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Irrationality Measures for the Series of Reciprocals from Recurrence Sequences

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Using Padé approximations of Heine's q-hypergeometric series we obtain new irrationality measures for the values of the series $\sum_{n=1}^{\infty} t^n/W_n$, where W_n is a Fibonacci or Lucas type arithmetical form satisfying the recurrence

$$W_{n+2} = rW_{n+1} + sW_n, \qquad r, s \in \mathbb{Q}^*$$

with initial values W_1, W_2 and $t \in \mathbb{Q}^*$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION AND RESULTS

Let W_n be the arithmetical form

$$W_n = W_n(r, s) = a\alpha^n + b\beta^n, \qquad n \in \mathbb{N}, \tag{1}$$

where

$$\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}, \qquad \beta = \frac{r - \sqrt{r^2 + 4s}}{2}. \tag{2}$$

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Thus (W_n) is a second-order recurrence sequence satisfying

$$W_{n+2} = rW_{n+1} + sW_n \qquad \forall n \in \mathbb{N}. \tag{3}$$

In the sequel we may suppose without a loss of generality that $|\alpha| > |\beta|$. Let $W_n \neq 0$ for all $n \in \mathbb{N}$, then

$$W(t) = W_{r,s}(t) = \sum_{n=1}^{\infty} \frac{t^n}{W_n} \tag{4}$$

determines the meromorphic function $W(t),\ t \in \mathbb{C}\backslash\{\alpha^{n+1}/\beta^n |\ n \in \mathbb{N}\}$ having the Mittag–Leffler expansion

$$W(t) = \sum_{n=1}^{m-1} \frac{t^n}{W_n} - \frac{t^m}{a\alpha^{m-1}} \sum_{n=0}^{\infty} \frac{\left(-\frac{b}{a}\left(\frac{\beta}{\alpha}\right)^{m-1}\right)^n}{t - \alpha\left(\frac{\alpha}{\beta}\right)^n}, \tag{5}$$

where $|\alpha/\beta|^m > |b/a|$.

Let $\mathbb{I}$ be an imaginary quadratic field, then we shall investigate some arithmetical properties of the values of the function $W_{r,s}(t),\ t \in \mathbb{I}^*\backslash\{\alpha^{n+1}/\beta^n |\ n \in \mathbb{N}\}$, when (W_n) is a Fibonacci or Lucas type solution of recurrence (3) and $r, s \in \mathbb{Q}^*$. By the Fibonacci and Lucas type solutions we mean the sequences (F_n) and (L_n), respectively, where

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n.$$

The Fibonacci (f_n) and Lucas (l_n) sequences, where

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+2} = f_{n+1} + f_n,$$

$$l_0 = 2, \qquad l_1 = 1, \qquad l_{n+2} = l_{n+1} + l_n$$

are included into the Fibonacci and Lucas type sequence classes, respectively.

Bézivin's [3, 5] general theorems on linear independence imply irrationality results for the class of series (4), where

$$\alpha \in \mathbb{I}_{\mathbb{Z}},\ |\alpha| > 1,\ a = \beta = 1,\ b \in \mathbb{I}^*\backslash\{-\alpha^n \,|\, n \in \mathbb{Z}^+\},\ t \in \mathbb{I}^*\backslash\{b\alpha^n \,|\, n \in \mathbb{Z}\}. \tag{6}$$

On the other hand, Borwein [6, 7] settling Erdös' conjecture proved quantitative irrationality of (4), when

$$\alpha \in \mathbb{Z},\ |\alpha| > 1,\ a = \beta = 1,\ b \in \mathbb{Q}^*\backslash\{-\alpha^n \,|\, n \in \mathbb{Z}^+\},\ t = \pm 1. \tag{7}$$

In Bundschuh and Väänänen [8] and Matala-aho and Väänänen [14], (6) and (7) are generalized to arbitrary number fields and in case (7) Borwein's irrationality measure 8.667 is improved to 4.311 in [8] and to 3.947 in [14].

André–Jeannin [1] proved the irrationality of the series $W_{r,\pm 1}$, when α and β are given by (2) and

$$r \in \mathbb{Z}\backslash\{0\}, \qquad t \in \mathbb{Z}\backslash\{0\}, \qquad |t| < |\alpha|,$$

thus including the Fibonacci sequence (f_n) case. Linear independence measures of q-exponential function and its derivatives in [8] imply an irrationality measure 8.621 for the Fibonacci case of (4). Approximation measure results in [14] give an irrationality measure 7.893 in all Fibonacci type cases of $W_{r,\pm 1}$ where $r \in \mathbb{Z}\backslash\{0\}$, if $s = 1$, and $r \in \mathbb{Z}\backslash\{0, \pm 1, \pm 2\}$, if $s = -1$.

Prévost [17] considered also the Lucas type cases of (4) proving a measure 21.12 for $W_{r,\pm 1}$, when $r \in \mathbb{Z}\backslash\{0\}$, while the results of Väänänen [18] imply a measure 47.33 in those cases.

The recent applications of Mahler's [16] and Nesterenko's [11, 15] methods give transcendence for several series similar to (4). Here we mention the following trancendental series [11, 16]:

$$\sum_{n=1}^{\infty} \frac{1}{f_n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{f_{2^n+1}}$$

as examples. About the transcendence character of series (4) it is known that, if α and β are algebraic numbers satisfying $\alpha\beta = 1$ and $t = 1$, then series (4) has transcendental value in the Lucas type case by Nesterenko's results [11, 15].

We shall state Theorem 1 only in the case $r > 0$, because $\alpha(-r, s) = -\beta(r, s)$ and $\beta(-r, s) = -\alpha(r, s)$ give $W_n(-r, s) = (-1)^n W_n(r, s)$. Thus the relation $W_{-r,s}(t) = W_{r,s}(-t)$ transfers the approximations from the case $r > 0$ to the case $r < 0$.

Let us set $r = R/d$ and $s = S/d$, where $d, R \in \mathbb{Z}^+$ and $S \in \mathbb{Z}\backslash\{0\}$.

THEOREM 1. *Let (W_n) be given by* (1), *where d, R and S satisfy*

$$R > |dS|^c - \frac{dS}{|dS|^c}, \qquad c = c(W) > 0 \tag{8}$$

with

$$c(F) = \frac{\pi^2}{\pi^2 - 3}, \qquad c(L) = \frac{3\pi^2}{2\pi^2 - 12}$$

and let $t \in \mathbb{I}^* \setminus \{\alpha^{n+1}/(d\beta^n) |\ n \in \mathbb{N}\}$. *Then there exist positive constants* C *and* N_0 *such that*

$$\left| \sum_{n=1}^{\infty} \frac{t^n}{W_n} - \frac{M}{N} \right| > |N|^{-m(W)-C \log\log|N|/\sqrt{\log|N|}}, \tag{9}$$

$$m(W) = \frac{\log(|\alpha|^2/|dS|)}{\log(|\alpha|^{1/c}/|dS|)} \tag{10}$$

for all $M, N \in \mathbb{Z}_{\mathbb{I}}$ *with* $|N| \geqslant N_0$.

First, we point out that our condition (8), e.g. in the Fibonacci type case allows irrationality results for series (4) also in the cases like $r = 2,\ s = 2$ and $r = 13,\ s = 6$ which are not covered by the earlier investigations. In [14] the cases $r \geqslant 4,\ s = 2$ and $r \geqslant 35,\ s = 6$ are allowed.

Let $r \in \mathbb{Z} \setminus \{0\}$, if $s = 1$, and $r \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$, if $s = -1$. Then in the case of Fibonacci type sequences (F_n) earlier irrationality measures 8.621, 7.893, and 10.21 proved in [8, 14, 17], respectively, will be improved considerably. In all the three papers cited the approximation formulae have connections to Padé approximations of the q-logarithm series

$$l_q(t) = \sum_{n=1}^{\infty} \frac{t^n}{1 - q^n}.$$

In the first two papers considerations are done in number fields and also the functional equation method, i.e. the iterations of the functional equation

$$(1 - t) l_q(qt) = (1 - t) l_q(t) - t$$

of $l_q(t)$ is used, while in the third paper only rational number field is used without the use of functional equation method. Also in this paper we use the Padé approximations of $l_q(z)$ (actually a more general q-series will be studied) without functional equation method. Now the corresponding functional equation

$$b(\alpha - t) W(\beta t/\alpha) = a(t - \alpha) W(t) + t \tag{11}$$

is used only to determine the analytic continuation (5) of series (4). Here we consider the coefficients of the denominator polynomial more carefully and the detailed study of the divisibility of the numbers F_n and the Fibonomials enables us to get a sharp irrationality measure 2.874 in the Fibonacci type case.

THEOREM 2F. *Let $r \in \mathbb{Z}\backslash\{0\}$, if $s = 1$, and $r \in \mathbb{Z}\backslash\{0, \pm 1, \pm 2\}$, if $s = -1$, and $t \in \mathbb{I}^*\backslash\{\alpha^{n+1}/\beta^n \mid n \in \mathbb{N}\}$. Then there exists a positive constant N_0 such that*

$$\left|\sum_{n=1}^{\infty} \frac{t^n}{F_n} - \frac{M}{N}\right| > |N|^{-2.874}, \tag{12}$$

for all $M, N \in \mathbb{Z}_{\mathbb{I}}$ with $|N| \geqslant N_0$.

In the Lucas type case we need to transform divisibility questions of the numbers L_n to the divisibility of cyclotomic polynomials and via that the earlier measures 21.12 by Prévost [17] and 47.33 by Väänänen [18] will be improved substantially.

THEOREM 2L. *If r, s, t are as in Theorem* 2F, *then there exists a positive constant N_0 such that*

$$\left|\sum_{n=1}^{\infty} \frac{t^n}{L_n} - \frac{M}{N}\right| > |N|^{-7.652}, \tag{13}$$

for all $M, N \in \mathbb{Z}_{\mathbb{I}}$ with $|N| \geqslant N_0$.

To prove Theorems 2 we shall need the following common factor results, which are known in the Fibonacci type case, see [9]. The Lucas type case seems to be new.

THEOREM 3. *Let $r, s \in \mathbb{Z}\backslash\{0\}$, then*

$$F_1 \cdots F_n | F_{h+1} \cdots F_{h+n} \qquad \forall h \in \mathbb{N}, \quad n \in \mathbb{Z}^+ \tag{14}$$

and

$$H_n H_{[n/2]} H_{[n/4]} \cdots | L_{h+1} \cdots L_{h+n} \qquad \forall h \in \mathbb{N}, \quad n \in \mathbb{Z}^+, \tag{15}$$

where

$$H_0 = H_1 = 1, \qquad H_m = L_1 \cdots L_{[m/2]}.$$

2. q-FACTORIALS AND W-NOMIALS

Let

$$(b, a)_0 = 1, \qquad (b, a)_n = (b - a)(b - aq) \ldots (b - aq^{n-1}), \quad n \in \mathbb{Z}^+$$

and $(a)_n = (1,a)_n$ be the q-series factorials, so, e.g. $(q)_n = (1-q)\dots(1-q^n)$. Further, the q-binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

satisfying

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}, \qquad 1 \leqslant k \leqslant n-1 \tag{16}$$

and it is well known (see [2]) that (16) implies

$$\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}[q] \qquad \text{and} \qquad \deg_q \begin{bmatrix} n \\ k \end{bmatrix} = k(n-k), \quad 0 \leqslant k \leqslant n. \tag{17}$$

Also we shall use the W-nomials (Fibonomials) defined by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_W = \begin{bmatrix} n \\ n \end{bmatrix}_W = 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_W = \frac{W_1 \cdots W_n}{W_1 \cdots W_k W_1 \cdots W_{n-k}}$$

for all $k, n \in \mathbb{N}$ with $1 \leqslant k \leqslant n-1$ for any recurrence sequence (3).

Let now $r, s \in \mathbb{Q}^*$ satisfy (8), then $D = r^2 + 4s > 0$ defines the (quadratic) field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ and its ring of integers $\mathbb{Z}_{\mathbb{K}}$. Set

$$\alpha = \frac{r + \sqrt{D}}{2}, \qquad \beta = \frac{r - \sqrt{D}}{2},$$

then $\bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$, i.e. α and β are field conjugates in $\mathbb{K}$ satisfying the equation $\alpha^2 - r\alpha - s = 0$.

We shall need several times the following special case of the symmetric polynomial behaviour in conjugate points such that, for the completeness, we shall include a proof of it.

LEMMA 1. *Let $P(x,y) \in \mathbb{Z}[x,y]$ be such that $P(x,y) = P(y,x)$. Then $P(\alpha,\beta) \in \mathbb{Q}$ and, if $\alpha, \beta \in \mathbb{Z}_{\mathbb{K}}$, then $P(\alpha,\beta) \in \mathbb{Z}$.*

Proof. Let $\gamma = P(\alpha,\beta)$. By taking the conjugates and using the symmetry we get

$$\bar{\gamma} = P(\bar{\alpha},\bar{\beta}) = P(\beta,\alpha) = P(\alpha,\beta) = \gamma,$$

which shows that $\gamma \in \mathbb{Q}$. If $\alpha, \beta \in \mathbb{Z}_{\mathbb{K}}$, then $\gamma = P(\alpha,\beta) \in \mathbb{Z}_{\mathbb{K}} \cap \mathbb{Q} = \mathbb{Z}$. ∎

By Lemma 1, F_n and $L_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

LEMMA 2. *Let $r, s \in \mathbb{Z}$ and let (F_n) be a Fibonacci type solution of recurrence* (3). *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_F \in \mathbb{Z} \qquad \forall k, n \in \mathbb{Z} \quad (0 \leqslant k \leqslant n). \tag{18}$$

Proof. When we set $q = \beta/\alpha$, then (16) shows that

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \alpha^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F + \beta^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F \qquad (1 \leqslant k \leqslant n-1). \tag{19}$$

Now $\alpha, \beta \in \mathbb{Z}_{\mathbb{K}}$ and by the definition $\begin{bmatrix} n \\ k \end{bmatrix}_F \in \mathbb{Q}$. Let us suppose that

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_F \in \mathbb{Z} \qquad \forall k \quad (0 \leqslant k \leqslant n-1) \tag{20}$$

for a given n. Then (19) shows that

$$\begin{bmatrix} n \\ k \end{bmatrix}_F \in \mathbb{Z}_{\mathbb{K}} \qquad \forall k \quad (0 \leqslant k \leqslant n)$$

proving (18) by induction. ∎

Also we note another proof of Lemma 2 in [9].

LEMMA 3. *Let $r \in \mathbb{Z}^+$ and $s \in \mathbb{Z} \setminus \{0\}$ satisfy* (8), *then*

$$\alpha > 1 \qquad \text{and} \qquad \alpha > |\beta|. \tag{21}$$

Proof. Now $d = 1$ and so condition (8) implies

$$\begin{aligned} \alpha &= \frac{r + \sqrt{r^2 + 4s}}{2} \\ &> \frac{1}{2}\left(|s|^c - \frac{s}{|s|^c} + \sqrt{|s|^{2c} - 2s + \frac{s^2}{|s|^{2c}} + 4s} \right) \\ &= \frac{1}{2}\left(|s|^c - \frac{s}{|s|^c} + |s|^c + \frac{s}{|s|^c} \right) = |s|^c. \end{aligned}$$

Thus

$$\log \alpha > c \log |s| \geqslant 0 \tag{22}$$

because $s \in \mathbb{Z}\backslash\{0\}$.

Now

$$x = \frac{\beta}{\alpha} = \frac{r - \sqrt{r^2 + 4s}}{r + \sqrt{r^2 + 4s}} = \frac{(r - \sqrt{r^2 + 4s})^2}{-4s}$$

and consequently $x > -1$ when $s > 0$. If $s < 0$, then the assumption $r > 1$ is needed for $x < 1$. The condition $r > 1$ follows from assumption (8) and $s \in \mathbb{Z}^-$. Hence $|x| < 1$. ∎

LEMMA 4. *Let $r, s \in \mathbb{Z}\backslash\{0\}$ and let (W_n) be a sequence defined by* (1) *such that $W_n \in \mathbb{Z}\backslash\{0\}$ for all $n \in \mathbb{N}$. Then there exist $M_n \in \mathbb{Z}^+$ such that*

$$\mathrm{LCM}[W_1, \ldots, W_n] | M_n \qquad \forall n \in \mathbb{Z}^+,$$

and

$$M_n \leqslant |\alpha|^{M(W)n^2 + O(n \log n)} \qquad \text{if } r \in \mathbb{Z}^+, \tag{23}$$

with

$$M(F) = 3/\pi^2, \qquad M(L) = 4/\pi^2.$$

Proof. Let

$$D_n(q) = \mathrm{LCM}[1 - q, 1 - q^2, \ldots, 1 - q^n]$$

for which it is proved in [12] that

$$\deg_q D_n(q) = \frac{3}{\pi^2} n^2 + O(n \log n). \tag{24}$$

Let us fix $n \in \mathbb{Z}^+$ and $d = \deg_q D_n(q)$. For a given $1 \leqslant i \leqslant n$ there exist a polynomial $B(q) \in \mathbb{Z}[q]$ such that

$$D_n(q) = (1 - q^i) B(q). \tag{25}$$

From (25) it follows that

$$q^d D_n(1/q) = (q^i - 1) q^{d-i} B(1/q) \tag{26}$$

and so

$$D_n^*(q) = q^d D_n(1/q) \tag{27}$$

is a common multiple of $1-q, 1-q^2, \ldots, 1-q^n$ and $\deg_q D_n^*(q) = d$. Hence

$$D_n^*(q) = \pm D_n(q). \tag{28}$$

Suppose first that $D_n^*(q) = D_n(q)$, which is equivalent to

$$D_n(q) = q^d D_n(1/q). \tag{29}$$

By (25) we get

$$\beta^d D_n(\alpha/\beta) = \frac{\alpha^i - \beta^i}{\alpha - \beta}(\beta - \alpha)\beta^{d-i} B(\alpha/\beta). \tag{30}$$

Let us denote

$$\gamma = \beta^d D_n(\alpha/\beta), \qquad \kappa = (\beta - \alpha)\beta^{d-i} B(\alpha/\beta).$$

Using (29) it follows easily that

$$\bar{\gamma} = \gamma, \qquad \bar{\kappa} = \kappa$$

giving the relation $\gamma = F_i \kappa$, where $\gamma, \kappa \in \mathbb{Z}$, proving the divisibility

$$F_i | \gamma. \tag{31}$$

Hence

$$\mathrm{LCM}[F_1, \ldots, F_n] | \beta^d D_n(\alpha/\beta). \tag{32}$$

In the case $D_n^*(q) = -D_n(q)$ the proof goes similarly.

From [12] we get the estimate

$$|D_n(q)| \leqslant |q|^{\frac{3}{\pi^2}n^2 + O(n \log n)}, \qquad |q| > 1. \tag{33}$$

Now $r \in \mathbb{Z}^+$ implies by (21) that $\alpha > |\beta|$ and thus

$$\mathrm{LCM}[F_1, \ldots, F_n] \leqslant |\beta|^d |\alpha/\beta|^{\frac{3}{\pi^2}n^2 + O(n \log n)} = |\alpha|^{\frac{3}{\pi^2}n^2 + O(n \log n)} \tag{34}$$

proving (23) in the Fibonacci type case.

From [4]—Démonstration du théorème—we get another proof to

$$M(F) = 3/\pi^2$$

and a proof for

$$M(L) = 4/\pi^2.$$

The proof of Bézivin needs slight modifications to get the upper bounds for least common multiples in the form of (23). ∎

Proof of Theorem 3. The divisibility property (14) for Fibonacci type numbers is proved in [9].

Let us start from the Fibonacci type numbers

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} = \beta^k \frac{x^k - 1}{\alpha - \beta} = \beta^{k-1} \prod_{d|k, d>1} \Phi_d(x), \qquad x = \alpha/\beta, \tag{35}$$

where $\Phi_d = \Phi_d(x)$ is the dth cyclotomic polynomial. Thus, e.g.

$$\begin{aligned} F_1 \cdots F_n &= \beta^{\binom{n}{2}} \Phi_2^{[n/2]} \Phi_3^{[n/3]} \cdots \Phi_n \\ &= E_2(\alpha, \beta)^{[n/2]} E_3(\alpha, \beta)^{[n/3]} \cdots E_n(\alpha, \beta), \end{aligned} \tag{36}$$

where $E_k = E_k(\alpha, \beta) = \beta^{\phi(k)} \Phi_k(\alpha/\beta)$. It is known [9] that

$$E_k(y, x) = E_k(x, y), \tag{37}$$

which implies

$$E_k(\alpha, \beta) \in \mathbb{Z} \qquad \forall k \in \mathbb{N} \quad (k \geqslant 2) \tag{38}$$

by Lemma 1. Also we have

$$E_2(\alpha, \beta)^{[n/2]+\lambda(2)} E_3(\alpha, \beta)^{[n/3]+\lambda(3)} \cdots E_n(\alpha, \beta) | F_{h+1} \cdots F_{h+n} \tag{39}$$

for all $h \in \mathbb{N}$, $n \in \mathbb{Z}^+$, where $\lambda(k) = 0$ or $\lambda(k) = 1$. First we note that (36) and (39) give another proof of (14).

Now we have to look for a common factor of the products

$$L_{h+1} \cdots L_{h+n}, \qquad h \in \mathbb{N}, \quad n \in \mathbb{Z}^+. \tag{40}$$

Clearly,

$$L_k = \alpha^k + \beta^k = \frac{F_{2k}}{F_k}. \tag{41}$$

Thus

$$L_{h+1} \cdots L_{h+n} = \frac{F_{2(h+1)}}{F_{h+1}} \cdots \frac{F_{2(h+n)}}{F_{h+n}}. \tag{42}$$

Let us study the factors E_l of (42), where $2 \leqslant l \leqslant [n/2]$. Let l be fixed and define

$$v_N = v_N(l) = \#\{L| \quad l|2L,\ h+1 \leqslant L \leqslant h+n\}$$

and

$$v_D = v_D(l) = \#\{L| \quad l|L,\ h+1 \leqslant L \leqslant h+n\}$$

to denote the numerator and denominator exponents v_N and v_D of E_l in (42). Because $2k+1|L$ if and only if $2k+1|2L$ it follows that

$$v_N(2k+1) = v_D(2k+1) = [n/(2k+1)] \quad \text{or} \quad [n/(2k+1)]+1. \tag{43}$$

The even indexed E_l, $l = 2k$, has the exponent $v_N(2k)$ in the numerator at least to $[n/k]$ and $v_D(2k)$ in the denominator at most to $[n/(2k)]+1$. If $v_D(2k) = [n/(2k)]+1$, then there exist $y = [n/(2k)]$ denominator index intervals $[L_j, L_{j+1} = L_j + l]$, $j = 1, \ldots, y$, containing the indices $L_j + k$. Thus there exist $y+1$ numerator indices $2L_i$, $i = 1, \ldots, y+1$, and y numerator indices $2(L_j + k)$, $j = 1, \ldots, y$, such that

$$l|2L_i \qquad \text{and} \qquad l|2(L_j+k) = 2L_j + l \tag{44}$$

giving

$$v_N(2k) \geqslant 2y+1 = v_D(2k) + [n/2k].$$

Hence

$$v_N(2k) - v_D(2k) \geqslant [n/2k] \tag{45}$$

in all the cases. By (42), (43) and (45) each product in (40) has the factor

$$G_n(L) = \prod_{k=1}^{[n/2]} E_{2k}^{\left[\frac{n}{2k}\right]}. \tag{46}$$

Let us denote

$$H_m = L_1 \cdots L_{[m/2]} \qquad \text{and} \qquad G_n^* = H_n H_{[n/2]} H_{[n/4]} \cdots.$$

We shall prove that

$$H_n G_{[n/2]}(L) | G_n(L) \qquad \forall n \in \mathbb{Z}^+. \tag{47}$$

Multiplication of the expressions

$$H_n = E_2^{N-[N/2]} E_4^{[N/2]-[N/4]} E_6^{[N/3]-[N/6]} \cdots E_{2N},$$

$$G_N(L) = E_2^{[N/2]} E_4^{[N/4]} E_6^{[N/6]} \cdots E_{2[N/2]}, \tag{48}$$

where $N = [n/2]$, shows (47), which implies by induction that

$$G_n^* | G_n(L) \qquad \forall n \in \mathbb{Z}^+. \quad \blacksquare \tag{49}$$

3. PADÉ APPROXIMATIONS

The series

$$f(z) = \sum_{n=0}^{\infty} \frac{(B)_n}{(C)_n} z^n$$

is a special case of Heine's q-series having the closed form (n, n) Padé approximations constructed in [13].

LEMMA 5 (Matala-aho [13]). *Let*

$$Q_n^*(z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} (Bq^{n-k+1})_k (Cq^n)_{n-k} (-z)^k, \tag{50}$$

and

$$R_n^*(z) = z^{2n+1} q^{n^2} \frac{(q)_n (B)_{n+1} (B, C)_n}{(C)_{2n+1}} \sum_{i=0}^{\infty} \frac{(q^{n+1})_i (Bq^{n+1})_i}{(q)_i (Cq^{2n+1})_i} z^i. \tag{51}$$

Then there exists a polynomial $P_n^*(z)$ of degree $\leqslant n$ such that

$$Q_n^*(z) f(z) - P_n^*(z) = R_n^*(z). \tag{52}$$

Now we shall set

$$B = -b/a, \qquad C = Bq, \qquad q = \beta/\alpha, \qquad z = t/\alpha.$$

Hence

$$W(t) = \frac{1}{a+b}(f(t/\alpha) - 1),$$

$$Q_n(t) = a^n \alpha^{\frac{3n^2+n}{2}} Q_n^*(t/\alpha), \qquad P_n(t) = a^n \alpha^{\frac{3n^2+n}{2}} (P_n^*(t/\alpha) - Q_n^*(t/\alpha))/(a+b)$$

and

$$R_n(t) = a^n \alpha^{\frac{3n^2+n}{2}} R_n^*(t/\alpha)/(a+b).$$

Thus we get the approximation formula

$$Q_n(t) W(t) - P_n(t) = R_n(t), \tag{53}$$

where

$$Q_n(t) = \sum_{k=0}^{n} q_{n,k} t^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F W_{n-k+1} \cdots W_{2n-k} (-s)^{\binom{k}{2}} (-t)^k, \tag{54}$$

$$P_n(t) = \sum_{k=0}^{n} p_{n,k} t^k, \qquad p_{n,k} = \sum_{i+j=k} q_{n,i}/W_j \tag{55}$$

and

$$\begin{aligned} R_n(t) &= (-ab)^n t^{2n+1} \beta^{n^2} \alpha^{\frac{n^2-3n}{2}-1} S_n(t), \\ S_n(t) &= \frac{(q)_n^2}{(a, -bq^{n+1})_{n+1}} \sum_{i=0}^{\infty} \frac{(q^{n+1})_i (a, -bq^{n+1})_i}{(q)_i (a, -bq^{2n+2})_i} \left(\frac{t}{\alpha}\right)^i. \end{aligned} \tag{56}$$

LEMMA 6. *Let $r, s \in \mathbb{Z}$ and let (W_n) be a sequence defined by* (1) *such that $W_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Then*

$$q_{nk} \in \mathbb{Z} \qquad \forall k, n \in \mathbb{N} \quad (0 \leqslant k \leqslant n). \tag{57}$$

Proof. By Lemma 2 the Fibonomials in

$$q_{nk} = \begin{bmatrix} n \\ k \end{bmatrix}_F W_{n-k+1} \cdots W_{2n-k} (-s)^{\binom{k}{2}} (-1)^k \tag{58}$$

are integers. ∎

LEMMA 7. *Let $r \in \mathbb{Z} \setminus \{0\}$, $s \in \mathbb{Z}$ and let $W_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Then there exist $G_n = G_n(W) \in \mathbb{Z}^+$ such that*

$$G_n | q_{nk} \qquad \forall k, n \in \mathbb{N} \quad (0 \leqslant k \leqslant n) \tag{59}$$

and

$$G_n \geqslant |\alpha|^{G(W) n^2 + O(n)} \qquad \text{if } r \in \mathbb{Z}^+, \tag{60}$$

with

$$G(F) = \tfrac{1}{2}, \qquad G(L) = \tfrac{1}{6}.$$

Proof. First, we note that

$$q_{nk} = \begin{bmatrix} n \\ k \end{bmatrix}_F \begin{bmatrix} 2n-k \\ n \end{bmatrix}_W W_1 \cdots W_n (-s)^{\binom{k}{2}} (-1)^k. \tag{61}$$

In the Fibonacci type $W = F$ cases the both Fibonomials in (61) are integers by Lemma 2. Thus

$$G_n(F) = F_1 \cdots F_n. \tag{62}$$

Also we could use result (14) to prove (62).

In the Lucas type cases the W-nomials are not integers and thus we have to look more carefully divisibility properties of Lucas type sequences. Clearly, result (15) of Theorem 3 gives a common factor

$$G_n(L) = H_n H_{[n/2]} H_{[n/4]} \ldots, \qquad H_m = L_1 \cdots L_{[m/2]} \tag{63}$$

for the coefficients

$$q_{nk} = \begin{bmatrix} n \\ k \end{bmatrix}_F L_{n-k+1} \cdots L_{2n-k} (-s)^{\binom{k}{2}} (-1)^k. \tag{64}$$

So, in the Fibonacci and Lucas type cases we have (60) by (62) and (63). ∎

Now we set $t = u/v$, where $u, v \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$,

$$q_n = v^n M_n G_n^{-1} Q_n(u/v), \qquad p_n = v^n M_n G_n^{-1} P_n(u/v),$$

$$r_n = v^n M_n G_n^{-1} R_n(u/v) \tag{65}$$

in order to get the numerical approximation forms

$$q_n W(u/v) - p_n = r_n, \qquad q_n,\ p_n \in \mathbb{Z}_{\mathbb{I}} \quad \forall n \in \mathbb{N} \tag{66}$$

by Lemmas 4–7.

LEMMA 8. *Let $r \in \mathbb{Z}^+$, $s \in \mathbb{Z}$ and let (W_n) be a sequence defined by* (1) *such that $W_n \in \mathbb{Z} \setminus \{0\}$ for all $n \in \mathbb{N}$. Then*

$$|q_n| \leqslant |\alpha|^{(3/2 + M(W) - G(W))n^2 + O(n \log n)} \tag{67}$$

and

$$|r_n| \leqslant |\beta|^{n^2} |\alpha|^{(1/2 + M(W) - G(W))n^2 + O(n \log n)}. \tag{68}$$

Proof. Using (21), (23), (54), (60) and estimates for q-binomials from [13] we get

$$|q_n| = v^n M_n G_n^{-1} \left| \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F W_{n-k+1} \cdots W_{2n-k} (-s)^{\binom{k}{2}} (-u/v)^k \right|$$

$$\leqslant (n+1) \max\{|u|, v\}^n 2^{2n} |\alpha|^{M(W)n^2 + O(n \log n) - G(W)n^2 + O(n)}. \tag{69}$$

$$\max_{0 \leqslant k \leqslant n} |\alpha|^{k(n-k) + \frac{n(3n-2k+1)}{2}} |s|^{\frac{k^2-k}{2}}.$$

Here

$$|s| = |\alpha\beta| < \alpha^2$$

by (21) and thus (69) implies

$$|q_n| \leqslant |\alpha|^{(M(W) - G(W) + 3/2)n^2 + O(n \log n)}.$$

In the remainder r_n the term $S_n(u/v)$ goes to the limit as $n \to \infty$. Thus (21), (23), (56) and (60) give the following estimation:

$$|r_n| = |ab|^n \left|\frac{u}{v}\right|^{2n+1} v^n M_n G_n^{-1} |\beta|^{n^2} |\alpha|^{\frac{n^2-3n}{2}-1} |S_n(u/v)|$$

$$\leqslant |\beta|^{n^2} |\alpha|^{O(n) + M(W)n^2 + O(n \log n) - G(W)n^2 + O(n) + n^2/2}. \quad \blacksquare \tag{70}$$

LEMMA 9. *Let (W_n) be a series defined by* (1) *such that $W_n \in \mathbb{Z} \setminus \{0\}$ for all $n \in \mathbb{N}$ and $abrst \neq 0$. Then*

$$q_n p_{n+1} - p_n q_{n+1} \neq 0 \qquad \forall n \in \mathbb{Z}^+. \tag{71}$$

Proof. By (53)–(56) we get

$$\Delta_n = Q_n P_{n+1} - P_n Q_{n+1} = R_n Q_{n+1} - R_{n+1} Q_n$$

$$= t^{2n+1} (-ab)^n \beta^{n^2} \alpha^{\frac{n^2-3n}{2}-1} \frac{(q)_n^2}{(a, -bq^{n+1})_{n+1}} W_{n+2} \cdots W_{2n+2}. \quad \blacksquare$$

4. PROOF OF THEOREMS 1 AND 2

The following Lemma 10 may be proved analogously to theorem in [14]. However, here we need to use only the imaginary quadratic field $\mathbb{I}$.

LEMMA 10. *Let $\Phi \in \mathbb{C}$ and $y > 1$. Let*

$$q_n\Phi - p_n = r_n, \qquad q_n,\ p_n \in \mathbb{Z}_{\mathbb{I}} \quad \forall n \in \mathbb{N}$$

be numerical approximation forms satisfying

$$q_n p_{n+1} - p_n q_{n+1} \neq 0, \tag{72}$$

$$|q_n| \leqslant y^{An^2+O(n\log n)} \tag{73}$$

and

$$|r_n| \leqslant y^{-Bn^2+O(n\log n)} \tag{74}$$

for all $n \in \mathbb{N}$ with some positive A and B. Then there exist positive constants C and N_0 such that

$$\left|\Phi - \frac{M}{N}\right| > |N|^{-(1+A/B)-C\log\log|N|/\sqrt{\log|N|}} \tag{75}$$

for all $M, N \in \mathbb{Z}_{\mathbb{I}}$ with $|N| \geqslant N_0$.

Thus $1 + A/B + \varepsilon$ is an irrationality measure of Φ for every $\varepsilon \in \mathbb{R}^+$.

Proof of Theorems 1 *and* 2. Let

$$c = c(W) = 1/(\tfrac{1}{2} + G(W) - M(W)),$$

where

$$\tfrac{1}{2} + G(W) - M(W) > 0,$$

which is the case at least when $W = F$ or $W = L$.

First, we suppose $r \in \mathbb{Z}^+$ and $s \in \mathbb{Z}\backslash\{0\}$. Condition (8) reads now as

$$r > |s|^c - \frac{s}{|s|^c} \tag{76}$$

and thus (22) gives

$$\log\alpha > c\log|s| \geqslant 0$$

yielding to

$$B = \tfrac{1}{2} + G(W) - M(W) - \log|s|/\log|\alpha| > 0. \tag{77}$$

Clearly,

$$A = \tfrac{3}{2} + M(W) - G(W). \tag{78}$$

Lemma 10 together with (77) and (78) gives

$$\begin{aligned} m(W) &= \frac{A+B}{B} \\ &= \frac{2 - \log|s|/\log\alpha}{1/2 + G(W) - M(W) - \log|s|/\log\alpha} = \frac{\log|\alpha|^2/|s|}{\log|\alpha|^{1/c}/|s|}. \end{aligned} \tag{79}$$

Suppose then that $r = R/d$ and $s = S/d$, where $d, R \in \mathbb{Z}^+$ and $S \in \mathbb{Z}\backslash\{0\}$. Now the recurrence is

$$dW_{n+2} = RW_{n+1} + SW_n,$$

where we set $W_n = V_n/d^n$. The sequence (V_n) satisfies

$$V_{n+2} = RV_{n+1} + dSV_n$$

and the corresponding series (4) is now

$$\sum_{n=1}^{\infty} \frac{(dt)^n}{V_n}.$$

Replace then r, s and t by R, dS and dt, respectively, in (4), (76) and (79) to get Theorem 1.

Using the values for $G(W)$ and $M(W)$ given by Lemmas 4 and 7 we get the numerical values

$$c(F) = \frac{1}{1/2 + 1/2 - 3/\pi^2} = \frac{\pi^2}{\pi^2 - 3} = 1.436706...$$

and

$$c(L) = \frac{1}{1/2 + 1/6 - 4/\pi^2} = \frac{3\pi^2}{2\pi^2 - 12} = 3.825819\ldots\ .$$

Note that, if $s = \pm 1$, then $m(W) = 2c(W)$. In this case we get irrationality measures

$$m(F) = \frac{2\pi^2}{\pi^2 - 3} = 2.87341\ldots, \qquad m(L) = \frac{3\pi^2}{\pi^2 - 6} = 7.65163\ldots\ . \quad \blacksquare$$

Finally, we note that in the Fibonacci type cases the value $c(F) = 1.437$ gives irrationality results in more general framework than the previous value $c(F) = 1.974$ in [14].

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