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Some Eisenstein Series Identities

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In this paper we use the theory of elliptic functions to provide different proofs of some Eisenstein series identities of Ramanujan from those given in a recent paper by B. C. Berndt, S. Bhargava, and F. G. Garvan (1995, *Trans. Amer. Math. Soc.* **347**, 4136–4244). From one of these identities we derive the inversion formula for the Borweins cubic theta functions via Venkatachanliengar's method. We also derive some striking Eisenstein series identities associated with the Borweins' cubic theta functions. © 2000 Academic Press

Key Words: Eisenstein series; elliptic functions; theta functions; Borweins' cubic theta functions; modular equations; hypergeometric functions.

1. INTRODUCTION

For $q=e^{2\pi i\tau}$, $\operatorname{Im}\tau>0$, Borwein *et al.* [6] introduced three functions, namely,

$$a(q)=\sum_{m,\,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q)=\sum_{m,\,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \qquad (\omega=e^{2\pi i/3}),$$

and

$$c(q)=\sum_{m,\,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

In terms of infinite products [5, 6, 9],

$$b(q)=\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}, \tag{1.1}$$

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and

$$c(q)=3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{1.2}$$

where as usual

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n).$$

Identities (1.1) and (1.2) were first discovered by the Borweins [5] and elementary proofs can be found in [6].

Set

$$L(q)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

$$M(q)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

and

$$N(q)=1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

In [4, Theorems 4.2–4.5], Berndt *et al.* proved the following identities of Ramanuajn, namely,

$$M(q)=a(q)(a^3(q)+8c^3(q)), \tag{1.3}$$

$$M(q^3)=\tfrac{1}{9}\,a(q)(9a^3(q)-8c^3(q)), \tag{1.4}$$

$$N(q)=a^6(q)-20a^3(q)\,c^3(q)-8c^3(q), \tag{1.5}$$

and

$$N(q^3)=a^6(q)-\tfrac{4}{3}\,a^3(q)\,c^3(q)+\tfrac{8}{27}\,c^3(q), \tag{1.6}$$

using Ramanujan's elliptic functions in the theory of Signature 3. Alternative proofs of (1.3) and (1.4) can also be found in [7], where the classical theory of elliptic functions and modular equations of degree 3 are employed.

Identities (1.3)–(1.6) have important applications. First, they are used to derive the interesting formulas [3]

$$j(\tau)=\frac{27a^3(q)(a^3(q)+8c^3(q))}{b^9(q)\,c^3(q)} \tag{1.7}$$

and

$$j(3\tau)=\frac{27a^3(q)(9a^3(q)-8b^3(q))}{b^3(q)\,c^9(q)}, \tag{1.8}$$

where j is the well-known modular j-invariant. Secondly, using (1.3), (1.4), and the identities of Ramanujan

$$q\,\frac{dL(q)}{dq}=\frac{L^2(q)-M(q)}{12}, \tag{1.9}$$

$$q\,\frac{dM(q)}{dq}=\frac{L(q)\,M(q)-N(q)}{12}, \tag{1.10}$$

and

$$q\,\frac{dN(q)}{dq}=\frac{L(q)\,N(q)-M^2(q)}{12}, \tag{1.11}$$

Chan [7] gave Venkatachaliengar's derivation of the Borwein inversion formula [5, 4],

$$a(q)={}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{c^3(q)}{a^3(q)}\right). \tag{1.12}$$

The main purpose of this paper is to provide new simple proofs of (1.3)–(1.6) by using the residue theorem of elliptic functions (Theorem 1 below). Using a similar method, we also derive many striking new identities associated with the Borwein functions, which we list below:

$$1+6\sum_{n=1}^{\infty}\left\{\frac{nq^{3n}}{1-q^{3n}}-\frac{5nq^{15n}}{1-q^{15n}}\right\}=\frac{1}{3}\{a(q)\,a(q^5)+2b(q)\,b(q^5)\}, \tag{1.13}$$

$$1+6\sum_{n=1}^{\infty}\left\{\frac{nq^{n}}{1-q^{n}}-\frac{5nq^{5n}}{1-q^{5n}}\right\}=a(q)\,a(q^5)+2c(q)\,c(q^5), \tag{1.14}$$

$$\begin{aligned}1+6\sum_{n=1}^{\infty}&\left\{\frac{nq^{n}}{1-q^{n}}-\frac{3nq^{2n}}{1-q^{2n}}-\frac{3nq^{3n}}{1-q^{3n}}+\frac{4nq^{4n}}{1-q^{4n}}+\frac{9nq^{6n}}{1-q^{6n}}+\frac{6nq^{12n}}{1-q^{12n}}\right\}\\&=a(q)\,a(q^4),\end{aligned} \tag{1.15}$$

$$\begin{aligned}1-6\sum_{n=1}^{\infty}&\left\{\frac{nq^{n}}{1-q^{n}}-\frac{4nq^{2n}}{1-q^{2n}}-\frac{9nq^{3n}}{1-q^{3n}}-\frac{16nq^{4n}}{1-q^{4n}}\right\}\\&=\frac{(q;q)_\infty^6\,(q^2;q^2)_\infty^3\,(q^{12};q^{12})_\infty}{(q^3;q^3)_\infty^2\,(q^6;q^6)_\infty^3\,(q^4;q^4)_\infty}=\frac{b^2(q)\,b(q^2)}{b(q^4)},\end{aligned} \tag{1.16}$$

$$1+3\sum_{n=1}^{\infty}\left\{\frac{2nq^n}{1-q^n}-\frac{2nq^{2n}}{1-q^{2n}}-\frac{9nq^{3n}}{1-q^{3n}}-\frac{9nq^{6n}}{1-q^{6n}}\right\}$$
$$=\frac{(q^3;q^3)_\infty^2(q^2;q^2)_\infty^{12}}{(q;q)_\infty^6(q^6;q^6)_\infty^4}=\frac{b^4(q^2)}{b^2(q)}, \tag{1.17}$$

$$1-6\sum_{n=1}^{\infty}\left\{\frac{nq^n}{1-q^n}-\frac{6nq^{3n}}{1-q^{3n}}+\frac{9nq^{9n}}{1-q^{9n}}\right\}$$
$$=\frac{(q;q)_\infty^6}{(q^3;q^3)_\infty^2}-9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}=b^2(q)-c^2(q^3), \tag{1.18}$$

$$7-24\sum_{n=1}^{\infty}\left\{\frac{2n^3q^n}{1-q^n}+\frac{7n^3q^{2n}}{1-q^{2n}}-\frac{81n^3q^{6n}}{1-q^{6n}}\right\}$$
$$+\left\{5+12\sum_{n=1}^{\infty}\left\{\frac{2nq^n}{1-q^n}-\frac{5nq^{2n}}{1-q^{2n}}-\frac{9nq^{6n}}{1-q^{6n}}\right\}\right\}^2$$
$$=32\frac{(q^3;q^3)_\infty^4(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{12}(q^6;q^6)_\infty^8}=32\frac{b^8(q^2)}{b^4(q)}, \tag{1.19}$$

$$277+120\sum_{n=1}^{\infty}\left\{\frac{10n^3q^n}{1-q^n}-\frac{81n^3q^{3n}}{1-q^{3n}}+\frac{625n^3q^{5n}}{1-q^{5n}}\right\}$$
$$-5\left\{7+12\sum_{n=1}^{\infty}\left\{\frac{2nq^n}{1-q^n}+\frac{9nq^{3n}}{1-q^{3n}}-\frac{25nq^{5n}}{1-q^{5n}}\right\}\right\}^2$$
$$=32\frac{(q;q)_\infty^{15}(q^{15};q^{15})_\infty}{(q^5;q^5)_\infty^3(q^3;q^3)_\infty^5}=32\frac{b^5(q)}{b(q^5)}, \tag{1.20}$$

$$5\left\{11+4\sum_{n=1}^{\infty}\left\{\frac{3nq^n}{1-q^n}+\frac{6nq^{3n}}{1-q^{3n}}-\frac{75nq^{15n}}{1-q^{15n}}\right\}\right\}^2$$
$$-317+120\sum_{n=1}^{\infty}\left\{\frac{n^3q^n}{1-q^n}-\frac{10n^3q^{3n}}{1-q^{3n}}-\frac{625n^3q^{15n}}{1-q^{15n}}\right\}$$
$$=288\frac{(q^3;q^3)_\infty^{15}(q^5;q^5)_\infty}{(q;q)_\infty^{15}(q^{15};q^{15})_\infty^3}=288\frac{c^5(q)}{c(q^5)}, \tag{1.21}$$

$$\left\{19-6\sum_{n=1}^{\infty}\left\{\frac{2nq^n}{1-q^n}-\frac{16nq^{2n}}{1-q^{2n}}-\frac{54nq^{3n}}{1-q^{3n}}+\frac{144nq^{6n}}{1-q^{6n}}\right\}\right\}^2$$
$$-233-24\sum_{n=1}^{\infty}\left\{\frac{13n^3q^n}{1-q^n}-\frac{32n^3q^{2n}}{1-q^{2n}}-\frac{243n^3q^{3n}}{1-q^{3n}}+\frac{2592n^3q^{6n}}{1-q^{6n}}\right\}$$
$$=128\frac{(q;q)_\infty^6(q^2;q^2)_\infty^6}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}=128b^2(q)\,b^2(q^2), \tag{1.22}$$

$$\left\{1+12\sum_{n=1}^{\infty}\left\{\frac{nq^n}{1-q^n}-\frac{nq^{2n}}{1-q^{2n}}-\frac{nq^{3n}}{1-q^{3n}}-\frac{nq^{6n}}{1-q^{6n}}\right\}\right\}^2$$

$$=1+24\sum_{n=1}^{\infty}\left\{\frac{n^3q^n}{1-q^n}-\frac{n^3q^{2n}}{1-q^{2n}}-\frac{n^3q^{3n}}{1-q^{3n}}+\frac{11n^3q^{6n}}{1-q^{6n}}\right\}, \tag{1.23}$$

$$\left\{1+12\sum_{n=1}^{\infty}\left\{\frac{nq^n}{1-q^n}-\frac{nq^{2n}}{1-q^{2n}}-\frac{2nq^{4n}}{1-q^{4n}}\right\}\right\}^2$$

$$=1+24\sum_{n=1}^{\infty}\left\{\frac{n^3q^n}{1-q^n}-\frac{n^3q^{2n}}{1-q^{2n}}+\frac{10n^3\,q^{4n}}{1-q^{4n}}\right\}. \tag{1.24}$$

Remark. Shortly after the completion of this paper, Chan informed the author that identity (1.14) had been discovered earlier by him and Liaw (unpublished).

In Section 2, we provide some basic facts about the theta function $\theta_1(z\mid q)$. In Section 3, we prove (1.3)–(1.6), (1.13), and (1.14); we also give a table to indicate how (1.15)–(1.24) are derived. Section 4 is devoted to the inversion formula for Borwein's function. It should be emphasized that our method can be used to derive Eisenstein series identities, rather than just to verify previously derived identities. This method thus provides deeper insight into the theory of Eisenstein series identities.

2. SOME BASIC FACTS ABOUT THE JACOBI THETA FUNCTION $\theta_1(z\mid q)$

In this section we will discuss some basic facts about the classical $\theta_1(z\mid q)$ defined by [12, p. 463]

$$\theta_1(z\mid q):=-iq^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{1/2n(n+1)}e^{(2n+1)\,iz}$$

$$=2q^{1/8}\sum_{n=0}^{\infty}(-1)^n\,q^{1/2n(n+1)}\sin(2n+1)\,z. \tag{2.1}$$

The Jacobi theta function $\theta_1(z\mid q)$ has the following product representation [12, p. 469]:

$$\theta_1(z\mid q)=2q^{1/8}\sin z(q;q)_\infty\,(qe^{2iz};q)_\infty\,(qe^{-2iz};q)_\infty$$

$$=-iq^{1/8}\,e^{iz}(q;q)_\infty\,(qe^{2iz};q)_\infty(e^{-2iz};q)_\infty. \tag{2.2}$$

From the above equation we readily find the special values of $\theta_1(z \mid q)$,

$$\theta_1'(0 \mid q) := \theta_1'(q) = 2q^{18}(q;q)_\infty^3,$$
$$\theta_1\left(\frac{\pi}{3} \,\middle|\, q\right) = \sqrt{3}\, q^{18}(q^3;q^3)_\infty, \tag{2.3}$$
$$\theta_1\left(\frac{\pi\tau}{3} \,\middle|\, q\right) = iq^{-124}(q^{13};q^{13})_\infty,$$

where the prime denotes a partial derivative with respect to z. In this paper, we will often use the following important expansion formula [12, p. 489],

$$\frac{\theta_1'}{\theta_1}(z \mid q) = \cot z + 4\sum_{n=1}^\infty \frac{q^n}{1-q^n}\sin 2nz, \tag{2.4}$$

and the well-known fact that

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 + \cdots. \tag{2.5}$$

We also need the following important identity

$$a(q) = 1 + 6\sum_{n=0}^\infty \left\{\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right\}, \tag{2.6}$$

and the Borwein cubic analogue of Jacobi's identity [4–7, 9]

$$a^3(q) = b^3(q) + c^3(q). \tag{2.7}$$

Identity (2.6) can be found in one of Ramanujan's letters to Hardy written from the nursing home, Fitzroy House, and was proved by Berndt in [1]. This identity was rediscovered by the Borweins [5].

Setting $z = \pi/3$ in (2.4) and after an elementary calculation, we find that

$$\begin{aligned}\frac{\theta_1'}{\theta}\left(\frac{\pi}{3} \,\middle|\, q\right) &= \frac{1}{\sqrt{3}}\left(1 + 6\sum_{n=0}^\infty \left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right)\\ &= \frac{1}{\sqrt{3}}\, a(q),\end{aligned} \tag{2.8}$$

by (2.6). Applying logarithmic differentiation to (2.2), we find that

$$\frac{\theta_1'}{\theta_1}(z\,|\,q)=i-2i\sum_{n=1}^{\infty}\frac{q^n e^{2iz}}{1-q^n e^{2iz}}+2i\sum_{n=0}^{\infty}\frac{q^n e^{-2iz}}{1-q^n e^{-2iz}}. \tag{2.9}$$

Replacing q by q^3 in the above equation, setting $z=\pi\tau$ in the resulting equation, and using (2.6), we deduce that

$$\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^3)=-\frac{2i}{3}-\frac{i}{3}a(q). \tag{2.10}$$

From the definition of $\theta_1(z\,|\,q)$ we readily find the functional equations

$$\theta_1(z+n\pi\,|\,q)=(-1)^n\,\theta_1(z\,|\,q),$$

$$\text{and}\qquad \theta_1(z+n\pi\tau\,|\,q)=(-1)^n\,q^{-n^2/2}e^{-2n\pi iz}\,\theta_1(z\,|\,q), \tag{2.11}$$

where n is any integer. Differentiating the above equations respectively with respect to z and then setting $z=0$ in the resulting equations, we find that

$$\theta_1'(n\pi\,|\,q)=(-1)^n\,\theta_1'(q),\qquad \text{and}\qquad \theta_1'(n\pi\tau\,|\,q)=(-1)^n\,q^{-n^2/2}\theta_1'(q). \tag{2.12}$$

By using the product representation of $\theta_1(z\,|\,q)$ and some elementary calculations, we find that

$$\theta_1(3z\,|\,q^3)=-\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}\,\theta_1(z\,|\,q)\,\theta_1\left(z+\frac{\pi}{3}\,\middle|\,q\right)\theta_1\left(z-\frac{\pi}{3}\,\middle|\,q\right), \tag{2.13}$$

and

$$\theta_1(z\,|\,q)=\frac{(q;q)_\infty}{(q^3;q^3)_\infty^3}\,\theta_1(z\,|\,q^3)\,\theta_1(z+\pi\tau\,|\,q^3)\,\theta_1(z-\pi\tau\,|\,q^3). \tag{2.14}$$

Logarithmically differentiating the above equations respectively with respect to z, we find that

$$\frac{\theta_1'}{\theta_1}\left(z+\frac{\pi}{3}\,\middle|\,q\right)+\frac{\theta_1'}{\theta_1}\left(z-\frac{\pi}{3}\,\middle|\,q\right)=3\,\frac{\theta_1'}{\theta_1}(3z\,|\,q^3)-\frac{\theta_1'}{\theta_1}(z\,|\,q) \tag{2.15}$$

and

$$\frac{\theta_1'}{\theta_1}(z+\pi\tau\mid q^3)+\frac{\theta_1'}{\theta_1}(z-\pi\tau\mid q^3)=\frac{\theta_1'}{\theta_1}(z\mid q)-\frac{\theta_1'}{\theta_1}(z\mid q^3). \tag{2.16}$$

Applying (2.4) and (2.5) to the right sides of the above equations, respectively, we find that

$$\frac{\theta_1'}{\theta_1}\left(z+\frac{\pi}{3}\,\middle|\, q\right)+\frac{\theta_1'}{\theta_1}\left(z-\frac{\pi}{3}\,\middle|\, q\right)=-\frac{8}{3}z+12\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{3n}}\sin 6nz$$
$$-4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz+O(z^3) \tag{2.17}$$

and

$$\frac{\theta_1'}{\theta_1}(z+\pi\tau\mid q^3)+\frac{\theta_1'}{\theta_1}(z-\pi\tau\mid q^3)=4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz$$
$$-4\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{3n}}\sin 2nz+O(z^3). \tag{2.18}$$

Differentiating the above two equations with respect to z respectively and then setting $z=0$ in the resulting equations, we obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{3}\,\middle|\, q\right)=-\frac{4}{3}+\sum_{n=1}^{\infty}\left(\frac{36nq^{3n}}{1-q^{3n}}-\frac{4nq^n}{1-q^n}\right)=\frac{1}{6}L(q)-\frac{3}{2}L(q^3) \tag{2.19}$$

and

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau\mid q^3)=4\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{nq^{3n}}{1-q^{3n}}\right)=\frac{1}{6}L(q^3)-\frac{1}{6}L(q). \tag{2.20}$$

3. PROOFS OF (1.4)–(1.7), (1.14) AND (1.15)

Our starting point is the following fundamental theorem of elliptic functions [8][p. 22, Theorem 2]:

THEOREM 1. *The sum of all the residues of an elliptic function vanishes in the period parallelogram.*

In this paper we call the above theorem the residue theorem. The residue theorem plays a key role in our derivations of the Eisenstein series identities. The idea is to use various quotients and products associated with the classical theta-function $\theta_1(z \mid q)$ to construct an elliptic function whose poles are known and then compute the residues of the elliptic function at these poles. We use L'Hôpital's rule to compute the residues of the elliptic function at simple poles and the method of logarithmic derivatives to compute the residues at poles of higher order. The residues at simple poles consist of theta functions and the residues at poles of higher order consist of theta functions and Lambert series. We set the sum of the residues to zero to obtain the identity involving theta functions and Lambert series. Therefore, in order to derive an Eisenstein series identities associated with the Borwein functions $a(q)$, $b(q)$, and $c(q)$, we should construct a suitable elliptic function whose residues at its poles are associated with the Borwein functions $a(q)$, $b(q)$, $c(q)$ and the Eisenstein series $L(q^n)$, $M(q^n)$ and $N(q^n)$, In this paper we will use the notation $\operatorname{Res}(f; \alpha)$ to denote the residue of $f(z)$ at α.

Without loss of generality we will only discuss the case of a function with a pole at the point 0. Let

$$f(z) = \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots,$$

and set

$$F(z) := z^n f(z), \qquad \phi(z) := \frac{F'(z)}{F(z)},$$

It is well known that

$$\operatorname{Res}(f; 0) = \frac{1}{(n-1)!} F^{(n-1)}(0).$$

By using the method of logarithmic differentiation, we readily find that:

$$n=2\colon \operatorname{Res}(f; 0) = F(0)\, \phi(0), \tag{3.1}$$

$$n=3\colon \operatorname{Res}(f; 0) = \tfrac{1}{2} F(0) \{\phi^2(0) + \phi'(0)\}, \tag{3.2}$$

$$\begin{aligned} n=5\colon \operatorname{Res}(f; 0) = \tfrac{1}{24} F(0) \{ &\phi^4(0) + 6\phi^2(0)\, \phi''(0) + 4\phi(0)\, \phi''(0) \\ &+ 3\phi'(0)^2 + \phi'''(0)\}, \end{aligned} \tag{3.3}$$

$$n=7:\ \mathrm{Res}(f;0)=\tfrac{1}{720}F(0)\{\phi^6(0)+15\phi^4(0)\,\phi'(0)+20\phi^3(0)\,\phi''(0)$$
$$+15\phi^2(0)\,\phi'''(0)+45\phi^2(0)\,\phi'(0)^2+60\phi(0)\,\phi'(0)\,\phi''(0)$$
$$+6\phi(0)\,\phi^{(4)}(0)+15\phi'(0)^3+15\phi'(0)\,\phi'''(0)$$
$$+10\phi''(0)^2+\phi^{(5)}(0)\}. \tag{3.4}$$

We will first indicate our method by proving the following formula of Ramanujan [2, p. 460, Entry 3(i)]:

$$a^2(q)=1+12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}. \tag{3.5}$$

Proof of (3.5). The above identity involves $a(q)$ and Eisenstein series $L(q)$ and $L(q^3)$. From (2.8) we know that the value of the logarithmic derivative of $\theta_1(z+\pi/3\,|\,q)$ at $z=0$ is associated with $a(q)$. From (2.19) we know that the value of the derivative of $\theta_1'/\theta_1\ (z+\pi/3\,|\,q)$ at $z=0$ is associated with $L(q)$ and $L(q^3)$. Therefore, by (3.2), our choice of the elliptic function should involve theta functions $\theta_1(z+\pi/3\,|\,q)$, with a pole of order 3 at $z=0$. After trying several functions without success we arrive at the function

$$f(z)=\frac{\theta_1^3\,(z+\pi/3\,|\,q)}{\theta_1^3(z\,|\,q)}.$$

By using (2.11), we verify that the above function is an elliptic function with a pole of order 3 at 0. We will use (3.2) to compute $\mathrm{Res}(f;0)$. Set

$$F(z):=z^3f(z),\qquad \text{and}\qquad \phi(z):=\frac{F'(z)}{F(z)}.$$

From the definition of $F(z)$, we find that

$$F(0)=\lim_{z\to 0}z^3f(z)=\frac{\theta_1^3\,(\pi/3\,|\,q)}{\theta_1'(q)^3}\neq 0. \tag{3.6}$$

From (2.4), (2.5), and the definition of $\phi(z)$ we find that

$$\phi(z)=\frac{3}{z}-3\frac{\theta_1'}{\theta_1}(z\,|\,q)+3\frac{\theta_1'}{\theta_1}\left(z+\frac{\pi}{3}\,\middle|\,q\right)$$
$$=z-12\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz+3\frac{\theta_1'}{\theta_1}\left(z+\frac{\pi}{3}\,\middle|\,q\right)+O(z^3). \tag{3.6}$$

From the above equation and using (2.4), (2.6), and (2.19) we readily find that

$$\phi(0) = 3\,\frac{\theta_1'}{\theta_1}\left(\frac{\pi}{3}\,\middle|\,q\right) = \sqrt{3}\,a(q),$$

$$\begin{aligned}\phi'(0) &= 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 3\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{3}\,\middle|\,q\right)\\ &= -3\left\{1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right\}.\end{aligned}$$

Substituting these identities into (3.2), we find that

$$\operatorname{Res}(f;0) = \frac{3}{2}\,F(0)\left\{a^2(q) - 1 - 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right\}.$$

The residue theorem of elliptic functions gives $\operatorname{Res}(f;0)=0$ and we complete the proof of (3.5), since $F(0)\neq 0$ by (3.6). Next we will prove the following two identities:

$$M(q) + 9M(q^3) = 10a^4(q), \tag{3.7}$$

$$7M(q) - 27M(q^3) = 60a^4(q) - 80a(q)\,b^3(q). \tag{3.8}$$

Proof. Identity (3.7) involves $M(q)$ and $M(q^3)$. From (2.4) and (2.5) we know that $\{(\theta_1'/\theta_1)(z\,|\,q) - (1/z)\}'''|_{z=0}$ is associated with $M(q)$ and $\{(\theta_1'/\theta_1)(3z^3\,|\,q^3) - (1/3z)\}'''|_{z=0}$ is associated with $M(q^3)$. Therefore, by (3.3), our choice of the elliptic function should therefore involve the theta functions $\theta_1(z\,|\,q)$ and $\theta_1(3z\,|\,q^3)$ with a pole of order 5 at $z=0$. We consider the elliptic function

$$f(z) = \frac{\theta_1(2z\,|\,q)\,\theta_1(3z\,|\,q^3)}{\theta_1^7(z\,|\,q)}.$$

By (2.11) it is easy to check that $f(z)$ is an elliptic function with period π and $\pi\tau$ and that 0 is its only pole of order 5. Set $F(z) := z^5 f(z)$, and $20\phi(z) := (F'/F)(z)$. From the definition of $f(z)$ we immediately have

$$F(0) = \frac{6\theta_1'(q^3)}{\theta_1'(q)^6} \neq 0. \tag{3.9}$$

By using (2.4) and (2.5) we find that

$$\begin{aligned}
\phi(z) &= \frac{5}{z} - 7\,\frac{\theta_1'}{\theta_1}(z \mid q) + 2\,\frac{\theta_1'}{\theta_1}(2z \mid q) + 3\,\frac{\theta_1'}{\theta_1}(3z \mid q^3) \\
&= \frac{5}{z} - 7\cot z + 2\cot 2z + 3\cot 3z \\
&\quad + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(2\sin 4nz - 7\sin 2nz) \\
&\quad + 12\sum_{n=1}^{\infty} \frac{q^{3n}}{1-q^{3n}}\sin 6nz \\
&= -2z - 2z^3 + 12\sum_{n=1}^{\infty} \frac{q^{3n}}{1-q^{3n}}\sin 6nz. \\
&\quad + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(2\sin 4nz - 7\sin 2nz) + O(z^5).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\phi'(0) &= -2\left\{1 + 12\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 36\,\frac{nq^{3n}}{1-q^{3n}}\right\} \\
&= -2a^2(q), \\
\phi'''(0) &= -12\left\{1 + 24\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n} + 216\,\frac{n^3q^{3n}}{1-q^{3n}}\right\} \\
&= -\frac{6}{5}M(q) - \frac{54}{5}M(q^3), \\
\phi(0) &= , \qquad \phi''(0) = 0.
\end{aligned}$$

Therefore, using (3.3), we find that

$$\begin{aligned}
\operatorname{Res}(f;0) &= \frac{1}{24}F(0)\{3\phi'(0)^2 + \phi'''(0)\} \\
&= 3\,\frac{\theta_1'(q^3)}{\theta_1'(q)^6}\left\{a^4(q) - \frac{1}{10}M(q) - \frac{9}{10}M(q^3)\right\}. \qquad (3.10)
\end{aligned}$$

Since $\operatorname{Res}(f;0) = 0$, we obtain (3.7).

To prove (3.8), we choose the elliptic function

$$f(z) = \frac{\theta_1^3(2z \mid q)}{\theta_1^6\,(z \mid q)\,\theta_1^2\,(3z \mid q^3)},$$

which is an elliptic function with periods π and $\pi\tau$. The poles of $f(z)$ are $\pi/3$ and $2\pi/3$, which are of order 2 and 0, which is of order 5. Let $F(z) := z^5 f(z)$ and $\phi(z) = (F'/F)(z)$.

We find that

$$F(0) = \frac{8}{9}\frac{1}{\theta'_1(q^3)^2\,\theta'_1(q)^3},$$

$$\begin{aligned}\phi(z) &= \frac{5}{z} - 6\,\frac{\theta'_1}{\theta_1}\,(z \mid q) + 6\,\frac{\theta'_1}{\theta_1}\,(2z \mid q) - 6\,\frac{\theta'_1}{\theta_1}\,(3z \mid q^3)\\ &= -4z + \frac{8}{3}\,(z^3) + O(z^5) - 24\sum_{n=1}^{\infty} = 20\,\frac{q^n}{1-q^n}\,(\sin 2nz - \sin 4nz)\\ &\quad - 24\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{3n}}\sin 6nz.\end{aligned}$$

Therefore,

$$\phi'(0) = 4a^2(q), \qquad \phi(0) = \phi''(0) = 0,$$

and

$$\phi'''(0) = -\tfrac{28}{5}\,M(q) + \tfrac{108}{5}\,M(q^3).$$

Hence, by (3.3), we obtain

$$\begin{aligned}\operatorname{Res}(f; 0) &= -\frac{1}{24}\,F(0)\{3\phi'(0)^2 + \phi'''(0)\}\\ &= \frac{1}{27\theta'_1(q^3)^2\,\theta'_1(q)^3}\left\{48a^4(q) - \frac{28}{5}\,M(q) + \frac{108}{5}\,M(q^3)\right\}.\end{aligned} \tag{3.11}$$

For computing $\operatorname{Res}(f; (\pi/3))$ we use a change of variables. By replacing z with $z + (\pi/3)$, $f(z)$ becomes

$$f_1(z) = -\frac{\theta_1^3\,(2z - \pi/3 \mid q)}{\theta_1^6\,(z + \pi/3 \mid q)\,\theta_1^2(3z \mid q^3)}.$$

Set $F_1(z) := z^2 f_1(z)$ and $\phi_1(z) := (F_1'/F_1)(z)$. Then

$$\begin{aligned}\phi_1(z) &= \frac{2}{z} - 6\frac{\theta_1'}{\theta_1}(3z \mid q^3) + 6\frac{\theta_1'}{\theta_1}\left(2z - \frac{\pi}{3} \,\middle|\, q\right) - 6\frac{\theta_1'}{\theta_1}\left(z + \frac{\pi}{3} \,\middle|\, q\right) \\ &= 6\frac{\theta_1'}{\theta_1}\left(2z - \frac{\pi}{3} \,\middle|\, q\right) - 6\frac{\theta_1'}{\theta_1}\left(z + \frac{\pi}{3} \,\middle|\, q\right) \\ &\quad + z - 24\sum_{n=1}^{\infty} \frac{q^{3n}}{1-q^{3n}} \sin 6z + O(z^3).\end{aligned}$$

From the above equation and (2.8) we immediately have

$$F_1(0) = \frac{1}{9\theta_1'(q^3)^2\, \theta_1^3\,(\pi/3 \mid q)}$$

and

$$\phi_1(0) = -12\frac{\theta_1'}{\theta_1}\left(\frac{\pi}{3} \,\middle|\, q\right) = -4\sqrt{3}a(q).$$

Substituting these identities into (3.1), we have

$$\operatorname{Res}\left(f; \frac{\pi}{3}\right) = \operatorname{Res}(f_1; 0) = -\frac{4\sqrt{3}\, a(q)}{9\theta_1^3\,(\pi/3 \mid q)\, \theta_1'(q^3)^2}. \tag{3.12}$$

Similarly, we find that

$$\operatorname{Res}\left(f; \frac{2\pi}{3}\right) = \operatorname{Res}\left(f; \frac{\pi}{3}\right) = -\frac{4\sqrt{3}\, a(q)}{9\theta_1^3\,(\pi/3 \mid q)\, \theta_1'(q^3)^2}. \tag{3.13}$$

Substituting (3.11), (3.12), and (3.13) into the identity

$$\operatorname{Res}(f; 0) + \operatorname{Res}\left(f; \frac{\pi}{3}\right) + \operatorname{Res}\left(f; \frac{2\pi}{3}\right) = 0,$$

and using the fact $\theta_1(\pi/3 \mid q) = \sqrt{3}\, q^{1/8}(q^3; q^3)_\infty$, we obtain (3.8). Solving simultaneously equations (3.7) and (3.8) and using (2.7), we obtain (1.3) and (1.4).

Next, we prove the following identities

$$N(q) + 27N(q^3) = 28a^6(q) - 56a^3(q)\, c^3(q), \tag{3.14}$$

$$243N(q^3) - 61N(q) = 182a^6(q) + 896a^3(q)\, c^3(q) + 560c^6(q). \tag{3.15}$$

Proof. Identity (3.14) involves $N(q)$ and $N(q^3)$. From (2.4) and (2.5) we know that $\{(\theta_1'/\theta_1\,(z\,|\,q)-(1/z)\}^{(5)}|_{z=0}$ is associated with $N(q)$ and $\{(\theta_1'/\theta_1)(z^3\,|\,q^3)-(1/z)\}^{(5)}|_{z=0}$ is associated with $N(q^3)$. Therefore, by (3.4), our choice of the elliptic function should involve theta functions $\theta_1(z\,|\,q)$ and $\theta_1(z\,|\,q^3)$, with a pole of order 7 at $z=0$. We consider the following elliptic function

$$f(z)=\frac{\theta_1(2z\,|\,q^3)\,\theta_1^2(z\,|\,q)}{\theta_1^{10}(z\,|\,q^3)}.$$

By using the second equation of (2.11) we know that $f(z)$ is an elliptic function with periods π and $3\pi\tau$, with pole 0 of order 7. Set $F(z):=z^7f(z)$ and $\phi(z)=:(F'/F)(z)$.

We have

$$F(0)=\frac{2\theta_1'(q)^2}{\theta_1'(q^3)^9}\neq 0,$$

and

$$\begin{aligned}
\phi(z)&=\frac{7}{z}+2\frac{\theta_1'}{\theta_1}(z\,|\,q)+2\frac{\theta_1'}{\theta_1}(2z\,|\,q^3)+10\frac{\theta_1'}{\theta_1}(z\,|\,q^3)\\
&=\frac{4}{3}z-\frac{8}{45}z^3-\frac{16}{135}z^5+O(z^7)\\
&\quad+8\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{3n}}(\sin 4\,nz-5\sin 2nz)\\
&\quad+8\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz.
\end{aligned}$$

Hence

$$\begin{aligned}
&\phi(0)=0, \qquad \phi''(0)=\phi^{(4)}(0)=0,\\
&\phi'(0)=\tfrac{4}{3}a^2(q),\\
&\phi'''(0)=-\tfrac{4}{15}M(q)-\tfrac{4}{5}M(q^3),
\end{aligned}$$

and

$$\phi^{(5)}(0)=-\tfrac{32}{63}\{N(q)+27N(q^3)\}.$$

Substituting the above equations into (3.4) and using the fact that $\mathrm{Res}(f;0)=0$, we find that

$$2N(q)+54N(q^3)=140a^6(q)-21a^2(q)\{M(q)+3M(q^3)\}.$$

Substituting (1.3) and (1.4) into the above equation and then using (2.7) in the resulting equation, we obtain (3.14).

To prove (3.15), we consider the elliptic function

$$f(z)=\frac{\theta_1^3(2z\,|\,q)}{\theta_1^9(z\,|\,q)\,\theta_1(3z\,|\,q^3)},$$

with periods π and $\pi\tau$. This function has simple poles at $\pi/3$ and $2\pi/3$ and a pole of order 7 at 0. By using L'Hôpital's rule, (2.3), and (2.12), we readily find that

$$\operatorname{Res}\left(f;\frac{\pi}{3}\right)=\lim_{z\to\pi/3}\left(z-\frac{\pi}{3}\right)f(z)=-\frac{q^{-9/8}}{182(q^3;q^3)_\infty^9},$$

$$\operatorname{Res}\left(f;\frac{2\pi}{3}\right)=\operatorname{Res}\left(f;\frac{\pi}{3}\right)=-\frac{q^{-9/8}}{182(q^3;q^3)_\infty^9}.$$

In order to compute $\operatorname{Res}(f;0)$ we set $F(z)=z^7f(z)$, $\phi(z)=(F'/F)(z)$. We find that

$$F(0)=\frac{q^{-9/8}}{48(q;q)_\infty^{18}\,(q^3;q^3)_\infty^3},$$

$$\begin{aligned}\phi(z)&=\frac{7}{z}+6\,\frac{\theta_1'}{\theta_1}(2z\,|\,q)-9\,\frac{\theta_1'}{\theta_1}(z\,|\,q)-3\,\frac{\theta_1'}{\theta_1}(3z\,|\,q^3)\\&=2z+\frac{14}{15}z^3+\frac{52}{45}z^5\\&\quad+12\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(2\sin 4nz-3\sin 2nz)\\&\quad-12\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{3n}}\sin 6nz+O(z^7).\end{aligned}$$

Hence

$$\begin{aligned}\phi(0)&=\phi''(0)=\phi^{(4)}(0)=0,\\\phi'(0)&=2a^2(q),\\\phi'''(0)&=-\tfrac{2}{5}\{13M(q)-27M(q^3)\},\\\phi^{(5)}(0)&=-\tfrac{21}{16}\{61N(q)-243N(q^3)\}.\end{aligned}$$

Substituting the above equations into (3.4), we find that

$$\operatorname{Res}(f;0)=\frac{q^{-9/8}}{(q;q)_\infty^{18}(q^3;q^3)_\infty^3}\times\{640a^6(q)-64a^2(q)(13M(q)-27M(q^3))+7(243N(q^3)-61N(q))\}. \tag{3.17}$$

Substituting (3.16) and (3.17) into the identity

$$\operatorname{Res}\left(f;\frac{\pi}{3}\right)+\operatorname{Res}\left(f;\frac{2\pi}{3}\right)+\operatorname{Res}(f;0)=0,$$

we deduce that

$$4(243N(q^3)-61N(q))=2240b^6(q)-630a^6(q)+63a^2(q)(13M(q)-27M(q^3)).$$

Using (1.3), (1.4), and (2.7) in the above equation, we obtain (3.15). By solving Eqs. (3.14) and (3.15) we obtain (1.4) and (1.5).

To prove (1.13), we consider the elliptic function

$$f(z)=\frac{\theta_1(z\mid q)\,\theta_1(2z\mid q^5)}{\theta_1^3(3z\mid q^{15})},$$

which is an elliptic function with periods π and $5\pi\tau$. 0 is its simple pole and $\pi/3$, $2\pi/3$ are its poles of order 3. It is easy to show that

$$\operatorname{Res}(f;0)=\frac{2\theta_1'(q)\,\theta_1'(q^5)}{7\theta_1'(q^{15})^3}. \tag{3.18}$$

To compute $\operatorname{Res}(f;(\pi/3))$ we use a change of variables. Replacing z by $z+(\pi/3)$, $f(z)$ becomes

$$f_1(z)=\frac{\theta_1\,(z+\pi/3\mid q)\,\theta_1\,(2z-\pi/3\mid q^5)}{\theta_1^3(3z\mid q^{15})}.$$

Set $F(z)=z^3 f_1(z)$, and $\phi(z)=(F'/F)(z)$. We have

$$F(0)=-\frac{\theta_1(\pi/3\mid q)\,\theta_1\,(\pi/3\mid q^5)}{27\theta_1'(q^{15})^3},$$

$$\phi(z)=\frac{\theta_1'}{\theta_1}\left(z+\frac{\pi}{3}\,\middle|\,q\right)+2\,\frac{\theta_1'}{\theta_1}\,(2z-\pi/3\mid q^5)+9z-36\sum_{n=1}^{\infty}\frac{q^{15n}}{1-q^{15n}}\sin 6nz+O(z^3).$$

Using (2.19) we find that

$$\phi(0) = \frac{1}{\sqrt{3}}(a(q)-2a(q^5)), \tag{3.19}$$

$$\begin{aligned}\phi'(0) = \frac{7}{3} - 4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\\ -16\sum_{n=1}^{\infty}\frac{nq^{5n}}{1-q^{5n}} - 72\sum_{n=1}^{\infty}\frac{nq^{15n}}{1-q^{15n}}.\end{aligned} \tag{3.20}$$

Therefore by (3.5) we have

$$\begin{aligned}\phi(0)^2 = -\frac{4}{3}a(q)\,a(q^5) + \frac{1}{3}\left\{5 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right.\\ \left. +48\sum_{n=1}^{\infty}\frac{nq^{5n}}{1-q^{5n}} - 144\sum_{n=1}^{\infty}\frac{nq^{15n}}{1-q^{15n}}\right\}.\end{aligned} \tag{3.21}$$

Using (3.2) we find that

$$\begin{aligned}\mathrm{Res}\left(f;\frac{\pi}{3}\right) &= \mathrm{Res}\,(f_1;0) = \frac{1}{2}F(0)\{\phi'(0)+\phi^2(0)\}\\ &= -q^{39/8}\frac{(q^3;q^3)_\infty}{36(q^{15};q^{15})_\infty^8}\\ &\quad\times\left\{1+6\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} - 30\sum_{n=1}^{\infty}\frac{nq^{15}}{1-q^{15n}} - \frac{1}{3}a(q)\,a(q^5)\right\}.\end{aligned} \tag{3.22}$$

It is easy to show that

$$\mathrm{Res}\left(f;\frac{\pi}{3}\right) = \mathrm{Res}\left(f;\frac{2\pi}{3}\right). \tag{3.23}$$

Combining (3.18), (3.22), (3.23) and the identity

$$\mathrm{Res}\left(f;\frac{\pi}{3}\right) + \mathrm{Res}\left(f;\frac{2\pi}{3}\right) + \mathrm{Res}(f;0) = 0,$$

we obtain (1.13).

To prove (1.14), we consider

$$f(z) = \frac{e^{-2iz}\theta_1(5z-5\pi\tau\,|\,q^{15})}{\theta_1^3(z\,|\,q^3)\,\theta_1^2(z-\pi\tau\,|\,q^3)}.$$

$f(z)$ is an elliptic function with periods π and $3\pi\tau$. $\pi\tau$ is its simple pole and 0 is its pole of order 3. We have

$$\operatorname{Res}(f;\pi\tau)=\frac{5i}{2}\,q^{1/2}\,\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty^6\,(q;q)_\infty^3}, \tag{3.24}$$

$$\operatorname{Res}(f;0)=-\frac{10i}{9}\,q^{-3/2}\,\frac{(q^5;q^5)_\infty}{(q^9;q^9)_\infty^9\,(q;q)_\infty^2} \times\left\{1+6\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-30\sum_{n=1}^{\infty}\frac{nq^{5n}}{1-q^{5n}}+a(q)\,a(q^5)\right\}. \tag{3.25}$$

Substituting (3.24) and (3.25) into the identity

$$\operatorname{Res}(f;\pi\tau)+\operatorname{Res}(f;0)=0,$$

we obtain (1.14).

Similarly, by applying the residue theorem, Theorem 1, to the elliptic functions

$$f_1(z)=\frac{\theta_1(2z-\pi/3\mid q^4)\,\theta_1(z+\pi/3\mid q)\,\theta_1^3(z+\pi/3\mid q^4)}{\theta_1(z+\pi/3\mid q^2)\,\theta_1^3(3z\mid q^{12})},$$

$$f_2(z)=\frac{\theta_1(2z\mid q)\,\theta_1(4z\mid q^4)}{\theta_1^3(z\mid q)\,\theta_1(2z\mid q^2)\,\theta_1(3z\mid q^3)},$$

$$f_3(z)=\frac{\theta_1(2z\mid q^2)\,\theta_1^2(z\mid q)}{\theta_1^3(z\mid q^6)\,\theta_1^5(z\mid q^2)},$$

$$f_4(z)=\frac{\theta_1(3z\mid q^3)\,\theta_1^2(2z\mid q^3)}{\theta_1(z\mid q^3)\,\theta_1^2(3z\mid q^9)\,\theta_1^2(z\mid q)},$$

$$f_5(z)=\frac{\theta_1(2z\mid q^2)\,\theta_1^4(z\mid q)}{\theta_1^3(3z\mid q^6)\,\theta_1^9(z\mid q^2)},$$

$$f_6(z)=\frac{\theta_1(2z\mid q)\,\theta_1(5z\mid q^5)}{\theta_1^6(z\mid q)\,\theta_1(3z\mid q^3)},$$

$$f_7(z)=\frac{\theta_1(2z\mid q^3)\,\theta_1(5z\mid q^{15})}{\theta_1(z\mid q)\,\theta_1^6(z\mid q^3)},$$

$$f_8(z)=\frac{\theta_1(2z\mid q)\,\theta_1^2(6z\mid q^6)}{\theta_1^3(z\mid q)\,\theta_1^2(2z\mid q^2)\,\theta_1^3(3z\mid q^3)},$$

$$f_9(z)=\frac{\theta_1(z\mid q)\,\theta_1(2z\mid q^6)}{\theta_1(z\mid q^2)\,\theta_1(z\mid q^3)\,\theta_1^5(z\mid q^6)},$$

$$f_{10}(z)=\frac{\theta_1(z\mid q)\,\theta_1(2z\mid q^4)}{\theta_1(z\mid q^2)\,\theta_1^6(z\mid q^4)},$$

respectively, we obtain identities (1.15)–(1.16). Finally, we list the periods and poles of $f_k(z)$, $1 \leqslant k \leqslant 10$, where $(\alpha)_n$ means that α is a pole of order n, in the following table.

Elliptic Functions	Periods	Poles	Identities
$f_1(z)$	$\pi, 4\pi\tau$	$(0)_3$	(1.15)
$f_2(z)$	$\pi, \pi\tau$	$(0)_3, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1$	(1.16)
$f_3(z)$	$\pi, 2\pi\tau$	$(0)_3, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1$	(1.17)
$f_4(z)$	$\pi, 3\pi\tau$	$(0)_3, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1, (\pi\tau)_1, (2\pi\tau)_1$	(1.18)
$f_5(z)$	$\pi, 2\pi\tau$	$(0)_5, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1$	(1.19)
$f_6(z)$	$\pi, \pi\tau$	$(0)_5, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1$	(1.20)
$f_7(z)$	$\pi, 3\pi\tau$	$(0)_5, (\pi\tau)_1, (2\pi\tau)_1$	(1.21)
$f_8(z)$	$\pi, \pi\tau$	$(0)_5, (\frac{\pi}{3})_1, (\frac{2\pi}{3})_1$	(1.22)
$f_9(z)$	$\pi, 6\pi\tau$	$(0)_5$	(1.23)
$f_{10}(z)$	$\pi, 4\pi\tau$	$(0)_5$	(1.24)

4. THE INVERSION FORMULA FOR THE BORWEIN CUBIC THETA FUNCTIONS

In this section we will use (1.3), (1.9), and the Eisenstein identity for $a^2(q)$ (Eq. (3.5)) to derive the inversion formula, (1.12), for the Borwein functions via Venkatachanliengar's method.

Proof. Define $z := a(q)$ and $x := (c^3(q)/a^3(q))$. From (3.7) we know that

$$x^{-1} = 1 + \frac{b^3(q)}{c^3(q)}.$$

Differentiating the above equation with respect to q and using (1.1), (1.2), and (3.5), we have

$$\begin{aligned} q\frac{dx}{dq} &= 3x^2\frac{b^3(q)}{c^3(q)}\left(\frac{1}{b(q)}\frac{db(q)}{dq} - \frac{1}{c(q)}\frac{dc(q)}{dq}\right) \\ &= x^2\frac{b^3(q)}{c^3(q)}\left(1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right) \\ &= x^2\frac{(1-x)}{x}z^2 \\ &= x(1-x)\,z^2. \end{aligned} \tag{4.1}$$

It is easy to show that

$$q(q;q)_\infty^{24} = \tfrac{1}{27}\, x(1-x)^3\, z^{12}.$$

Thus by using the method of logarithmic derivatives and the above equation we have

$$\begin{aligned}
L(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} &= q\,\frac{d}{dq} \log(q(q;q)_\infty^{24}) \\
&= \frac{d}{dx} \log\left(\frac{1}{27}\, x(1-x)^3\, z^{12}\right) q\,\frac{dx}{dq} \\
&= x(1-x)\, z^2 \left(\frac{12}{z}\frac{dz}{dx} + \frac{1}{x} - \frac{3}{1-x}\right) \\
&= 12x(1-x)\, z\,\frac{dz}{dx} + (1-4z)\, z^2. \qquad (4.2)
\end{aligned}$$

From the above equation we immediately have

$$\frac{dL(q)}{dx} = 2(7-16x)\, z\,\frac{dz}{dx} + 12x(1-x)\, z\,\frac{d^2z}{dx^2} + 12x(1-x)\left(\frac{dz}{dx}\right)^2 - 4z^2. \qquad (4.3)$$

Thus we have

$$\begin{aligned}
q\,\frac{dL(q)}{dq} &= \frac{dL(q)}{dx}\, q\,\frac{dx}{dq} \\
&= 2(7-16x)\, z^3\,\frac{dz}{dx} + 12x^2(1-x)^2\, z^3\,\frac{d^2z}{dx^2} \\
&\quad + 12x^2(1-x)^2\, z^2\left(\frac{dz}{dx}\right)^2 - 4z^4 x(1-x). \qquad (4.4)
\end{aligned}$$

From (4.2) we also have

$$L^2(q) = 144x^2(1-x)^2\, z^2\left(\frac{dz}{dx}\right)^2 + 24x(1-x)(1-4x)\, z^3\,\frac{dz}{dx} + (1-4x)^2\, z^4. \qquad (4.5)$$

Equation (1.3) can be written as

$$M(q) = z^4(1+8x). \qquad (4.6)$$

Substituting (4.4), (4.5), and (4.6) into (1.3), and performing a little reduction, we obtain

$$x(1-x)\frac{d^2z}{dx^2}+(1-2x)\frac{dz}{dx}-\frac{2}{9}z=0. \tag{4.7}$$

From the above differential equation we can easily obtain the following important result (see [7] for details)

$$z={}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;x). \tag{4.8}$$

This completes the proof.

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