} \right\}, \\
\Gamma_1(N) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \gamma \equiv 0 \pmod{N} \right\}.
\end{aligned}$$

We recall  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathcal{H}$  via Mobius transformations, that is  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$ .

Additionally we let  $(A : \tau) = \gamma\tau + \delta$ .

We recall a weakly holomorphic modular form of integral weight  $k$  on a congruence subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  is a holomorphic function on  $\mathcal{H}$  such that

1. if  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ , then  $f(A\tau) = (\gamma\tau + \delta)^k f(\tau)$ ,
2. if  $B \in \mathrm{SL}_2(\mathbb{Z})$  then  $(B : \tau)^{-k} F(Bz)$  has an expansion of the form  $\sum_{n=n_0}^{\infty} a_n \exp(2\pi i n z / N)$ .

When  $k$  is a half integer, we require  $\Gamma \subset \Gamma_0(4)$  and replace (1) with  $f(A\tau) = \left(\frac{\gamma}{\delta}\right)^{2k} \epsilon(\delta)^{-2k} (\gamma\tau + \delta)^k f(\tau)$ . Here  $\left(\frac{m}{n}\right)$  is the Jacobi symbol extended to all integers  $n$  by

$$\begin{aligned} \left(\frac{0}{\pm 1}\right) &= 1, \\ \left(\frac{m}{n}\right) &= \begin{cases} \left(\frac{m}{|n|}\right) & \text{if } m > 0, \text{ or, } m < 0 \text{ and } n > 0 \\ -\left(\frac{m}{|n|}\right) & \text{if } m < 0 \text{ and } n < 0 \end{cases}, \end{aligned}$$

and  $\epsilon(\delta)$  is 1 when  $\delta \equiv 1 \pmod{4}$  and is  $i$  otherwise.

A harmonic weak Maass form satisfies the transformation law in (1), but the condition of holomorphic is replaced with being smooth and annihilated by the weight  $k$  hyperbolic Laplacian operator,

$$\Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

where  $\tau = x + iy$ , and condition (2) is replaced with  $(B : \tau)^{-k} F(Bz)$  having at most linear exponential growth as  $z \rightarrow i\infty$ .

If  $f$  is a harmonic weak Maass form of weight  $2 - k$  on  $\Gamma_1(N)$ , then  $f$  can be written as  $f = f^+ + f^-$ , where  $f^+$  and  $f^-$  have expansions of the form

$$f^+(\tau) = \sum_{n=n_0}^{\infty} a(n)q^n, \quad f^-(\tau) = \sum_{n=1}^{\infty} b(n)\Gamma(k-1, 4\pi ny)q^{-n}.$$

Here  $\Gamma$  is the incomplete Gamma function given by  $\Gamma(y, x) = \int_x^{\infty} e^{-t} t^{y-1} dt$ . We call  $f^+$  the holomorphic part and  $f^-$  the non-holomorphic part. The non-holomorphic part is often written instead as an integral of the form

$$f^-(\tau) = \int_{-\bar{\tau}}^{\infty} g(z)(-i(z + \tau))^{k-2} dz.$$

The rest of the article is organized as follows. In Section 3 we give the transformation formulas for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $N(a, c; \tau)$  and prove  $N(a, c; \tau)$  is a harmonic weak Maass form. In Section 4 we recognize  $P(a, c; \tau)$  as a modular form and give its transformation formulas. In Section 5 we give the transformation formulas for  $N_7(k; \tau)$  and prove it is a harmonic weak Maass form. Additionally we recognize the products in the dissection of  $\mathcal{O}(\zeta_7; \tau)$  as modular forms. In Section 6 we determine the orders at cusps for our various functions. In Section 7 we demonstrate that Theorem 1.1 reduces to verifying a certain modular function is zero, which will follow by the valence formula.

### 3. Transformations formulas for $N(a, c; \tau)$

For a matrix  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we have,  $\nu(A)$ , the  $\eta$ -multiplier defined by

$$\eta(A\tau) = \nu(A) \sqrt{\gamma\tau + \delta} \eta(\tau),$$

where  $\eta(\tau)$  is Dedekind's eta-function,

$$\eta(\tau) = q^{\frac{1}{24}} (q; q)_{\infty}.$$

A convenient form for the  $\eta$ -multiplier when  $\gamma \neq 0$ , which can be found in [9], is

$$\nu(A) = \begin{cases} \left(\frac{\delta}{|\gamma|}\right) \exp\left(\frac{\pi i}{12}((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) - 3\gamma)\right) & \text{if } \gamma \equiv 1 \pmod{2}, \\ \left(\frac{\gamma}{\delta}\right) \exp\left(\frac{\pi i}{12}((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) + 3\delta - 3 - 3\gamma\delta)\right) & \text{if } \delta \equiv 1 \pmod{2}. \end{cases} \quad (3.1)$$

For an integer  $m$  we let

$${}_m A = \begin{pmatrix} \alpha & m\beta \\ \gamma/m & \delta \end{pmatrix}, \quad {}_m A = \begin{pmatrix} m\alpha & \beta \\ \gamma & \delta/m \end{pmatrix}.$$

Our transformation formulas for  $N(a, c; \tau)$  are easily deduced by the transformations of  $\tilde{\mu}(u, v; \tau)$ . The following essential properties are from Theorem 1.11 of [16]. If  $k, l, m, n$  are integers then

$$\begin{aligned} & \tilde{\mu}(u + k\tau + l, v + m\tau + n; \tau) \\ &= (-1)^{k+l+m+n} \exp(\pi i \tau (k - m)^2 + 2\pi i (k - m)(u - v)) \tilde{\mu}(u, v; \tau). \end{aligned} \quad (3.2)$$

If  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  then

$$\tilde{\mu}\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; A\tau\right) = \nu(A)^{-3} \exp\left(-\frac{\pi i \gamma (u - v)^2}{\gamma\tau + \delta}\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}(u, v; \tau). \quad (3.3)$$

**Proposition 3.1.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . If  $\gamma$  is even then

$$N(a, c; A\tau) = \nu({}^2A)^{-3} (-1)^{\beta + \frac{\alpha-1}{2}} \exp \left( -\pi i \left( \frac{2a^2\gamma\delta}{c^2} + \frac{2a}{c} + \frac{\alpha\beta}{2} - \frac{2a\delta}{c} \right) \right) \\ \exp \left( -\pi i \tau \left( \frac{2a^2\gamma^2}{c^2} - \frac{2a\gamma}{c} + \frac{1}{2} \right) \right) \sqrt{\gamma\tau + \delta} \tilde{\mu} \left( \frac{2a\delta}{c} + \frac{2a\gamma\tau}{c}, \tau; 2\tau \right).$$

If  $\gamma$  odd and  $\delta$  even

$$N(a, c; A\tau) = 2^{-\frac{1}{2}} \nu({}_2A)^{-3} \exp \left( -\pi i \left( \frac{2a^2\gamma\delta}{c^2} - \frac{2a\beta\gamma}{c} + \frac{\alpha\beta}{2} \right) \right) \\ \exp \left( -\pi i \tau \left( \frac{2a^2\gamma^2}{c^2} - \frac{2a\alpha\gamma}{c} + \frac{\alpha^2}{2} \right) \right) \sqrt{\gamma\tau + \delta} \tilde{\mu} \left( \frac{a\delta}{c} + \frac{a\gamma\tau}{c}, \frac{\alpha\tau}{2} + \frac{\beta}{2}; \frac{\tau}{2} \right).$$

**Proof.** The proofs are lengthy, but straightforward calculations from applying the transformation formula and identities for  $\tilde{\mu}(u, v; \tau)$ . First for  $\gamma$  even we have  $A \in \Gamma_0(2)$ , so  ${}^2A \in \mathrm{SL}_2(\mathbb{Z})$ , and

$$2A\tau = \frac{2\alpha\tau + 2\beta}{\gamma\tau + \delta} = \frac{\alpha(2\tau) + 2\beta}{\frac{\gamma}{2}(2\tau) + \delta} = {}^2A(2\tau).$$

We then apply (3.3) with  $A \mapsto {}^2A$ ,  $\tau \mapsto 2\tau$ ,  $u = \frac{2a(\gamma\tau + \delta)}{c}$ ,  $v = (\alpha\tau + \beta)(\gamma\tau + \delta)$  to get

$$N(a, c; A\tau) \\ = \exp \left( -\frac{\pi i A\tau}{2} \right) \tilde{\mu} \left( \frac{2a}{c}, A\tau; \frac{\alpha(2\tau) + 2\beta}{\frac{\gamma}{2}(2\tau) + \delta} \right) \\ = \nu({}^2A)^{-3} \sqrt{\gamma\tau + \delta} \exp \left( -\frac{\pi i A\tau}{2} \right) \exp \left( \frac{-\pi i \gamma}{2(\gamma\tau + \delta)} \left( \frac{2a(\gamma\tau + \delta)}{c} - (\alpha\tau + \beta) \right)^2 \right) \\ \tilde{\mu} \left( \frac{2a(\gamma\tau + \delta)}{c}, \alpha\tau + \beta; 2\tau \right) \\ = \nu({}^2A)^{-3} \sqrt{\gamma\tau + \delta} \exp \left( -\frac{\pi i A\tau(1 + \gamma(\alpha\tau + \beta))}{2} \right) \exp \left( -\pi i \tau \left( \frac{2a^2\gamma^2}{c^2} - \frac{2a\alpha\gamma}{c} \right) \right) \\ \exp \left( -\pi i \left( \frac{2a^2\gamma\delta}{c^2} - \frac{2a\beta\gamma}{c} \right) \right) \tilde{\mu} \left( \frac{2a(\gamma\tau + \delta)}{c}, \alpha\tau + \beta; 2\tau \right) \\ = \nu({}^2A)^{-3} \sqrt{\gamma\tau + \delta} \exp \left( -\pi i \tau \left( \frac{2a^2\gamma^2}{c^2} - \frac{2a\alpha\gamma}{c} + \frac{\alpha^2}{2} \right) \right) \\ \exp \left( -\pi i \left( \frac{2a^2\gamma\delta}{c^2} - \frac{2a\beta\gamma}{c} + \frac{\alpha\beta}{2} \right) \right) \tilde{\mu} \left( \frac{2a(\gamma\tau + \delta)}{c}, \alpha\tau + \beta; 2\tau \right).$$

We note that  $\alpha$  is odd and so we apply (3.2) with  $\tau \mapsto 2\tau$ ,  $u = \frac{2a(\gamma\tau+\delta)}{c}$ ,  $k = l = 0$ ,  $v = \tau$ ,  $m = \frac{\alpha-1}{2}$ ,  $n = \beta$ , and simplify to obtain

$$N(a, c; A\tau) = \nu({}^2A)^{-3}(-1)^{\beta+\frac{\alpha-1}{2}} \exp\left(-\pi i \left(\frac{2a^2\gamma\delta}{c^2} + \frac{2a}{c} + \frac{\alpha\beta}{2} - \frac{2a\delta}{c}\right)\right) \\ \exp\left(-\pi i \tau \left(\frac{2a^2\gamma^2}{c^2} - \frac{2a\gamma}{c} + \frac{1}{2}\right)\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\delta}{c} + \frac{2a\gamma\tau}{c}, \tau; 2\tau\right).$$

When  $\gamma$  is odd and  $\delta$  is even, we instead have  ${}_2A \in \mathrm{SL}_2(\mathbb{Z})$  and  $2A\tau = {}_2A(\frac{\tau}{2})$ . We then apply (3.3) and simplify to obtain the result. We omit the details.  $\square$

We use the first case to determine on which subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  that  $N(a, c; \tau)$  is a Maass form and we use the second case to determine orders at cusps.

**Corollary 3.2.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2) \cap \Gamma_0(c)$ , then

$$N(a, c; A\tau) = \nu({}^2A)^{-3}(-1)^{\beta+\frac{\alpha-1}{2}+\frac{\alpha\gamma}{c}} \exp\left(-\pi i \left(\frac{-2a^2\gamma\delta}{c^2} + \frac{2a(1-\delta)}{c} + \frac{\alpha\beta}{2}\right)\right) \\ \sqrt{\gamma\tau + \delta} N(a\delta, c; \tau).$$

**Proof.** This follows from Proposition 3.1 and (3.2) applied with  $l = \frac{\alpha\gamma}{c}$ .  $\square$

**Corollary 3.3.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2) \cap \Gamma_0(c^2) \cap \Gamma_1(c)$ , then

$$N(a, c; A\tau) = \nu({}^2A)^{-3} \sqrt{\gamma\tau + \delta} (-1)^{\beta+\frac{\alpha-1}{2}} i^{-\alpha\beta} N(a, c; \tau).$$

**Proof.** Using (3.2) we deduce that  $N(a + c, c; \tau) = N(a, c; \tau)$ , so with  $\delta \equiv 1 \pmod{c}$  we have  $N(a\delta, c; \tau) = N(a, c; \tau)$ .  $\square$

**Corollary 3.4.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(4) \cap \Gamma_1(c) \cap \Gamma_0(16) \cap \Gamma_0(c^2)$ , then

$$N(a, c; A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} N(a, c; \tau).$$

In particular  $N(a, c; \tau)$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  on  $\Gamma_1(4) \cap \Gamma_1(c) \cap \Gamma_0(16) \cap \Gamma_0(c^2)$ .

**Proof.** Noting that  $\alpha \equiv 1 \pmod{4}$ , for the transformation we must verify that

$$\left(\frac{\gamma}{\delta}\right) = \nu(^2A)^{-3} i^\beta. \quad (3.4)$$

Since  $A \in \Gamma_0(16)$ , we have  $^2A \in \Gamma_0(8)$ , and so applying (3.1) yields

$$\begin{aligned} \nu(^2A)^{-3} i^\beta &= \left(\frac{\gamma/2}{\delta}\right) \exp\left(\frac{-\pi i}{4} \left((\alpha + \delta)\frac{\gamma}{2} - 2\beta\delta\left(\frac{\gamma^2}{4} - 1\right) + 3\delta - 3 - \frac{3\gamma\delta}{2} - 2\beta\right)\right) \\ &= \left(\frac{\gamma}{\delta}\right) \left(\frac{2}{\delta}\right) \exp\left(\frac{-\pi i}{4} (2\beta(\delta - 1) + 3(\delta - 1))\right) \\ &= \left(\frac{\gamma}{\delta}\right) \left(\frac{2}{\delta}\right) \exp\left(-3\pi i \frac{(\delta - 1)}{4}\right) \\ &= \left(\frac{\gamma}{\delta}\right) \left(\frac{2}{\delta}\right) (-1)^{\frac{\delta-1}{4}} \\ &= \left(\frac{\gamma}{\delta}\right) (-1)^{\frac{\delta^2-1}{8}} (-1)^{\frac{\delta-1}{4}} \\ &= \left(\frac{\gamma}{\delta}\right) (-1)^{\frac{(\delta-1)(\delta+3)}{8}}. \end{aligned}$$

But  $\delta \equiv 1, 5 \pmod{8}$  so that  $\frac{(\delta-1)(\delta+3)}{8}$  is even. Thus (3.4) holds.

The growth condition at the cusps is satisfied by  $N(a, c; \tau)$ , as if we take  $\alpha/\gamma$  with  $\gcd(\alpha, \gamma) = 1$ , then we can take a matrix  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and apply the transformations in Proposition 3.1. Clearly we can apply the first case when  $\gamma$  is even. We note when  $\gamma$  is odd that we may choose  $\delta$  odd, as if it is not, then we can replace  $\beta$  by  $\beta + \alpha$  and  $\delta$  by  $\delta + \gamma$ .

Writing the weight  $\frac{1}{2}$  hyperbolic Laplacian as

$$\Delta_{\frac{1}{2}} = -4y^{\frac{3}{2}} \frac{\partial}{\partial \tau} \sqrt{y} \frac{\partial}{\partial \bar{\tau}},$$

we see  $\Delta_{\frac{1}{2}}$  annihilates  $q^{-\frac{1}{4}} \mu(\frac{2a}{c}, \tau; 2\tau)$  as it is holomorphic in  $\tau$ . By Lemma 1.8 of [16] we have

$$\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{1}{4}} R(\frac{2a}{c} - \tau; 2\tau) = -i \exp(-\frac{\pi i \tau}{2} - \pi y) A(\bar{\tau}) = -i \exp(-\frac{\pi i \bar{\tau}}{2}) A(\bar{\tau}),$$

where  $A(\bar{\tau})$  is a series giving a function holomorphic in  $\bar{\tau}$ . Thus  $\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{1}{4}} R(\frac{2a}{c} - \tau; 2\tau)$  is anti-holomorphic, and so  $\Delta_{\frac{1}{2}} q^{-\frac{1}{4}} R(\frac{2a}{c} - \tau; 2\tau) = 0$ . Therefore  $\Delta_{\frac{1}{2}}$  annihilates  $N(a, c; \tau)$ .  $\square$

**Proposition 3.5.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ , then the non-holomorphic part of  $N(a, c; \tau)$  is given by

$$\frac{i\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n (\zeta_c^{-2an} - \zeta_c^{2an}) \Gamma(\tfrac{1}{2}; 4\pi y n^2) q^{-n^2}.$$

**Proof.** We recall

$$\begin{aligned} N(a, c; \tau) &= q^{-\frac{1}{4}} \tilde{\mu}(\tfrac{2a}{c}, t; 2\tau) \\ &= q^{-\frac{1}{4}} \mu(\tfrac{2a}{c}, t; 2\tau) + \frac{iq^{-\frac{1}{4}}}{2} R(\tfrac{2a}{c} - \tau; 2\tau) \\ &= q^{-\frac{1}{4}} \mu(\tfrac{2a}{c}, t; 2\tau) + \frac{i\zeta_c^{-a}}{2} + \frac{\zeta_c^{-a}}{\sqrt{2}} \int_{-\bar{\tau}}^{i\infty} \frac{g_{0, \frac{1}{2} - \frac{2a}{c}}(2z)}{\sqrt{-i(z + \tau)}} dz. \end{aligned}$$

Thus the non-holomorphic part is

$$\begin{aligned} &\frac{\zeta_c^{-a}}{\sqrt{2}} \int_{-\bar{\tau}}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} n \exp(2\pi i z n^2 + 2\pi i n(\tfrac{1}{2} - \tfrac{2a}{c}))}{\sqrt{-i(z + \tau)}} dz \\ &= \frac{\zeta_c^{-a}}{\sqrt{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n n \zeta_c^{-2an} \int_{-\bar{\tau}}^{i\infty} \frac{e^{2\pi i z n^2}}{\sqrt{-i(z + \tau)}} dz. \end{aligned}$$

We exchange the order of the integral and series and use the substitution  $z = \frac{-t}{2\pi i n^2} - \tau$  so that

$$\begin{aligned} &\frac{\zeta_c^{-a}}{\sqrt{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n n \zeta_c^{-2an} \int_{-\bar{\tau}}^{i\infty} \frac{e^{2\pi i z n^2}}{\sqrt{-i(z + \tau)}} dz \\ &= -\frac{\zeta_c^{-a}}{\sqrt{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n n \zeta_c^{-2an}}{2\pi i n^2} \int_{4\pi n^2 y}^{\infty} e^{-t-2\pi i n^2 \tau} \sqrt{\frac{2\pi n^2}{t}} dt \\ &= \frac{i\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n \operatorname{sgn}(n) \zeta_c^{-2an} q^{-n^2} \Gamma(\tfrac{1}{2}; 4\pi n^2 y) \\ &= \frac{i\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n (\zeta_c^{-2an} - \zeta_c^{2an}) q^{-n^2} \Gamma(\tfrac{1}{2}; 4\pi n^2 y). \quad \square \end{aligned}$$

#### 4. Transformations for $P(a, c; \tau)$

The function  $P(a, c; \tau)$  can be written in terms of known modular forms. In particular. As in [10] we have the Klein forms given by

$$t_{(a_1, a_2)} = -q^{B_2(a_1) - \frac{1}{12}} \exp(\pi i a_2(a_1 - 1)) \frac{[\zeta; q]_\infty}{(q; q)_\infty^2},$$

where  $B_2(x) = x^2 - x + \frac{1}{6}$ ,  $a_1$  and  $a_2$  are rational, and  $\zeta = \exp(2\pi i(a_1\tau + a_2))$ . For

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

$$t_{(a_1, a_2)}(A\tau) = (\gamma\tau + \delta)^{-1} t_{(a_1, a_2) \cdot A}(\tau),$$

and for integers  $b_1$  and  $b_2$

$$t_{(a_1+b_1, a_2+b_2)}(\tau) = (-1)^{b_1 b_2 + b_1 + b_2} \exp(-\pi i(b_1 a_2 - b_2 a_1)) t_{(a_1, a_2)}(\tau). \quad (4.1)$$

Additionally,  $t_{(a_1, a_2)}(\tau)$  is holomorphic on  $\mathcal{H}$  and has no zeros nor poles on  $\mathcal{H}$ . Thus  $t_{(a_1, a_2)}(\tau)$  is a modular form of weight  $-1$  on some subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ .

We note that

$$P(a, c; \tau) = \frac{-\zeta_c^a \eta(2\tau)}{\eta(\tau)^2 t_{0, \frac{2a}{c}}(2\tau)}.$$

**Proposition 4.1.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2) \cap \Gamma_0(c^2) \cap \Gamma_1(c)$ , then

$$P(a, c; A\tau) = \frac{\nu(2A)}{\nu(A)^2} \sqrt{\gamma\tau + \delta} P(a, c; \tau) = \nu(2A)^{-2} (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta} \sqrt{\gamma\tau + \delta} P(a, c; \tau).$$

**Proof.** We have

$$\begin{aligned} P(a, c; A\tau) &= \frac{-\zeta_c^a \nu(2A)}{\nu(A)^2} \sqrt{\gamma\tau + \delta} \frac{\eta(2\tau)}{\eta(\tau)^2 t_{\frac{a\gamma}{c}, \frac{2a\delta}{c}}(2\tau)} \\ &= \frac{-\zeta_c^a \nu(2A)}{\nu(A)^2} \sqrt{\gamma\tau + \delta} \frac{\eta(2\tau)}{\eta(\tau)^2 t_{0, \frac{2a}{c}}(2\tau)} \\ &= \frac{\nu(2A)}{\nu(A)^2} \sqrt{\gamma\tau + \delta} P(a, c; \tau), \end{aligned}$$

where in the second line we have applied (4.1) with  $a_1 = 0$ ,  $a_2 = \frac{2a}{c}$ ,  $b_1 = \frac{a\gamma}{c}$ , and  $b_2 = \frac{2a(\delta-1)}{c}$ .

The second identity of the Proposition follows by a lengthy calculation with (3.1) to verify that on  $\Gamma_0(2)$  we indeed have  $\frac{\nu(2A)}{\nu(A)^2} = \nu(2A)^{-3} (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta}$ . We omit these calculations, as they are rather long but straightforward.  $\square$



**Proposition 4.2.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(4) \cap \Gamma_1(c) \cap \Gamma_0(16) \cap \Gamma_0(c^2)$ , then

$$P(a, c; A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} P(a, c; \tau).$$

In particular  $P(a, c; \tau)$  is a weight  $\frac{1}{2}$  modular form on  $\Gamma_1(4) \cap \Gamma_1(c) \cap \Gamma_0(16) \cap \Gamma_0(c^2)$ .

**Proof.** As in the proof of [Corollary 3.4](#) we have  $\nu(^2A)^{-3}i^\beta = (\frac{\gamma}{\delta})$ , so the identity holds.  $\square$

**Corollary 4.3.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ , then  $\mathcal{M}(a, c; \tau)$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  on  $\Gamma_1(4) \cap \Gamma_1(c) \cap \Gamma_0(16) \cap \Gamma_0(c^2)$ .

**Proposition 4.4.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Then the non-holomorphic part of  $\mathcal{M}(a, c; \tau)$  is given by

$$\frac{1}{\sqrt{\pi}} \frac{(1 - \zeta_c^a)}{(1 + \zeta_c^a)} \sum_{n=1}^{\infty} (-1)^n (\zeta_c^{-2an} - \zeta_c^{2an}) \Gamma(\tfrac{1}{2}; 4\pi y n^2) q^{-n^2}.$$

## 5. Transformations for $N_7(k; \tau)$ and additional products

We work with a more general function than  $N_7(k; \tau)$ . For integers  $a$  and  $c$ ,  $c > 0$ , and  $c \nmid a$ , we define

$$M(a, c; \tau) = q^{-\frac{1}{2}(\frac{a}{c} - \frac{1}{2})^2} \tilde{\mu}\left(\frac{a\tau}{c}, \frac{\tau}{2}; \tau\right).$$

We see that  $N_7(k; \tau) = M(k, 7; 98\tau)$ .

**Proposition 5.1.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ , then

$$M(a, c; A\tau) = \nu(A)^{-3} \exp\left(-\pi i \alpha \beta \left(\frac{a}{c} - \frac{1}{2}\right)^2\right) \exp\left(-\pi i \alpha^2 \tau \left(\frac{a}{c} - \frac{1}{2}\right)^2\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{a\alpha\tau + a\beta}{c}, \frac{\alpha\tau + \beta}{2}; \tau\right).$$

**Proof.** As with the transformations for  $N(a, c; \tau)$ , this follows from [\(3.3\)](#) and elementary rearrangements.  $\square$

**Proposition 5.2.** Suppose  $k$  is an integer and  $7 \nmid k$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(14) \cap \Gamma_0(98)$ , then

$$N_7(k; \tau) = \nu(^2A)^{-3} (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta} \sqrt{\gamma\tau + \delta} N_7(k; \tau).$$

**Proof.** By Proposition 5.1 we have

$$\begin{aligned} N_7(k; A\tau) &= M(k, 7; {}^{98}A 98\tau) \\ &= \nu({}^{98}A)^{-3} i^{-\alpha\beta} \exp\left(-\pi i 98\alpha^2\tau \left(\frac{k}{7} - \frac{1}{2}\right)^2\right) \sqrt{\gamma\tau + \delta} \\ &\quad \tilde{\mu}(14k\alpha\tau + 14k\beta, 49\alpha\tau + 49\beta; 98\tau) \\ &= \nu({}^{98}A)^{-3} (-1)^\beta \sqrt{\gamma\tau + \delta} \exp\left(-\pi i 98\alpha^2\tau \left(\frac{k}{7} - \frac{1}{2}\right)^2\right) \tilde{\mu}(14k\alpha\tau, 49\alpha\tau; 98\tau). \end{aligned}$$

But  $\alpha \equiv 1 \pmod{14}$ , so applying (3.2) with  $\tau \mapsto 98\tau$ ,  $u = 14k\tau$ ,  $v = 49\tau$ ,  $k = \frac{k(\alpha-1)}{7}$ ,  $l = 0$ ,  $m = \frac{\alpha-1}{2}$ ,  $n = 0$ , and simplifying yields

$$\begin{aligned} &\tilde{\mu}(14k\alpha\tau, 49\alpha\tau; 98\tau) \\ &= \tilde{\mu}\left(14k\tau + \frac{k(\alpha-1)}{7}98\tau, 49\tau + \frac{(\alpha-1)}{2}98\tau; 98\tau\right) \\ &= (-1)^{\frac{\alpha-1}{2}} \exp\left(98\pi i \tau \alpha^2 \left(\frac{k}{7} - \frac{1}{2}\right)^2 - 98\tau \left(\frac{k}{7} - \frac{1}{2}\right)^2\right) \tilde{\mu}(14k\tau, 49\tau; 98\tau). \end{aligned}$$

Thus

$$N_7(k; \tau) = \nu({}^{98}A)^{-3} (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta} \sqrt{\gamma\tau + \delta} N_7(k; \tau).$$

However, by Theorem 1.64 of [15] we find that  $\frac{\eta(98\tau)^3}{\eta(2\tau)^3}$  is a modular function on  $\Gamma_0(98)$ , so that  $\nu({}^{98}A)^3 = \nu(^2A)^3$  for  $A \in \Gamma_0(98)$ .  $\square$

**Corollary 5.3.** Suppose  $k$  is in integer and  $7 \nmid k$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(28) \cap \Gamma_0(784)$ , then

$$N_7(k; A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} N_7(k; \tau).$$

In particular,  $N_7(k; \tau)$  is a harmonic weak Maass form of weight  $\frac{1}{2}$  on  $\Gamma_1(28) \cap \Gamma_0(784)$ .

**Proof.** As in the proof of Corollary 3.4, on  $\Gamma_1(28) \cap \Gamma_0(784)$  we have  $\nu(^2A)^{-3} (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta} = \left(\frac{\gamma}{\delta}\right)$ , so that the transformation is correct.

One can again use Lemma 1.8 of [16] to see that the weight  $\frac{1}{2}$  hyperbolic Laplacian annihilates  $N_7(k; \tau)$ . For the condition at the cusps, one examines the transformations of

$M(a, c; 98\tau)$  under  $\mathrm{SL}_2(\mathbb{Z})$ , which we do in Proposition 6.4 of the next section to compute orders at cusps.  $\square$

**Proposition 5.4.** *For  $1 \leq k \leq 6$ , the non-holomorphic part of  $N_7(k; \tau)$  is given by*

$$\begin{aligned} & \frac{i}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n q^{-(7n+k)^2} \Gamma\left(\frac{1}{2}; 4\pi y(7n+k)^2\right) \\ & - \frac{i}{2\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n q^{-(7n-k)^2} \Gamma\left(\frac{1}{2}; 4\pi y(7n-k)^2\right) \end{aligned}$$

**Proof.** We note that

$$\begin{aligned} N_7(k; \tau) &= q^{-(k-7/2)^2} \tilde{\mu}(14k\tau, 49\tau; 98\tau) \\ &= q^{-(k-7/2)^2} \mu(14k\tau, 49\tau; 98\tau) + \frac{iq^{-(k-7/2)^2}}{2} R\left(\left(\frac{k}{7} - \frac{1}{2}\right) 98\tau; 98\tau\right) \\ &= q^{-(k-7/2)^2} \mu(14k\tau, 49\tau; 98\tau) + \frac{\zeta_{14}^{-k}}{2} \int_{-98\tau}^{i\infty} \frac{g_{\frac{k}{7}, \frac{1}{2}}(z)}{\sqrt{-i(z+98\tau)}} dz \\ &= q^{-(k-7/2)^2} \mu(14k\tau, 49\tau; 98\tau) + \frac{7\zeta_{14}^{-k}}{\sqrt{2}} \int_{-\tau}^{i\infty} \frac{g_{\frac{k}{7}, \frac{1}{2}}(98z)}{\sqrt{-i(z+\tau)}} dz, \end{aligned}$$

where the second to last line follows from Theorem 1.16 of [16]. Thus the non-holomorphic part of  $N_7(k; \tau)$  is given by

$$\frac{7\zeta_{14}^{-k}}{\sqrt{2}} \int_{-\tau}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} \left(n + \frac{k}{7}\right) \exp\left(98\pi iz\left(n + \frac{k}{7}\right)^2 + \pi i\left(n + \frac{k}{7}\right)\right)}{\sqrt{-i(z+\tau)}} dz.$$

We exchange the integral with the series, use the substitution  $z = \frac{-t}{98\pi i(n + \frac{k}{7})^2} - \tau$ , and simplify to find that

$$\begin{aligned} & \frac{7\zeta_{14}^{-k}}{\sqrt{2}} \int_{-\tau}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} \left(n + \frac{k}{7}\right) \exp\left(98\pi iz\left(n + \frac{k}{7}\right)^2 + \pi i\left(n + \frac{k}{7}\right)\right)}{\sqrt{-i(z+\tau)}} dz \\ &= \frac{i}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sgn}(7n+k) q^{-(7n+k)^2} \Gamma\left(\frac{1}{2}; 4\pi y(7n+k)^2\right) \\ &= \frac{i}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n q^{-(7n+k)^2} \Gamma\left(\frac{1}{2}; 4\pi y(7n+k)^2\right) \\ & - \frac{i}{2\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n q^{-(7n-k)^2} \Gamma\left(\frac{1}{2}; 4\pi y(7n-k)^2\right). \quad \square \end{aligned}$$

We will write the additional products in terms of the function  $f_{N,\rho}$ , defined for integers  $N$  and  $\rho$  with  $N \geq 1$  and  $N \nmid \rho$  by

$$f_{N,\rho}(\tau) = q^{\frac{(N-2\rho)^2}{8N}} (q^\rho, q^{N-\rho}, q^N; q^N)_\infty.$$

The transformations of  $f_{N,\rho}(\tau)$  were studied by Biagioli in [1]. Lemma 2.1 of [1] states that for  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$  we have

$$f_{N,\rho}(A\tau) = \nu({}^N A)^3 (-1)^{\rho\beta + \lfloor \rho\alpha/N \rfloor + \lfloor \rho/N \rfloor} \exp\left(\frac{\pi i \alpha \beta \rho^2}{N}\right) \sqrt{\gamma\tau + \delta} f_{N,\rho\alpha}(\tau),$$

where  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ . We note that  $f_{N,\rho}(\tau) = f_{N,N+\rho}(\tau) = f_{N,-\rho}(\tau)$ .

**Proposition 5.5.** *If  $F(\tau) = \eta(98\tau)^{r_0} \prod_{k=1}^7 f_{98,7k}(\tau)^{r_k}$ , and  $A \in \Gamma_1(14) \cap \Gamma_0(98)$  then*

$$F(A\tau) = \nu({}^{98} A)^{r_0+3R} (-1)^{(\frac{\alpha-1}{2}+\beta)S} i^{\alpha\beta S} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} F(\tau),$$

where  $R = r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7$  and  $S = r_1 + r_3 + r_5 + r_7$ .

**Proof.** This follows from the transformation formula for  $f_{N,\rho}$  after a few simplifications. We note that  $7k\alpha \equiv 7k \pmod{98}$ , and by writing  $\alpha = 1 + 14\frac{(\alpha-1)}{14}$  we see that

$$\begin{aligned} \frac{7k\alpha}{98} &= \frac{7k}{98} + \frac{k(\alpha-1)}{14}, \\ \left\lfloor \frac{7k\alpha}{98} \right\rfloor &= \frac{k(\alpha-1)}{14} \equiv \frac{k(\alpha-1)}{2} \pmod{2}. \end{aligned}$$

Thus

$$\begin{aligned} F(A\tau) &= \nu({}^{98} A)^{r_0+3(r_1+r_2+r_3+r_4+r_5+r_6+r_7)} (-1)^{(\beta+\frac{\alpha-1}{2})\sum_{k=1}^7 kr_k} \exp\left(\frac{\pi i \alpha \beta}{2} \sum_{k=1}^7 k^2 r_k\right) \\ &= \nu({}^{98} A)^{r_0+3(r_1+r_2+r_3+r_4+r_5+r_6+r_7)} (-1)^{(\beta+\frac{\alpha-1}{2})(r_1+r_3+r_5+r_7)} i^{\alpha\beta(r_1+r_3+r_5+r_7)}. \quad \square \end{aligned}$$

**Corollary 5.6.** *If  $F(\tau)$  is as in Proposition 5.5,  $r_0 + R = 1$ ,  $r_0 + 3R \equiv -3 \pmod{24}$ ,  $1 + S \equiv 0 \pmod{4}$ , and  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(28) \cap \Gamma_0(784)$  then*

$$F(A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} F(\tau).$$

In particular  $F(\tau)$  is a modular form of weight  $\frac{1}{2}$  on  $\Gamma_1(28) \cap \Gamma_0(784)$ .

**Proof.** We note

$$F(A\tau) = \nu(^{98}A)^{-3}(-1)^{\beta}i^{-\beta}\sqrt{\gamma\tau + \delta}F(\tau),$$

but as in the proof [Corollary 5.3](#) we know  $\nu(^{98}A)^{-3} = \nu(^2A)^{-3}$  and as in the proof of [Corollary 3.4](#) we know  $(\frac{\gamma}{\delta}) = \nu(^2A)^{-3}i^{\beta}$ .  $\square$

## 6. Orders at cusps

For a non-negative real number  $x$ , we let  $[x]$  denote the greatest integer less than or equal to  $x$  and  $\{x\}$  the fractional part of  $x$ . That is,  $x = [x] + \{x\}$ ,  $[x] \in \mathbb{Z}$ , and  $0 \leq \{x\} < 1$ .

**Proposition 6.1.** *If  $u, v, w, x \in \mathbb{R}$  and  $0 \leq u, w < 1$ , then the lowest power of  $q$  appearing in the expansion of  $\mu(u\tau + v, w\tau + x; \tau)$  is  $\nu(u, w)$ , where*

$$\nu(u, w) = \begin{cases} \frac{u+w}{2} - \frac{1}{8} & \text{if } u + w \leq 1, \\ \frac{7}{8} - \frac{u+w}{2} & \text{if } u + w > 1. \end{cases}$$

**Proof.** We have

$$\mu(u\tau + v, w\tau + x; \tau) = \frac{\exp(\pi iv)q^{\frac{u}{2}}}{\vartheta(w\tau + x; \tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2} + nw} \exp(2\pi inx)}{1 - \exp(2\pi iv)q^{u+n}}.$$

But

$$\begin{aligned} \frac{1}{\vartheta(w\tau + x; \tau)} &= \frac{i \exp(\pi ix)q^{\frac{w}{2} - \frac{1}{8}}}{(q, \exp(2\pi ix)q^w, \exp(-2\pi ix)q^{1-w}; q)_{\infty}} \\ &= \begin{cases} \frac{i \exp(\pi ix)q^{-\frac{1}{8}}}{(1 - \exp(2\pi ix))} (1 + \dots) & \text{if } w = 0, \\ i \exp(\pi ix)q^{\frac{w}{2} - \frac{1}{8}} (1 + \dots) & \text{if } w \neq 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2} + nw} \exp(2\pi inx)}{1 - \exp(2\pi iv)q^{u+n}} \\ &= \frac{1}{1 - \exp(2\pi iv)q^u} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2} + nw} \exp(2\pi inx)}{1 - \exp(2\pi iv)q^{u+n}} \\ &\quad - \exp(-2\pi iv)q^{-u} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2} - nw} \exp(-2\pi inx)}{1 - \exp(-2\pi iv)q^{n-u}} \\ &= \frac{1}{1 - \exp(2\pi iv)q^u} - \exp(2\pi ix)q^{1+w} (1 + \dots) + \exp(-2\pi i(v+x))q^{1-u-w} (1 + \dots) \end{aligned}$$

$$= \begin{cases} \frac{1}{1-\exp(2\pi i v)}(1+\dots) & \text{if } u = 0, \\ 1+\dots & \text{if } u \neq 0 \text{ and } u+w < 1, \\ (1+\exp(-2\pi i(v+x)))(1+\dots) & \text{if } u+w = 1, \\ \exp(-2\pi i(v+x))q^{1-u-w}(1+\dots) & \text{if } u+w > 1. \end{cases}$$

The result then follows after examining the various cases.  $\square$

**Corollary 6.2.** *If  $f(\tau) = q^\alpha \tilde{\mu}(u\tau + v, w\tau + x; \tau)$  is a harmonic weak Maass form, with  $u, v, w, x \in \mathbb{R}$ , then the lowest power of  $q$  appearing in the expansion of the holomorphic part of  $f(\tau)$  is at least  $\alpha + \tilde{\nu}(u, w)$ , where*

$$\tilde{\nu}(u, w) = \frac{1}{2}([\![u]\!] - [\![w]\!])^2 + ([\![u]\!] - [\![w]\!])(\{u\} - \{w\}) + k(u, w),$$

and

$$k(u, w) = \begin{cases} \nu(\{u\}, \{w\}) & \{u\} - \{w\} \neq \pm \frac{1}{2} \\ \min(\frac{1}{8}, \nu(\{u\}, \{w\})) & \{u\} - \{w\} = \pm \frac{1}{2} \end{cases}.$$

**Proof.** By (3.2) we have

$$\begin{aligned} & \tilde{\mu}(u\tau + v, w\tau + x; \tau) \\ &= (-1)^{[\![u]\!] + [\![w]\!] } \exp\left(\pi i \tau ([\![u]\!] - [\![v]\!] )^2 + 2\pi i ([\![u]\!] - [\![v]\!] ) (\{u\} \tau + v - \{v\} \tau - w)\right) \\ & \quad \tilde{\mu}(\{u\} \tau + v, \{w\} \tau + x; \tau). \end{aligned}$$

However, the lowest power of  $q$  appearing in the  $q$ -expansion of  $\mu(\{u\} \tau + v, \{w\} \tau + x; \tau)$  is  $\nu(\{u\}, \{w\})$ . Now  $R((\{u\} - \{w\})\tau + v - x; \tau)$  contributes nothing to the holomorphic part unless  $\{u\} - \{w\} = \pm \frac{1}{2}$ , in which class it contributes a constant multiple of  $q^{\frac{1}{8}}$ . This completes the proof.  $\square$

We recall for a modular form  $f$  on some congruence subgroup  $\Gamma$ , the invariant order at  $\infty$  is the least power of  $q$  appearing in the  $q$ -expansion at  $i\infty$ . That is, if

$$f(\tau) = \sum_{m=m_0}^{\infty} a(m) \exp(2\pi i \tau m/N),$$

and  $a(m_0) \neq 0$ , then the invariant order is  $m_0/N$ . For a modular form, this is always a finite number. For a harmonic weak Maass form, we cannot take such an expansion, however we can do so for the holomorphic part. If  $f$  is a modular form of weight  $k$ ,

$\gcd(\alpha, \gamma) = 1$ , and  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , then the invariant order of  $f$  at the cusp  $\frac{\alpha}{\gamma}$  is the invariant order at  $\infty$  of  $(A : \tau)^{-k} f(A\tau)$ . In the same fashion, if  $f$  is a harmonic weak Maass form, then the invariant order of the holomorphic part of  $f$  at the cusp  $\frac{\alpha}{\gamma}$

is the invariant order at  $\infty$  of  $(A : \tau)^{-k} f(A\tau)$ . This value is independent of the choice of  $A$ .

**Proposition 6.3.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Suppose  $\alpha$  and  $\gamma$  are non-negative integers and  $\gcd(\alpha, \gamma) = 1$ . If  $\gamma$  is even, then the invariant order of the holomorphic part of  $N(a, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least*

$$-\frac{a^2\gamma^2}{c^2} + \frac{a\gamma}{c} - \frac{1}{4} + 2\tilde{\nu}\left(\frac{a\gamma}{c}, \frac{1}{2}\right).$$

*If  $\gamma$  is odd, then the invariant order of  $N(a, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least*

$$-\frac{a^2\gamma^2}{c^2} + \frac{a\alpha\gamma}{c} - \frac{\alpha^2}{4} + \frac{1}{2}\tilde{\nu}\left(\frac{2a\gamma}{c}, \alpha\right).$$

**Proof.** We take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . By Proposition 3.1, if  $\gamma$  is even, then

$$N(a, c; A\tau) = \varepsilon \exp\left(-\pi i \tau \left(\frac{2a^2\gamma^2}{c^2} - \frac{2a\gamma}{c} + \frac{1}{2}\right)\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\delta}{c} + \frac{2a\gamma\tau}{c}, \tau; 2\tau\right),$$

and if  $\gamma$  odd then

$$\begin{aligned} N(a, c; A\tau) \\ = \varepsilon \exp\left(-\pi i \tau \left(\frac{2a^2\gamma^2}{c^2} - \frac{2a\alpha\gamma}{c} + \frac{\alpha^2}{2}\right)\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{a\delta}{c} + \frac{a\gamma\tau}{c}, \frac{\alpha\tau}{2} + \frac{\beta}{2}; \frac{\tau}{2}\right), \end{aligned}$$

for some constant  $\varepsilon$ . The proposition now follows from Corollary 6.2.  $\square$

**Proposition 6.4.** *Suppose  $a$ ,  $c$  and  $m$  are integers,  $c > 0$ ,  $m > 0$ ,  $c \nmid a$ , and  $M(a, c; m\tau)$  is a harmonic weak Maass form. Suppose  $\alpha$  and  $\gamma$  are non-negative integers and  $\gcd(\alpha, \gamma) = 1$ . Then the invariant order of the holomorphic part of  $M(a, c; m\tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least*

$$-\frac{g^2x^2}{2m}\left(\frac{a}{c} - \frac{1}{2}\right)^2 + \frac{g^2}{m}\tilde{\nu}\left(\frac{ax}{c}, \frac{x}{2}\right),$$

where  $g = \gcd(m, \gamma)$  and  $x = m\alpha/g$ .

**Proof.** We take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . We set  $u = \gamma/g$ , so that  $x$  and  $u$  are relatively prime. We then take  $L = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . Next we set

$$B = L^{-1} \begin{pmatrix} m\alpha & m\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & m\beta v - \delta y \\ 0 & m/g \end{pmatrix}.$$

The result then follows [Corollary 6.2](#) as [Proposition 5.1](#) gives

$$\begin{aligned} M(a, c; mA\tau) &= M(a, c; L(B\tau)) \\ &= \varepsilon \sqrt{\gamma\tau + \delta} \exp \left( -\pi i x^2 \left( \frac{a}{c} - \frac{1}{2} \right)^2 B\tau \right) \tilde{\mu} \left( \frac{axB\tau + ay}{c}, \frac{xB\tau + y}{2}; B\tau \right), \end{aligned}$$

for some constant  $\varepsilon$ .  $\square$

While we have only verified that  $M(a, c; m\tau)$  is a harmonic weak Maass form for  $c = 7$  and  $m = 98$ , it is clear other values will give harmonic weak Maass forms as well.

**Proposition 6.5.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Suppose  $\alpha$  and  $\gamma$  are non-negative integers and  $\gcd(\alpha, \gamma) = 1$ . If  $\gamma$  is even, then the invariant order of  $P(a, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is*

$$\left\{ \frac{a\gamma}{c} \right\} - \left\{ \frac{a\gamma}{c} \right\}^2.$$

*If  $\gamma$  is odd, then the invariant order of  $P(a, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is*

$$\frac{1}{4} \left\{ \frac{a\gamma}{c} \right\} - \frac{1}{4} \left\{ \frac{a\gamma}{c} \right\}^2 - \frac{1}{16}.$$

**Proof.** We take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . If  $\gamma$  is even, then we have

$$P(a, c; A\tau) = \frac{\varepsilon \eta(2A\tau)}{\eta(A\tau)^2 t_{0, \frac{2a}{c}}(2A\tau)} = \frac{\varepsilon \sqrt{\gamma\tau + \delta} \eta(2\tau)}{\eta(\tau)^2 t_{\frac{a\gamma}{c}, \frac{2a\delta}{c}}(2\tau)} = \frac{\varepsilon \sqrt{\gamma\tau + \delta} \eta(2\tau)}{\eta(\tau)^2 t_{\{\frac{a\gamma}{c}\}, \frac{2a\delta}{c}}(2\tau)},$$

where  $\varepsilon$  is some constant. The result then follows as the lowest power of  $q$  appearing in the expansion of

$$\frac{\eta(2\tau)}{\eta(\tau)^2 t_{\{\frac{a\gamma}{c}\}, \frac{2a\delta}{c}}(2\tau)}$$

is  $\left\{ \frac{a\gamma}{c} \right\} - \left\{ \frac{a\gamma}{c} \right\}^2$ . The case when  $\gamma$  is odd follows by similar calculations, but we instead choose  $A \in \mathrm{SL}_2(\mathbb{Z})$  with  $\delta$  even and use that

$$2A\tau = \frac{2a(\tau/2) + \beta}{\gamma(\tau/2) + \delta/2}.$$

We omit the details.  $\square$



The following is Lemma 3.2 of [1].

**Proposition 6.6.** *Suppose  $\alpha$  and  $\gamma$  are non-negative integers and  $\gcd(\alpha, \gamma) = 1$ . Then the invariant order of  $f_{N,\rho}(\tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is*

$$\frac{\gcd(N, \gamma)^2}{2N} \left( \left\{ \frac{\alpha\rho}{\gcd(N, \gamma)} \right\} - \frac{1}{2} \right)^2.$$

## 7. Proof of Theorem 1.1

To begin, we consider the equation

$$\begin{aligned} \mathcal{M}(1, 7; \tau) = & \frac{A(-16, -8, -8)F_0^{15}}{F_1^3 F_2^3 F_3^4 F_4 F_5 F_6 F_7^2} + \frac{A(-1, 1, -1)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6^2} + \frac{A(1, -1, -1)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_6^2} + \frac{A(-5, -1, -3)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5^2} \\ & + \frac{A(20, 8, 12)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(-3, -5, -1)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(4, 4, 0)F_0^{15}}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} + \frac{A(3, 5, 1)F_0^{15} F_3^2}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} \\ & + \frac{A(2, -4, 2)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(18, 6, 10)F_0^{15} F_5}{F_1^3 F_2^3 F_3^3 F_4 F_6 F_7^2} + \frac{A(-12, -2, -8)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(18, 6, 10)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} \\ & + \frac{A(-18, -6, -10)F_0^{15} F_3^2}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(-10, -2, -6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(-12, -2, -8)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} \\ & + \frac{A(12, 10, 6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(-30, -6, -18)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(-14, -6, -10)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(16, 2, 10)F_0^{15} F_6}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} \\ & + \frac{A(1, -1, 1)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(37, -1, 25)F_0^{15}}{2F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(-1, -3, -1)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(-2, 2, -2)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_6 F_7^2} \\ & + \frac{A(1, -1, 1)F_0^3 F_3^3 F_4 F_5}{2F_1^3 F_6 F_7^2} + \frac{A(1, -1, 1)F_0^3 F_2 F_3 F_4}{F_1^3 F_6 F_7^2} + \frac{A(-2, 2, -2)F_0^3 F_2 F_3 F_5}{F_1^3 F_3 F_6 F_7^2} \\ & + \frac{A(8, -8, 8)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_6 F_7^2} + \frac{A(-8, 8, -8)F_0^{15} F_5}{F_1^3 F_2^3 F_3^3 F_4 F_6 F_7^2} + \frac{A(-40, 8, -32)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(28, -2, 20)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6^2} \\ & + \frac{A(20, 6, 12)F_0^{15} F_3}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} + \frac{A(8, -8, 8)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_7^2} + \frac{A(-8, -6, -4)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(-4, -4, -2)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} \\ & + \frac{A(2, -2, 2)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(-6, 2, -6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_6 F_7^2} + \frac{A(-10, -2, -8)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(4, 4, 2)F_0^{15} F_3}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} \\ & + \frac{A(10, 2, 8)F_0^{15} F_3^2}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} + \frac{A(10, 2, 6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(24, 12, 16)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_6 F_7^2} + \frac{A(8, 8, 6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} \\ & + \frac{A(-8, -8, -6)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} + \frac{A(-24, -12, -16)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(-24, -8, -18)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(0, -4, 0)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_7^2} \\ & + \frac{A(16, 4, 10)F_0^{15}}{F_1^3 F_2^3 F_4 F_5 F_6 F_7^2} + \frac{A(-40, -8, -24)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(2, 4, 0)F_0^{15} F_4}{F_1^3 F_2^3 F_3^3 F_5 F_6 F_7^2} + \frac{A(18, 4, 12)F_0^{15} F_5}{F_1^3 F_2^3 F_3^3 F_4 F_6 F_7^2} \\ & + \frac{A(40, 8, 24)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(-18, -4, -12)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(6, -4, 4)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} \\ & + \frac{A(-18, -4, -12)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(22, 4, 12)F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2} + \frac{A(-8, 0, -6)F_0^3}{F_4 F_7} + \frac{A(6, 4, 4)F_0^3}{F_6 F_7} \\ & + \frac{A(8, 2, 4)F_0^3}{F_2 F_7} - iA(-8, 0 - 6)N_7(2; \tau) - iA(6, 4, 4)N_7(3; \tau) \\ & - iA(8, 2, 4)N_7(1; \tau), \end{aligned} \tag{7.1}$$

where  $F_k = f_{98,7k}(\tau)$  for  $1 \leq k \leq 6$  and  $F_0 = \eta(98\tau)$ . By Corollaries 4.3, 5.3, and 5.6 this is an identity between two harmonic weak Maass forms on  $\Gamma_0(784) \cap \Gamma_1(28)$ . We see

(7.1) is the identity in Theorem 1.1 with the appropriate non-holomorphic parts added to each side. However using Proposition 4.4, and rewriting the series with  $n \mapsto 7n \pm k$  for  $k = 1, 2, 3$ , we find that the non-holomorphic part of the left hand side is equal to the non-holomorphic part of the right hand side as given by Proposition 5.4. Therefore (7.1) is equivalent to Theorem 1.1. Furthermore, by subtracting the left hand side of (7.1) from the right hand side, we see (7.1) is equivalent to verifying a weakly holomorphic modular form on  $\Gamma_1(28) \cap \Gamma_0(784)$  is zero. In fact we can work on a larger group. By Corollary 3.3 and Propositions 4.1, 5.2, and 5.5 each individual term in (7.1) satisfies

$$f(A\tau) = \nu(A)^{-3}(-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta} f(\tau),$$

for  $A \in \Gamma_0(98) \cap \Gamma_1(14)$ . We subtract  $\mathcal{M}(1, 7; \tau)$  from both sides and divide by  $\frac{F_0^{15}}{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2}$  to obtain the equivalent identity

$$\begin{aligned} 0 = & A(-16, -8, -8) + \frac{A(-1, 1, -1)F_4 F_7^2}{F_5^2 F_6} + \frac{A(1, -1, -1)F_3 F_5 F_7^2}{F_4^2 F_6^2} + \frac{A(-5, -1, -3)F_2 F_6 F_7^2}{F_4^2 F_5^2} \\ & + \frac{A(20, 8, 12)F_3 F_6 F_7}{F_4 F_5^2} + \frac{A(-3, -5, -1)F_3 F_4^2}{F_5 F_6^2} + \frac{A(4, 4, 0)F_3^3 F_7}{F_4^2 F_6^2} + \frac{A(3, 5, 1)F_3^5}{F_4^2 F_5 F_6^2} + \frac{A(2, -4, 2)F_4 F_7}{F_5 F_6} \\ & + \frac{A(18, 6, 10)F_3 F_5^2 F_7}{F_4^2 F_6^2} + \frac{A(-12, -2, -8)F_3 F_6}{F_4 F_5} + \frac{A(18, 6, 10)F_3 F_4^2}{F_6^2 F_7} + \frac{A(-18, -6, -10)F_3^5}{F_4^2 F_6^2 F_7} \\ & + \frac{A(-10, -2, -6)F_2 F_6 F_7}{F_4^2 F_5} + \frac{A(-12, -2, -8)F_2 F_3^2 F_6}{F_4^2 F_5^2} + \frac{A(12, 10, 6)F_4}{F_6} + \frac{A(-30, -6, -18)F_3 F_6}{F_4 F_7} \\ & + \frac{A(-14, -6, -10)F_3 F_7}{F_5^2} + \frac{A(16, 2, 10)F_3^2 F_6^2}{F_4^2 F_5^2} + \frac{A(1, -1, 1)F_3^2 F_4}{F_5 F_6 F_7} + \frac{A(37, -1, 25)F_2 F_6}{2F_4^2} \\ & + \frac{A(-1, -3, -1)F_2 F_3}{F_5 F_6} + \frac{A(-2, 2, -2)F_2^2 F_5}{F_4^2 F_7} + \frac{A(1, -1, 1)F_2^3 F_3^6 F_4^3 F_5^2}{2F_0^{12} 2F_6^2} + \frac{A(1, -1, 1)F_4^4 F_3^4 F_4 F_5}{F_0^{12} 2F_7} \\ & + \frac{A(-2, 2, -2)F_2^5 F_3^2 F_4^4 F_5^3}{F_0^2 2F_6 F_7} + \frac{A(8, -8, 8)F_4 F_5}{F_6 F_7} + \frac{A(-8, 8, -8)F_3 F_4^4}{F_4^2 F_6^2 F_7} + \frac{A(-40, 8, -32)F_3}{F_5} \\ & + \frac{A(28, -2, 20)F_3^2 F_7^2}{F_4^2 F_6^2} + \frac{A(20, 6, 12)F_3^4 F_7}{F_4^2 F_5 F_6^2} + \frac{A(8, -8, 8)F_2 F_5 F_6}{F_4 F_7} + \frac{A(-8, -6, -4)F_2 F_3 F_4 F_7}{F_5^2 F_6^2} \\ & + \frac{A(-4, -4, -2)F_4^2}{F_6^2} + \frac{A(2, -2, 2)F_3 F_4 F_7}{F_5 F_6} + \frac{A(-6, 2, -6)F_3^2 F_5 F_7}{F_4^2 F_6^2} + \frac{A(-10, -2, -8)F_3^2 F_4^2}{F_5 F_6^2 F_7} \\ & + \frac{A(4, 4, 2)F_3^4}{F_4^2 F_6^2} + \frac{A(10, 2, 8)F_3^6}{F_4^2 F_5 F_6^2 F_7} + \frac{A(10, 2, 6)F_2 F_3 F_6 F_7}{F_4^2 F_5^2} + \frac{A(24, 12, 16)F_4^2 F_5}{F_6^2 F_7} + \frac{A(8, 8, 6)F_3 F_4}{F_5 F_6} \\ & + \frac{A(-8, -8, -6)F_3^2 F_6}{F_4 F_5 F_7} + \frac{A(-24, -12, -16)F_3^3 F_4}{F_5^2 F_6 F_7} + \frac{A(-24, -8, -18)F_2 F_7}{F_5 F_6} + \frac{A(0, -4, 0)F_2 F_3 F_6}{F_4^2 F_5} \\ & + \frac{A(16, 4, 10)F_2 F_3^3 F_6}{F_4^2 F_5^2 F_7} + \frac{A(-40, -8, -24)F_3 F_4}{F_6 F_7} + \frac{A(2, 4, 0)F_3 F_4^2 F_7}{F_5^2 F_6^2} + \frac{A(18, 4, 12)F_3^3 F_5}{F_4^2 F_6^2 F_7} \\ & + \frac{A(40, 8, 24)F_3^2}{F_5^2} + \frac{A(-18, -4, -12)F_3^3 F_7^2}{F_4^2 F_5 F_6^2} + \frac{A(6, -4, 4)F_2}{F_6} + \frac{A(-18, -4, -12)F_2 F_3 F_6}{F_4^2 F_7} \\ & + \frac{A(22, 4, 12)F_2 F_3^2}{F_5 F_6 F_7} + \frac{A(-8, 0, -6)F_3^3 F_2^2 F_3^3 F_5 F_6}{F_0^{12} F_7} + \frac{A(6, 4, 4)F_1^3 F_3^3 F_3^3 F_4 F_5}{F_0^{12} F_7} \\ & + \frac{A(8, 2, 4)F_1^3 F_2^2 F_3^3 F_4 F_5 F_6}{F_0^{12} F_7} - \frac{iA(-8, 0, -6)F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2 N_7(2; \tau)}{F_0^{15}} \\ & - \frac{iA(6, 4, 4)F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2 N_7(3; \tau)}{F_0^{15}} - \frac{iA(8, 2, 4)F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2 N_7(1; \tau)}{F_0^{15}} \\ & - \frac{F_1^3 F_2^3 F_3^3 F_4 F_5 F_6 F_7^2 \mathcal{M}(1, 7; \tau)}{F_0^{15}}. \end{aligned} \tag{7.2}$$

We let  $RHS$  denote the right hand side of (7.2) and let  $\Gamma = \Gamma_0(98) \cap \Gamma_1(14)$ . We know  $RHS$  is a modular function on  $\Gamma$ .

The valence formula, for modular functions can be stated as follows. Suppose  $f$  is a modular function on some congruence subgroup  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ . Suppose  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we then have a cusp  $\zeta = A(\infty) = \frac{\alpha}{\gamma}$ . We let  $\mathrm{ord}(f; \zeta)$  denote the invariant order of  $f$  at  $\zeta$ . We define the width of  $\zeta$  with respect to  $\Gamma$  as  $\mathrm{width}_\Gamma(\zeta) := w$ , where  $w$  is the least positive integer such that  $A \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} A^{-1} \in \Gamma$ . We then define the order of  $f$  at  $\zeta$  with respect to  $\Gamma$  as  $\mathrm{ORD}_\Gamma(f; \zeta) = \mathrm{ord}(f; \zeta) \mathrm{width}_\Gamma(\zeta)$ . For  $z \in \mathcal{H}$  we let  $\mathrm{ord}(f; z)$  denote the order of  $f$  at  $z$  as a meromorphic function. We then define the order of  $f$  at  $z$  with respect to  $\Gamma$  as  $\mathrm{ORD}_\Gamma(f; z) = \mathrm{ord}(f; z)/m$  where  $m$  is the order of  $z$  as a fixed point of  $\Gamma$  (so  $m = 1, 2$ , or  $3$ ). If  $f$  is not the zero function and  $\mathcal{D} \subset \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$  is a fundamental domain for the action of  $\Gamma$  on  $\mathcal{H}$  along with a complete set of inequivalent cusps for the action, then

$$\sum_{\zeta \in \mathcal{D}} \mathrm{ORD}_\Gamma(f; \zeta) = 0.$$

A complete set of inequivalent cusps, along with their widths, for  $\Gamma_0(98) \cap \Gamma_1(14)$  is

cusps	0	$\frac{1}{14}$	$\frac{3}{38}$	$\frac{2}{25}$	$\frac{1}{12}$	$\frac{3}{35}$	$\frac{2}{23}$	$\frac{3}{28}$	$\frac{5}{42}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{5}{28}$	$\frac{3}{14}$	$\frac{8}{35}$	$\frac{5}{21}$	$\frac{2}{7}$	$\frac{17}{56}$
width	98	1	49	98	49	2	98	1	1	49	2	1	1	2	2	2	1
cusps	$\frac{11}{35}$	$\frac{9}{28}$	$\frac{5}{14}$	$\frac{18}{49}$	$\frac{37}{98}$	$\frac{8}{21}$	$\frac{19}{49}$	$\frac{11}{28}$	$\frac{3}{7}$	$\frac{22}{49}$	$\frac{16}{35}$	$\frac{45}{98}$	$\frac{13}{28}$	$\frac{10}{21}$	$\frac{27}{56}$	$\frac{29}{56}$	$\frac{11}{21}$
width	2	1	1	2	1	2	2	1	2	2	2	1	1	2	1	1	2
cusps	$\frac{15}{28}$	$\frac{19}{35}$	$\frac{4}{7}$	$\frac{13}{21}$	$\frac{9}{14}$	$\frac{39}{56}$	$\frac{5}{7}$	$\frac{16}{21}$	$\frac{27}{35}$	$\frac{11}{14}$	$\frac{6}{7}$	$\frac{37}{42}$	$\frac{13}{14}$	$\infty$			
width	1	2	2	2	1	1	2	2	2	1	2	1	1	1			

We let  $\mathcal{D}$  denote these cusps along with a fundamental region of the action of  $\Gamma$ .

We note  $RHS$  has no poles on  $\mathcal{H}$ , but it may have zeros on  $\mathcal{H}$ . We take a lower bound on the orders at the non-infinite cusps by taking the minimum order of each of the individual summands in (7.2), which we compute with Propositions 6.3, 6.4, 6.5, and 6.6. This lower bound yields

$$\sum_{\zeta \in \mathcal{D}} \mathrm{ORD}_\Gamma(RHS; \zeta) \geq \mathrm{ord}(RHS, \infty) - 109.$$

However, we can expand  $RHS$  as a  $q$ -series and find the coefficients of  $RHS$  are zero to at least  $q^{110}$ . Thus

$$\sum_{\zeta \in \mathcal{D}} \mathrm{ORD}_\Gamma(RHS; \zeta) \geq 1,$$

and so  $RHS$  must be identically zero by the valence formula. This proves (7.2), which is equivalent to (7.1), which is equivalent to Theorem 1.1. Thus Theorem 1.1 holds.

It is very fortunate that each of the 7 terms of the 7-dissection of  $\mathcal{M}(1, 7)$  transforms the same as  $\mathcal{M}(1, 7)$  under  $\Gamma_0(98) \cap \Gamma_1(14)$ . In general the terms of a dissection are harmonic Maass forms, but possibly with respect to a smaller subgroup than  $\mathcal{M}(a, c)$ .

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