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Some new continued fraction approximation of Euler's constant

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A B S T R A C T

In this paper, using continued fraction, some quicker classes of sequences convergent to Euler's constant are provided. Finally, for demonstrating the superiority of our new convergent sequences over DeTemple's sequence, Vernescu's sequence and Mortici's sequences, some numerical computations are also given.

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1. Introduction

An important concern in the theory of mathematical constants is to define some new sequences which have higher convergent speed towards some fundamental constants. Those constants and new sequences play an important role in many fields of mathematics and nature science, such as special functions, theory of probability, physics, applied statistics, number theory, and analysis.

It is well known that one of the most useful convergent sequences is

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.1}$$

whose limit is known as Euler's constant, denoted by

$$\gamma = 0.577215\ldots.$$

So far, many researchers have devoted great efforts and achieved much in the area of improving the convergent rate of the sequence $(\gamma_n)_{n\geq 1}$. Among them, there are many inspiring achievements. For example, in [12–14,16], the estimate

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n} \tag{1.2}$$

was established with interesting geometric interpretations. In [1,2], a faster convergent sequence $(D_n)_{n\geq 1}$ to γ was introduced, which is defined as

$$D_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right). \tag{1.3}$$

DeTemple also concluded that the speed of the new sequence to γ is the same as the speed of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < D_n - \gamma < \frac{1}{24n^2}. \tag{1.4}$$

In [15], Vernescu presented a new modification,

$$V_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \tag{1.5}$$

and the estimate was provided as

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}. \tag{1.6}$$

In both (1.3) and (1.5), only slight modifications are made to Euler's sequence (1.1), but the convergent rates are significantly improved from n^{-1} to n^{-2}.

Moreover, Mortici obtained some even faster convergent sequences than (1.1), (1.3) and (1.5).

In [5], two sequences were defined as follows,

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6}\,)n} - \ln\left(n + \frac{1}{\sqrt{6}}\right), \tag{1.7}$$

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6}\,)n} - \ln\left(n - \frac{1}{\sqrt{6}}\right). \tag{1.8}$$

Both of (1.7) and (1.8) had been proved to converge to γ as n^{-3}.

Furthermore,

$$\mu_n(a,b) = \sum_{k=1}^{n} \frac{1}{k} + \ln\left(e^{a/(n+b)} - 1\right) - \ln a \tag{1.9}$$

was introduced by Mortici in [7], where $a, b \in \mathbb{R}$, $a > 0$. They also proved that among the sequences $(\mu_n(a,b))_{n\geq 1}$, in the case of $a = \sqrt{2}/2$, $b = (2+\sqrt{2}\,)/4$, the privileged sequence offers the best approximations of γ, since

$$\lim_{n\to\infty} n^3\left(\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right) - \gamma\right) = \frac{\sqrt{2}}{96}. \tag{1.10}$$

Recently, in [3,4], we also provided some approximations of Euler's constant. A new important sequence was defined as follows,

$$L_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \cdots}}}}, \tag{1.11}$$

where $a_1 = 1/2$, $a_2 = 1/6$, $a_3 = -1/6$, $\ldots$. Two special sequences were provided as

$$L_n^{(2)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{a_1}{n + a_2}, \tag{1.12}$$

$$L_n^{(3)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{a_1}{n + \frac{a_2 n}{n + a_3}}. \tag{1.13}$$

These two sequences converge more quickly than all other sequences mentioned above, since for any $n \in \mathbb{N}$,

$$\frac{1}{72(n+1)^3} < \gamma - L_n^{(2)} < \frac{1}{72n^3} \quad \text{and} \quad \frac{1}{120(n+1)^4} < L_n^{(3)} - \gamma < \frac{1}{120(n-1)^4}.$$

It is these works that motivate our study. In this paper, our main goal is to modify the sequence based on the early works of DeTemple, Mortici and Lu, and provide new convergent sequence with higher speed and relatively simple form.

The rest of this paper is arranged as follows. In Section 2, we provide the main results. In Section 3, the proofs of the main results are given. In Section 4, we give some numerical computations which demonstrate the superiority of our new convergent sequence over DeTemple's sequence, Vernescu's sequence and Mortici's sequence.

2. The main results

Theorem 2.1. *For any fixed $k, s \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, we have the following convergent sequence for Euler's constant,*

$$r_{n,k}^{(s)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k} \ln\left(1 + \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots + a_s}}}}\right), \tag{2.1}$$

where

$$a_1 = \frac{k}{2}, \qquad a_2 = \frac{2-3k}{12}, \qquad a_3 = \frac{3k^2+4}{12(3k-2)},$$
$$a_4 = -\frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{20(3k-2)(3k^2+4)}, \quad \ldots.$$

Furthermore, let

$$r_{n,k}^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k} \ln\left(1 + \frac{a_1}{n}\right); \tag{2.2}$$

$$r_{n,k}^{(2)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k} \ln\left(1 + \frac{a_1}{n + a_2}\right); \tag{2.3}$$

$$r_{n,k}^{(3)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k} \ln\left(1 + \cfrac{a_1}{n + \frac{a_2 n}{n + a_3}}\right); \tag{2.4}$$

$$r_{n,k}^{(4)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k} \ln\left(1 + \cfrac{a_1}{n + \cfrac{a_2 n}{n + \frac{a_3 n}{n + a_4}}}\right). \tag{2.5}$$

We also have

$$\lim_{n\to\infty} n^2\left(r_{n,k}^{(1)} - \gamma\right) = -\frac{1}{12} + \frac{k}{8}; \tag{2.6}$$

$$\lim_{n\to\infty} n^3\left(r_{n,k}^{(2)} - \gamma\right) = -\frac{k^2}{96} - \frac{1}{72}; \tag{2.7}$$

$$\lim_{n\to\infty} n^4\left(r_{n,k}^{(3)} - \gamma\right) = \frac{1}{5760} \frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{2 - 3k}; \tag{2.8}$$

$$\lim_{n\to\infty} n^5\left(r_{n,k}^{(4)} - \gamma\right) = \frac{1}{115\,200} \frac{15k^6 + 120k^4 + 1520k^2 + 2304}{3k^2 + 4}. \tag{2.9}$$

Using Theorem 2.1, we have a conclusion as follows:

Corollary 2.1. *For $s = 1$, the fastest possible sequence $(r_{n,k}^{(1)})_{n\ge 1}$ is obtained only for $k = 1$ and*

$$\lim_{n\to\infty} n^2\big(r_{n,1}^{(1)} - \gamma\big) = \frac{1}{24}. \tag{2.10}$$

For $s = 2$, the fastest possible sequence $(r_{n,k}^{(2)})_{n\ge 1}$ is obtained only for $k = 1$ and

$$\lim_{n\to\infty} n^3\big(r_{n,1}^{(2)} - \gamma\big) = -\frac{7}{288}. \tag{2.11}$$

For $s = 3$, the fastest possible sequence $(r_{n,k}^{(3)})_{n\ge 1}$ is obtained only for $k = 2$ and

$$\lim_{n\to\infty} n^4\big(r_{n,2}^{(3)} - \gamma\big) = -\frac{1}{180}. \tag{2.12}$$

For $s = 4$, the fastest possible sequence $(r_{n,k}^{(4)})_{n\ge 1}$ is obtained only for $k = 1$ and

$$\lim_{n\to\infty} n^5\big(r_{n,1}^{(4)} - \gamma\big) = \frac{3959}{806\,400}. \tag{2.13}$$

It is easy to see that $r_{n,1}^{(1)} = D_n$ and (2.2) is equivalent to (1.3). Comparing with DeTemple's sequence $(D_n)_{n\ge 1}$, Vernescu's sequence $(V_n)_{n\ge 2}$, Mortici's sequences $(u_n)_{n\ge 1}$, $(v_n)_{n\ge 1}$ and $\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4})$, $r_{n,2}^{(3)}$ improve the rate of convergence from n^{-2} and n^{-3} to n^{-4}. In fact, using Theorem 2.1, we can obtain other convergent sequences which are faster than $r_{n,2}^{(3)}$.

Furthermore, for $r_{n,1}^{(2)}$ and $r_{n,2}^{(3)}$, similarly to (1.2), (1.4) and (1.6), we also have the following inequalities:

Theorem 2.2. *For all natural numbers n,*

$$\frac{7}{288(n+1)^3} < \gamma - r_{n,1}^{(2)} < \frac{7}{288n^3}; \tag{2.14}$$

$$\frac{1}{180(n+1)^4} < \gamma - r_{n,2}^{(3)} < \frac{1}{180n^4}. \tag{2.15}$$

For obtaining Theorem 2.1, we need the following lemma which was used in [6–11] and very useful for construction of convergent sequence.

Lemma 2.1. *If $(x_n)_{n\ge 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{2.16}$$

with $s > 1$, then

$$\lim_{n\to\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.17}$$

Lemma 2.1 was firstly proved by Mortici in [9]. From Lemma 2.1, we can see that the speed of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value s satisfying (2.16).

3. Proofs of the main results

3.1. Proof of Theorem 2.1

Based on the argument of Theorem 2.1 in [10] or Theorem 5 in [11], we need to find the value of $a_1 \in R$ which produces the most accurate approximation of the form

$$r_{n,k}^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k}\ln\left(1 + \frac{a_1}{n}\right). \tag{3.1}$$

To measure the accuracy of this approximation, a method is to say that an approximation (3.1) is better as $r_{n,k}^{(1)} - \gamma$ faster converges to zero. Using (3.1), we have

$$r_{n,k}^{(1)} - r_{n+1,k}^{(1)} = -\frac{1}{n+1} - \ln n - \frac{1}{k}\ln\left(1 + \frac{a_1}{n}\right) + \ln(n+1) + \frac{1}{k}\ln\left(1 + \frac{a_1}{n+1}\right). \tag{3.2}$$

Developing in power series in $1/n$, we have

$$r_{n,k}^{(1)} - r_{n+1,k}^{(1)} = \frac{k-2a_1}{2k}\frac{1}{n^2} + \frac{3a_1 + 3a_1^2 - 2k}{3k}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{3.3}$$

From Lemma 2.1, we know that the speed of convergence of the sequence $(r_{n,k}^{(1)})_{n\geq 1}$ is even higher as the value s satisfying (2.16). Thus, using Lemma 2.1, we have:

(i) If $a_1 \neq k/2$, then the convergent rate of sequence $(r_{n,k}^{(1)} - \gamma)_{n\geq 1}$ is $1/n$, since

$$\lim_{n\to\infty} n\left(r_{n,k}^{(1)} - \gamma\right) = \frac{k-2a_1}{2k} \neq 0.$$

(ii) If $a_1 = k/2$, then from (3.3), we have

$$r_{n,k}^{(1)} - r_{n+1,k}^{(1)} = \left(-\frac{1}{6} + \frac{k}{4}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),$$

and the rate of convergence of the sequence $(r_{n,k}^{(1)} - \gamma)_{n\geq 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2\left(r_{n,k}^{(1)} - \gamma\right) = -\frac{1}{12} + \frac{k}{8}.$$

We know that the fastest possible sequence $(r_{n,k}^{(1)})_{n\geq 1}$ is obtained only for $a_1 = k/2$.

Next, we define the sequence $(r_{n,k}^{(2)})_{n\geq 1}$ by the relation

$$r_{n,k}^{(2)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k}\ln\left(1 + \frac{k/2}{n+a_2}\right). \tag{3.4}$$

Using the similar method from (3.1)–(3.3), we have

$$\begin{aligned} r_{n,k}^{(2)} - r_{n+1,k}^{(2)} &= \left(a_2 - \frac{1}{6} + \frac{k}{4}\right)\frac{1}{n^3} \\ &\quad + \left(\frac{1}{4} - \frac{3k}{8} - \frac{3a_2k}{4} - \frac{3a_2}{2} - \frac{3a_2^2}{2} - \frac{k^2}{8}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \end{aligned} \tag{3.5}$$

The fastest possible sequence $(r_{n,k}^{(2)})_{n\geq 1}$ is obtained when $a_2 = 1/6 - k/4$, and we have

$$\lim_{n\to\infty} n^3\left(r_{n,k}^{(2)} - \gamma\right) = -\frac{k^2}{96} - \frac{1}{72},$$

and the rate of convergence is n^{-3}.

Moreover, we define the third sequence with the conclusions above,

$$r_{n,k}^{(3)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \frac{1}{k}\ln\left(1 + \frac{k/2}{n + \frac{n(1/6-k/4)}{n+a_3}}\right). \tag{3.6}$$

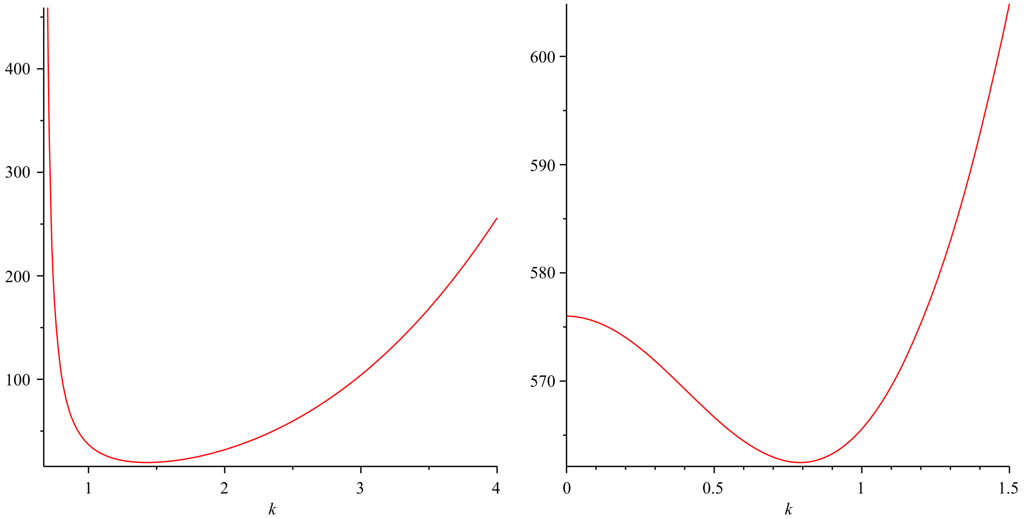
Then we can get the equation below,

$$\begin{aligned} r_{n,k}^{(3)} - r_{n+1,k}^{(3)} &= \left(\frac{3a_3k}{8} - \frac{a_3}{4} - \frac{1}{24} - \frac{k^2}{32}\right)\frac{1}{n^4} \\ &\quad + \left(\frac{k^2}{12} + \frac{11a_3}{18} - \frac{11a_3k}{12} - \frac{a_3^2k}{2} + \frac{a_3^2}{3} + \frac{17}{135}\right)\frac{1}{n^5} + O\left(\frac{1}{n^6}\right). \end{aligned} \tag{3.7}$$

Taking $a_3 = (3k^2+4)/(36k-24)$, we obtain the fastest sequence $(r_{n,k}^{(3)})_{n\geq 1}$ and the convergent rate is n^{-4}, since

$$\lim_{n\to\infty} n^4\left(r_{n,k}^{(3)} - \gamma\right) = \frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{5760(2-3k)}.$$

By induction, it is easy to obtain that $a_4 = -\frac{15k^4-30k^3+60k^2-104k+96}{20(3k-2)(3k^2+4)}, \ldots,$ and other sequence can be obtained in the same way.



(a) Value of $f_3(k)$ (b) Value of $f_4(k)$

Fig. 1. Values of $f_3(k)$ and $f_4(k)$.

3.2. Proof of Corollary 2.1

From (2.6), it is very easy to have the conclusion that when $k = 1$, $f_1(k) = |-1/12 + k/8| = 1/24$, which is the minimum value. Then, the proof of (2.10) is completed.

From (2.7), we have

$$f_2(k) = \left|-\frac{1}{96}k^2 - \frac{1}{72}\right| = \frac{1}{96}k^2 + \frac{1}{72} \quad \text{and} \quad f_2'(k) = \frac{k}{48} \geq 0.$$

Since the value of $f_2(k)$ increases as $k > 0$ increasing, we get the conclusion that when $k = 1$, we have the minimum value of $f_2(k)$. Then the proof of (2.11) is finished.

From (2.8) and (a) in Fig. 1, we take the minimum value of

$$f_3(k) = \left|\frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{5670(3k - 2)}\right|$$

as $k = 2$, so that the proof of (2.12) is completed.

From (2.9), it is easy to see from (b) in Fig. 1 that we obtain the minimum value of $f_4(k) = |(15k^6 + 120k^4 + 1520k^2 + 2304)/(115\,200(3k^2 + 4))|$ as $k = 1$.

3.3. Proof of Theorem 2.2

Based on the argument of Theorem in [1] or the method in [2], first, we prove (2.14). It is easy to have

$$\gamma - r_{n,1}^{(2)} = \sum_{k=n}^{\infty} \left(r_{k+1,1}^{(2)} - r_{k,1}^{(2)}\right) = \sum_{k=n}^{\infty} f(k), \tag{3.8}$$

where

$$f(k) = \frac{1}{k+1} - \ln\left(1 + \frac{1}{k}\right) - \ln\left(1 + \frac{1/2}{k + 11/12}\right) + \ln\left(1 + \frac{1/2}{k - 1/12}\right).$$

Next, we have

$$f'(x) = -\frac{6048x^2 + 6528x + 935}{x(12x+5)(12x-1)(12x+17)(12x+11)(x+1)^2}. \tag{3.9}$$

For the upper bound in (2.14), for $x \geq 1$, we have

$$-f'(x) \leq \frac{7}{24(x+\frac{1}{2})^5}. \tag{3.10}$$

Since $f(\infty) = 0$, we have

$$\begin{aligned} f(k) &= -\int_k^{\infty} f'(x)dx \leq \frac{7}{24} \int_k^{\infty} \left(x + \frac{1}{2}\right)^{-5} dx \\ &= \frac{7}{96}\left(k + \frac{1}{2}\right)^{-4} \leq \frac{7}{96} \int_k^{k+1} x^{-4} dx, \end{aligned} \tag{3.11}$$

where we use the following fact

$$\int_k^{k+1} x^{-4} dx - \left(k + \frac{1}{2}\right)^{-4} = \frac{40k^4 + 80k^3 + 51k^2 + 11k + 1}{3k^3(k+1)^3(2k+1)^4} > 0$$

in the last inequality in (3.11). Combining (3.8) and (3.11), we have

$$\gamma - r_{n,1}^{(2)} \leq \sum_{k=n}^{\infty} \frac{7}{96} \int_k^{k+1} x^{-4} dx = \frac{7}{96} \int_n^{\infty} x^{-4} dx = \frac{7}{288n^3}. \tag{3.12}$$

For the lower bound, by (3.9), for $x \geq 0$, we have

$$-f'(x) \geq \frac{7}{24(x+1)^5}. \tag{3.13}$$

By (3.13), we have

$$f(k) = -\int_k^\infty f'(x)dx \geq \frac{7}{24}\int_k^\infty (x+1)^{-5}dx$$
$$= \frac{7}{96}(k+1)^{-4} \geq \frac{7}{96}\int_{k+1}^{k+2} x^{-4}dx. \tag{3.14}$$

Combining (3.8) and (3.14), we have

$$\gamma - r_n^{(2)} \geq \sum_{k=n}^\infty \frac{7}{96}\int_{k+1}^{k+2} x^{-4}dx = \frac{7}{96}\int_{n+1}^\infty x^{-4}dx = \frac{7}{288(n+1)^3}. \tag{3.15}$$

Combining (3.12) and (3.15), we complete the proof of (2.14).

Next, we prove (2.15). It is easy to have

$$\gamma - r_{n,2}^{(3)} = \sum_{k=n}^\infty \left(r_{k+1,2}^{(3)} - r_{k,2}^{(3)}\right) = \sum_{k=n}^\infty g(k), \tag{3.16}$$

where

$$g'(x) = -\frac{1}{(x+1)^2(3x^2+9x+7)(3x^2+3x+1)}. \tag{3.17}$$

For the upper bound in (2.15), for $x \geq 0$, we have

$$-g'(x) \leq \frac{1}{9(x+1/2)^6}. \tag{3.18}$$

Since $g(\infty) = 0$, by (3.18), we have

$$g(k) = -\int_k^\infty g'(x)dx \leq \frac{1}{9}\int_k^\infty \left(x+\frac{1}{2}\right)^{-6}dx$$
$$= \frac{1}{45}\left(k+\frac{1}{2}\right)^{-5} \leq \frac{1}{45}\int_k^{k+1} x^{-5}dx. \tag{3.19}$$

Combining (3.16) and (3.19), we have

$$\gamma - r_{n,2}^{(3)} \leq \sum_{k=n}^\infty \frac{1}{45}\int_k^{k+1} x^{-5}dx = \frac{1}{45}\int_n^\infty x^{-5}dx = \frac{1}{180n^4}. \tag{3.20}$$

For the lower bound, for $x \geq 0$, we have

$$-g'(x) \geq \frac{1}{9(x+1)^6}. \tag{3.21}$$

By (3.21), we have

$$g(k) = -\int_k^\infty g'(x)dx \geq \frac{1}{9}\int_k^\infty (x+1)^{-6}dx = \frac{1}{45}(k+1)^{-5} \geq \frac{1}{45}\int_{k+1}^{k+2} x^{-5}dx. \tag{3.22}$$

Combining (3.16) and (3.22), we have

$$\gamma - r_{n,2}^{(3)} \geq \sum_{k=n}^\infty \frac{1}{45}\int_{k+1}^{k+2} x^{-5}dx = \frac{1}{45}\int_{n+1}^\infty x^{-5}dx = \frac{1}{180(n+1)^4}. \tag{3.23}$$

Combining (3.20) and (3.23), we complete the proof of (2.15).

4. Numerical computation

In this section, we give Tables 1 and 2 to demonstrate the superiority of our new convergent sequence $(r_{n,1}^{(2)})_{n\geq1}$, $(r_{n,2}^{(3)})_{n\geq1}$ and $(r_{n,1}^{(4)})_{n\geq1}$ over DeTemple's sequence $(D_n)_{n\geq1}$, Vernescu's $(V_n)_{n\geq1}$, Mortici's sequences $(\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}))_{n\geq1}$, and Lu's sequences $(L_n^{(2)})_{n\geq1}$ and $(L_n^{(3)})_{n\geq1}$.

Table 1
Simulations for D_n, V_n, $\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4})$, $L_n^{(2)}$ and $L_n^{(3)}$.

n	$D_n - \gamma$	$V_n - \gamma$	$\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}) - \gamma$	$L_n^{(2)} - \gamma$	$L_n^{(3)} - \gamma$
10	3.7733×10^{-4}	-8.3250×10^{-4}	1.1807×10^{-5}	-1.2832×10^{-5}	8.2941×10^{-7}
25	6.4061×10^{-5}	-1.3331×10^{-4}	8.6183×10^{-7}	-8.6169×10^{-7}	2.1317×10^{-8}
50	1.6337×10^{-5}	-3.3332×10^{-5}	1.1265×10^{-7}	-1.0941×10^{-7}	1.3331×10^{-9}
100	4.1252×10^{-6}	-8.3333×10^{-6}	1.4402×10^{-8}	-1.3782×10^{-8}	8.3329×10^{-11}
250	6.6401×10^{-7}	-1.3333×10^{-6}	9.3431×10^{-10}	-8.8616×10^{-10}	2.1333×10^{-12}
1000	4.1625×10^{-8}	-8.3333×10^{-8}	1.4698×10^{-11}	-1.3878×10^{-11}	8.3333×10^{-15}

Table 2
Simulations for $r_{n,1}^{(2)}$, $r_{n,2}^{(3)}$, and $r_{n,1}^{(4)}$.

n	$r_{n,1}^{(2)} - \gamma$	$r_{n,2}^{(3)} - \gamma$	$r_{n,1}^{(4)} - \gamma$
10	-2.2748×10^{-5}	-4.5329×10^{-7}	4.3237×10^{-8}
25	-1.5152×10^{-6}	-1.3126×10^{-8}	4.7845×10^{-10}
50	-1.9192×10^{-7}	-8.5398×10^{-10}	1.5330×10^{-11}
100	-2.4147×10^{-8}	-5.4455×10^{-11}	4.8499×10^{-13}
250	-1.5515×10^{-9}	-1.4109×10^{-12}	5.0029×10^{-15}
1000	-2.4290×10^{-11}	-5.5444×10^{-15}	4.9035×10^{-18}

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