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Generalized Dedekind sums and equidistribution mod 1

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Dedekind sums are well-studied arithmetic sums, with values uniformly distributed mod 1. Based on their relation to certain modular forms, Dedekind sums may be defined as functions on the cusp set of $\mathrm{SL}(2,\mathbb{Z})$. We present a compatible notion of Dedekind sums, which we name Dedekind symbols, for any non-cocompact lattice $\Gamma < \mathrm{SL}(2,\mathbb{R})$, and prove the corresponding equidistribution mod 1 result. The latter part builds up on a paper of Vardi, who first connected exponential sums of Dedekind sums to Kloosterman sums.

1. Introduction

In this article, we introduce a function for non-cocompact lattices of $\mathrm{SL}(2,\mathbb{R})$ that relates to, and actually generalizes, the classical Dedekind sums

$$s(a;c) = \sum_{n=1}^{c-1} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right), \qquad (c \in \mathbb{N},\ a \in \mathbb{Z},\ (a,c)=1),$$

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where

$$x \mapsto ((x)) \;:=\; \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \qquad (\{x\} = \text{ fractional part of } x \in \mathbb{R})$$

is the odd and periodic "sawtooth" function of expectancy zero.

There is a ubiquitous character to the Dedekind sums, as they appear in a wide range of contexts. The name alone hinges on their relation to the logarithm of the Dedekind η-function

$$\eta(z) \;=\; e\left(\frac{z}{24}\right) \prod_{n \geq 1} (1 - e(nz)) \qquad \left(e(z) = e^{2\pi i z}\right)$$

defined on the upper half-plane $\mathbb{H}$, a classical player in the theories of modular forms, elliptic curves, and theta functions. More precisely, for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$,

$$\log \eta(\gamma z) - \log \eta(z) \;=\; \frac{1}{2}(\operatorname{sign}(c))^2 \log\left(\frac{cz + d}{i \operatorname{sign}(c)}\right) + \frac{\pi i}{12}\Phi(\gamma), \tag{1.1}$$

where the defect $\Phi(\gamma)$ arising from the ambiguity of the principal branch of the logarithm is given by

$$\Phi(\gamma) \;=\; \begin{cases} b/d & c = 0, \\ \frac{a+d}{c} - 12\operatorname{sign}(c) s(a; |c|) & c \neq 0. \end{cases} \tag{1.2}$$

While this is not obvious at first glance, the values of Φ are always integers. The latter fact, as many other fundamental properties pertaining to Dedekind sums, may be found in the monograph [RadG72]. Dedekind's original proof of the transformation formula (1.1) is of analytic nature, but it can also be deduced by purely topological arguments. Atiyah [Ati87] discusses this approach and offers an overview of the appearance of $\log \eta$ and the Dedekind sums in various contexts of number theory, topology and geometry, exhibiting no less than seven equivalent characterizations of $\log \eta$ across these different fields!

An alternative presentation of the Dedekind sums consists in defining $s(a; c)$ as a function on the cusp set of $\mathrm{SL}(2, \mathbb{Z})$, which can be identified with the extended rational line $\mathbb{Q} \cup \{\infty\}$. This identification can then be exploited to study some of their properties via continued fraction expansions, as is done in [KM94]. We propose a modified construction. Let Γ_∞ denote the stabilizer subgroup of $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ at ∞, that is,

$$\Gamma_\infty \;=\; \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \right\} \;=\; \pm \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}.$$

There is a one-to-one correspondence between the cusp set of Γ, i.e. $\{\gamma(\infty) : \gamma \in \Gamma\}$ and the quotient Γ/Γ_∞. We can thus express (signed) Dedekind sums via the assignment

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_\infty \;\mapsto\; \frac{1}{12}\frac{a+d}{c} - \frac{1}{12}\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \left(\overset{(1.2)}{=}\ \operatorname{sign}(c)s(a;|c|)\right).$$

This map descends to the double coset $\Gamma_\infty\backslash\,\Gamma/\,\Gamma_\infty$. In fact, this is simply a manifestation of the periodicity of the Dedekind sums, since, for each integer m,

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix}$$

and

$$\operatorname{sign}(c)s(a+mc;|c|) \;=\; \operatorname{sign}(c)s(a;|c|).$$

We call the resulting double coset function on $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ the Dedekind symbol for $\mathrm{SL}(2,\mathbb{Z})$. Our main result is that this construction may be generalized to any non-cocompact lattice $\Gamma < \mathrm{SL}(2,\mathbb{R})$.

Theorem 1. *Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a non-cocompact lattice. For each cusp $\mathfrak{a}$ and each constant $k_0 \in \mathbb{R}$, there exists a continuous family $\{\eta_\mathfrak{a} = \eta_\mathfrak{a}(k_0,k)\}_{k\in\mathbb{R}_{>0}}$ of nowhere-vanishing functions on $\mathbb{H}$ such that, for each $\gamma \in \Gamma$,*

$$\log\eta_\mathfrak{a}(\gamma z) - \log\eta_\mathfrak{a}(z) \;=\; \frac{k}{2}\log\left(-(cz+d)^2\right) + 2\pi i k\Phi_\mathfrak{a}(\gamma) \tag{1.3}$$

where $\Phi_\mathfrak{a}(\gamma)$ is defined by the above formula and real-valued, and log *denotes the principal branch of the logarithm. Moreover, each such function satisfies a generalized Kronecker first limit formula*

$$\lim_{s\to 1}\left(E_\mathfrak{a}(z,s) - \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})^{-1}}{s-1}\right) = -k_0 - \ln\left(y\,|\eta_\mathfrak{a}(z)|^{2/k}\right), \tag{1.4}$$

where $E_\mathfrak{a}(z,s)$ are the non-holomorphic Eisenstein series.

Corollary 1. *Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a non-cocompact lattice. There exists a nowhere-vanishing real-weight cusp form that generalizes Dedekind's η-function.*

Theorem 2. *Let $\sigma_\mathfrak{a} \in \mathrm{SL}(2,\mathbb{R})$ be a scaling matrix, i.e. $\sigma_\mathfrak{a}(\infty) = \mathfrak{a}$ and $\left(\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}\right)_\infty = \pm\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1 \end{smallmatrix}\right)$. The function $\mathcal{S}_\mathfrak{a} : \Gamma \to \mathbb{R}$, given by*

$$\mathcal{S}_\mathfrak{a}(\gamma) = \begin{cases} \mathfrak{a} & c = 0, \\ \dfrac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\dfrac{a+d}{c} - \Phi_\infty\begin{pmatrix} a & * \\ c & d \end{pmatrix} & c \neq 0, \end{cases}$$

where

$$\begin{pmatrix} a & * \\ c & d \end{pmatrix} = \sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a}$$

and $\Phi_\infty : \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \to \mathbb{R}$ *is defined by (1.3), factors through the double coset* $\Gamma_{\mathfrak{a}}\backslash\Gamma/\Gamma_{\mathfrak{a}}$ *and does not depend on the particular choice of the scaling* $\sigma_{\mathfrak{a}}$*. We call* $\mathcal{S}_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}\backslash\Gamma/\Gamma_{\mathfrak{a}} \to \mathbb{R}$ *the Dedekind symbol for* Γ *at its cusp at* $\mathfrak{a}$*.*

Remark 1. If $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $k_0 = 2\gamma_2 - 2\ln(2)$ where γ_e is the Euler–Mascheroni constant, $k = 1/2$, then $\eta_\infty = \eta$ the Dedekind η-function, $\mathcal{S}_\infty \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) = \mathrm{sign}(c)s(a;|c|)$, and (1.4) is Kronecker's first limit formula.

Goldstein [Gol73,Gol74] derived formally the functions $\log \eta_{\mathfrak{a}}$ from the Fourier expansion of Eisenstein series, and used them to give explicit formulas of Dedekind sums for certain principal congruence subgroups. We present a softer construction that differs from Goldstein's in that it does not rely on explicit Fourier coefficients and that we motivate the analytic existence of the functions $\log \eta_{\mathfrak{a}}$. Finally, the definition of Dedekind symbols as double coset functions is new, as is the generalization to all cofinite Fuchsian groups.

The second part of this paper concerns the distribution of values of the Dedekind symbols $\mathcal{S}_{\mathfrak{a}}$ mod 1. The statistics of Dedekind sums have been extensively studied; we know that their values become equidistributed mod 1 [Var87], and that this result extends to the graph $\left(\frac{a}{c}, s(a;c)\right)$ [Mye88]. Bruggeman studied the distribution of $s(a;c)/c$ [Bru89] and Vardi showed that $s(a;c)/\log c$ has a limiting Cauchy distribution as $c \to \infty$ [Var93]. The focus later shifted to the distribution of mean values of Dedekind sums [CFKS96, Zha96].

We will generalize Vardi's equidistribution mod 1 result to the Dedekind symbols; for any $k \in \mathbb{R}_{>0}$, the sequence of values

$$\{ks(a;c)\}_{\substack{0\le a<c \\ (a,c)=1}}$$

becomes equidistributed mod 1 as $c \to \infty$ [Var87, Thm. 1.6]. Our motivation, and the building block of Vardi's proof, is the striking identity

$$\sum_{\substack{a=1 \\ (a,c)=1}}^{c-1} e\left(12s(a;c)\right) \overset{(1.2)}{=} \sum_{\substack{a=1 \\ (a,c)=1}}^{c-1} e\left(\frac{a+d}{c}\right) \underbrace{e^{-2\pi i\Phi(\gamma)}}_{\equiv 1} = S(1,1;c)$$

relating Dedekind sums to Kloosterman sums.

Theorem 3. *Let* $V = \mathrm{vol}(\Gamma\backslash\mathbb{H})$*. Then*

$$\sum_{\substack{0\le a<c \\ \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}}} e\left(\mathcal{S}_{\mathfrak{a}}(\gamma)\right) = e\left(-\frac{1}{4}\right) S_{\mathfrak{a}}\left(\left\lceil\frac{V}{4\pi}\right\rceil, \left\lceil\frac{V}{4\pi}\right\rceil, c, \chi\right),$$

where $S_{\mathfrak{a}}(m,n,c,\chi)$ *are the Kloosterman–Selberg sums*

$$S_{\mathfrak{a}}(m,n,c,\chi) = \sum_{\substack{0\le a<c \\ \left(\begin{smallmatrix} a & * \\ c & d\end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}} e\left(\frac{(m-\alpha)a+(n-\alpha)d}{c}\right)\overline{\chi\begin{pmatrix} a & * \\ c & d\end{pmatrix}}$$

for the multiplier system $\chi(\Gamma) = e\left(\Phi_\infty(\Gamma) - \frac{1}{4}\right)$ *arising from the transformation for the* η_∞*-function for* $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$*, and where* $\alpha = \left\lceil \frac{V}{4\pi}\right\rceil - \frac{V}{4\pi} \in [0,1)$ *is uniquely determined by* $\chi\left(\begin{smallmatrix} 1 & 1 \\ & 1\end{smallmatrix}\right) = e(-\alpha)$.

Theorem 4. *For each* $k \in \mathbb{R}_{>0}$*, the sequence of values*

$$\left\{ k\mathcal{S}_{\mathfrak{a}}(\gamma) : \begin{array}{c} \left(\begin{smallmatrix} a & * \\ c & *\end{smallmatrix}\right) = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} \\ 0 \le a < c \end{array} \right\}$$

becomes equidistributed mod 1 as $c \to \infty$.

Our proof is intrinsically similar to that of [Var87], relying on the spectral theory of Kloosterman sums and, more particularly, the work of Selberg [Sel65] and Goldfeld & Sarnak [GS83] to prove non-trivial bounds for sums of Kloosterman sums. Whereas the results in [GS83] generalize immediately to the setting of cofinite Fuchsian groups, one needs to work for the trivial bound on sums of Kloosterman sums, which is exactly the counting function for double coset representatives indexed according to the ordering given above. In doing so, we recovered a result of Good [Goo83, Thm. 4]. We record a *direct* proof of this result and discuss its optimality.

Theorem 5. *Let* $\Gamma < \mathrm{SL}(2,\mathbb{R})$ *be a non-cocompact lattice with a cusp at* $\mathfrak{a}$*. Let* $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2,\mathbb{R})$ *be a scaling matrix such that* $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$ *and* $\left(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\right)_\infty = \pm\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1\end{smallmatrix}\right) =: B$*. Let* $X > 0$*. Then the double coset counting function*

$$N_{\mathfrak{a}}(X) \ = \ \#\left\{ B\begin{pmatrix} a & * \\ c & *\end{pmatrix} B \ : \ \begin{array}{c} 0<c\le X \\ 0\le a<c, \end{array} \ \left(\begin{smallmatrix} a & * \\ c & *\end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \right\}$$

is finite, and, as $X \to \infty$,

$$N_{\mathfrak{a}}(X) = \frac{X^2}{\pi V} + \sum_{1/2<\sigma_j<1} \tau_j X^{2\sigma_j} + O\left(X^{4/3+\varepsilon}\right)$$

for any $\varepsilon > 0$*, where the sum runs over all exceptional eigenvalues* $\lambda_j = \sigma_j(1-\sigma_j) < \frac{1}{4}$ *for* Γ *and each* $\tau_j = \tau_j(\sigma_j)$ *is a constant.*

2. Preliminaries

2.1. *Non-cocompact lattices of* $\mathrm{SL}(2,\mathbb{R})$

Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a lattice, with the assumption that $-I \in \Gamma$. The projection $\overline{\Gamma} < \mathrm{PSL}(2,\mathbb{R})$ acts properly discontinuously on the upper half-plane $\mathbb{H}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \ \mapsto \ \frac{az+b}{cz+d}.$$

If moreover Γ is non-cocompact, then $\Gamma\backslash\mathbb{H}$ admits a finite number of inequivalent cusps. In practice, it is most useful to work with the cusp at ∞. There is a standard change of coordinates to achieve this. In fact, for each cusp $\mathfrak{a}$, there exists a scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2,\mathbb{R})$ that verifies

(1) $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$
(2) $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\right)_{\infty} = \pm\begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}$

These two conditions do not determine a scaling matrix uniquely, but up to right multiplication by any matrix of the form $\pm\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$, $x \in \mathbb{R}$.

2.2. *Double coset decomposition*

Let $B := \pm\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1 \end{smallmatrix}\right)$. The trivial computation

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} = \begin{pmatrix} a+mc & * \\ c & d+nc \end{pmatrix},$$

shows that the lower left matrix entry c depends only on the double coset $B\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)B$, and that a and d are determined up to integer multiples of c. Each double coset for which $c \neq 0$ has then a unique representative of the form

$$B\begin{pmatrix} a & * \\ c & d \end{pmatrix}B, \qquad c > 0, \quad 0 \leq a,\ d < c.$$

Moreover, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} a & b' \\ c & d' \end{smallmatrix}\right)$ two matrices of determinant 1, one has $\gamma^{-1}\gamma' = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, any double coset $B\gamma B$, $\gamma \in \Gamma$, for which $c \neq 0$ is really only determined by the left column of the representative γ. For any scaling matrix $\sigma_{\mathfrak{a}}$, and any $x > 0$, there are at most finitely many double cosets $B\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)B$ such that $\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ and $0 < c \leq x$ [Shi71, Lm. 1.24]. We thus have the double coset decomposition

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \;=\; B\cup\bigcup_{c>0}\;\bigcup_{\substack{0\le a<c\\ \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}} B\begin{pmatrix} a & * \\ c & * \end{pmatrix}B.$$

2.3. Eisenstein series

The Eisenstein series for Γ at its cusp at $\mathfrak{a}$ is defined by

$$E_{\mathfrak{a}}(z,s) \;:=\; E_{\mathfrak{a}}(z,s;\Gamma) \;=\; \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathfrak{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$$

where $z\in\mathbb{H}$, $s=\sigma+it\in\mathbb{C}$. The series converges absolutely and uniformly on compact subsets for $\sigma>1$. As a function of z, it is Γ-invariant, non-holomorphic and satisfies

$$\Delta E_{\mathfrak{a}}(z,s) \;=\; s(1-s)E_{\mathfrak{a}}(z,s),$$

where Δ is the hyperbolic Laplacian $\Delta = -y^2\left(\partial_{xx}+\partial_{yy}\right)$. Eisenstein series admit a Fourier expansion in each cusp, which takes the form

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \;=\; \delta_{\mathfrak{ab}}y^s + \varphi_{\mathfrak{ab}}(s)y^{1-s} + O\left(e^{-2\pi y}\right)$$

as $y\to\infty$ where

$$\varphi_{\mathfrak{ab}}(s) \;=\; \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{c>0}\frac{\#\{\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} : a\in[0,c)\}}{c^{2s}}$$

[Iwa02, Thm. 3.4]. In the definition above, $\Gamma(s)$ denotes the classical Gamma function, which is holomorphic on the complex plane except for simple poles at every non-positive integer.

Eisenstein series famously admit a meromorphic continuation to the whole complex s-plane, which follows from the meromorphic continuation of $\varphi_{\mathfrak{ab}}$ [Sel56]. In particular, $\varphi_{\mathfrak{ab}}(s)$ is holomorphic in the half-plane $\sigma\ge 1/2$ except for possibly finitely many simple poles $\sigma_j\in(1/2,1)$ coming from the residual spectrum of Γ, and a simple pole at $s=1$ of residue

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}.$$

Moreover, away from the real line, $\varphi_{\mathfrak{ab}}(s)$ is bounded in the half-plane $\sigma\ge 1/2$ [Sel89, Eq. (8.6)].

2.4. Selberg–Kloosterman sums

An automorphic form of real weight $k \in \mathbb{R}_{>0}$ has prescribed transformation

$$f(\gamma z) \;=\; \chi_k(\gamma)(cz+d)^k f(z),$$

under Γ, where the multiplier $\chi_k(\Gamma)$ needs to satisfy

(1) For all $\gamma \in \Gamma$, $|\chi_k(\gamma)| = 1$
(2) $\chi_k(-I) = e^{-\pi i k}$
(3) $\chi_k(\gamma_1\gamma_2) j(\gamma_1\gamma_2, z)^k = \chi_k(\gamma_1)\chi_k(\gamma_2) j(\gamma_1, \gamma_2 z)^k j(\gamma_2, z)^k$

where $j(\gamma, z) := cz + d$ and $\arg(z) \in (-\pi, \pi]$, to be consistent with the determination of $\arg(cz + d)$ such that

$$(cz+d)^k \;=\; |cz+d|^k \, e^{ik \arg(cz+d)}$$

is uniquely determined.

Fix a scaling $\sigma_{\mathfrak{a}}$. Under the action of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$, we have the periodicity relation $f(z + 1) = \chi_k\left(\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)\right) f(z)$, which yields the Fourier series expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(y) e\left((n-\alpha_k)x\right)$$

for f, where $\alpha_k \in [0, 1)$ is uniquely determined by $\chi_k\left(\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)\right) = e(-\alpha_k)$. The famous problem of estimating the order of magnitude of Fourier coefficients of such a cusp form can be reduced to estimate sums of the generalized Kloosterman sums

$$S(m, n\,; c, \chi_k) \;:=\; \sum_{\substack{0\le a<c \\ \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}} \overline{\chi_k\left(\begin{pmatrix} a & * \\ c & d \end{pmatrix}\right)}\, e\left(\frac{(m-\alpha_k)a + (n-\alpha_k)d}{c}\right)$$

[Sel65]. In the generality of our setting, one knows that as $x \to \infty$,

$$\sum_{0<c\le x} \frac{S(m, n\,; c, \chi_k)}{c} \;=\; \sum_{1/2<\sigma_j<1} \tau_j(m, n) x^{2\sigma_j - 1} + O\left(x^{1/3+\varepsilon}\right) \qquad (\varepsilon > 0) \quad (2.1)$$

where the sum runs over the exceptional eigenvalues $\lambda_j = \sigma_j(1 - \sigma_j) < \frac{1}{4}$ for Γ [GS83, Pro79].

3. Proof of Theorem 1

Set $V := \mathrm{vol}(\Gamma\backslash\mathbb{H})$. Consider the Laurent expansion of the Eisenstein series $E_{\mathfrak{a}}(z,s)$ at its first pole, i.e.

$$E_{\mathfrak{a}}(z,s) = \frac{V^{-1}}{s-1} + K_{\mathfrak{a}}(z) + O(s-1)$$

as $s \to 1$. Using the Fourier expansion of the Eisenstein series, we have [JO05, Eq. (4.7)]

$$K_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z) \;=\; \sum_{n<0} k_{\mathfrak{ab}}(n) e(n\overline{z}) + \delta_{\mathfrak{ab}} y + k_{\mathfrak{ab}}(0) - V^{-1} \ln y + \sum_{n>0} k_{\mathfrak{ab}}(n) e(nz) \quad (3.1)$$

with $k_{\mathfrak{ab}}(-n) = \overline{k_{\mathfrak{ab}}(n)}$, and $k_{\mathfrak{ab}}(n) \ll |n|^{1+\varepsilon}$, for any $\varepsilon > 0$, with an implied constant depending only on Γ and ε. Moreover, setting

$$K_{\mathfrak{a}}(z) \;=\; \lim_{\substack{s\to 1\\ s\in\mathbb{R}_{>1}}} \left(E_{\mathfrak{a}}(z,s) - \frac{V^{-1}}{s-1} \right),$$

we can see that $K_{\mathfrak{a}}(z)$ is Γ-invariant, real-valued and real-analytic. A simple computation yields

$$\Delta K_{\mathfrak{a}}(z) = \frac{-1}{V}$$

and from that observation we construct the harmonic function

$$H_{\mathfrak{a}}(z; k_0) \;:=\; V K_{\mathfrak{a}}(z) + \ln \mathfrak{Im}(z) + k_0$$

for any real constant $k_0 \in \mathbb{R}$. Let $F_{\mathfrak{a}} : \mathbb{H} \to \mathbb{C}$ denote the holomorphic function with real part $\mathfrak{Re} F_{\mathfrak{a}}(z; k_0) = H_{\mathfrak{a}}(z; k_0)$. Observe that $F_{\mathfrak{a}}$ won't be automorphic, as the perturbation from $K_{\mathfrak{a}}$ to $H_{\mathfrak{a}}$ induces the logarithmic defect

$$H_{\mathfrak{a}}(z; k_0) - H_{\mathfrak{a}}(\gamma z; k_0) \;=\; \ln |cz + d|^2 .$$

By analogy with Dedekind's transformation formula for $\log \eta$, we want to consider the RHS as the real part of the *principal* branch of the logarithm, that is, the branch with $-\pi < \arg(z) \le \pi$. Hence

$$\mathfrak{Re}\left(F_{\mathfrak{a}}(z; k_0) - F_{\mathfrak{a}}(\gamma z; k_0)\right) \;=\; \mathfrak{Re} \log(-(cz+d)^2).$$

Lemma 3.1. *The function*

$$\Phi_{\mathfrak{a}}(\gamma, z; k_0) := \frac{1}{4\pi i} \left(F_{\mathfrak{a}}(z; k_0) - F_{\mathfrak{a}}(\gamma z; k_0) - \log\left(-(cz+d)^2\right) \right)$$

is real-valued, constant in z and k_0, i.e. $\Phi_{\mathfrak{a}}(\gamma, z; k_0) = \Phi_{\mathfrak{a}}(\gamma)$ for all $z \in \mathbb{H}$ and $k_0 \in \mathbb{R}$, and it factors through $\overline{\Gamma} < \mathrm{PSL}(2, \mathbb{R})$, where $\overline{\Gamma}$ is the image of Γ under the canonical projection $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$.

Proof. In fact,

$$\mathfrak{Im}\left(\Phi_{\mathfrak{a}}(\gamma, z; k_0)\right) = -\frac{1}{4\pi}\left(H_{\mathfrak{a}}(z; k_0) - H_{\mathfrak{a}}(\gamma z; k_0) - \ln|cz + d|^2\right) = 0.$$

By the Open Mapping Theorem, the function $\Phi_{\mathfrak{a}}(\gamma, z; k_0)$ is therefore constant in z. It is immediate that $\Phi_{\mathfrak{a}}(-\gamma) = \Phi_{\mathfrak{a}}(\gamma)$ for all $\gamma \in \Gamma$. The Fourier series for $K_{\mathfrak{a}}(z)$ given by (3.1) yields

$$\begin{aligned}\mathfrak{Re}\left(F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z; k_0)\right) &= V K_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z) + \ln y + k_0 \\ &= V \mathfrak{Re}\left(\delta_{\mathfrak{a}\mathfrak{b}} \frac{z}{i} + k_{\mathfrak{a}\mathfrak{b}}(0) + 2\sum_{n>0} k_{\mathfrak{a}\mathfrak{b}}(n) e(nz)\right) + k_0,\end{aligned}$$

hence, by another application of the Open Mapping Theorem,

$$F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z; k_0) = V\left(\delta_{\mathfrak{a}\mathfrak{b}} \frac{z}{i} + k_{\mathfrak{a}\mathfrak{b}}(0) + 2\sum_{n>0} k_{\mathfrak{a}\mathfrak{b}}(n) e(nz)\right) + k_0 + i\tau \tag{3.2}$$

for a real constant τ. It follows that $F_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z; k_0) - F_{\mathfrak{a}}(\gamma\sigma_{\mathfrak{b}} z; k_0)$ is independent of k_0 and τ. $\square$

Define, for each positive scalar $k \in \mathbb{R}_{>0}$, the function

$$\eta_{\mathfrak{a}}(z; k_0, k) \;=\; e^{-kF_{\mathfrak{a}}(z;k_0)/2}.$$

It is nowhere-vanishing, with $\log \eta_{\mathfrak{a}}(z; k_0, k) = -kF_{\mathfrak{a}}(z; k_0)/2$ and

$$\begin{aligned}\log \eta_{\mathfrak{a}}(\gamma z; k) - \log \eta_{\mathfrak{a}}(z; k) &\equiv \log \eta_{\mathfrak{a}}(\gamma z; k_0, k) - \log \eta_{\mathfrak{a}}(z; k_0, k) \\ &= \frac{k}{2} \log\left(-(cz + d)^2\right) + 2\pi i k \Phi_{\mathfrak{a}}(\gamma).\end{aligned}$$

The Kronecker first limit formula

$$V K_{\mathfrak{a}}(z) = \mathfrak{Re}\left(F_{\mathfrak{a}}(z; k_0)\right) - \ln y - k_0 = -k_0 - \ln\left(y\,|\eta_{\mathfrak{a}}(z; k_0, k)|^{2/k}\right)$$

follows directly from our definitions.

4. Proof of Theorem 2

Observe that

$$\log(-(cz+d)^2) \;=\; 2\left(\log(cz+d) - \frac{\pi i}{2}\mathrm{sign}(c(-d))\right),$$

where the symbol $c(-d) = c$ if $c \neq 0$ and $-d$ otherwise. Consider the associated function

$$\rho_{\mathfrak{a}}(\gamma) = \Phi_{\mathfrak{a}}(\gamma) - \frac{1}{4}\mathrm{sign}(c(-d))$$

for each $\gamma \in \Gamma$. Note that contrarily to $\Phi_{\mathfrak{a}}$, $\rho_{\mathfrak{a}}$ does not descend to $\overline{\Gamma} < \mathrm{PSL}(2,\mathbb{R})$. By definition,

$$\rho_{\mathfrak{a}}(\gamma) \;=\; \frac{1}{4\pi i}\left(F_{\mathfrak{a}}(z) - F_{\mathfrak{a}}(\gamma z) - 2\log(cz+d)\right)$$

for each $\gamma \in \Gamma$.

Lemma 4.1. *The map $\rho_{\mathfrak{a}}$ is a quasimorphism, i.e. the arising 2-cocycle*

$$d\rho_{\mathfrak{a}}(\gamma_1,\gamma_2) = \rho_{\mathfrak{a}}(\gamma_1\gamma_2) - \rho_{\mathfrak{a}}(\gamma_1) - \rho_{\mathfrak{a}}(\gamma_2)$$

is uniformly bounded. Moreover, $d\rho_{\mathfrak{a}}$ does not depend on the cusp $\mathfrak{a}$.

Proof. We can compute that

$$d\rho_{\mathfrak{a}}(\gamma_1,\gamma_2) \;=\; \frac{1}{2\pi i}\left(\log j(\gamma_1,\gamma_2 z) + \log j(\gamma_2,z) - \log j(\gamma_1\gamma_2,z)\right). \tag{4.1}$$

The RHS of (4.1) does not depend on the cusp $\mathfrak{a}$ and takes values in the set $\{0,\pm 1\}$. □

Lemma 4.2. *Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a non-cocompact lattice with a cusp at ∞ and $\Gamma_\infty \cong B$. Then $\rho_\infty\left(\left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)\right) = \frac{V}{4\pi}$.*

Proof. We record a more general transformation rule. Let $\gamma_{\mathfrak{a}}$ generate $\Gamma_{\mathfrak{a}}$, i.e. $\Gamma_{\mathfrak{a}} \cong \langle \pm\gamma_{\mathfrak{a}}\rangle$. By definition of the scaling $\sigma_{\mathfrak{a}}$, $\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)$. Then, by (3.2), $F_{\mathfrak{a}}(\sigma_{\mathfrak{a}}z) - F_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}}z) = iV$. Under the given hypotheses, we thus have $\rho_\infty\left(\left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)\right) = \frac{1}{4\pi i}(iV - 0) = \frac{V}{4\pi}$. □

We are now ready to prove (2). Let $\gamma_{\mathfrak{a}}$ generate $\Gamma_{\mathfrak{a}}$, i.e. $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}^m\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix}1&m\\&1\end{smallmatrix}\right) =: T_m$ for all $m \in \mathbb{Z}$. Take any $\gamma \in \Gamma$ such that $\gamma' = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$ has lower-left entry $c \neq 0$. Let $m,n \in \mathbb{Z}$. Then

$$\mathcal{S}_{\mathfrak{a}}\left(\gamma_{\mathfrak{a}}^m\gamma\gamma_{\mathfrak{a}}^n\right) = \frac{V}{4\pi}\frac{(a+mc)+(d+nc)}{c} - \Phi_\infty\left(\begin{pmatrix}a+mc & * \\ c & d+nc\end{pmatrix}\right),$$

where, using Lemma 4.1 and Lemma 4.2,

$$\Phi_\infty\left(\begin{pmatrix} a+mc & * \\ c & d+nc \end{pmatrix}\right) = \rho_\infty\left(T_m\gamma' T_n\right) + \frac{1}{4}\mathrm{sign}(c) =$$

$$= \rho_\infty(T_m) + \rho_\infty(T_n) + \rho_\infty(\gamma') + \frac{1}{4}\mathrm{sign}(c) + d\rho_\infty(T_m\gamma', T_n) + d\rho_\infty(T_m, \gamma')$$

$$= \frac{V}{4\pi}(m+n) + \Phi_\infty\left(\begin{pmatrix} a & * \\ c & d \end{pmatrix}\right) + d\rho_\infty(T_m\gamma', T_n) + d\rho_\infty(T_m, \gamma').$$

It should be clear from (4.1) that the last two terms vanish. We can conclude that $\mathcal{S}_{\mathfrak{a}}\left(\gamma_{\mathfrak{a}}^m \gamma \gamma_{\mathfrak{a}}^n\right) = \mathcal{S}_{\mathfrak{a}}(\gamma)$ for all $m, n \in \mathbb{Z}$.

Finally, we show that $\mathcal{S}_{\mathfrak{a}}$ does not depend on the choice of scaling $\sigma_{\mathfrak{a}}$. Fix $x \in \mathbb{R}$. Set $\sigma'_{\mathfrak{a}} = \sigma_{\mathfrak{a}} n_x$ with $n_x = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$. Let $\gamma \in \Gamma$, $\overline{\gamma} = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$. The definition of $\mathcal{S}_{\mathfrak{a}}(\gamma)$ does not depend on $\sigma_{\mathfrak{a}}$ if, for $\Phi_\infty : \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \to \mathbb{R}$ and $\Phi'_\infty : (\sigma'_{\mathfrak{a}})^{-1}\Gamma\sigma'_{\mathfrak{a}} \to \mathbb{R}$, we have $\Phi_\infty(\overline{\gamma}) = \Phi'_\infty(n_x^{-1}\overline{\gamma}n_x)$ or, $\rho_\infty(\overline{\gamma}) = \rho'_\infty(n_x^{-1}\overline{\gamma}n_x)$, which is equivalent to

$$F_\infty(z) - F_\infty(\overline{\gamma}z) - 2\log j(\overline{\gamma}, z) = F_\infty(n_x^{-1}z) - F_\infty(n_x^{-1}\overline{\gamma}z) - 2\log j(\overline{\gamma}, z).$$

Using (3.2), it is easy to see that this equality holds (note that both n_x and I are trivially scalings for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$).

5. Proof of Corollary 1

We obtain a family of functions $\eta_{\mathfrak{a}}, \eta_{\mathfrak{b}}, \ldots, \eta_{\mathfrak{m}}$, one for each Γ-inequivalent cusp of $\Gamma\backslash\mathbb{H}$. For the $\eta_{\mathfrak{a}}$-function in cusp $\mathfrak{b}$, we have

$$\begin{aligned} |\eta_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z; k_0, k)| &= e^{-\frac{k}{2}(VK_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) + \ln \mathfrak{Im}(\sigma_{\mathfrak{b}}z) + k_0)} \\ &= |j(\sigma_{\mathfrak{b}}, z)|^k\, e^{-k_0k/2} e^{-\frac{k}{2}(VK_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) + \ln y)}. \end{aligned}$$

Using once more the Fourier expansion [JO05, Eq. (4.7)], we have

$$\left|j(\sigma_{\mathfrak{b}}, z)^{-k}\eta_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z; k_0, k)\right| = O\left(e^{-\frac{kV}{2}\delta_{\mathfrak{a}\mathfrak{b}}y}\right)$$

as $y \to \infty$. Then the product function $\eta = \eta_{\mathfrak{a}}\eta_{\mathfrak{b}}\cdots\eta_{\mathfrak{m}}$ is nowhere-vanishing and has exponential decay in each cusp.

6. Proof of Theorem 3

Observe that for each $\gamma \in \Gamma$,

$$\eta_{\mathfrak{a}}(\gamma z; k) = \eta_{\mathfrak{a}}(z; k)\,(cz + d)^k\, e^{2\pi i k\rho_{\mathfrak{a}}(\gamma)}.$$

Lemma 6.1. *For each $k > 0$, $\chi_{\mathfrak{a},k}(\Gamma) = e(k\rho_{\mathfrak{a}}(\Gamma))$ defines a multiplier for Γ of weight k.*

Proof. We have indeed that

$$e\left(kd\rho_{\mathfrak{a}}(\gamma_1,\gamma_2)\right) = \frac{j(\gamma_1,\gamma_2 z)^k j(\gamma_2,z)^k}{j(\gamma_1\gamma_2,z)^k}$$

for all $\gamma_1,\gamma_2 \in \Gamma$ by (4.1). We can also check directly from the definitions that

$$\rho_{\mathfrak{a}}(-I) = \Phi_{\mathfrak{a}}(-I) - \frac{1}{4} = \frac{-1}{2\pi i}\log i - \frac{1}{4} = -\frac{1}{2},$$

hence $\chi_{\mathfrak{a},k}(-I) = e^{-\pi i k}$. $\square$

Corollary 6.2. *Let $k \in \mathbb{R}_{>0}$. Every cofinite non-cocompact Fuchsian group admits a multiplier system of weight k.*

Let $k \in \mathbb{R}_{>0}$. Consider the function $\eta_\infty(\cdot;k)$ for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$. We have, by Lemma 4.2,

$$\eta_\infty(z+1;k) = \chi_{\infty,k}\left(\begin{pmatrix} 1 & 1 \\ & 1\end{pmatrix}\right)\eta_\infty(z;k) = e\left(k\frac{V}{4\pi}\right)\eta_\infty(z;k).$$

Therefore $\eta_\infty(z;k)$ has a Fourier series at infinity of the form

$$\eta_\infty(x+iy;k) = \sum_{n\in\mathbb{Z}} a_n(y)e\left((n-\alpha_k)x\right)$$

with $\alpha_k = \left\lceil k\frac{V}{4\pi}\right\rceil - k\frac{V}{4\pi}$, where $\lceil x\rceil$ denotes the smallest integer $\geq x$. The associated generalized Kloosterman sums are

$$S_{\mathfrak{a}}(m,n;c,\chi_{\infty,k}) = \sum_{\left(\begin{smallmatrix} a & * \\ c & d\end{smallmatrix}\right)\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}/B} \overline{\chi_{\infty,k}\left(\begin{pmatrix} a & * \\ c & d\end{pmatrix}\right)}e\left(\frac{(m-\alpha_k)a+(n-\alpha_k)d}{c}\right),$$

which we recover by unfolding the definition of the Dedekind symbol $\mathcal{S}_{\mathfrak{a}}$ in the exponential sum

$$e\left(\frac{k}{4}\right)\sum e\left(k\mathcal{S}_{\mathfrak{a}}(\gamma)\right) \;=\; S_{\mathfrak{a}}\left(\left\lceil\frac{kV}{4\pi}\right\rceil, \left\lceil\frac{kV}{4\pi}\right\rceil, c, \chi_{\infty,k}\right),$$

where the sum is indexed over all double coset representatives γ of $\Gamma_{\mathfrak{a}}\backslash\Gamma/\Gamma_{\mathfrak{a}}$ such that $\left(\begin{smallmatrix} a & * \\ c & d\end{smallmatrix}\right) = \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ with $0 \leq a < c$.

7. Proof of Theorem 5

Recall that (cf. Section 2.3)

$$\varphi_{\mathfrak{a}}(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{c>0} \frac{\#\{\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} : a \in [0,c)\}}{c^{2s}} =: \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)} Z_{\mathfrak{a}}(s) \tag{7.1}$$

where, from the theory of Eisenstein series, we have that the Dirichlet series $Z_{\mathfrak{a}}(s)$ converges absolutely in the half-plane $\mathfrak{Re}(s) > 1$, and admits a meromorphic continuation to the whole plane. On $\mathfrak{Re}(s) = 1$, $Z_{\mathfrak{a}}(s)$ has a single, simple pole, at $s = 1$, with residue

$$\operatorname*{Res}_{s=1} Z_{\mathfrak{a}}(s) = \frac{1}{\pi} \operatorname*{Res}_{s=1} \varphi_{\mathfrak{a}}(s) = \frac{1}{\pi V}.$$

By the Wiener–Ikehara Theorem [Ike31],

$$N_{\mathfrak{a}}(X) := \# \left\{ \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} : \begin{array}{c} 0 \le a < c \\ c \le X \end{array} \right\} \sim \frac{X^2}{\pi V}$$

as $X \to \infty$. To deduce a more precise asymptotic, we need the meromorphic continuation of the scattering matrix $(\varphi_{\mathfrak{a}\mathfrak{b}}(s))_{\mathfrak{a},\mathfrak{b}}$ to $\mathfrak{Re}(s) \ge 1/2$. (However, the 'quality' of the attained error term will also limited by the fact that we can not push past the critical line $\mathfrak{Re}(s) = \frac{1}{2}$.) Integrating by parts, for any s with $\mathfrak{Re}(s) > 1$,

$$Z_{\mathfrak{a}}(s) = \lim_{x\to\infty} \left(\frac{N_{\mathfrak{a}}(X)}{X^{2s}} + 2s \int_0^X N_{\mathfrak{a}}(u) u^{-2s-1} du \right) = 2s N_{\mathfrak{a}}^{\sim}(2s),$$

where $N_{\mathfrak{a}}^{\sim}(s)$ denotes the Mellin transform of $N_{\mathfrak{a}}(y)$. By the Mellin Inversion Theorem, we recover the Perron formula

$$N_{\mathfrak{a}}(X) \ = \ \frac{1}{2\pi i} \int_{(\sigma_\varepsilon)} Z_{\mathfrak{a}}(s) \frac{X^{2s}}{s} ds \ := \ \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_\varepsilon - iT}^{\sigma_\varepsilon + iT} Z_{\mathfrak{a}}(s) \frac{X^{2s}}{s} ds,$$

where we fix $\sigma_\varepsilon = 1 + \varepsilon > 1$. To shift the line of integration, we need some control on the growth of $Z_{\mathfrak{a}}(s)$ along vertical lines in $\mathfrak{Re}(s) \le \sigma_\varepsilon$. Away from the real line, $\varphi_{\mathfrak{a}}(s)$ is bounded in the half-plane $\mathfrak{Re}(s) \ge 1/2$. On the line $\mathfrak{Re}(s) = \sigma_\varepsilon$, $|Z_{\mathfrak{a}}(s)| = O(1)$, while on $\mathfrak{Re}(s) = 1/2$, we have by Stirling's formula for the Γ-function that $|Z_{\mathfrak{a}}(s)| = O\left(|t|^{1/2}\right)$ as $|t| \to \infty$. By the Phragmén–Lindelöf principle, there is a linear function $g(\sigma)$ with $g(1/2) = 1/2$ and $g(\sigma_\varepsilon) = 0$ such that

$$|Z_{\mathfrak{a}}(\sigma + it)| = O\left(|t|^{g(\sigma)}\right) \tag{7.2}$$

for all $1/2 \leq \sigma \leq \sigma_\varepsilon$ as $|t| \to \infty$. Let T be a large parameter depending on X such that $T \to \infty$ as $X \to \infty$. Then

$$N_{\mathfrak{a}}(X) = \frac{1}{2\pi i} \int\limits_{\substack{\mathfrak{Re}(s)=\sigma_\varepsilon \\ |\mathfrak{Im}(s)| \leq T}} Z_{\mathfrak{a}}(s) \frac{X^{2s}}{s} ds + O\left(\frac{X^{2\sigma_\varepsilon}}{T}\right)$$

as $X \to \infty$. We then apply the Residue Theorem to the rectangular path of integration with edges $C_1 = \left[\frac{1}{2} + iT, \frac{1}{2} - iT\right], C_2 = \left[\frac{1}{2} - iT, \sigma_\varepsilon - iT\right]$, etc. and obtain using (7.2) that

$$N_{\mathfrak{a}}(X) = \frac{X^2}{\pi V} + \sum_{1/2 < \sigma_j < 1} \tau_j X^{2\sigma_j} + O\left(\frac{X^{2+\varepsilon}}{T} + T^{1/2} X\right)$$

as $X \to \infty$. The error term is minimized by choosing $T = X^{2/3}$, which gives $O\left(X^{4/3+\varepsilon}\right)$.

The above argument has the advantage to apply in the generality of the Fuchsian group setting. On the other hand, it will not produce optimal error terms. To see this, take $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Then the counting function $N_\infty(X)$ is precisely the partial sum

$$N_\infty(X) = \sum_{n=1}^{\lfloor X \rfloor} \sum_{\substack{a=1 \\ (a,n)=1}}^{n} 1 = \sum_{n=1}^{\lfloor X \rfloor} \phi(n)$$

for the Euler totient function ϕ. On the other hand, we have

$$Z_\infty(s) = \sum_{n \geq 1} \frac{\phi(n)}{n^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Then $Z_\infty(s)$ has no poles in $[1/2, 1)$, and, upon assuming the Riemann Hypothesis, there are no poles in $(1/4, 1)$. Under the Riemann Hypothesis, we can improve our estimate to

$$N_\infty(X) = \frac{3}{\pi^2} X^2 + O\left(X^{1+\varepsilon}\right)$$

for any $\varepsilon > 0$. However, by way of algebraic manipulations involving the Möbius function, we have the well-known elementary estimate

$$N_\infty(X) = \frac{3}{\pi^2} X^2 + O(X \ln X).$$

Moreover, Montgomery conjectured for the maximum order of magnitude of the remainder term

$$R_\infty(X) = N_\infty(X) - \frac{3}{\pi^2} X^2$$

that $R_\infty(X) \ll X \ln \ln X$ should hold [Mon87].

8. Proof of Theorem 4

By Theorem 3 and Weyl's criterium for equidistribution mod 1 [Wey16], the statement of Theorem 4 is equivalent to

$$\sum_{c \le X} S_{\mathfrak{a}}\left(\left\lceil \frac{kV}{4\pi} \right\rceil, \left\lceil \frac{kV}{4\pi} \right\rceil, c, \chi_{\infty,k}\right) = o\left(N_{\mathfrak{a}}(X)\right)$$

(for each $k > 0$) as $X \to \infty$. This is established by comparing the asymptotics of (2.1) and Theorem 5.

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