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# REPRESENTATIONS OF INTEGERS BY CERTAIN $2k$ -ARY QUADRATIC FORMS

DONGXI YE

*Dedicated to my newborn nephew Haolin Li*

ABSTRACT. Suppose  $k$  is a positive integer. In this work, we establish formulas for the number of representations of integers by the quadratic forms

$$x_1^2 + \cdots + x_k^2 + m(x_{k+1}^2 + \cdots + x_{2k}^2)$$

for  $m \in \{2, 4\}$ .

## 1. INTRODUCTION

In the long history of number theory, one of the classical problems is to give an explicit formula for the number of ways that one can express a positive integer  $n$  as a sum of  $2k$  squares, that is, the number of integral solutions of

$$x_1^2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k}^2 = n,$$

which we denote by  $\mathcal{R}_k(n)$ . It is known from the theory of modular forms that in general,

$$\mathcal{R}_k(n) = \Delta_k(n) + \mathcal{E}_k(n)$$

where  $\Delta_k(n)$  is a divisor function and  $\mathcal{E}_k(n)$  is a function of order substantially lower than that of  $\Delta_k(n)$ . Formulas for  $\mathcal{R}_k(n)$  in this fashion have been found and studied by various mathematicians. For  $k = 1, 2, 3$  and  $4$ , i.e., sums of 2, 4, 6 and 8 squares, (reformulated) formulas for  $\mathcal{R}_k(n)$  are originally due to Jacobi [16],

$$(1.1) \quad \mathcal{R}_1(n) = 4 \sum_{d|n} \left( \frac{-4}{d} \right),$$

$$(1.2) \quad \mathcal{R}_2(n) = 8 \sum_{d|n} d - 32 \sum_{d|\frac{n}{4}} d,$$

$$(1.3) \quad \mathcal{R}_3(n) = -4 \sum_{d|n} \left( \frac{-4}{d} \right) d^2 + 16 \sum_{d|n} \left( \frac{-4}{n/d} \right) d^2,$$

$$(1.4) \quad \mathcal{R}_4(n) = 16 \sum_{d|n} d^3 - 32 \sum_{d|\frac{n}{2}} d^3 + 256 \sum_{d|\frac{n}{4}} d^3$$

where, here and throughout this work,  $(\cdot)$  denotes the Kronecker symbol. The result for  $k = 5$ , i.e., sum of 10 squares, was due (without proof) in part to Eisenstein [12], and fully described (without proof) by Liouville [19]. The results for  $1 \leq k \leq 9$  were all proved by Glaisher [13]. In around

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1916, this classical problem was “completely” solved (without proof) by Ramanujan [28], [29, Eqs. (145)–(147)]. To state Ramanujan’s result, we need the well-known Dedekind eta function

$$\eta(\tau) := q^{1/24} \prod_{j=1}^{\infty} (1 - q^j)$$

where, here and throughout this paper,  $\tau$  denotes a complex number with positive imaginary part and  $q = e^{2\pi i\tau}$ . For brevity, throughout this work, we write  $\eta_m$  for  $\eta(m\tau)$  for any positive integer  $m$ . In addition, we define  $\chi_D(\cdot)$  to be the Kronecker symbol  $(\frac{D}{\cdot})$ , and define  $\sigma_k(n)$ ,  $\sigma_{k,\chi_D}^{\infty}(n)$  and  $\sigma_{k,\chi_D}^0(n)$  to be the divisor functions

$$\begin{aligned}\sigma_k(n) &= \sum_{d|n} d^k, \\ \sigma_{k,\chi_D}^{\infty}(n) &= \sum_{d|n} \chi_D(d) d^k, \\ \sigma_{k,\chi_D}^0(n) &= \sum_{d|n} \chi_D(n/d) d^k\end{aligned}$$

with the convention that they are defined to be 0 if  $n$  is not a positive integer. Now we reformulate and summarize Ramanujan’s result in Theorem 1.1 below.

**Theorem 1.1** (Ramanujan). *Suppose  $k$  is a positive integer. Then there are unique rational numbers  $c_{j,k}$  depending on  $j$  and  $k$  such that*

$$\begin{aligned}\mathcal{R}_k(n) &= \begin{cases} -\frac{2k}{B_k} \left( \frac{(-1)^{k/2} \sigma_{k-1}(n) - (1 + (-1)^{k/2}) \sigma_{k-1}(n/2) + 2^k \sigma_{k-1}(n/4)}{-1 + 2^k} \right) & \text{if } k \text{ is even,} \\ -\frac{2k}{B_{k,4}} \left( \frac{\sigma_{k-1,\chi_{-4}}^{\infty}(n) + (-1)^{(k-1)/2} 2^{k-1} \sigma_{k-1,\chi_{-4}}^0(n/4)}{1 + \delta_{k,1}} \right) & \text{if } k \text{ is odd} \end{cases} \\ &\quad + \sum_{1 \leq j \leq (k-1)/4} c_{j,k} a_{j,k}(n)\end{aligned}$$

where, here and throughout this paper,  $\delta_{\cdot,\cdot}$  denotes the Kronecker delta,  $B_k$  is the  $k$ th ordinary Bernoulli number,  $B_{k,4}$  is the  $k$ th generalized Bernoulli number of order 4 defined via

$$\frac{t}{e^{4t} - 1} \sum_{j=1}^4 \chi_{-4}(j) e^{jt} = \sum_{n=0}^{\infty} B_{n,4} \frac{t^n}{n!},$$

and the numbers  $a_{j,k}(n)$  are defined via

$$\sum_{n=0}^{\infty} a_{j,k}(n) q^n = \frac{\eta_2^{10k}}{(\eta_1 \eta_4)^{4k}} \times \frac{(\eta_1 \eta_4)^{24j}}{\eta_2^{48j}}.$$

Theorem 1.1 was proved first by Mordell [23] utilizing the theory of modular forms. An elementary proof was given by Cooper in [10] by making skillful use of Ramanujan’s  ${}_1\psi_1$  formula.

In recent work [11], Cooper, Kane and the author extended Ramanujan’s results and determined equivalently the number of integral solutions of

$$x_1^2 + \cdots + x_k^2 + p \binom{x_{k+1}^2 + \cdots + x_{2k}^2}{2} = n$$

for  $p \in \{3, 7, 11, 23\}$  along the “divisor function + lower order term” fashion. Motivated by the work of Ramanujan, and Cooper *et al.*, in this work we aim to establish analogous formulas for the number of representations of integers by the quadratic forms

$$x_1^2 + \cdots + x_k^2 + m(x_{k+1}^2 + \cdots + x_{2k}^2)$$

for  $m \in \{2, 4\}$ .

This work is organized as follows. In Section 2, we state our main results, and as illustrations, we also present some examples that follow from the general case we obtain. Proofs will be given in Section 3. In the last section, we conclude this work with some remarks, which explain the existence of these Ramanujan-Mordell type formulas.

## 2. STATEMENT OF RESULTS

Let us denote by  $r(1^k m^k; n)$  the number of integral solutions of the equation

$$x_1^2 + \cdots + x_k^2 + m(x_{k+1}^2 + \cdots + x_{2k}^2) = n.$$

The main results of this work are summarized in the following theorem.

**Theorem 2.1.** *Suppose  $k$  is a positive integer.*

(1) *Let  $\ell_2$  be defined by*

$$\ell_2 = \begin{cases} \frac{k-1}{2} & \text{if } k \text{ is odd,} \\ \frac{k-2}{2} & \text{if } k \text{ is even.} \end{cases}$$

*Then there are unique rational numbers  $c_{j,k,2}$  depending on  $j$  and  $k$  such that*

$$\begin{aligned} & r(1^k 2^k; n) \\ &= \begin{cases} -\frac{k}{B_{k,8}} \left( \frac{2\sigma_{k-1,\chi_{-2}}^\infty(n) + 2(-8)^{(k-1)/2} \sigma_{k-1,\chi_{-2}}^0(n)}{1 + \delta_{k,1}} \right) & \text{if } k \text{ is odd,} \\ -\frac{k}{B_k} \left( \frac{(-1)^{\frac{k}{2}} \sigma_{k-1}(n) - (-1)^{\frac{k}{2}} \sigma_{k-1}\left(\frac{n}{2}\right) - 2^{\frac{k}{2}} \sigma_{k-1}\left(\frac{n}{4}\right) + 8^{\frac{k}{2}} \sigma_{k-1}\left(\frac{n}{8}\right)}{2^{\frac{k}{2}-1}(2^k - 1)} \right) & \text{if } k \text{ is even} \end{cases} \\ &+ \sum_{j=1}^{\ell_2} c_{j,k,2} a_{j,k,2}(n) \end{aligned}$$

*where  $\chi_{-2}$ ,  $\sigma_k(n)$ ,  $\sigma_{k,\chi_{-2}}^\infty(n)$  and  $\sigma_{k,\chi_{-2}}^0(n)$  are as defined in Section 1, and  $B_{k,8}$  is the  $k$ th generalized Bernoulli of order 8 defined via*

$$\frac{t}{e^{8t} - 1} \sum_{j=1}^8 \chi_{-2}(j) e^{jt} = \sum_{n=0}^{\infty} B_{n,8} \frac{t^n}{n!},$$

*and the numbers  $a_{j,k,2}(n)$  are given by*

$$\sum_{n=0}^{\infty} a_{j,k,2}(n) q^n = \frac{(\eta_2 \eta_4)^{3k}}{(\eta_1 \eta_8)^{2k}} \times \left( \frac{\eta_1 \eta_8}{\eta_2 \eta_4} \right)^{8j}.$$

(2) *Let  $\ell_4$  be defined by*

$$\ell_4 = \begin{cases} k-2 & \text{if } k \text{ is odd,} \\ k-1 & \text{if } k \text{ is even.} \end{cases}$$

*Then there are unique rational numbers  $c_{j,k,4}$  depending on  $j$  and  $k$  such that*

$$r(1^k 4^k; n)$$

$$\begin{aligned}
 &= \begin{cases} 2\sigma_0^\infty(n) - 2\sigma_0^\infty\left(\frac{n}{2}\right) + 4\sigma_0^\infty\left(\frac{n}{4}\right) & \text{if } k = 1, \\ -\frac{k}{B_{k,4}} \left( (-1)^{\frac{k+1}{2}} \sigma_{k-1}^\infty(n) - (-1)^{\frac{k+1}{2}} \sigma_{k-1}^\infty\left(\frac{n}{2}\right) + 2\sigma_{k-1}^\infty\left(\frac{n}{4}\right) \right. \\ \quad \left. - (-1)^{\frac{k+1}{2}} \sigma_{k-1}^0(n) + 2^{k-1} \sigma_{k-1}^0\left(\frac{n}{2}\right) - 2^{2k-1} \sigma_{k-1}^0\left(\frac{n}{4}\right) \right) & \text{if } k \geq 3 \text{ and } k \text{ is odd,} \\ -\frac{k}{B_k} \left( \frac{(-1)^{\frac{k}{2}} \sigma_{k-1}(n) - (-1)^{\frac{k}{2}} \sigma_{k-1}\left(\frac{n}{2}\right) - 2^k \sigma_{k-1}\left(\frac{n}{8}\right) + 4^k \sigma_{k-1}\left(\frac{n}{16}\right)}{2^{k-1}(2^k - 1)} \right) & \text{if } k \text{ is even} \end{cases} \\
 &+ \sum_{j=1}^{\ell_4} c_{j,k,4} a_{j,k,4}(n)
 \end{aligned}$$

where the divisor functions  $\sigma_k(n)$ ,  $\sigma_{k,\chi_{-4}}^\infty(n)$  and  $\sigma_{k,\chi_{-4}}^0(n)$  are as defined in Section 1, and the numbers  $a_{j,k,4}(n)$  are given by

$$\sum_{n=0}^{\infty} a_{j,k,4}(n) q^n = \frac{(\eta_2 \eta_8)^{5k}}{(\eta_1 \eta_4^2 \eta_{16})^{2k}} \times \frac{(\eta_1 \eta_4 \eta_{16})^{4j}}{(\eta_2 \eta_8)^{6j}}.$$

For  $k = 1, 2, 3$  or  $4$ , Theorem 2.1 gives the following analogues of (1.1)–(1.4).

$$(2.1) \quad r(1^1 2^1; n) = 2 \sum_{d|n} \left( \frac{-2}{d} \right),$$

$$(2.2) \quad r(1^1 4^1; n) = 2 \sum_{d|n} \left( \frac{-4}{d} \right) - 2 \sum_{d|\frac{n}{2}} \left( \frac{-4}{d} \right) + 4 \sum_{d|\frac{n}{4}} \left( \frac{-4}{d} \right),$$

$$(2.3) \quad r(1^2 2^2; n) = 4 \sum_{d|n} d - 4 \sum_{d|\frac{n}{2}} d + 8 \sum_{d|\frac{n}{4}} d - 32 \sum_{d|\frac{n}{8}} d,$$

$$(2.4) \quad r(1^2 4^2; n) = 2 \sum_{d|n} d - 2 \sum_{d|\frac{n}{2}} d + 8 \sum_{d|\frac{n}{8}} d - 32 \sum_{d|\frac{n}{16}} d + 2a_{1,2,4}(n),$$

$$(2.5) \quad r(1^3 2^3; n) = -\frac{2}{3} \sum_{d|n} \left( \frac{-2}{d} \right) d^2 + \frac{16}{3} \sum_{d|n} \left( \frac{-2}{n/d} \right) d^2 + \frac{4}{3} a_{1,3,2}(n),$$

$$\begin{aligned}
 (2.6) \quad r(1^3 4^3; n) &= -2 \sum_{d|n} \left( \frac{-4}{d} \right) d^2 + 2 \sum_{d|\frac{n}{2}} \left( \frac{-4}{d} \right) d^2 - 4 \sum_{d|\frac{n}{4}} \left( \frac{-4}{d} \right) d^2 \\
 &+ 2 \sum_{d|n} \left( \frac{-4}{n/d} \right) d^2 - 8 \sum_{d|\frac{n}{2}} \left( \frac{-4}{n/2d} \right) d^2 + 64 \sum_{d|\frac{n}{4}} \left( \frac{-4}{n/4d} \right) d^2 + 6a_{1,3,4}(n),
 \end{aligned}$$

$$(2.7) \quad r(1^4 2^4; n) = 4 \sum_{d|n} d^3 - 4 \sum_{d|\frac{n}{2}} d^3 - 16 \sum_{d|\frac{n}{4}} d^3 + 64 \sum_{d|\frac{n}{8}} d^3 + 4a_{1,4,2}(n),$$

$$(2.8) \quad r(1^4 4^4; n) = \sum_{d|n} d^3 - \sum_{d|\frac{n}{2}} d^3 - 16 \sum_{d|\frac{n}{8}} d^3 + 256 \sum_{d|\frac{n}{16}} d^3 + 7a_{1,4,4}(n) - 12a_{2,4,4}(n).$$

The formula (2.1) was proved by Shen [31]. The formula (2.2) was due in part to Ramanujan [7, Entry 25(i), (iii), p. 40], [8, Entry 18, p. 152]. The formulas (2.3) and (2.4) were stated without proof by Liouville [20, 21] and proved by Pepin [26, 27], Bachmann [6], and Alaca *et al.* [1]. The formulas (2.5), and (2.6)–(2.8) are due to Alaca *et al.* [2] and Alaca *et al.* [3, 4, 5], respectively.

Now if we consider the generating function of  $r(1^k m^k; n)$ , we note that

$$\sum_{n=0}^{\infty} r(1^k m^k; n) q^n = (\theta(\tau) \theta(m\tau))^k$$

where  $\theta(\tau)$  is Ramanujan's theta function defined by

$$(2.9) \quad \theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{\eta_2^5}{\eta_1^2 \eta_4^2}$$

where the  $\eta$ -quotient representation after the second equality is due to Jacobi [16]. In view of that  $r(1^k m^k; n)$  is the  $n$ th Fourier coefficient of  $(\theta(\tau) \theta(m\tau))^k$  and the  $\eta$ -quotient representation (2.9), Theorem 2.1 is equivalent to the following theorem.

**Theorem 2.2.** *Suppose  $k$  is a positive integer. Let  $E_k(\tau)$  be the normalized Eisenstein series of weight  $k$  on  $SL_2(\mathbb{Z})$  defined by*

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

(1) Let  $\ell_2$  be as defined in Theorem 2.1(1). Let  $F_{k,2}(\tau)$  be defined by

$$F_{k,2}(\tau) = \begin{cases} \frac{E_{k,\chi_{-2}}^{\infty}(\tau) + (-8)^{(k-1)/2} E_{k,\chi_{-2}}^0(\tau)}{1 + \delta_{k,1}} & \text{if } k \text{ is odd,} \\ \frac{(-1)^{\frac{k}{2}} E_k(\tau) - (-1)^{\frac{k}{2}} E_k(2\tau) - 2^{\frac{k}{2}} E_k(4\tau) + 8^{\frac{k}{2}} E_k(8\tau)}{2^{\frac{k}{2}} (2^k - 1)} & \text{if } k \text{ is even,} \end{cases}$$

where  $E_{k,\chi_{-2}}^{\infty}(\tau)$  and  $E_{k,\chi_{-2}}^0(\tau)$  are Eisenstein series of weight  $k$  on  $\Gamma_0(8)$  with character  $\chi_{-2}$  defined by

$$E_{k,\chi_{-2}}^{\infty}(\tau) = 1 - \frac{2k}{B_{k,8}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi_{-2}}^{\infty}(n) q^n$$

and

$$E_{k,\chi_{-2}}^0(\tau) = \delta_{k,1} - \frac{2k}{B_{k,8}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi_{-2}}^0(n) q^n,$$

and let  $x_2 = x_2(\tau)$  be defined by

$$x_2(\tau) = \left( \frac{\eta_1 \eta_8}{\eta_2 \eta_4} \right)^8.$$

Then there are unique rational numbers  $c_{j,k,2}$  depending on  $j$  and  $k$  such that

$$(2.10) \quad (\theta(\tau) \theta(2\tau))^k = F_{k,2}(\tau) + (\theta(\tau) \theta(2\tau))^k \sum_{j=1}^{\ell_2} c_{j,k,2} x_2^j.$$

(2) Let  $\ell_4$  be as defined in Theorem 2.1(2). Let  $F_{k,4}(\tau)$  be defined by

$$F_{k,4}(\tau) = \begin{cases} \frac{1}{2} \left( E_{1,\chi_2}^\infty(\tau) - E_{1,\chi_2}^\infty(2\tau) + 2E_{1,\chi_2}^\infty(4\tau) \right) & \text{if } k = 1, \\ \frac{1}{2} \begin{pmatrix} (-1)^{\frac{k+1}{2}} E_{k,\chi_{-2}}^\infty(\tau) - (-1)^{\frac{k+1}{2}} E_{k,\chi_{-2}}^\infty(2\tau) + 2E_{k,\chi_{-2}}^\infty(4\tau) \\ -(-1)^{\frac{k+1}{2}} E_{k,\chi_{-2}}^0(\tau) + 2^{k-1} E_{k,\chi_{-2}}^0(2\tau) - 2^{2k-1} E_{k,\chi_{-2}}^0(4\tau) \end{pmatrix} & \text{if } k \geq 3 \text{ and is odd,} \\ \frac{(-1)^{\frac{k}{2}} E_k(\tau) - (-1)^{\frac{k}{2}} E_k(2\tau) - 2^k E_k(8\tau) + 4^k E_k(16\tau)}{2^k(2^k - 1)} & \text{if } k \text{ is even,} \end{cases}$$

where  $E_{k,\chi_{-4}}^\infty(\tau)$  and  $E_{k,\chi_{-4}}^0(\tau)$  are Eisenstein series of weight  $k$  on  $\Gamma_0(4)$  with character  $\chi_{-4}$  defined by

$$E_{k,\chi_{-4}}^\infty(\tau) = 1 - \frac{2k}{B_{k,4}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi_{-4}}^\infty(n) q^n$$

and

$$E_{k,\chi_{-4}}^0(\tau) = \delta_{k,1} - \frac{2k}{B_{k,4}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi_{-4}}^0(n) q^n,$$

and let  $x_4 = x_4(\tau)$  be defined by

$$x_4(\tau) = \frac{(\eta_1 \eta_4 \eta_{16})^4}{(\eta_2 \eta_8)^6}.$$

Then there are unique rational numbers  $c_{j,k,4}$  depending on  $j$  and  $k$  such that

$$(2.11) \quad (\theta(\tau)\theta(4\tau))^k = F_{k,4}(\tau) + (\theta(\tau)\theta(4\tau))^k \sum_{j=1}^{\ell_4} c_{j,k,4} x_4^j.$$

### 3. PROOF OF RESULTS

This section is devoted to proving Theorem 2.2. The proof hinges on the following preliminary results.

**Lemma 3.1.** Let  $E_k(\tau)$ ,  $E_{k,\chi_{-2}}^\infty(\tau)$ ,  $E_{k,\chi_{-2}}^0(\tau)$ ,  $E_{k,\chi_{-4}}^\infty(\tau)$  and  $E_{k,\chi_{-4}}^0(\tau)$  be as defined in Theorem 2.2. Then under the transformation  $\tau \rightarrow \tau + \frac{1}{2}$ , the following identities hold.

When  $k$  is even,

$$E_k\left(\tau + \frac{1}{2}\right) = -E_k(\tau) + (2^k + 2)E_k(2\tau) - 2^k E_k(4\tau);$$

when  $k$  is odd,

$$E_{k,\chi_{-2}}^\infty\left(\tau + \frac{1}{2}\right) = -E_{k,\chi_{-2}}^\infty(\tau) + 2E_{k,\chi_{-2}}^\infty(2\tau),$$

$$E_{k,\chi_{-2}}^0\left(\tau + \frac{1}{2}\right) = -E_{k,\chi_{-2}}^0(\tau) + 2^k E_{k,\chi_{-2}}^0(2\tau),$$

$$E_{k,\chi_{-4}}^\infty\left(\tau + \frac{1}{2}\right) = -E_{k,\chi_{-4}}^\infty(\tau) + 2E_{k,\chi_{-4}}^\infty(2\tau),$$

$$E_{k,\chi_{-4}}^0\left(\tau + \frac{1}{2}\right) = -E_{k,\chi_{-4}}^0(\tau) + 2^k E_{k,\chi_{-4}}^0(2\tau).$$

*Proof.* The proofs are similar to that of [11, Lemma 3.2], so we omit the details.  $\square$

**Lemma 3.2.** Let  $\text{ord}_z(f)$  denote the order of vanishing of  $f(\tau)$  at  $z$ . Let  $F_{k,2}(\tau)$  and  $F_{k,4}(\tau)$  be as defined in Theorem 2.2. Then we have

$$(3.1) \quad \text{ord}_{1/2}(F_{k,2}) = \begin{cases} \frac{1}{2} & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even,} \end{cases}$$

$$(3.2) \quad \text{ord}_{1/2}(F_{k,4}) = \begin{cases} 2 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* By the well-known transformation formulas for Let  $E_k(\tau)$ ,  $E_{k,\chi_{-2}}^\infty(\tau)$  and  $E_{k,\chi_{-2}}^0(\tau)$ , see e.g., [17] and [30, pp. 79–83], and Lemma 3.1, we can deduce that for  $k$  odd,

$$\begin{aligned} & \left[ E_{k,\chi_{-2}}^\infty(\tau) + (-8)^{(k-1)/2} E_{k,\chi_{-2}}^0(\tau) \right] \Big|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= (2\tau + 1)^{-k} \left[ E_{k,\chi_{-2}}^\infty \left( \frac{1}{2} + \frac{-1}{4\tau + 2} \right) + (-8)^{(k-1)/2} E_{k,\chi_{-2}}^0 \left( \frac{1}{2} + \frac{-1}{4\tau + 2} \right) \right] \\ &= (2\tau + 1)^{-k} \left\{ -E_{k,\chi_{-2}}^\infty \left( \frac{-1}{4\tau + 2} \right) + 2E_{k,\chi_{-2}}^\infty \left( \frac{-1}{2\tau + 1} \right) \right. \\ & \quad \left. + (-8)^{(k-1)/2} \left[ -E_{k,\chi_{-2}}^0 \left( \frac{-1}{4\tau + 2} \right) + 2^k E_{k,\chi_{-2}}^0 \left( \frac{-1}{2\tau + 1} \right) \right] \right\} \\ &= \frac{i2^k}{8^{1/2}} E_{k,\chi_{-2}}^0 \left( \frac{\tau}{2} + \frac{1}{4} \right) - \frac{2i}{8^{1/2}} E_{k,\chi_{-2}}^0 \left( \frac{\tau}{4} + \frac{1}{8} \right) \\ & \quad + \frac{i8^{1/2}(-8)^{(k-1)/2}}{4^k} \left[ E_{k,\chi_{-2}}^\infty \left( \frac{\tau}{2} + \frac{1}{4} \right) - E_{k,\chi_{-2}}^\infty \left( \frac{\tau}{4} + \frac{1}{8} \right) \right] \\ &= Cq^{1/4} + O(q^{1/2}) \end{aligned}$$

for some nonzero constant  $C$  as  $\tau \rightarrow i\infty$ . For  $k = 2$ ,

$$\begin{aligned} & \left[ -E_2(\tau) + E_2(2\tau) - 2E_2(4\tau) + 8E_2(8\tau) \right] \Big|_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= (2\tau + 1)^{-2} \left[ -E_2 \left( \frac{1}{2} + \frac{-1}{4\tau + 2} \right) + E_2 \left( 1 + \frac{-1}{2\tau + 1} \right) \right. \\ & \quad \left. - 2E_2 \left( 2 + \frac{-1}{(2\tau + 1)/2} \right) + 8E_2 \left( 4 + \frac{-1}{(2\tau + 1)/4} \right) \right] \\ &= (2\tau + 1)^{-2} \left[ E_2 \left( \frac{-1}{4\tau + 2} \right) - 5E_2 \left( \frac{-1}{2\tau + 1} \right) \right. \\ & \quad \left. + 2E_2 \left( \frac{-1}{(2\tau + 1)/2} \right) + 8E_2 \left( \frac{-1}{(2\tau + 1)/4} \right) \right] \\ &= 4E_2(4\tau + 2) + \frac{12}{\pi i(2\tau + 1)} - 5E_2(2\tau + 1) - \frac{30}{\pi i(2\tau + 1)} \\ & \quad + \frac{1}{2}E_2 \left( \tau + \frac{1}{2} \right) + \frac{6}{\pi i(2\tau + 1)} + \frac{1}{2}E_2 \left( \frac{\tau}{2} + \frac{1}{4} \right) + \frac{12}{\pi i(2\tau + 1)} \end{aligned}$$

$$\begin{aligned}
 &= 4E_2(4\tau + 2) - 5E_2(2\tau + 1) + \frac{1}{2}E_2\left(\tau + \frac{1}{2}\right) + \frac{1}{2}E_2\left(\frac{\tau}{2} + \frac{1}{4}\right) \\
 &= Cq^{1/2} + O(q)
 \end{aligned}$$

for some nonzero constant  $C$  as  $\tau \rightarrow i\infty$ , and for  $k \geq 4$  and even,

$$\begin{aligned}
 &\left[(-1)^{\frac{k}{2}}E_k(\tau) - (-1)^{\frac{k}{2}}E_k(2\tau) - 2^{\frac{k}{2}}E_k(4\tau) + 8^{\frac{k}{2}}E_k(8\tau)\right] \Big|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\
 &= (2\tau + 1)^{-k} \left[ (-1)^{\frac{k}{2}}E_k\left(\frac{1}{2} + \frac{-1}{4\tau + 2}\right) - (-1)^{\frac{k}{2}}E_k\left(1 + \frac{-1}{2\tau + 1}\right) \right. \\
 &\quad \left. - 2^{\frac{k}{2}}E_k\left(2 + \frac{-1}{(2\tau + 1)/2}\right) + 8^{\frac{k}{2}}E_k\left(4 + \frac{-1}{(2\tau + 1)/4}\right) \right] \\
 &= (2\tau + 1)^{-k} \left[ -(-1)^{\frac{k}{2}}E_k\left(\frac{-1}{4\tau + 2}\right) + (-1)^{\frac{k}{2}}(2^k + 2)E_k\left(\frac{-1}{2\tau + 1}\right) \right. \\
 &\quad \left. - (-1)^{\frac{k}{2}}2^kE_k\left(\frac{-1}{(2\tau + 1)/2}\right) - (-1)^{\frac{k}{2}}E_k\left(\frac{-1}{2\tau + 1}\right) \right. \\
 &\quad \left. - 2^{\frac{k}{2}}E_k\left(\frac{-1}{(2\tau + 1)/2}\right) + 8^{\frac{k}{2}}E_k\left(\frac{-1}{(2\tau + 1)/4}\right) \right] \\
 &= -(-1)^{\frac{k}{2}}2^kE_k(4\tau + 2) + (-1)^{\frac{k}{2}}(2^k + 2)E_k(2\tau + 1) - (-1)^{\frac{k}{2}}E_k\left(\tau + \frac{1}{2}\right) \\
 &\quad - (-1)^{\frac{k}{2}}E_k(2\tau + 1) - 2^{-\frac{k}{2}}E_k\left(\tau + \frac{1}{2}\right) + 2^{-\frac{k}{2}}E_k\left(\frac{\tau}{2} + \frac{1}{4}\right) \\
 &= Cq^{1/2} + O(q)
 \end{aligned}$$

for some nonzero constant  $C$  as  $\tau \rightarrow i\infty$ . Together with the fact that the width of  $\frac{1}{2}$  of  $\Gamma_0(8)$  is 2, the above observations conclude (3.1).

Making use of corresponding transformation formulas of  $E_{k,\chi_{-4}}^\infty(\tau)$  and  $E_{k,\chi_{-4}}^0(\tau)$  and Lemma 3.1 together with the fact that the width of  $\frac{1}{2}$  of  $\Gamma_0(16)$  is 4, we can prove (3.2) in a similar fashion, so we omit the details.  $\square$

**Lemma 3.3.** *If  $f(\tau) = \prod_{d|N} \eta_d^{r_d}$  for some positive integer  $N$  with  $k = \frac{1}{2} \sum_{d|N} r_d \in \mathbb{Z}$ , with the additional properties that*

$$\sum_{d|N} dr_d \equiv 0 \pmod{24}$$

and

$$\sum_{d|N} \frac{N}{d} r_d \equiv 0 \pmod{24},$$

then  $f(\tau)$  satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau)$$

for every  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ . Here the character  $\chi$  is defined by Jacobi symbol  $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$  where  $s = \prod_{d|N} d^{r_d}$ .

*Proof.* See Gordon and Hughes [14], or Newman [24, 25].  $\square$

**Lemma 3.4.** *Let  $a$ ,  $c$  and  $N$  be positive integers with  $c|N$  and  $\gcd(a, c) = 1$ . If  $f(\tau) = \prod_{d|N} \eta_d^{r_d}$  satisfies the conditions of Lemma 3.3 for  $N$ , then the order of vanishing  $\text{ord}_{a/c}(f)$  of  $f(\tau)$  at the cusp  $a/c$  is*

$$\frac{N}{24} \sum_{d|N} \frac{\gcd(c, d)^2 r_d}{\gcd(c, N/c) cd}.$$

*Proof.* See Biagioli [9], Ligozat [18] or Martin [22]. □

*Proof of Theorem 2.2.* Let  $m \in \{2, 4\}$ . Let  $\ell_m$  be as defined in Theorem 2.1. Consider the functions

$$f_m(\tau) = \frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m(\tau)^{\ell_m}} \quad \text{and} \quad g_m(\tau) = \frac{1}{x_m(\tau)}.$$

Both  $f_m(\tau)$  and  $g_m(\tau)$  are analytic on the upper half plane  $\mathbb{H}$ . Employing the transformation formulas for  $E_k(\tau)$ ,  $E_{k,\chi_{-2}}^\infty(\tau)$ ,  $E_{k,\chi_{-2}}^0(\tau)$ ,  $E_{k,\chi_{-4}}^\infty(\tau)$  and  $E_{k,\chi_{-4}}^0(\tau)$ , see e.g., [17] and [30, pp. 79–83], and Lemma 3.3 we may verify that both  $f_m(\tau)$  and  $g_m(\tau)$  are invariant under  $\Gamma_0(4m)$  and  $\begin{pmatrix} 0 & 1 \\ -4m & 0 \end{pmatrix}$ . Therefore, both  $f_m(\tau)$  and  $g_m(\tau)$  are invariant under  $\Gamma_0(4m)^+$ , the group obtained

from  $\Gamma_0(4m)$  by adjoining its Fricke involution  $\begin{pmatrix} 0 & 1 \\ -4m & 0 \end{pmatrix}$ . Let us analyze the behavior at  $\tau = \infty$ .

By observing the  $q$ -expansions, we find that  $f_m(\tau)$  has rational coefficients, and

$$f_m(\tau) = \frac{1 + O(q)}{(1 + O(q))^k q^{\ell_m} (1 + O(q))^{\ell_m}} = q^{-\ell_m} + O(q^{-\ell_m+1}).$$

Therefore  $f_m(\tau)$  has a pole of order  $\ell_m$  at  $\infty$ . Similarly, we note that  $g_m(\tau)$  has a simple pole at  $\tau = \infty$ . It implies that there exist rational constants  $a_{1,k,m}, \dots, a_{\ell_m,k,m}$  such that the function

$$h_m(\tau) := f_m(\tau) - \sum_{j=1}^{\ell_m} a_{j,k,m} g_m(\tau)^j$$

has no pole at  $\tau = \infty$ , that is,

$$h_m(\tau) = a_{0,k,m} + O(q) \quad \text{as } \tau \rightarrow \infty$$

for some constant  $a_0$ . Let us consider the behavior of  $h_m(\tau)$  at  $\tau = \frac{1}{2}$ . By Lemma 3.2, we have

$$\text{ord}_{1/2}(F_{k,m}) = \begin{cases} \frac{1}{2} & \text{if } m = 2 \text{ and } k \text{ is odd,} \\ 1 & \text{if } m = 2 \text{ and } k \text{ is even,} \\ 2 & \text{if } m = 4 \text{ and } k \text{ is odd,} \\ 1 & \text{if } m = 4 \text{ and } k \text{ is even.} \end{cases}$$

Moreover, by Lemma 3.4 together with the  $\eta$ -quotient representation (2.9) of  $\theta(\tau)$ ,

$$\theta(\tau) = \frac{\eta_2^5}{\eta_1^2 \eta_4^2},$$

we can compute that

$$\text{ord}_{\mathfrak{c}_m}(\theta(\tau)\theta(m\tau)) = \begin{cases} \frac{1}{2} & \text{if } m = 2 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 1 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 0 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}, \end{cases}$$

and

$$\text{ord}_{\mathfrak{c}_m}(x_m) = \begin{cases} -1 & \text{if } m = 2 \text{ or } 4 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 0 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}. \end{cases}$$

Here  $\mathfrak{c}_m$  denotes a cusp of  $\Gamma_0(4m)$ . It is clear that  $\text{ord}_{\mathfrak{c}_m}(F_{k,m}) \geq 0$  for  $m = 4$  and  $\mathfrak{c}_m = \frac{1}{4}$ . Thus we summarize that

$$\text{ord}_{\mathfrak{c}_m}(h_m) = 0 \quad \text{if } m = 2 \text{ or } 4 \text{ and } \mathfrak{c}_m = \frac{1}{2},$$

and

$$\text{ord}_{\mathfrak{c}_m}(h_m) \geq 0 \quad \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}.$$

Since the set of inequivalent cusps of  $\Gamma_0(4m)^+$  is  $\{\infty, \frac{1}{2}\}$  if  $m = 2$ , or is  $\{\infty, \frac{1}{2}, \frac{1}{4}\}$  if  $m = 4$ , it follows that  $h_m(\tau)$  is holomorphic on  $X(\Gamma_0(4m)^+) = \Gamma_0(4m)^+ \backslash \mathbb{H}$ , and thus  $h_m(\tau)$  is a constant, that is,  $h_m(\tau) \equiv a_{0,k,m}$ . Moreover, since  $\text{ord}_{1/2}(h_m) = 0$ ,  $h_m(\tau)$  does not vanish at  $\frac{1}{2}$  and thus  $a_{0,k,m} \neq 0$ . Therefore, we have

$$f_m(\tau) = \sum_{j=0}^{\ell_m} a_{j,k,m} g_m(\tau)^j,$$

which is equivalent to

$$F_{k,m}(\tau) = (\theta(\tau)\theta(m\tau))^k \sum_{j=0}^{\ell_m} a_{j,k,m} x_m^{\ell-j} = (\theta(\tau)\theta(m\tau))^k \sum_{j=0}^{\ell_m} b_{j,k,m} x_m^j,$$

where  $b_{j,k,m} = a_{\ell_m-j,k,m}$ . Equating the constant term shows that  $b_{0,k,m} = 1$ . Now take  $b_{j,k,m} = -c_{j,k,m}$  to complete the proof.  $\square$

#### 4. CONCLUDING REMARKS

In this section, we conclude this work with some remarks on the essence of existence of (2.10) and (2.11), and an explanation of why the upper indices of the sums on the right hand sides of (2.10) and (2.11) cannot be improved further based on the functions  $x_m(\tau)$  we use.

- (1) The essence of existence of (2.10) and (2.11) is that for  $l \in \{2, 4\}$ ,  $X(\Gamma_0(4m)^+)$  is of genus zero, and the function  $\frac{1}{x_m(\tau)}$  is a Hauptmodul of  $X(\Gamma_0(4m)^+)$ , i.e., a generator of the function field  $\mathbb{C}(X(\Gamma_0(4m)^+))$ . Since the function  $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k}$  is in  $\mathbb{C}(X(\Gamma_0(4m)^+))$ , and the locations of poles of it are the same as the locations of zeros of  $\frac{1}{x_m(\tau)}$ , then  $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k}$  is a polynomial in  $x_m(\tau)$ .

In general, for a positive integer  $m$ , we may obtain identities for  $(\theta(\tau)\theta(m\tau))^k$  similar to (2.10) and (2.11) if we could construct a function  $F_k(\tau)$  by a linear combination of Eisenstein series of weight  $k$  such that  $\frac{F_k(\tau)}{(\theta(\tau)\theta(m\tau))^k}$  is invariant under some genus zero discrete subgroup  $\Gamma$  of  $\text{SL}_2(\mathbb{R})$ , and could construct a generator  $\psi(\tau)$  for the function field  $\mathbb{C}(X(\Gamma))$  such that the locations of zeros of  $\psi(\tau)$  are the same as that of  $(\theta(\tau)\theta(m\tau))^k$ .

- (2) We now explain that with the Hauptmodul  $\frac{1}{x_m(\tau)}$  we use in this work, the upper indices  $\ell_m$  cannot be improved any further, i.e., cannot be smaller. From the proof of Theorem 2.2, we can first note that for  $k$  fixed, the size of  $\ell_m$  is determined by the order of vanishing of the function  $F_{k,m}(\tau)$  at  $\frac{1}{2}$ ; the higher the order is, the smaller  $\ell_m$  will be. Then according to the proof of Lemma 3.2,  $\text{ord}_{1/2}(F_{k,m})$  is directly related to the definition of  $F_{k,m}(\tau)$  as a linear combination of Eisenstein series. Thus it is natural ask whether one could redefine  $F_{k,m}(\tau)$  to have higher order at  $\frac{1}{2}$ . In our cases, this is impossible. For example, for  $m = 2$ , and  $k \geq 4$  and even, first we know that  $(\theta(\tau)\theta(2\tau))^k$  and  $x_2(\tau)$  are modular forms of weight

$k$  on  $\Gamma_0(8)$  with trivial character by Lemma 3.3. Then we must have  $F_{k,2}(\tau)$  be a linear combination of Eisenstein series of weight  $k$  on  $\Gamma_0(8)$  with trivial character, thus we must have

$$F_{k,2}(\tau) = C_1 E_k(\tau) + C_2 E_k(2\tau) + C_3 E_k(4\tau) + C_4 E_k(8\tau)$$

for some constants  $C_1, \dots, C_4$  since  $E_k(m\tau)$  for  $m \in \{1, 2, 4, 8\}$  are linearly independent Eisenstein series of weight  $k$  on  $\Gamma_0(8)$  with trivial character and the dimension of the space spanned by such Eisenstein series is 4. According to the proof of Lemma 3.2, in order to have  $\text{ord}_{1/2}(F_{k,2}) \geq 1$ , we must have  $C_4 = 0$ . In addition, since  $x_2(\tau)$  is invariant under  $\Gamma_0(8)^+$ , then  $\frac{F_{k,2}(\tau)}{(\theta(\tau)\theta(2\tau))^k}$  must also be invariant under  $\Gamma_0(8)^+$ . Following such modularity, we can deduce that  $C_1 = 0$  and  $C_2 = 2^{-k/2}C_3$ , and thus we have

$$F_{k,2}(\tau) = C_3 \left( 2^{-k/2} E_k(2\tau) + E_k(4\tau) \right).$$

However, similar to the proof of Lemma 3.2, we can show that the order of vanishing of  $2^{-k/2} E_k(2\tau) + E_k(4\tau)$  at  $\frac{1}{2}$  is 0. This demonstrates our claim for the case  $m = 2$ , and  $k \geq 4$  and even. For the cases  $m = 2$  and  $k = 2$ , and  $m = 2$  or 4 and  $k = 1$ , the formulas we obtained do not involve any lower order term. For the other cases, similar arguments can be applied by the facts that

- (i) the space of Eisenstein series of weight 2 on  $\Gamma_0(16)$  is spanned by  $mE_2(m\tau) - E_2(\tau)$  for  $m|16$  and  $m \neq 1$ , and  $\sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) \sigma_1(n)q^n$ ;
- (ii) the space of Eisenstein series of even weight  $k \geq 4$  on  $\Gamma_0(16)$  is spanned by  $E_k(m\tau)$  for  $m|16$  and  $\sum_{n=1}^{\infty} \chi_{-4}(n) \sigma_{k-1}(n)q^n$ ;
- (iii) the space of Eisenstein series of odd weight  $k \geq 3$  on  $\Gamma_0(8)$  with character  $\chi_{-2}$  is spanned by  $E_{k,\chi_{-2}}^{\infty}(\tau)$  and  $E_{k,\chi_{-2}}^0(\tau)$ ;
- (iv) the space of Eisenstein series of odd weight  $k \geq 3$  on  $\Gamma_0(16)$  with character  $\chi_{-4}$  is spanned by  $E_{k,\chi_{-4}}^{\infty}(m\tau)$  and  $E_{k,\chi_{-4}}^0(m\tau)$  for  $m|4$ .

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