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A characterization of regular tetrahedra in $\mathbb{Z}^3$

Eugen J. Ionascu [1]

Department of Mathematics, Columbus State University, Columbus, GA 31907, United States

A R T I C L E I N F O

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A B S T R A C T

Text. In this note we characterize all regular tetrahedra whose vertices in $\mathbb{R}^3$ have integer coordinates. The main result is a consequence of the characterization of all equilateral triangles having integer coordinates [R. Chandler, E.J. Ionascu, A characterization of all equilateral triangles in $\mathbb{Z}^3$, Integers 8 (2008), Article A19]. Previous work on this topic begun in [E.J. Ionascu, A parametrization of equilateral triangles having integer coordinates, J. Integer Seq. 10 (2007), Article 07.6.7]. We will use this characterization to point out some corollaries. The number of such tetrahedra whose vertices are in the finite set $\{0, 1, 2, \ldots, n\}^3$, $n \in \mathbb{N}$, is related to the sequence A103158 in the Online Encyclopedia of Integer Sequences [Neil J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at: http://www.research.att.com/~njas/sequences/, 2005].

Video. For a video summary of this paper, please visit http://www.youtube.com/watch?v=LT3aAUUFMFk.

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1. Introduction

In this paper we give a solution to the following system of Diophantine equations:

$$\begin{cases} (x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=(x_a-x_c)^2+(y_a-y_c)^2+(z_a-z_c)^2, \\ (x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=(x_a-x_d)^2+(y_a-y_d)^2+(z_a-z_d)^2, \\ (x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=(x_b-x_c)^2+(y_b-y_c)^2+(z_b-z_c)^2, \\ (x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=(x_b-x_d)^2+(y_b-y_d)^2+(z_b-z_d)^2, \\ (x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2=(x_c-x_d)^2+(y_c-y_d)^2+(z_c-z_d)^2 \end{cases} \tag{1}$$

E-mail address: ionascu_eugen@colstate.edu.

[1] Honorific Member of the Romanian Institute of Mathematics "Simion Stoilow".

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where x_a, y_a, z_a, x_b, y_b, z_b, x_c, y_c, z_c, x_d, y_d, z_d are in $\mathbb{Z}$. If $A(x_a, y_a, z_a)$, $B(x_b, y_b, z_b)$, $C(x_c, y_c, z_c)$, and $D(x_d, y_d, z_d)$ are considered as points in $\mathbb{R}^3$ the Diophantine system (1) represents exactly the condition that makes $ABCD$ a regular tetrahedron with vertices having integer coordinates. Clearly (1) is invariant under translations in $\mathbb{R}^3$ by vectors having integer coordinates. So, without loss of generality we are interested in the case, $A = O$, where O is the origin. Since each of the faces OBC, OCD and OBD is an equilateral triangle it is necessary to solve for instance the following system:

$$\begin{cases} x_b^2 + y_b^2 + z_b^2 = x_c^2 + y_c^2 + z_c^2, \\ x_b^2 + y_b^2 + z_b^2 = (x_b - x_c)^2 + (y_b - y_c)^2 + (z_b - z_c)^2. \end{cases} \tag{2}$$

This problem was solved in [1]. We recall the following facts from [1,3]. Every equilateral triangle in $\mathbb{R}^3$ whose vertices have integer coordinates, after a translation that brings one of its vertices to the origin, must be contained in a plane with a normal vector (a, b, c) $(a, b, c \in \mathbb{Z})$ satisfying the Diophantine equation

$$a^2 + b^2 + c^2 = 3d^2 \tag{3}$$

where $d \in \mathbb{N}$. Moreover, a, b, c, and d can be chosen so that the sides of such a triangle have length of the form $d\sqrt{2(m^2 - mn + n^2)}$ with m and n integers. A more precise statement is the next parametrization of these triangles.

Theorem 1.1. *Let a, b, c, d be odd integers such that $a^2 + b^2 + c^2 = 3d^2$ and $\gcd(a, b, c) = 1$. Then for every $m, n \in \mathbb{Z}$ the points $P(u, v, w)$ and $Q(x, y, z)$, whose coordinates are given by*

$$\begin{cases} u = m_u m - n_u n, \\ v = m_v m - n_v n, \\ w = m_w m - n_w n, \end{cases} \quad \text{and} \quad \begin{cases} x = m_x m - n_x n, \\ y = m_y m - n_y n, \\ z = m_z m - n_z n, \end{cases} \tag{4}$$

with

$$\begin{cases} m_x = -\frac{1}{2}\big[db(3r + s) + ac(r - s)\big]/q, & n_x = -(rac + dbs)/q, \\ m_y = \frac{1}{2}\big[da(3r + s) - bc(r - s)\big]/q, & n_y = (das - bcr)/q, \\ m_z = (r - s)/2, & n_z = r, \end{cases}$$

and

$$\begin{cases} m_u = -(rac + dbs)/q, & n_u = -\frac{1}{2}\big[db(s - 3r) + ac(r + s)\big]/q, \\ m_v = (das - rbc)/q, & n_v = \frac{1}{2}\big[da(s - 3r) - bc(r + s)\big]/q, \\ m_w = r, & n_w = (r + s)/2, \end{cases} \tag{5}$$

where $q = a^2 + b^2$ and (r, s) is a suitable solution of $2q = s^2 + 3r^2$ that makes all the numbers in (5) integers, together with the origin $(O(0, 0, 0))$ forms an equilateral triangle in $\mathbb{Z}^3$ contained in the plane

$$\mathcal{P}_{a,b,c} := \big\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \;\big|\; a\alpha + b\beta + c\gamma = 0\big\}$$

and having sides-lengths equal to $d\sqrt{2(m^2 - mn + n^2)}$.

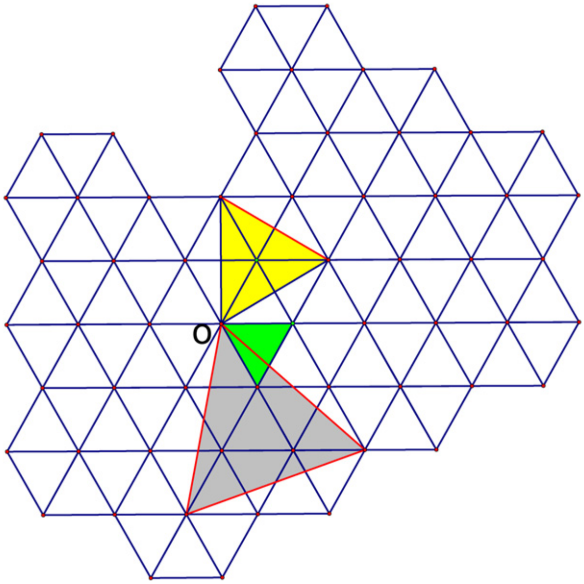

Fig. 1. Lattice generated by (4) with $m, n \in \mathbb{Z}$ in the plane $\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) \in \mathbb{R} \mid a\alpha + b\beta + c\gamma = 0\}$.

Conversely, there exists a choice of the integers r and s such that given an arbitrary equilateral triangle in $\mathbb{R}^3$ whose vertices, one at the origin and the other two having integer coordinates and contained in the plane $\mathcal{P}_{a,b,c}$, then there also exist integers m and n such that the two vertices not at the origin are given by (4) *and* (5)*.*

In Fig. 1 we show a few triangles that can be obtained, using (4), for various values of m and n. Out of all these equilateral triangles only a few are faces of regular tetrahedra with integer coordinates. The restriction comes from the following characterization given as Proposition 5.3 in [3].

Proposition 1.1. *A regular tetrahedron having side lengths l and with integer coordinates exists, if and only if $l = \lambda\sqrt{2}$ for some $\lambda \in \mathbb{N}$.*

Given a $\lambda \in \mathbb{N}$, we are interested in finding a characterization of all such tetrahedra having side lengths equal to $\lambda\sqrt{2}$. Let us denote this class of objects by T_λ and its subset of tetrahedra that have the origin as a vertex by T^0_λ. Obviously we have $\mathrm{T}_\lambda = \bigcup_{v\in\mathbb{Z}^3}(\mathrm{T}^0_\lambda + v)$ but this union is not a partition. In what follows we will use the notation $\mathrm{T} := \bigcup_{\lambda\in\mathbb{N}}(\mathrm{T}_\lambda)$. This latter union is, on the other hand, a partition.

Next we are going to concentrate on T^0_λ. It turns out that T^0_λ is numerous if λ is divisible by many prime factors of the form $6k+1$, $k \in \mathbb{N}$, as one can see from the following. First we recall Euler's $6k+1$ theorem (see [5, pp. 568] and [2, pp. 56]).

Proposition 1.2. *An integer t can be written as $s^2 - sr + r^2$ for some $s, r \in \mathbb{Z}$ if and only if in the prime factorization of t, 2 and the primes of the form $6k-1$ appear to an even exponent.*

The following result seems to be known but we could not find a reference for it.

Proposition 1.3. *Given $k = 3^\alpha(a^2)b$ where a does not contain any prime factors of the form $6\ell+1$, b is the product of such primes, say $b = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$, then the number of representations of k, $k = m^2 - mn + n^2$,*

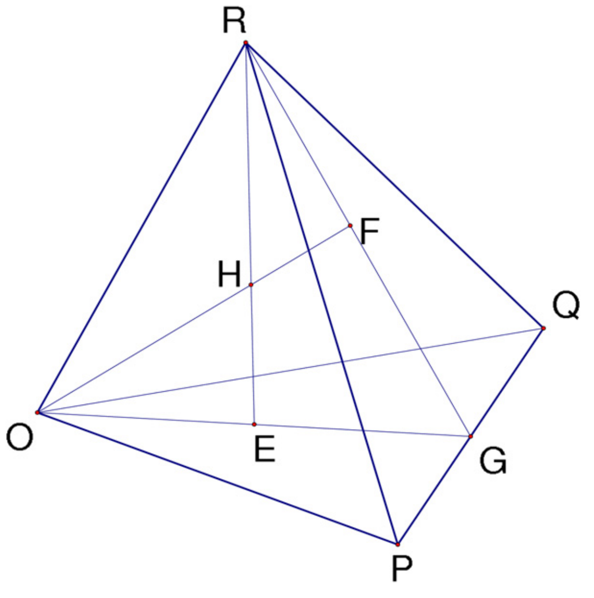

Fig. 2. Regular tetrahedron.

counting all possible pairs (m, n) *of integers that satisfy this equation, is equal to*

$$6(r_1+1)(r_2+1)\cdots(r_j+1).$$

For a proof of this proposition one can use arguments similar to those in the proof of Gauss's Theorem about the number of representations of a number as sums of two squares of integers [5]. In this case one has to replace Gaussian integers with Eisenstein integers, i.e. the ring of numbers of the form $m+n\omega$, $m, n \in \mathbb{Z}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. Related to Proposition 1.3, we found a conjecture in a paper posted on the mathematics archives (see [4]). This article refers to the number of representations of a number as m^2+mn+n^2 with $m, n \in \mathbb{N}$.

2. Main results

Let us begin by refining the argument used in the proof of Proposition 5.3 in [3]. That proposition referred to Fig. 2 representing a regular tetrahedron whose vertices are given by the origin, $P(u, v, w)$, $Q(x, y, z)$ and R:

By the characterization of equilateral triangles in Theorem 1.1 we may assume that the coordinates of P and Q are given by (4) and (5).

If E denotes the center of the face $\triangle OPQ$, then from Theorem 1.1 we know that $\frac{\vec{ER}}{|\vec{ER}|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{3}d}(a, b, c)$ for some $a, b, c, d, m, n \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, d odd satisfying $a^2+b^2+c^2 = 3d^2$ and $l = d\sqrt{2(m^2-mn+n^2)}$. Let us denote m^2-mn+n^2 by $\zeta(m, n)$. The coordinates of E are $(\frac{u+x}{3}, \frac{y+v}{3}, \frac{z+w}{3})$.

From the Pythagorean theorem one can find easily that $|\overline{RE}| = l\sqrt{\frac{2}{3}}$. Since $\vec{OR} = \vec{OE} + \vec{ER}$, the coordinates of R must be given by

$$\left(\frac{u+x}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}a, \frac{y+v}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}b, \frac{z+w}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}c\right)$$

or

$$\left(\frac{u+x}{3} \pm \frac{2a\sqrt{\zeta(m,n)}}{3}, \frac{y+v}{3} \pm \frac{2b\sqrt{\zeta(m,n)}}{3}, \frac{z+w}{3} \pm \frac{2c\sqrt{\zeta(m,n)}}{3}\right).$$

Since these coordinates are assumed to be integers we see that $\zeta(m,n)$ must be a perfect square, say k^2, $k \in \mathbb{N}$.

It is worth mentioning that this leads to the Diophantine equation

$$m^2 - mn + n^2 = k^2 \tag{6}$$

whose positive solutions are known as Eisenstein triples, or Eisenstein numbers. A primitive solution of (6) is one for which $\gcd(m,n) = 1$. These triples can be characterized in a similar way to the Pythagorean triples.

Theorem 2.1. *Every primitive solution of the Diophantine equation* (6) *is in one of the two forms*:

$$m = t^2 - s^2, \qquad n = 2st - s^2 \quad \text{and} \quad k = s^2 - st + t^2, \quad \text{with } t > s, \quad \text{or}$$
$$m = 2st - s^2, \qquad n = 2st - t^2 \quad \text{and} \quad k = s^2 - st + t^2, \quad \text{with } 2t > s > t/2, \tag{7}$$

where $s, t \in \mathbb{N}$, $\gcd(s,t) = 1$, *and* $s + t \not\equiv 0 \pmod 3$. *Conversely, every triple given by one of the alternatives in* (7) *is a primitive solution of* (6)*.*

We leave the proof of this theorem to the reader.

Next, let us introduce the notation $\Omega(k) := \{(m,n) \in \mathbb{Z}^2\colon\ \zeta(m,n) = k^2\}$. If the primes dividing k are all of the form $p \equiv 2 \pmod 3$ then we have simply

$$\Omega(k) = \bigl\{(k,0), (k,k), (0,k), (-k,0), (-k,-k), (0,-k)\bigr\}$$

but in general this set can be a much larger as one can see from Proposition 1.3. For instance if $k = 7$ then one can check the nontrivial representation $49 = \zeta(8,3)$. For each solution $(m,n) \in \Omega(k)$, in general, one can find eleven more by applying the following transformations: $(m,n) \overset{\tau_1}{\rightsquigarrow} (m-n,m)$, $(m,n) \overset{\tau_2}{\rightsquigarrow} (m,n-m)$, then their permutations $(m,n) \overset{\tau_3}{\rightsquigarrow} (n,m)$, $(m,n) \overset{\tau_4}{\rightsquigarrow} (m,m-n)$, $(m,n) \overset{\tau_5}{\rightsquigarrow} (n-m,n)$, and finally the reflections into the origin of all the above maps $(m,n) \overset{\tau_6}{\rightsquigarrow} (-m,-n)$, $(m,n) \overset{\tau_7}{\rightsquigarrow} (n-m,-m)$, $(m,n) \overset{\tau_8}{\rightsquigarrow} (-m,m-n)$, $(m,n) \overset{\tau_9}{\rightsquigarrow} (-n,-m)$, $(m,n) \overset{\tau_{10}}{\rightsquigarrow} (-m,n-m)$, $(m,n) \overset{\tau_{11}}{\rightsquigarrow} (m-n,-n)$.

Hence, the coordinates of R depend on $(m,n) \in \Omega(k)$ and two possible choices of signs:

$$R = \left(\frac{\substack{(m_x+m_u)m \\ -(n_x+n_u)n \\ \pm 2ak}}{3}, \frac{\substack{(m_y+m_v)m \\ -(n_y+n_v)n \\ \pm 2bk}}{3}, \frac{\substack{(m_z+m_w)m \\ -(n_z+n_w)n \\ \pm 2ck}}{3}\right), \quad (m,n) \in \Omega(k). \tag{8}$$

We would like to show that for every primitive solution of Eq. (3), every $k \in \mathbb{N}$, and every $(m,n) \in \Omega(k)$ one has either integer coordinates in (8) or can choose the signs in order to accomplish this. A primitive solution of (3) is a solution that satisfies the conditions $\gcd(a,b,c) = 1$. We begin with a few lemmas.

Lemma 2.1. *Every primitive solution* (a,b,c,d) *of* (3) *must satisfy*

(i) $a \equiv \pm 1 \pmod 6$, $b \equiv \pm 1 \pmod 6$ *and* $c \equiv \pm 1 \pmod 6$,
(ii) $a^2 + b^2 \equiv 2 \pmod 6$.

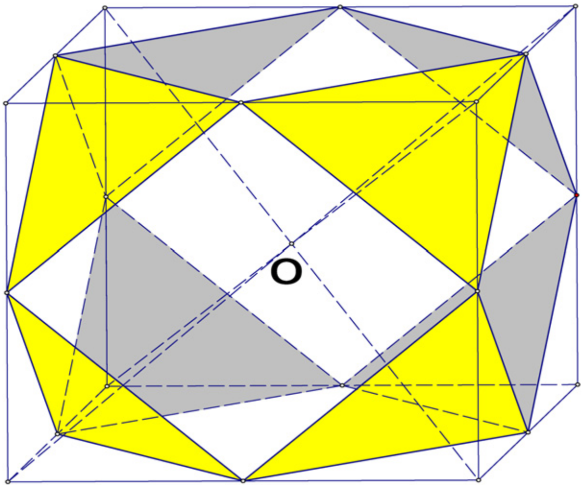

Fig. 3. All regular tetrahedra in T_1^0.

Proof. One can see that all numbers a, b, c and d must be odd since $\gcd(a,b,c)=1$. Then an odd perfect square is congruent to only 1 or 3 modulo 6. By way of contradiction if we have for instance $a \equiv 3 \pmod 6$ then 3 divides $a^2 = 3d^2 - (b^2+c^2)$. This implies that 3 divides b^2+c^2 which is possible only if 3 divides b and c. Therefore that would contradict the assumption $\gcd(a,b,c)=1$. Hence, (i) is shown and (ii) is just a simple consequence of (i). □

Let us observe that (ii) in Lemma 2.1 implies in particular that $q = a^2 + b^2$ is coprime to 3.

Lemma 2.2. *Let $k \in \mathbb{N}$ and $(m,n) \in \Omega(k)$.*

(a) *Then m and n must satisfy $k \equiv \pm(m+n) \pmod 3$.*
(b) *If $k \equiv 0 \pmod 3$ then $m \equiv n \equiv 0 \pmod 3$.*

Proof. (a) This part follows immediately from the identities

$$k^2 \equiv 4k^2 = 4m^2 - 4mn + 4n^2 = 3n^2 + (2m-n)^2 \equiv (2m-n)^2 \equiv (m+n)^2 \pmod 3.$$

(b) For the second part let us observe that if $k \equiv 0 \pmod 3$ then by part (a) we must have $m = 3t - n$ for some $t \in \mathbb{Z}$. This in turn gives $9k'^2 = k^2 = m^2 - mn + n^2 = 3(3t^2 - 3nt + n^2)$ which shows that 3 divides $3t^2 - 3nt + n^2$. So, finally n must be divisible by 3 and then so is m. □

Lemma 2.3. *Let $q = a^2 + b^2$ with (a,b,c,d) a primitive solution of (3) and $r, s \in \mathbb{Z}$ a solution of the equation $2q = s^2 + 3r^2$. Then $1 \equiv a^2 \equiv b^2 \equiv c^2 \equiv s^2 \pmod 3$.*

Proof. Since $2c^2 = 6d^2 - 2q \equiv -s^2 - 3r^2 \equiv -s^2 \pmod 3$ then we have $2c^2 \equiv 2s^2 \pmod 3$ which together with Lemma 2.1 gives the desired conclusion. □

For $a = b = c = 1$, we have 8 elements in T_1^0 as shown in Fig. 3, one for each triangular face of a cuboctahedron (Archimedean solid A_1 or also known as a truncated cube):

The number of tetrahedra in T_λ^0 in general depends on λ. For instance, if $\lambda = 3$ then we calculated that $|\mathrm{T}_3^0| = 40$.

In general, the faces of an element in T^0_λ must have normal vectors that satisfy (3). But these equations can be very different and one has to go search far enough to find such an example. For instance, the tetrahedron $OABC$ where $A = (376, -841, 2265)$, $B = (-1005, -2116, 701)$, $C = (1411, -1965, 356)$ has the four faces with normal vectors

$$(-187, 113, 73), \quad \text{satisfying } 187^2 + 113^2 + 73^2 = 3\big(133^2\big),$$

$$(-343, -253, -37), \quad \text{satisfying } 343^2 + 253^2 + 37^2 = 3(247)^2,$$

$$(19, 41, 151), \quad \text{satisfying } 19^2 + 41^2 + 151^2 = 3(91)^2 \quad \text{and}$$

$$(391, -2461, 1661), \quad \text{satisfying } 391^2 + 2461^2 + 1661^2 = 3(1729)^2.$$

Theorem 2.2. *Every tetrahedron in T^0_λ can be obtained by taking as one of its faces an equilateral triangle having the origin as a vertex and the other two vertices given by* (4) *and* (5) *with a, b, c and d odd integers satisfying* (3) *with d a divisor of λ, and then completing it with the fourth vertex as in* (8) *for some $(m, n) \in \Omega(\lambda/d)$.*

Conversely, if we let a, b, c and d be a primitive solution of (3)*, let $k \in \mathbb{N}$ and $(m, n) \in \Omega(k)$, then the coordinates of the point R in* (8) *are*

(a) *all integers, if $k \equiv 0 \pmod 3$ regardless of the choice of signs or*
(b) *integers, precisely for only one choice of the signs if $k \not\equiv 0 \pmod 3$.*

Proof. The first part of the theorem follows from the arguments at the beginning of this section. For the second part, the case in which k is a multiple of 3 follows from Lemma 2.2.

Let us look into a situation in which $k \not\equiv 0 \pmod 3$. First we are going to analyze the third coordinate of R. Since (5) gives $m_z + m_w = \frac{r-s}{2} + r = \frac{3r-s}{2}$ and $n_z + n_w = \frac{3r+s}{2}$ we see that

$$2\big[(m_z + m_w)m - (n_z + n_w)n \pm 2ck\big] \equiv -s(m + n) \pm ck \pmod 3. \tag{9}$$

By Lemmas 2.2 and 2.3 we get that

$$-s(m + n) \pm ck \equiv \pm k(s \pm c) \equiv 0 \pmod 3$$

only for one choice of signs. So, the last coordinate of R satisfies the statement of our theorem in part (b). If the coordinates of R are (x_R, y_R, z_R) (which by (8) are rational numbers) we must have $x_R^2 + y_R^2 + z_R^2 = \ell^2$. So, it is enough to analyze just one other coordinate of R. From (5) we obtain that $2q(m_x + m_u) = -db(3r + s) - ac(r - s) - 2(rac + dbs) \equiv acs \pmod 3$, and $2q(n_x + n_u) = -2(rac + dbs) - db(s - 3r) - ac(r + s) \equiv -acs \pmod 3$. Therefore,

$$2q\big[(m_x + m_u)m - (n_x + n_u)n \pm 2ak\big] \equiv acs(m + n) \pm qak \pmod 3.$$

But $q \equiv -1 \pmod 3$ by Lemma 2.1, and so

$$2q\big[(m_x + m_u)m - (n_x + n_u)n \pm 2ak\big] \equiv a\big(cs(m + n) \mp k\big) \pmod 3.$$

Because $c^2 \equiv 1 \pmod 3$, we can multiply the above relation by c to get

$$2qc\big[(m_x + m_u)m - (n_x + n_u)n \pm 2ak\big] \equiv -a\big[-s(m + n) \pm ck\big] \pmod 3,$$

which shows, based on (9), that $(m_x + m_u)m - (n_x + n_u)n \pm 2ak$ is divisible by 3 if and only if z_R is an integer. Hence we have proved the last part of Theorem 2.2. $\square$

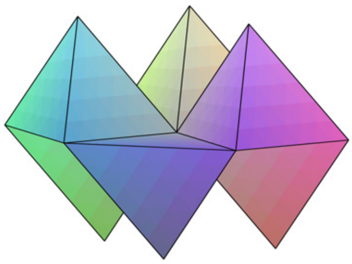

Fig. 4. Regular tetrahedra in the case $k \not\equiv 0 \pmod 3$.

Fig. 4 shows how our tetrahedra look if we start from equilateral triangles as bases contained in a plane and sharing a vertex. This configuration is as expected according to Theorem 2.2.

Remark. The alternating behavior that can be observed in Fig. 4 follows from Theorem 2.2 part (b). Indeed, we can start with an equilateral triangle contained in a plane $\mathfrak{P}$ which is one of the tiles and the generator of a regular triangular tessellation. By Theorem 2.2 part (b) each such tile from this tessellation is the face of a regular tetrahedron in T that is located in one and only one of the sides of $\mathfrak{P}$. We denote the two half spaces by $\mathfrak{P}_+$ and $\mathfrak{P}_-$. Let us take one such tetrahedron, say in $\mathfrak{P}_+$, and translate this tetrahedron such that the fourth vertex, R, becomes the origin. The other vertices give rise to three vertices of integer coordinates in $\mathfrak{P}_-$ that are each the fourth vertex R' for three other tetrahedra living in $\mathfrak{P}_-$ as in Fig. 4. This type of translation can be repeated with one of the tetrahedra in $\mathfrak{P}_-$ and one obtains the alternating pattern of tetrahedra as in Fig. 4.

Proposition 2.1. *For a regular tetrahedron in* T^0_λ *each face lies in a plane with a normal vector* $n_i = (a_i, b_i, c_i) \in \mathbb{Z}^3$, $i = 1, 2, 3, 4$, *which satisfy*

$$\begin{aligned} a_i^2 + b_i^2 + c_i^2 = 3d_i^2, &\quad d_i \text{ an odd integer dividing } \lambda,\ 1 \leqslant i \leqslant 4, \\ a_i a_j + b_i b_j + c_i c_j + d_i d_j = 0, &\quad 1 \leqslant i < j \leqslant 4. \end{aligned} \tag{10}$$

Proof. We are going to refer to Fig. 2. Let E be the center of the face OPQ, F be the center of the face RPQ, G the midpoint of the side $\overline{PQ}$, and H the center of the tetrahedron $ORPQ$ (O being the origin). Since O, R, E, F, G, and H are all coplanar, and F and E are the intersections of the medians on each corresponding face we have $\frac{RF}{RG} = \frac{2}{3}$ and $\frac{EG}{EO} = \frac{1}{2}$. By Menelaus's theorem $\frac{HO}{HF} \cdot \frac{RF}{RG} \cdot \frac{EG}{EO} = 1$. This gives $\frac{HO}{HF} = 3$ and from here we have $HO = HR = \frac{3}{4}\sqrt{\frac{2}{3}}\ell = \sqrt{\frac{3}{8}}\ell$. Using the cosine law in the triangle ORH gives: $OR^2 = 2HR^2 - 2HR^2 \cos \widehat{OHR}$ which gives

$$\cos \widehat{OHR} = -\frac{1}{3}. \tag{11}$$

So, if the normal exterior unit vectors of the faces RPQ and OPQ are, say $\frac{(a_1,b_1,c_1)}{\sqrt{3}d_1}$ and $\frac{(a_2,b_2,c_2)}{\sqrt{3}d_2}$, using (11) gives $a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 = 0$. The rest of the identities follow similarly or from what we have mentioned so far. $\square$

Remark. If we consider the matrix

$$MT := \frac{1}{2}\begin{bmatrix} \frac{a_1}{d_1} & \frac{b_1}{d_1} & \frac{c_1}{d_1} & 1 \\ \frac{a_2}{d_2} & \frac{b_2}{d_2} & \frac{c_2}{d_2} & 1 \\ \frac{a_3}{d_3} & \frac{b_3}{d_3} & \frac{c_3}{d_3} & 1 \\ \frac{a_4}{d_4} & \frac{b_4}{d_4} & \frac{c_4}{d_4} & 1 \end{bmatrix} \tag{12}$$

the restrictions in (10) can be reformulated in terms of MT by requiring it to be orthogonal, i.e. $MT(MT)^t = (MT)^t MT = I$. It is known that the set of orthogonal matrices form a group. We also want to point out that in (10) only two of the exterior normal vectors n_i are essential. The other two can be computed from the given ones.

An interesting question here is whether or not every two vectors (a, b, c) and (a', b', c') which satisfy the conditions

$$a^2 + b^2 + c^2 = 3d^2, \qquad {a'}^2 + {b'}^2 + {c'}^2 = 3{d'}^2, \qquad aa' + bb' + cc' + dd' = 0 \tag{13}$$

for some $d, d' \in \mathbb{Z}$, correspond to a tetrahedron in T containing two faces normal to the given vectors. It turns out that the answer is a positive one and the proof of it follows from Theorem 4 in [3].

Another corollary of Theorem 2.2 about solutions of a particular case of the Diophantine system (13) is given next.

Corollary 2.1. *For every odd integer $d > 1$, the Diophantine system*

$$a^2 + b^2 + c^2 = 3d^2, \qquad {a'}^2 + {b'}^2 + {c'}^2 = 3d^2, \qquad aa' + bb' + cc' = -d^2 \tag{14}$$

always has nontrivial solutions (*one trivial solution is $a = b = c = d$ and $a' = b' = -d$, $c = d$.*)

Proof. We have shown in [3] that the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ always has nontrivial solutions. We then take $m = n = 1$ in Theorem 1.1 to obtain an equilateral triangle contained in the plane of normal vector (a, b, c). By Theorem 2.2, case $k = 1$, we can complete this equilateral triangle to a regular tetrahedron in T. Taking the normals to the new faces will give four normals and say $(\hat{a}, \hat{b}, \hat{c})$ is one of them. Let us assume that $\gcd(\hat{a}, \hat{b}, \hat{c}) = 1$. We know that there is a $\hat{d}$ such that $\hat{a}^2 + \hat{b}^2 + \hat{c}^2 = 3\hat{d}^2$. The side length of this tetrahedron is, by our choice of m and n, equal to $d\sqrt{2}$. Therefore, from Theorem 1.1 applied to the plane normal to $(\hat{a}, \hat{b}, \hat{c})$, we get that $d\sqrt{2} = \hat{d}\sqrt{2(u^2 - uv + v^2)}$ for some $u, v \in \mathbb{Z}$, which implies that $\hat{d}$ divides d. Hence we can adjust $(\hat{a}, \hat{b}, \hat{c})$, if necessary, by a multiplicative factor in order to satisfy (14). $\square$

Supplementary material

The online version of this article contains additional supplementary material.
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References

[1] R. Chandler, E.J. Ionascu, A characterization of all equilateral triangles in $\mathbb{Z}^3$, Integers 8 (2008), Article A19.
[2] R. Guy, Unsolved Problems in Number Theory, Springer-Verlag, 2004.
[3] E.J. Ionascu, A parametrization of equilateral triangles having integer coordinates, J. Integer Seq. 10 (2007), Article 07.6.7.
[4] U.P. Nair, Elementary results on the binary quadratic form $a^2 + ab + b^2$, arXiv:0408107v1.
[5] K. Rosen, Elementary Number Theory, fifth ed., Addison–Wesley, 2004.