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Selberg Zeta Functions over Function Fields

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We compute explicitly the Selberg trace formula for principal congruence subgroups of $PGL(2, \mathbb{F}_q[t])$ which is the modular group in positive characteristic cases. We also express the Selberg zeta function as a determinant of the Laplacian which is composed of both discrete and continuous spectra. All factors are calculated explicitly, and they are rational functions in q^{-s}.

Key Words: Selberg trace formula; Ihara–Selberg zeta function; function field; Ramanujan graph/diagram.

1. INTRODUCTION

The aim of this paper is to give a new explicit example of the Selberg trace formula and the Selberg zeta function. We treat principal congruence subgroups Γ of $\Gamma(1) = PGL(2, \mathbb{F}_q[t])$ with $\mathbb{F}_q$ the finite field. The group $\Gamma(1)$ is naturally introduced as an analog of the standard modular group $PSL(2, \mathbb{Z})$ in view of the number theoretic analogy between algebraic number fields and function fields over finite fields. In place of the upper half space which the standard modular group acts on, the group $\Gamma(1)$ operates on the so-called Bruhat–Tits tree X.

As an analog of non-compact arithmetic manifold such as the modular surface, our Γ supplies an infinite arithmetic graph which is called a Ramanujan diagram by Morgenstern [M1] [M2].

Our results can be regarded as a generalization of those works on the Ihara–Selberg zeta functions which treated finite graphs [VN] [ST].

We survey the theory of Bruhat–Tits tree and the harmonic analysis on $\Gamma\backslash X$ in Section 2. Next Section 3 is devoted to the construction of the Selberg trace formula and the explicit calculation of each term. The final expression is in Theorem 3.2. The adjacency operator T_Γ has both discrete and continuous spectra, and the continuous ones can be described in terms of the suitable Eisenstein series. Finally in Section 4 we express the Selberg

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zeta function as the determinant of T_Γ, and obtain its rationality in q^{-s} (Theorem 4.1).

2. PRELIMINARIES

Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[t]$ the ring of polynomials in t over $\mathbb{F}_q$, and $k = \mathbb{F}_q(t)$ its quotient field. The valuation at $1/t$ on k, which corresponds to infinity, is defined by $v_\infty(f/g) = \deg g - \deg f$, where f and g are polynomials in $\mathbb{F}_q[t]$, and the norm is given by $|a|_\infty = q^{-v_\infty(a)} (a \in k)$. Let k_∞ be the completion of k with respect to this norm $|\cdot|_\infty$, and r_∞ be the ring of local integers. Then $k_\infty = \mathbb{F}_q((t^{-1}))$ is the field of Laurent series in the uniformizer t^{-1} over $\mathbb{F}_q$, and $r_\infty = \mathbb{F}_q[[t^{-1}]]$ is the ring of Taylor series in t^{-1} over $\mathbb{F}_q$. If an element a in k_∞ is written as $\sum_{i=n}^{\infty} a_i t^{-i} (a_n \neq 0)$, then $v_\infty(a) = n$ and $|a|_\infty = q^{-n}$.

For a ring R we let $PGL(2, R)$ be the group of 2×2 invertible matrices over R divided by its center. Throughout this paper we put $G = PGL(2, k_\infty)$ and $K = PGL(2, r_\infty)$. Note that K is a maximal compact subgroup of G. We will study the homogeneous space G/K. As is described in [Se, II.1.1], we can endow G/K with the structure of the $q+1$ regular tree X. Given a graph Y, we write $V(Y)$ or simply Y for the set of vertices of Y and $E(Y)$ for the set of edges of Y. Then $G/K = V(X)$. The tree X has a natural distance d, namely, if u and v are adjacent in X we let $d(u, v) = 1$. From the way of construction of the tree X, the neighbors of the vertex $gK (g \in G)$ are the $q+1$ cosets $gs_iK (i = 1, ..., q+1)$, where

$$\{s_1, ..., s_{q+1}\} = \left\{ \begin{pmatrix} t^{-1} & \alpha \\ 0 & 1 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q \right\} \cup \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Let B be the subgroup of G of upper triangular matrices. Since $G = BK$, we have

$$G/K \simeq B/B \cap K,$$

so we can take the following set of matrices

$$\left\{ \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z}, x \in k_\infty, x \bmod t^n r_\infty \right\} \tag{1}$$

as a complete set of representatives of $X = G/K$. Hence from the viewpoint of the analogy to the upper half plane $\mathbb{H}$, it is convenient that if $g \in X$ is equivalent to $\left(\begin{smallmatrix} t^n & x \\ 0 & 1 \end{smallmatrix}\right)$ ($x \bmod t^n r_\infty$), we call x the x-coordinate of g and call

t^n or simply n the y-coordinate of g. Using elementary divisors, we also see that every element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ can be written in the form

$$g = k \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} k', \tag{2}$$

where $k, k' \in K$ and $n = v_\infty(\det g) - 2 \min\{v_\infty(x) \mid x = a, b, c, d\}$.

The group G acts on the tree X as a group of automorphisms. The action of G on X can be extended to the boundary of X, which is defined by the set of equivalence classes of half-lines, with two half-lines being equivalent if they differ in a finite graph. We see ∂X can be identified with $\mathbb{P}^1(k_\infty)$. The action of G on $\partial X = \mathbb{P}^1(k_\infty)$ is the usual fractional linear transformation, which is induced from its matrix action on X.

From the view of actions on X and ∂X, we classify elements in G which act without inversions on X, as follows:

(1) identity

(2) hyperbolic: elements which have no fixed vertices on X. (Then from Lemma 2.1 they have two fixed points on ∂X.)

(3) elliptic: elements which have fixed vertices on X and no fixed points on ∂X.

(4) parabolic: elements which have fixed vertices on X and a fixed point on ∂X.

(5) split hyperbolic: elements which have fixed vertices on X and two fixed points on ∂X.

The following Lemma summarizes the properties of hyperbolic elements.

Lemma 2.1 (Tits) [Se, p. 63]. *Suppose $P \in G$ is hyperbolic (i.e., has no fixed vertex on X). Let the degree of P be defined by* $\deg P = \min\{d(v, Pv) \mid v \in V(X)\}$. *We put*

$$T_P = \{v \in V(X) \mid d(v, Pv) = \deg P\}.$$

Then

(1) *T_P is the vertex set of an infinite path in X.*

(2) *P induces a shift by the distance* $\deg P$ *on T_P.*

(3) *If a vertex u is of distance d from T_P then $d(u, Pu) = \deg P + 2d$.*

Let Γ be a discrete subgroup of G which acts without inversions on X. Then it naturally gives rise to a quotient graph $\Gamma \backslash X$. For example we see the case when $\Gamma = \Gamma(1) := PGL(2, \mathbb{F}_q[t])$, which acts without inversions. Throughout this paper we put for $n \in \mathbb{Z}$

$$\sigma_n := \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \in V(X), \qquad e_n = (\sigma_n, \sigma_{n+1}) \in E(X), \tag{3}$$

where we let $e = (v, u)$ mean that vertices v and u are adjacent by the edge e. Let $\Gamma_0(1) = PGL(2, \mathbb{F}_q)$, $B_0(1) = \{ (\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \in \Gamma(1) \mid a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \}$, and for every $m \geqslant 1$, $\Gamma_m(1) = \{ (\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \in \Gamma(1) \mid a, d \in \mathbb{F}_q^\times, \deg(b) \leqslant m \}$. Then for the action of $\Gamma(1)$ on the tree X, the following theorem is known:

THEOREM 2.1 [Se, II.1.6].

(1) *The infinite path $(\sigma_0, \sigma_1, \sigma_2, ...)$ is a quotient graph $\Gamma(1) \backslash X$.*

(2) *For every $m \geqslant 0$, $\Gamma_m(1)$ is the stabilizer of σ_m. Moreover B_0 is the stabilizer of e_0, and $\Gamma_m(1)$ is the stabilizer of e_m for $m \geqslant 1$.*

If Γ is a principal congruence subgroup

$$\Gamma(A) = \{ \gamma \in PGL(2, \mathbb{F}_q[t]) \mid \gamma \equiv I \pmod{A} \} \qquad (A \in \mathbb{F}_q[t]),$$

which is normal in $\Gamma(1)$, a quotient graph $\Gamma(A) \backslash X$ can be found in [L1] or in detail in [M2]. For other congruence subgroups, see also [GN]. In general if Γ is a discrete subgroup of G of finite covolume (i.e., a lattice in G), Lubotzky [Lu, Theorem 6.1.] shows that the quotient graph $\Gamma \backslash X$ is the union of a finite graph $\mathscr{F}_0$ together with finitely many infinite half lines (which are called ends).

The classification of conjugacy classes in $\Gamma = \Gamma(1)$ is known analogously to the case of $PSL(2, \mathbb{Z})$. We denote the conjugacy class of γ in Γ by $\{\gamma\}_\Gamma$. We write D for the subset of $\mathbb{F}_q[t]$ consisting of monic and square-free polynomials of even degree and M for the subset of $\mathbb{F}_q[t]$ consisting of monic polynomials. The mapping $d \mapsto k(\sqrt{d})$ establishes a one-to-one correspondence between D and the set of real quadratic function fields. If $\omega = x + y\sqrt{d}$ is a quadratic irrational function, where $x, y \in k$, ω satisfies the quadratic equation:

$$C\omega^2 - B\omega + A = 0 \qquad A, B, C \in \mathbb{F}_q[t].$$

If we require $g.c.d(A, B, C) = 1$ and that the coefficient of the highest power in t of $2Cy$ is 1, then the polynomials A, B, C are uniquely determined, so we write $\omega = \{A, B, C\}$. In this setting the discriminant of ω is defined by $B^2 - 4AC = 4C^2y^2d$. If two quadratic irrational functions ω_1, ω_2 are equivalent under the equivalence relation $\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}$ with $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in PGL(2, \mathbb{F}_q[t])$, then we say they are $PGL(2, \mathbb{F}_q[t])$-equivalent. If

ω_1 and ω_2 are $PGL(2, \mathbb{F}_q[t])$-equivalent, then they have the discriminant. For $d \in D$, $l \in M$ we put $\mathcal{O}_{l\sqrt{d}} = \mathbb{F}_q[t] + \mathbb{F}_q[t]\, l\sqrt{d}$, which is an order in $k(\sqrt{d})$.

PROPOSITION 2.1 [Ak]. *Let q be an odd prime power and α be a generator of $\mathbb{F}_q^\times$. We write $C^\times$ for a complete set of representatives of equivalence classes in $\mathbb{F}_q^\times - \{1\}$ defined by the relation $ab = 1$ and C_+ for a complete set of representatives of equivalence classes in $\mathbb{F}_q$ defined by the relation $a + b = 0$. Let $h_{l\sqrt{d}}$ be the narrow class number of the order $\mathcal{O}_{l\sqrt{d}}$ in $k(\sqrt{d})$ and $\varepsilon_{l\sqrt{d}} = t_0 + u_0 l\sqrt{d}$ a fundamental unit of $\mathcal{O}_{l\sqrt{d}}$. For every real quadratic irrational function $\omega = \{A, B, C\}$ of discriminant dl^2, we put*

$$\gamma_\omega = \begin{pmatrix} (t_0 + Bu_0)/2 & -Au_0 \\ Cu_0 & (t_0 - Bu_0)/2 \end{pmatrix}. \tag{4}$$

Then a complete set of representatives of conjugacy classes of $\Gamma = PGL(2, \mathbb{F}_q[t])$ is given by the following five types of elements: One of them consists only of the identity element and the others are the following four types

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}_\Gamma (x \in I), \quad \left\{\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\right\}_\Gamma (c \in C^\times), \quad \left\{\begin{pmatrix} a/2 & \alpha/4 \\ 1 & a/2 \end{pmatrix}\right\}_\Gamma (a \in C_+),$$

and $\{\gamma_\omega^n\}_\Gamma$ $(d \in D, l \in M, n = 1, 2, ...)$, where ω runs through a complete set of representative of Γ-equivalence classes of the real quadratic irrational functions of discriminant dl^2. The number of this complete set is $h_{l\sqrt{d}}$.

Remark. The elements of the above four types are parabolic, split hyperbolic, elliptic, and hyperbolic, respectively. A hyperbolic element γ_ω stabilizes $\omega \in \partial X$ and its conjugate $\omega' \in \partial X$.

Let Γ be a lattice in G. We consider $\mathbb{C}$-valued functions defined on the set of vertices $V(X)$. If a function f on X satisfies $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$ and $g \in V(X)$, f is called an automorphic function for Γ. In the following of this section we will study the harmonic analysis of automorphic functions for Γ, namely, just functions on the quotient graph $\Gamma \backslash X$. We denote the stabilizer of $v \in V(\Gamma\backslash X)$ (resp. $e \in E(\Gamma\backslash X)$) in Γ by Γ_v (resp. Γ_e). The graph $\Gamma\backslash X$ can be made into a measure space by a Haar measure of G. If we normalize it so that the volume of K is 1, it yields an atomic measure m on $\Gamma\backslash X$ that assigns to a vertex $v \in V(\Gamma\backslash X)$ the measure

$$m(v) = |\Gamma_v|^{-1}$$

(see [Se, II.1.5]). For later use we put

$$m(e) = |\Gamma_e|^{-1},$$

where $e \in E(\Gamma \backslash X)$. For automorphic functions f_1, f_2 for Γ, we define as usual the inner product $\langle \cdot, \cdot \rangle_{\Gamma \backslash X}$, sometimes simply written as $\langle \cdot, \cdot \rangle$, by

$$\langle f_1, f_2 \rangle_{\Gamma \backslash X} := \int_{\Gamma \backslash X} f_1(g) \cdot \bar{f}_2(g)\, dg = \sum_{v \in V(\Gamma \backslash X)} f_1(v) \cdot \bar{f}_2(v)\, m(v),$$

and denote the space of all square integrable functions on $\Gamma \backslash X$ by $L^2(\Gamma \backslash X)$. Now we define a natural operator on X, which we call the adjacency operator or Laplacian, by

$$(Tf)(v) := \sum_{(v,\, u) \in E(X)} f(u) \qquad (f : X \to \mathbb{C}). \tag{5}$$

It induces an operator on $\Gamma \backslash X$, sometimes denoted clearly by T_Γ, and we see that T_Γ is represented as

$$(T_\Gamma f)(v) = \sum_{e = (v,\, u) \in E(\Gamma \backslash X)} \frac{m(e)}{m(v)} f(u) \qquad (f : \Gamma \backslash X \to \mathbb{C}). \tag{6}$$

It is known that T is a self-adjoint operator and $\|T\| \leqslant q+1$ (see e.g. [M1]). We assume a function $f_\lambda : X \to \mathbb{C}$ satisfies $Tf_\lambda = \lambda f_\lambda$ and we consider its alternating function $\tilde{f}_\lambda$ which is equal to $f_\lambda(v)$ if the distance between v and σ_0 is even, and is equal to $-f_\lambda(v)$ otherwise. Then $\tilde{f}_\lambda$ satisfies $T\tilde{f}_\lambda = -\lambda \tilde{f}_\lambda$.

Next we define an important function $\psi_s(g)$ $(g \in G,\, s \in \mathbb{C})$ by

$$\psi_s(g) := |\det(g)|_\infty^s\, h((0, 1)\, g)^{-2s}, \tag{7}$$

where we denote $h((x, y)) := \sup\{|x|_\infty, |y|_\infty\}$. It can be checked that $\psi_s(g)$ is K-right and N-left invariant, where $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in G\}$, and satisfies

$$(T\psi_s)(g) = (q^s + q^{1-s})\, \psi_s(g). \tag{8}$$

Let Γ be a non-uniform (i.e., finite co-volume but not co-compact) lattice in G. As usual in the theory of automorphic functions, we will define cusp forms and the Eisenstein series for each cusp.

Let $\kappa_1, ..., \kappa_\mu$ be a complete set of inequivalent cusps for Γ. Throughout we take a quotient graph $\Gamma\backslash X$ which contains the end corresponding to $\infty \in \partial X$. Let Γ_{κ_i} be the stabilizer in Γ of κ_i and take an element $\tilde{\kappa}_i \in G$ such that $\tilde{\kappa}_i \infty = \kappa_i$. Let f be an automorphic function for Γ, then f can be expanded in the Fourier series at each cusp. Fix a cusp κ_i, and we say that $f \in L^2(\Gamma\backslash X)$ is cuspidal at κ_i if the constant term of the Fourier expansion at κ_i vanishes, i.e., for all $g \in G$

$$\int_{(\tilde{\kappa}_i^{-1}\Gamma_{\kappa_i}\tilde{\kappa}_i \cap N)\backslash N} f(\tilde{\kappa}_i ng)\, dn = 0,$$

where the invariant measure on $(\tilde{\kappa}_i^{-1}\, \Gamma_{\kappa_i}\, \tilde{\kappa}_i \cap N)\backslash N$ is normalized so that the total measure of $(\tilde{\kappa}_i^{-1}\, \Gamma_{\kappa_i}\, \tilde{\kappa}_i \cap N)\backslash N$ is equal to 1. Let us denote the end attached to a cusp κ_i by $(a_{L_i}, a_{L_i+1}, ...)$, where we always assume that $a_{L_i} \in \mathscr{F}_0$ is the *foot*, i.e., a_{L_i} has more than two neighbors in $\Gamma\backslash X$. Here L_i is defined by $\tilde{\kappa}_i^{-1} a_{L_i} = \sigma_{L_i}$. To each cusp κ_i, we see from the way of the action of Γ_{κ_i} on X that for $n \geqslant L_i$

$$f\left(\tilde{\kappa}_i \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\tilde{\kappa}_i \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{9}$$

for all $x \in k_\infty$. Therefore if f is cuspidal at κ_i, then f vanishes on the end at κ_i. A function $f \in L^2(\Gamma\backslash X)$ is called a cusp form if it is an eigenfunction of the operator T_Γ and it is cuspidal at all cusps. In particular, cusp forms are supported by the finite graph $\mathscr{F}_0$. Hence the number of cusp forms is finite.

Next we review the eigen condition on an end $(a_L, a_{L+1}, ...)$. In detail see [E2]. Let $f \in L^2(\Gamma\backslash X)$ be an eigenfunction of T_Γ with eigenvalue λ. Then for $n \geqslant L+1$

$$\lambda f(a_n) = (T_\Gamma f)(a_n) = q\, f(a_{n-1}) + f(a_{n+1}). \tag{10}$$

We assume $0 \leqslant \lambda < q+1 (\lambda \neq 2\sqrt{q})$ and solving this difference equation we have that

$$\begin{aligned}(x - x')\, f(a_n) = x^{n-L-1}&(f(a_{L+1})\, x - f(a_L)\, q) \\ &- x'^{n-L-1}(f(a_{L+1})\, x' - f(a_L)\, q),\end{aligned}$$

where x, x' are roots of $u^2 - \lambda u + q = 0$. If $2\sqrt{q} < \lambda < q+1$ then x and x' are real, and we take $x > x'$, so $x' < \sqrt{q} < x$. Since typically $m(a_n) =$ (a constant) $\times\, q^{-n}$, $f \in L^2(\Gamma\backslash X)$ must satisfy $f(a_n) = o(q^{n/2})$ as $n \to \infty$.

Therefore we have $f(a_{L+1})\,x - f(a_L)\,q = 0$, so that

$$(x - x')\,f(a_n) = -x'^{\,n-L-1}(f(a_{L+1})\,x' - f(a_L)\,q).$$

Next if $0 \leqslant \lambda < 2\sqrt{q}$, then $x' = \bar{x}$, $|x| = \sqrt{q}$. For $f \in L^2(\Gamma\backslash X)$, we can check that f must vanishes on the end. From the Fourier coefficients condition (see e.g. [L1, p. 239 (1.2)]) we see that in fact f is cuspidal at this cusp. Finally if $\lambda = 2\sqrt{q}$, then by solving (10) directly we see that f must vanish on the end.

Now we define the Eisenstein series for a cusp κ_i by

$$E_i(g, s) = \sum_{\gamma \in \Gamma_{\kappa_i}\backslash\Gamma} \psi_s(\tilde{\kappa}_i^{-1}\gamma g), \tag{11}$$

where ψ_s is as in (7) and $s \in \mathbb{C}$. From (8) this function satisfies

$$(TE_i)(g, s) = (q^s + q^{1-s})\,E_i(g, s) \tag{12}$$

for $g \in X$. It is obvious that $E_i(g, s)$ is invariant under Γ, so it can be expanded as a Fourier series at each cusp κ_j. When Γ is a principal congruence group $\Gamma(A)$, Li ([L1]) computes concretely the Fourier series expansion of $E_i(g, s)$ for each cusp κ_j, which can be expressed in terms of the L-functions associated to the characters χ on $\mathbb{F}_q[t]$ mod A. In particular the constant term of the Fourier series of $E_i(g, s)$ at a cusp κ_j is of the form:

$$\delta_{ij} q^{ns} + \varphi_{ij}(s)\,q^{n(1-s)}, \tag{13}$$

where δ_{ij} is Kronecker's δ. We define the matrix $\Phi(s) := (\varphi_{ij}(s))$, which is called the scattering matrix of Γ. Its determinant $\varphi(s) := \det \Phi(s)$ is called the scattering determinant of Γ. As for $\varphi_{ij}(s)$ and $\Phi(s)$, we summarize the results which will be necessary later.

Theorem 2.2 [L1, p. 241, p. 242, p. 249]. Let $\Gamma = \Gamma(A)(A \in \mathbb{F}_q[t])$. *Then the function $\varphi_{ij}(s)$ is a rational function in q^{-2s}, and for fixed $g \in X$ the function $E_i(g, s)$ is a rational function in q^{-s}. Moreover $\varphi_{ij}(s)$ and $E_i(g, s)$ with $g \in X$ fixed are holomorphic on $Re(s) \geqslant 1/2$ except for simple poles at $s = 1 + n\pi i/\log q (n \in \mathbb{Z})$. The matrix $\Phi(s)$ is symmetric and satisfies the functional equation*

$$\Phi(s) \cdot \Phi(1 - s) = I. \tag{14}$$

Let $\Gamma = \Gamma(A)$. Using the function field analogue of the Ramanujan conjecture proved by Drinfeld ([Dr]), Morgenstern ([M2]) deduces that the eigenvalues of T except for $\lambda = \pm(q+1)$ satisfy $|\lambda| \leqslant 2\sqrt{q}$. Thus $\Gamma\backslash X$ is a Ramanujan diagram. For the definition of a Ramanujan diagram see [M1].

Let $L^1(G)$ be the space of all integrable functions on G, and $C(K\backslash G/K)$ be the space of all continuous functions f on G such that $f(kgk')=f(g)$ for all $g\in G$ and $k, k'\in K$. Observing that a function on X is a K-right invariant function on G, we define an integral operator L_k on L^2-functions on X by

$$(L_k f)(g)=\int_X k(g,g')\, f(g')\, dg',$$

where the kernel function $k(g,g')$ is represented as $k(g,g')=F(g'^{-1}g)$ for some $F\in L^1(G)\cap C(K\backslash G/K)$. Note that $k(g,g')$ is determined by $d(g,g')$, where d is the distance function on the tree X. So we write $k(g,g')=k(d(g,g'))$ as a one-variable function. If we let $cg=\sigma_0$, $cg'=\sigma_n (n\geqslant 1)$ by some element $c\in G$, then we have $k(g,g')=k(d(g,g'))=k(n)=F(\sigma_n)$. Assume f is a function on X satisfying $Tf=\lambda f$, where T is the adjacency operator (5) and $\lambda\in\mathbb{C}$. Then it is seen that there exists a constant $\Lambda(\lambda)$ depending only on k and λ, and not on the individual function f, such that $L_k f=\Lambda(\lambda)\, f$. We call the transformation $k\mapsto\Lambda(\lambda)$ the Selberg transform. The Selberg transform and its inverse transform can be explicitly computed as in the following proposition.

Proposition 2.2 [VN, p. 428]. *If we put $s=\frac{1}{2}+ir$, $\lambda=q^s+q^{1-s} (s, r\in\mathbb{C})$ and set $\Lambda(\lambda)=h(r)$, then the Selberg transform of the kernel function k is given by the following formulas:*

$$\Lambda(\lambda)=h(r)=\sum_{n=-\infty}^{\infty} c(n)\, q^{inr},$$

$$c(n)=q^{|n|/2}\left(k(|n|)+\sum_{m=1}^{\infty}(q^m-q^{m-1})\, k(|n|+2m)\right) \qquad (n\in\mathbb{Z}).$$

Conversely, for given $\Lambda(\lambda)=h(r)$ we have the inverse Selberg transform as

$$c(n)=\frac{\log q}{2\pi}\int_{-\pi/\log q}^{\pi/\log q} h(r)\, q^{-inr}\, dr;\ (c(n)=c(-n)),$$

$$k(|n|)=q^{-|n|/2}\left(c(|n|)-(q-1)\sum_{m=1}^{\infty} c(|n|+2m)\, q^{-m}\right)$$

for $n\in\mathbb{Z}$.

Remark. Let F be a function in $C(K\backslash G/K)$. Note that the volume of $K\sigma_n K$ is equal to $q^n+q^{n-1} (n\geqslant 1)$. This is the number of the vertices of X which are n-distant from σ_0. So we have

$$\int_G |F(g)|\, dg=|F(\sigma_0)|+\frac{q+1}{q}\sum_{n\geqslant 1} q^n\, |F(\sigma_n)|.$$

Hence $F \in L^1(G)$ if and only if

$$\sum_{n \geqslant 1} q^n \, |F(\sigma_n)| < \infty. \tag{15}$$

We note that from Proposition 2.2 it follows that for a sequence $c(n) \in \mathbb{C} (n \in \mathbb{Z})$ satisfying $c(n) = c(-n)$ and

$$\sum_{n \geqslant m} q^{n/2} \, |c(n)| = O(q^{-m}), \qquad m \to \infty, \tag{16}$$

put $h(r) = \sum_{n=-\infty}^{\infty} c(n) \, q^{inr}$ then the function $h(r)$ is the Selberg transform of some function $F \in L^1(G) \cap C(K \backslash G/K)$ with

$$\sum_{n \geqslant m} q^n \, |F(\sigma_n)| = O(q^{-m}), \qquad m \to \infty.$$

3. TRACE FORMULA

Throughout this section let q be an odd prime power and Γ be a principal congruence group $\Gamma(A) (A \in \mathbb{F}_q[t])$ with $\deg(A) = a \geqslant 1$. We consider the following integral operator L_k on $L^2(\Gamma \backslash X)$:

$$(L_k f)(g) := \int_X k_F(g, g') \, f(g') \, dg' \qquad (f \in L^2(\Gamma \backslash X)), \tag{17}$$

where $F \in L^1(G) \cap C(K \backslash G/K)$ and $k_F(g, g') := F(g'^{-1} g)$ as introduced in Section 2. This integral operator L_k can be written as

$$(L_k f)(g) = \int_{\Gamma \backslash X} K_F(g, g') \, f(g) \, dg, \tag{18}$$

where $K_F(g, g') = \sum_{\gamma \in \Gamma} k_F(g, \gamma g')$. Note that $K_F(g, g')$ is a Γ-left invariant and K-right invariant function on $G \times G$. Throughout we simply write $k(g, g')$ (resp. $K(g, g')$) for $k_F(g, g')$ (resp. $K_F(g, g')$). In this section we will compute explicitly the trace of this integral operator L_k, which we call the Selberg trace formula for Γ.

LEMMA 3.1. *Let $F \in L^1(G) \cap C(K \backslash G/K)$. Then $K(g, g')$ converges absolutely on $G \times G$. Moreover, $K(g, g')$ is bounded on $C \times V(\Gamma \backslash X)$ and on $V(\Gamma \backslash X) \times C$, where C is any finite subset of $V(\Gamma \backslash X)$.*

Proof. Fix $g, g' \in G$. We have

$$\begin{aligned} \sum_{\gamma \in \Gamma} |F(g'^{-1} \gamma^{-1} g)| &= \sum_{\gamma \in \Gamma} \int_{g'^{-1} \gamma^{-1} g K} |F(g_1)| \, dg_1 = \sum_{\gamma \in \Gamma} \int_{\gamma^{-1} g K} |F(g'^{-1} g_1)| \, dg_1 \\ &= |\Gamma \cap g K g^{-1}| \cdot \int_{\Gamma g K} |F(g'^{-1} g_1)| \, dg_1, \end{aligned}$$

which and $F \in L^1(G)$ imply that $K(g, g')$ converges absolutely. Moreover, let $g \in C$, $g' \in G$, then since $|\Gamma \cap gKg^{-1}| < \infty$ on C, we have that $K(g, g')$ is bounded on $C \times V(\Gamma \backslash X)$. Next if we apply the above argument to $\tilde{F}(g) := F(g^{-1})$, we see that $K(g, g')$ is bounded on $V(\Gamma \backslash X) \times C$. ∎

The following lemma shows how $K(g, g')$ grows in the neighborhood of cusps. These are necessary for proving Theorem 3.1.

LEMMA 3.2. *Let $F \in L^1(G) \cap C(K \backslash G/K)$ and $g, g' \in V(\Gamma \backslash X)$. Assume that $\sum_{j \geqslant l} q^j |F(\sigma_j)| = O(q^{-l})$ as $l \to \infty$. Then $K(g, g') - \sum_{\gamma \in \Gamma_{\kappa_i}} k(g, \gamma g')$ is bounded when g and g' tend to a common cusp κ_i, and $K(g, g')$ is bounded when g and g' tend to different cusps.*

Proof. For the first assertion, we can assume that both g and g' tend to the cusp ∞. So we put $g = \sigma_n$, $g' = \sigma_m (n, m \geqslant 1)$. Then

$$\sum_{\gamma \in \Gamma - \Gamma_\infty} |k(g, \gamma g')| = \sum_{\gamma \in \Gamma - \Gamma_\infty} |k(\sigma_n, \gamma \sigma_m)| = \sum_{\gamma \in \Gamma - \Gamma_\infty} \int_{\sigma_m^{-1} \gamma^{-1} \sigma_n K} |F(g)| \, dg.$$

For $\gamma, \gamma' \in \Gamma - \Gamma_\infty$ one sees that $\sigma_m^{-1} \gamma^{-1} \sigma_n K = \sigma_m^{-1} \gamma'^{-1} \sigma_n K$ if and only if $\gamma'^{-1} \in \gamma^{-1} \cdot (\Gamma \cap \sigma_n K \sigma_n^{-1})$. Here from (2) we can check that for all $\gamma \in \Gamma(1) - B_{\Gamma(1)}$ $(B_{\Gamma(1)} := B \cap \Gamma(1))$

$$\sigma_m^{-1} \gamma \sigma_n \in K \sigma_j K, \tag{19}$$

where $j \geqslant m + n$. Namely if $\gamma \notin \Gamma_\infty$, the larger n and m are, the longer the distance between σ_n and $\gamma \sigma_m$ becomes. From these and Theorem 2.1 we have

$$\begin{aligned} &\sum_{\gamma \in \Gamma - \Gamma_\infty} \int_{\sigma_m^{-1} \gamma^{-1} \sigma_n K} |F(g)| \, dg \\ &\quad \leqslant |\Gamma \cap \sigma_n K \sigma_n^{-1}| \int_{\bigcup_{j \geqslant m+n} K \sigma_j K} |F(g)| \, dg \\ &\quad \leqslant |\Gamma(1) \cap \sigma_n K \sigma_n^{-1}| \, \frac{q+1}{q} \sum_{j \geqslant m+n} q^j \, |F(\sigma_j)| \\ &\quad = (q^2 - 1) \, q^n \sum_{j \geqslant m+n} q^j \, |F(\sigma_j)|. \end{aligned}$$

Thus we have the first assertion.

Next we may assume that g tends to ∞ and g' tends to another cusp $\kappa (\kappa \neq \infty)$. At present since $\Gamma = \Gamma(A) \subset \Gamma(1)$, from Theorem 2.1 we can take an element $\tilde{\kappa}$ in $\Gamma(1)$ such that $\tilde{\kappa} \infty = \kappa$. We now consider the behavior of $\sum_{\gamma \in \Gamma} k(g, \gamma \tilde{\kappa} g') = \sum_{\gamma \in \Gamma} F(g'^{-1} \tilde{\kappa}^{-1} \gamma^{-1} g)$ as g, g' tend to ∞. Here if we assume that $\tilde{\kappa}^{-1}\gamma^{-1}$ belongs to $B_{\Gamma(1)}$, i.e., $\tilde{\kappa}^{-1}\gamma^{-1}$ is a stabilizer of ∞, then we have $\gamma^{-1}\infty=\kappa$. But this contradicts the fact ∞ and κ are not equivalent with respect to Γ. Therefore $\tilde{\kappa}^{-1}\gamma^{-1}\notin B_{\Gamma(1)}$. Similarly using (19), we have that as $n, m\to\infty$

$$\sum_{\gamma\in\Gamma} |F(\sigma_m^{-1}\tilde{\kappa}^{-1}\gamma^{-1}\sigma_n)| \leqslant |\Gamma(A)\cap\sigma_n K\sigma_n^{-1}| \int_{\substack{\bigcup K\sigma_j K\\ j\geqslant m+n}} |F(g)|\,dg,$$

so we have the second assertion. ∎

LEMMA 3.3. *Let $F\in L^1(G)\cap C(K\backslash G/K)$ and $n, m>a$. Then we have*

$$\sum_{x\in A\mathbb{F}_q[t]} k(\sigma_n, n(x)\,\sigma_m)=q^{((n+m)/2)+1-a}\,c(n-m), \tag{21}$$

where $A\mathbb{F}_q[t]=\{Af(t)\mid f(t)\in\mathbb{F}_q[t]\}$ and $n(x)=\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$.

Proof. First by using (2) we decompose the matrix $\sigma_m^{-1}\,n(x)\,\sigma_n=\left(\begin{smallmatrix}t^{n-m} & t^{-m}x\\ 0 & 1\end{smallmatrix}\right)$ to the form $k\left(\begin{smallmatrix}t^{N(l)} & 0\\ 0 & 1\end{smallmatrix}\right)k'\,(k, k'\in K)$, where $l=\deg(x)$. We can check that

$$N(l)=\begin{cases}|n-m| & (l\leqslant\sup\{n, m\})\\ 2l-n-m & (l>\sup\{n, m\}).\end{cases}$$

This can be easily seen also from the geometric viewpoint. Hence by $F\in C(K\backslash G/K)$ and Proposition 2.2 we have that

$$\begin{aligned}\sum_{x\in A\mathbb{F}_q[t]} k(\sigma_n, n(x)\,\sigma_m) &= \sum_{x\in A\mathbb{F}_q[t]} F(\sigma_n^{-1}n(-x)\,\sigma_n)\\ &= q^{\sup\{n, m\}+1-a}\,F(\sigma_{|n-m|})\\ &\quad + q^{\sup\{n, m\}+1-a}\sum_{j=1}^{\infty}(q^j-q^{j-1})\,F(\sigma_{|n-m|+2j})\\ &= q^{\sup\{n, m\}+1-a}q^{-(|n-m|)/2}\,c(n-m)\\ &= q^{((n+m)/2)+1-a}c(n-m). \qquad ∎\end{aligned}$$

The following theorem is particularly important and is the first step to the Selberg trace formula for Γ.

THEOREM 3.1. *Let $\Gamma = \Gamma(A)(A \in \mathbb{F}_q[t], \deg(A) = a \geqslant 1)$ and $\mu = \mu_\Gamma$ denote the number of inequivalent cusps for Γ. Assume that a sequence $c(n) \in \mathbb{C}$, $n \in \mathbb{Z}$ satisfies $c(n) = c(-n)$ and (16): $\sum_{n \geqslant m} q^{n/2} |c(n)| = O(q^{-m})$. We define the kernel function $H(g, g')$ by*

$$H(g, g') = \sum_{i=1}^{\mu} \frac{q \log q}{4\pi q^a} \int_{-\pi/\log q}^{\pi/\log q} h(r)\, E_i(g, s)\, E_i(g', \bar{s})\, dr \qquad \left(s = \frac{1}{2} + ir\right). \tag{22}$$

Then $\hat{K}(g, g') := K(g, g') - H(g, g')$ is bounded on $\Gamma \backslash X$. Let $\mathscr{D}$ be the set of all discrete $L^2(\Gamma \backslash X)$-eigenfunctions of T_Γ. Then for any $f \in \mathscr{D}$ we have

$$\int_{\Gamma \backslash X} H(g, g')\, f(g')\, dg' = 0. \tag{23}$$

Proof. From (13) and Theorem 2.2 we have for $Re(s) = 1/2$, $E_i(g, s) = O(q^{n/2})$ as g tends to a cusp. Combining this, the result of L^2-eigencondition on an end noted in Section 2 and Morgenstern's result in Section 2, we see that for any $f \in \mathscr{D}$ the inner product $\langle E_i(g, s), f \rangle (Re(s) = 1/2)$ makes sense. Let us assume $T_\Gamma f = \lambda f$. Then from (12) the equation $(q^s + q^{1-s}) \langle E_i(g, s), f \rangle = \langle T_\Gamma E_i(g, s), f \rangle = \langle E_i(g, s), T_\Gamma f \rangle = \bar{\lambda} \langle E_i(g, s), f \rangle$ holds. But we must have $\langle E_i(g, s), f \rangle = 0$ since $q^s + q^{1-s}$ is not a constant. This proves (23).

Next we consider the behavior of the function $H(g, g')$ as g and g' tend to a common cusp to κ_i. Then since from (9) the value on an end is given by the constant term of the Fourier series expansion, we have

$$\begin{aligned} &\sum_{l=1}^{\mu} E_l(\tilde{\kappa}_i g, s)\, E_l(\tilde{\kappa}_i g', \bar{s}) \\ &\quad = \sum_{l=1}^{\mu} (\delta_{li} q^{ns} + \varphi_{li}(s)\, q^{n(1-s)})(\delta_{li} q^{m\bar{s}} + \varphi_{li}(\bar{s})\, q^{m(1-\bar{s})}), \end{aligned} \tag{24}$$

where we put $\tilde{\kappa}_i g = \sigma_n$, $\tilde{\kappa}_i g' = \sigma_m$ $(n, m > a)$. By (14) the scattering matrix $\Phi(s)$ on $Re(s) = 1/2$ is unitary, so the last equation (24) is equal to

$$q^{ns} q^{m\bar{s}} + q^{n(1-s)} q^{m(1-\bar{s})} \tag{25}$$

$$+ \varphi_{ii}(s)\, q^{n(1-s)} q^{m\bar{s}} + \varphi_{ii}(\bar{s})\, q^{ns} q^{m(1-\bar{s})} \tag{26}$$

$$\begin{aligned} &= q^{(n+m)/2} (q^{(n-m)\, ir} + q^{-(n-m)\, ir} \\ &\quad + \varphi_{ii}(s)\, q^{-(n+m)\, ir} + \varphi_{ii}(\bar{s})\, q^{(n+m)\, ir}). \end{aligned}$$

It follows from Proposition 2.2 that

$$\frac{q\log q}{4\pi q^a}\int_{-\pi/\log q}^{\pi/\log q} h(r)\, q^{(n+m)/2}(q^{(n-m)\,ir}+q^{-(n-m)\,ir})\,dr$$

$$=\frac{q\log q}{2\pi q^a}\int_{-\pi/\log q}^{\pi/\log q} h(r)\, q^{(n+m)/2}\, q^{(n-m)\,ir}\,dr$$

$$=q^{(n+m)/2+1-a}c(n-m). \tag{27}$$

On the other hand, from Theorem 2.2 we see that

$$\frac{q\log q}{4\pi q^a}\int_{-\pi/\log q}^{\pi/\log q} h(r)\, q^{(n+m)/2}(\varphi_{ii}(s)\, q^{-(n+m)\,ir}+\varphi_{ii}(\bar{s})\, q^{(n+m)\,ir})\,dr \tag{28}$$

$$=C\cdot q^{(n+m)/2}c(n+m), \tag{29}$$

where C is a constant. Recalling the assumption, we see that for $n, m>a$ (29) is bounded. Besides, as g, g' tend to a common cusp κ_i, we have

$$\sum_{\gamma\in\Gamma_{\kappa_i}} k(g,\gamma g')=\sum_{\gamma\in\Gamma_{\kappa_i}} k(\tilde{\kappa}_i\sigma_n,\gamma\tilde{\kappa}_i\sigma_m)$$

$$=\sum_{\gamma'\in\tilde{\kappa}_i^{-1}\Gamma_{\kappa_i}\tilde{\kappa}_i} k(\sigma_n,\gamma'\sigma_m)=\sum_{x\in A\mathbb{F}_q[t]} k(\sigma_n, n(x)\,\sigma_m),$$

where we put $g=\tilde{\kappa}_i\sigma_m$, $g'=\tilde{\kappa}_i\sigma_n$. Therefore from Lemma 3.2, Lemma 3.3 and (27) we can conclude that

$$\hat{K}(g,g')=\left(\sum_{\gamma\in\Gamma} k(g,\gamma g')-\sum_{\gamma\in\Gamma_{\kappa_i}} k(g,\gamma g')\right)+\left(\sum_{\gamma\in\Gamma_{\kappa_i}} k(g,\gamma g')-H(g,g')\right)$$

is bounded as g and g' tend to the same cusp κ_i.

Next let us assume that g tends to a cusp κ_i and that g' tends to a different cusp $\kappa_j (i\neq j)$, and examine the behavior of

$$\sum_{l=1}^{\mu} E_l(\tilde{\kappa}_i g, s)\, E_l(\tilde{\kappa}_j g', \bar{s})$$

$$=\sum_{l=1}^{\mu} (\delta_{li}q^{ns}+\varphi_{li}(s)\, q^{n(1-s)})(\delta_{lj}q^{m\bar{s}}+\varphi_{lj}(\bar{s})\, q^{m(1-\bar{s})}),$$

as $n, m \to \infty$. In this case the fact that $\Phi(s)$ on $Re(s) = 1/2$ is unitary implies that $H(g, g')$ remains no terms of the form (25), but only terms of the form (26). Hence $H(g, g')$ is bounded. Therefore from Lemma 3.2 we have that $\hat{K}(g, g') = K(g, g') - H(g, g')$ is bounded as g and g' tend to different cusps.

When g tends to a cusp and g' remains in a finite subset of $V(\Gamma\backslash X)$ or when g remains in a finite subset of $V(\Gamma\backslash X)$ and g' tends to a cusp, by Lemma 3.1 and the assumption we can easily check that $\hat{K}(g, g')$ is bounded. This completes the proof. ∎

Remark. In the case $\Gamma = \Gamma(1) = PGL(2, \mathbb{F}_q[t])$, Efrat [E2] derives explicitly the spectral decomposition of $L^2(\Gamma\backslash X)$. The discrete spectrum of the adjacency operator T_Γ on $L^2(\Gamma\backslash X)$ consists only of two trivial eigenvalues $\lambda = q+1, -(q+1)$. They correspond to the poles of the Eisenstein series $E_\infty(g, s)$ at $s = 1, 1 - \pi i/\log q$, respectively.

Next we define the integral operator $\hat{L}_k$ on $L^2(\Gamma\backslash X)$ having the kernel function $\hat{K}(g, g')$ in Theorem 3.1, by

$$(\hat{L}_k f)(g) = \int_{\Gamma\backslash X} \hat{K}(g, g')\, f(g')\, dg',$$

which is the *discrete part* of the operator L_k defined in (17). Let $\{\lambda_1, \lambda_2, ..., \lambda_M\}$ be the set of eigenvalues of T_Γ. Then from Theorem 3.1 we obtain the following formula with respect to the trace of $\hat{L}_k$,

$$Tr(\hat{L}_k) = \int_{\Gamma\backslash X} \hat{K}(g, g)\, dg = \sum_{n=1}^{M} h(r_n), \tag{30}$$

where h is the Selberg transform of k as in Proposition 2.2. We call this formula (30) the Selberg trace formula for Γ. As usual, the term $\int_{\Gamma\backslash X} \hat{K}(g, g)\, dg$ can be divided into the sum over the conjugacy classes of Γ. For $\gamma \in \Gamma$ let $\{\gamma\}_\Gamma$ be the conjugacy class of γ in Γ and $\Gamma(\gamma)$ be the centralizer of γ in Γ. The elements in $\Gamma(\gamma)$ have the same fixed points as γ on $X \cup \partial X$. Note from Theorem 2.1 we see that if $\deg(A) \geqslant 1$ then $\Gamma = \Gamma(A)$ has no elliptic elements. We rewrite the trace formula (30) as the following: Put

$$C(I) = \int_{\Gamma(I)\backslash X} k(g, Ig)\, dg,$$

$$C(H) = \sum_{\{\gamma\}_\Gamma} \int_{\Gamma(\gamma)\backslash X} k(g, \gamma g)\, dg,$$

$$C(P,n)=\sum_{\{\gamma\}_\Gamma}\int_{\Gamma(\gamma)\backslash\tilde{\kappa}_i Y_n} k(g,\gamma g)\,dg,$$

$$E(n)=\int_{\mathcal{S}_n} H(g,g)\,dg,$$

$$C(P)=\lim_{n\to\infty}\,(C(P,n)-E(n)).$$

Here for $C(H)$ the sum is taken over the conjugacy classes of hyperbolic elements in Γ, and for $C(P,n)$ the sum is taken over the conjugacy classes of parabolic elements in Γ and note that they are divided into the classes corresponding to inequivalent cusps $\{\kappa_1,\kappa_2,...,\kappa_\mu\}$. We let Y_n be the subgraph of X such that the y-coordinates of the vertices in Y_n are less than or equal to n, and

$$\mathcal{S}_n=\Gamma\backslash X\cap\left(\bigcap_{i=1}^{\mu}\tilde{\kappa}_i Y_n\right). \tag{31}$$

As we know later, $C(P)$ has a finite evaluation, so the Selberg trace formula (30) can be expressed as

$$\sum_{n=1}^{M} h(r_n)=C(I)+C(H)+C(P). \tag{32}$$

Now we will compute the explicit expression of the integrals $C(I)$, $C(H)$, $C(P,n)$, $E(n)$ and the final form of the Selberg trace formula.

PROPOSITION 3.1. *Let $vol(\Gamma\backslash X)$ be the total measure of $\Gamma\backslash X$, i.e., $\int_{\Gamma\backslash X} dm(v)$, and we denote the set of primitive hyperbolic conjugacy classes of Γ by $\mathfrak{P}_\Gamma$. Then we have*

$$C(I)=vol(\Gamma\backslash X)\,k(0), \tag{33}$$

$$C(H)=\sum_{\{P\}\in\mathfrak{P}_\Gamma}\sum_{l=1}^{\infty}\frac{\deg P}{q^{l\deg P/2}}\,c(l\deg P), \tag{34}$$

$$C(P,n)=\mu\left\{(n-a+1)\,c(0)+\sum_{m=1}^{\infty} c(2m)-\frac{c(0)}{q-1}\right\}+o(1)\qquad(n\to\infty), \tag{35}$$

where $\deg P$ *is as in Lemma* 2.1.

Proof. For $C(I)$, we have

$$C(I)=\int_{\Gamma\backslash X} k(0)\,dg = vol(\Gamma\backslash X)\,k(0).$$

For $C(H)$, we note that $C(H)$ may be written as

$$C(H)=\sum_{\{P\}\in\mathfrak{P}_\Gamma}\sum_{l=1}^{\infty}\int_{\Gamma(P)\backslash X} k(g, P^l g)\,dg,$$

since for a hyperbolic element $\gamma\in\Gamma$, elements in $\Gamma(\gamma)$ have the same fixed points as those of γ on ∂X and from Proposition 2.1 $\Gamma(\gamma)$ is a cyclic group. Now a combinatorial computation yields

$$\int_{\Gamma(P)\backslash X} k(g, P^l g)\,dg$$
$$=\deg P\cdot k(l\deg P)+\deg P\cdot(q-1)\sum_{m=1}^{\infty} q^{m-1}k(2m+l\deg P).$$

Using the Selberg transform, this is given by

$$\frac{\deg P}{q^{l\deg P/2}}\,c(l\deg P),$$

hence we obtain (34).

Next we will compute $C(P,n)$. Note that we can write

$$\begin{aligned} C(P,n)&=\sum_{i=1}^{\mu}\sum_{I\neq\gamma\in\Gamma_{\kappa_i}}\int_{\Gamma_{\kappa_i}\backslash\tilde{\kappa}_i Y_n} k(g,\gamma g)\,dg\\ &=\sum_{i=1}^{\mu}\sum_{I\neq\gamma\in\Gamma_\infty}\int_{\Gamma_\infty\backslash Y_n} k(g,\gamma g)\,dg,\end{aligned}$$

where $\Gamma_\infty=\{n(x)=\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\in\Gamma(A)\}$. Now we compute the following integral

$$\sum_{I\neq\gamma\in\Gamma_\infty}\int_{\Gamma_\infty\backslash Y_n} k(g,\gamma g)\,dg=\sum_{0\neq x\in A\mathbb{F}_q[T]}\int_{\Gamma_\infty\backslash Y_n} F(g^{-1}n(x)\,g)\,dg. \qquad (36)$$

It is easily seen that the set of matrices

$$\left\{g=\begin{pmatrix} t^m & x\\ 0 & 1\end{pmatrix}\in V(X)\,\middle|\, m\in\mathbb{Z},\ |x|_\infty<q^a\right\}$$

is a complete set of representatives of $\Gamma_\infty\backslash X$. The number of vertices in $\Gamma_\infty\backslash X$ with y-coordinate t^m is

$$\begin{cases} 1 & (m \geqslant a) \\ q^{a-1-m} & (m < a), \end{cases}$$

and from Theorem 2.1 the number of stabilizers of the vertex $\left(\begin{smallmatrix} t^m & x \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty\backslash X$ is

$$\begin{cases} q^{m+1-a} & (m \geqslant a) \\ 1 & (m < a), \end{cases}$$

(in detail see [M2]). Hence the total measure of all the vertices in $\Gamma_\infty\backslash X$ with y-coordinate t^m is q^{a-1-m}. For $g = \left(\begin{smallmatrix} t^m & x \\ 0 & 1 \end{smallmatrix}\right)$, we have $g^{-1}n(x')\,g = \left(\begin{smallmatrix} 1 & t^{-m}x' \\ 0 & 1 \end{smallmatrix}\right)$, which is independent of x. Combining the above facts, the equation (36) is given by

$$\begin{aligned} &\sum_{0 \neq x \in A\mathbb{F}_q[T]} \left\{ \sum_{n \geqslant m} q^{a-1-m} F\left(\begin{pmatrix} 1 & t^{-m}x \\ 0 & 1 \end{pmatrix}\right) \right\} \\ &\quad = \sum_{n \geqslant m} q^{a-1-m} \left\{ \sum_{0 \neq x \in A\mathbb{F}_q[T]} F\left(\begin{pmatrix} 1 & t^{-m}x \\ 0 & 1 \end{pmatrix}\right) \right\}. \end{aligned} \tag{37}$$

By using (2), the integral $\sum_{0 \neq x \in A\mathbb{F}_q[T]} F(\left(\begin{smallmatrix} 1 & t^{-m}x \\ 0 & 1 \end{smallmatrix}\right))$ can be computed as if $m \geqslant a$

$$q^{m-a+1} \left\{ F(\sigma_0) + \sum_{j=1}^{\infty} (q^j - q^{j-1})\, F(\sigma_{2j}) \right\} - F(\sigma_0), \tag{38}$$

if $m < a$

$$\begin{aligned} &(q-1)\, F(\sigma_{2(a-m)}) + \sum_{j=1}^{\infty} (q^{j+1} - q^j)\, F(\sigma_{2(a-m+j)}) \\ &\quad = q \left\{ F(\sigma_{2(a-m)}) + \sum_{q=1}^{\infty} (q^j - q^{j-1})\, F(\sigma_{2(a-m+j)}) \right\} - F(\sigma_{2(a-m)}). \end{aligned} \tag{39}$$

Here we separate the sum of (37) as the following form: $\sum_{n \geqslant m} = \sum_{n \geqslant m \geqslant a} + \sum_{a > m}$. Then the first sum $\sum_{n \geqslant m \geqslant a}$ is equal to

$$\begin{aligned} &\sum_{n \geqslant m \geqslant a} \left\{ F(\sigma_0) + \sum_{j=1}^{\infty} (q^j - q^{j-1})\, F(\sigma_{2j}) \right\} - q^{a-1} \sum_{n \geqslant m \geqslant a} q^{-m} F(\sigma_0) \\ &\quad = (n-a+1)\, c(0) - \frac{1}{q-1} \left(1 - \frac{1}{q^{n-a+1}}\right) F(\sigma_0). \end{aligned}$$

The second sum $\sum_{a>m}$ is equal to

$$\sum_{m<a} q^{a-m} \left\{ F(\sigma_{2(a-m)}) + \sum_{q=1}^{\infty} (q^j - q^{j-1})\, F(\sigma_{2(a-m)+2j}) \right\}$$

$$- \sum_{m<a} q^{a-1-m} F(\sigma_{2(a-m)})$$

$$= \sum_{m<a} c(2(a-m)) - \sum_{m<a} q^{a-1-m} F(\sigma_{2(a-m)})$$

$$= \sum_{m\geqslant 1} c(2m) - \sum_{m\geqslant 1} q^{m-1} F(\sigma_{2m})$$

$$= \sum_{m\geqslant 1} c(2m) - \frac{c(0) - F(\sigma_0)}{q-1}.$$

Therefore the Eq. (37) becomes

$$(n-a+1)\, c(0) + \sum_{m\geqslant 1} c(2m) - \frac{c(0)}{q-1} + o(1) \qquad (n \to \infty).$$

Thus we have the formula (35). ∎

Next we will give the explicit computation of $E(n)$. For this purpose we use the following lemma.

LEMMA 3.4. *Let f, g be functions on $\Gamma\backslash X$, and $n \geqslant a$. Then we have the Green's formula*

$$\int_{\mathscr{S}_n} \{ f(T_\Gamma g) - (T_\Gamma f)\, g \}\, dm(v)$$

$$= \sum_{i=1}^{\mu} \{ f(\tilde{\kappa}_i \sigma_n)\, g(\tilde{\kappa}_i \sigma_{n+1}) - f(\tilde{\kappa}_i \sigma_{n+1})\, g(\tilde{\kappa}_i \sigma_n) \}\, m(\tilde{\kappa}_i e_n),$$

where $\mathscr{S}_n$ is as in (31).

Proof. For a function f on $\Gamma\backslash X$ we put

$$\hat{f}(v) := \begin{cases} f(v) & (v \in \mathscr{S}_n) \\ 0 & (v \in \Gamma\backslash X - \mathscr{S}_n). \end{cases}$$

Then for $\hat{f}$, $\hat{g}$ we have

$$\langle \hat{f}, T_\Gamma \hat{g} \rangle_{\Gamma\backslash X} = \langle T_\Gamma \hat{f}, \hat{g} \rangle_{\Gamma\backslash X}, \tag{40}$$

since T_Γ is self-adjoint on $\Gamma\backslash X$. Now we have

$$\int_{\mathscr{S}_n} \{f(T_\Gamma g)-(T_\Gamma f)\, g\}\, dm(v)$$

$$= \sum_{v\in\mathscr{S}_n} f(v) \left\{ \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\in\mathscr{S}_n}} \frac{m(e)}{m(v)} g(u) + \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\notin\mathscr{S}_n}} \frac{m(e)}{m(v)} g(u) \right\} m(v)$$

$$- \sum_{v\in\mathscr{S}_n} \left\{ \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\in\mathscr{S}_n}} \frac{m(e)}{m(v)} f(u) + \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\notin\mathscr{S}_n}} \frac{m(e)}{m(v)} f(u) \right\} g(v)\, m(v)$$

$$= \sum_{v\in\mathscr{S}_n} f(v) \left\{ \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\in\mathscr{S}_n}} \frac{m(e)}{m(v)} g(u) \right\} m(v)$$

$$+ \sum_{i=1}^{\mu} f(\tilde{\kappa}_i\sigma_n)\, m(\tilde{\kappa}_i e_n)\, g(\tilde{\kappa}_i\sigma_{n+1})$$

$$- \sum_{v\in\mathscr{S}_n} \left\{ \sum_{\substack{e=(v,u)\in E(\Gamma\backslash X)\\ u\in\mathscr{S}_n}} \frac{m(e)}{m(v)} f(u) \right\} g(v)\, m(v)$$

$$- \sum_{i=1}^{\mu} m(\tilde{\kappa}_i e_n)\, f(\tilde{\kappa}_i\sigma_{n+1})\, g(\tilde{\kappa}_i\sigma_n)$$

$$= \int_{\mathscr{S}_n} \{\hat{f}(T_\Gamma \hat{g})-(T_\Gamma \hat{f})\, \hat{g}\}\, dm(v)$$

$$+ \sum_{i=1}^{\mu} f(\tilde{\kappa}_i\sigma_n)\, m(\tilde{\kappa}_i e_n)\, g(\tilde{\kappa}_i\sigma_{n+1}) - \sum_{i=1}^{\mu} m(\tilde{\kappa}_i e_n)\, f(\tilde{\kappa}_i\sigma_{n+1})\, g(\tilde{\kappa}_i\sigma_n).$$

Here from the definition of $\hat{f}$, $\hat{g}$ and (40), it follows that

$$\int_{\mathscr{S}_n} \{\hat{f}(T_\Gamma \hat{g})-(T_\Gamma \hat{f})\, \hat{g}\}\, dm(v) = \int_{\Gamma\backslash X} \{\hat{f}(T_\Gamma \hat{g})-(T_\Gamma \hat{f})\, \hat{g}\}\, dm(v) = 0,$$

so we have the assertion. ∎

For functions f, g on $\Gamma\backslash X$, let us define the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{S}_n}$ by

$$\langle f, g\rangle_{\mathscr{S}_n} = \int_{\mathscr{S}_n} f\cdot \bar{g}\, dm(v),$$

where $\mathscr{S}_n$ is as in (31). From Lemma 3.4 we will derive the inner product formula for the Eisenstein series on $\mathscr{S}_n$. Using this formula we can give the explicit computation of

$$E(n) = \sum_{i=1}^{\mu} \int_{\mathscr{S}_n} \frac{q \log q}{4\pi\, q^a} \left(\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\, E_i\left(g, \frac{1}{2} + ir\right) E_i\left(g, \frac{1}{2} - ir\right) dr \right) dg. \tag{41}$$

Proposition 3.2. *Let $\Phi(s)$ be the scattering matrix for Γ and $\varphi(s)$ be its determinant* $\det \Phi(s)$. *Then we have*

$$E(n) = \mu\left(n + \frac{1}{2}\right) c(0) - \frac{1}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r) \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr$$
$$+ Tr\, \Phi\left(\frac{1}{2}\right)\left(\frac{1}{2} c(0) + \sum_{m=1}^{\infty} c(2m)\right) + o(1) \qquad (n \to \infty). \tag{42}$$

$$C(P) = \left(\mu - Tr\, \Phi\left(\frac{1}{2}\right)\right)\left(\frac{1}{2} c(0) + \sum_{m=1}^{\infty} c(2m)\right)$$
$$+ \frac{1}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r) \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr - \mu\left(a + \frac{1}{q-1}\right) c(0). \tag{43}$$

Proof. We will apply $f = E_i(g, s)$, $g = E_j(g, s')(s, s' \in \mathbb{C})$ to Lemma 3.4. By (9) we have $E_i(\tilde{\kappa}_l \sigma_n, s) = \delta_{il} q^{ns} + \varphi_{il}(s)\, q^{n(1-s)} (n \geqslant a)$. From Theorem 2.1 we see that the measure $m(e_n)(n \geqslant a)$ is given by $q^{-(n-a+1)}$(see [M2, p. 117 (5)]). Hence if we put $s = \sigma + ir$, $s' = \bar{s} = \sigma - ir(\sigma, r \in \mathbb{R})$, we have the following product formula

$$\langle E_i(\cdot, s), E_j(\cdot, s) \rangle_{\mathscr{S}_n}$$
$$= q^{a-1}\left(\frac{\delta_{ij} q^{(n+1)(2\sigma-1)} - \sum_{l=1}^{\mu} \varphi_{il}(s)\, \varphi_{jl}(\bar{s})\, q^{n(1-2\sigma)}}{q^{2\sigma-1} - 1} \right.$$
$$\left. + \frac{\varphi_{ji}(\bar{s})\, q^{(n+1)2ir} - \varphi_{ij}(s)\, q^{n(-2ir)}}{q^{2ir} - 1} \right),$$

whenever $\langle E_i(\cdot, s), E_j(\cdot, s) \rangle_{\mathscr{S}_n}$ is well defined and $s = \bar{s} \neq 1/2$. If we set the column vector $\mathscr{E}(g, s) := {}^t(E_1(g, s), ..., E_\mu(g, s))$, this inner product formula may be written in the matrix form as

$$\begin{aligned}
&\langle \mathscr{E}(\,\cdot\,, s), {}^t\mathscr{E}(\,\cdot\,, s)\rangle_{\mathscr{S}_n} \\
&= q^{a-1}\left(\frac{q^{(n+1)(2\sigma-1)} - \Phi(s)\,\Phi(\bar{s})\,q^{n(1-2\sigma)}}{q^{2\sigma-1}-1} + \frac{\Phi(\bar{s})\,q^{(n+1)\,2ir} - \Phi(s)\,q^{n(-2ir)}}{q^{2ir}-1}\right), \\
&= q^{a-1}\left(\frac{q^{(n+1)(2\sigma-1)} - q^{(n+1)(1-2\sigma)}}{q^{2\sigma-1}-1} + \frac{q^{1-2\sigma} - \Phi(s)\,\Phi(\bar{s})}{q^{2\sigma-1}-1}\,q^{n(1-2\sigma)}\right. \\
&\quad \left. + \frac{\Phi(\bar{s})\,q^{(n+1)\,2ir} - \Phi(s)\,q^{n(-2ir)}}{q^{2ir}-1}\right).
\end{aligned} \tag{44}$$

Next we must evaluate this formula on the line $Re(s) = 1/2$. For $s = \sigma + ir(\sigma, r \in \mathbb{R})$ we have

$$\begin{aligned}
&\lim_{\sigma \to 1/2} \frac{q^{(n+1)(2\sigma-1)} - q^{(n+1)(1-2\sigma)}}{q^{2\sigma-1}-1} \\
&\qquad = \lim_{\sigma \to 1/2} \frac{2(n+1)(\log q)(q^{(n+1)(2\sigma-1)} + q^{(n+1)(1-2\sigma)})}{2(\log q)\,q^{2\sigma-1}} \\
&\qquad = 2(n+1).
\end{aligned}$$

On the other hand, the functional equation $\Phi(\frac{1}{2}+ir)\,\Phi(\frac{1}{2}-ir) = I(r \in \mathbb{R})$ implies

$$\Phi'\left(\frac{1}{2}+ir\right)\cdot\Phi^{-1}\left(\frac{1}{2}+ir\right) = \Phi'\left(\frac{1}{2}-ir\right)\cdot\Phi^{-1}\left(\frac{1}{2}-ir\right).$$

Using these equations, we obtain

$$\begin{aligned}
&\lim_{\sigma \to 1/2} \frac{q^{1-2\sigma} - \Phi(s)\,\Phi(\bar{s})}{q^{2\sigma-1}-1}\,q^{n(1-2\sigma)} \\
&\qquad = -1 - \frac{1}{\log q}\,\Phi'\left(\frac{1}{2}+ir\right)\Phi^{-1}\left(\frac{1}{2}+ir\right).
\end{aligned}$$

Hence on $Re(s) = 1/2$ the inner formula (44) is given by

$$\begin{aligned}
&\left\langle \mathscr{E}\left(\,\cdot\,, \frac{1}{2}+ir\right), {}^t\mathscr{E}\left(\,\cdot\,, \frac{1}{2}-ir\right)\right\rangle_{\mathscr{S}_n} \\
&\qquad = q^{a-1}\left(2n+1 - \frac{1}{\log q}\,\Phi'\left(\frac{1}{2}+ir\right)\Phi^{-1}\left(\frac{1}{2}+ir\right)\right. \\
&\qquad\quad \left. + \frac{\Phi\,(\frac{1}{2}-ir)\,q^{(n+1)\,2ir} - \Phi(\frac{1}{2}+ir)\,q^{n(-2ir)}}{q^{2ir}-1}\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{i=1}^{\mu} \int_{\mathscr{S}_n} E_i\left(g, \frac{1}{2}+ir\right) E_i\left(g, \frac{1}{2}-ir\right) dr \\
&= Tr \left\langle \mathscr{E}\left(\cdot, \frac{1}{2}+ir\right), {}^t\mathscr{E}\left(\cdot, \frac{1}{2}-ir\right)\right\rangle_{\mathscr{S}_n} \\
&= q^{a-1}\left((2n+1)\,\mu - \frac{1}{\log q}\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right. \\
&\quad \left. + Tr\,\frac{\Phi\left(\frac{1}{2}-ir\right) q^{(n+1)\,2ir} - \Phi\left(\frac{1}{2}+ir\right) q^{n(-2ir)}}{q^{2ir}-1}\right),
\end{aligned}$$

by using $\varphi(s) = \det\,\Phi(s)$ and

$$\frac{\varphi'}{\varphi}(s) = Tr\,\Phi'(s)\,\Phi^{-1}(s).$$

Now we consider the matrix form integral

$$I(n) := \frac{\log q}{4\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{\Phi\left(\frac{1}{2}-ir\right) q^{(n+1)\,2ir} - \Phi\left(\frac{1}{2}+ir\right) q^{n(-2ir)}}{q^{2ir}-1}\,dr.$$

We put

$$I_1(n) := \frac{\log q}{4\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{\Phi\left(\frac{1}{2}-ir\right) q^{(n+1)\,2ir} - \Phi\left(\frac{1}{2}\right)}{q^{2ir}-1}\,dr,$$

$$I_2(n) := \frac{\log q}{4\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{\Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{1}{2}+ir\right) q^{n(-2ir)}}{q^{2ir}-1}\,dr,$$

then $I(n) = I_1(n) + I_2(n)$. Recalling Theorem 2.2 and noting that $h(r)$ is analytic on $|Im(r)| < 1/2$ by (16), we move the contour in the integration $I_1(n)$ to $Im(r) = b$, where b is positive and sufficiently small. Put

$$C_1 = \left\{ r \in \mathbb{C} \,\middle|\, -\frac{\pi}{\log q} \leqslant Re(r) \leqslant \frac{\pi}{\log q},\ Im(r) = b \right\},$$

then $I_1(n)$ is equal to

$$
\begin{aligned}
I_1(n) &= \frac{\log q}{4\pi} \int_{C_1} h(r) \frac{-\Phi\left(\frac{1}{2}\right)}{q^{2ir}-1}\, dr + O(q^{-2(n+1)b}) \\
&= \frac{\log q}{4\pi} \Phi\left(\frac{1}{2}\right) \int_{C_1} h(r) \left(\sum_{m \geqslant 0} q^{2imr} \right) dr + O(q^{-2(n+1)b}) \\
&= \frac{1}{2} \Phi\left(\frac{1}{2}\right) \left(c(0) + \sum_{m \geqslant 1} c(2m) \right) + o(1) \qquad (n \to \infty),
\end{aligned}
$$

since $|q^{2ir}| < 1$ on C_1. Similarly we move the contour in the integration $I_2(n)$ to $Im(r) = -b$, where b is as above. Put

$$
C_2 = \left\{ r \in \mathbb{C} \,\middle|\, -\frac{\pi}{\log q} \leqslant Re(r) \leqslant \frac{\pi}{\log q},\ Im(r) = -b \right\},
$$

we have

$$
\begin{aligned}
I_2(n) &= \frac{\log q}{4\pi} \int_{C_2} h(r) \frac{\Phi\left(\frac{1}{2}\right)}{q^{2ir}-1}\, dr + O(q^{-2nb}) \\
&= \frac{\log q}{4\pi} \Phi\left(\frac{1}{2}\right) \int_{C_2} h(r) \left(\sum_{m \geqslant 1} q^{2imr} \right) dr + O(q^{-2nb}) \\
&= \frac{1}{2} \Phi\left(\frac{1}{2}\right) \sum_{m \geqslant 1} c(2m) + o(1) \qquad (n \to \infty)
\end{aligned}
$$

since $|q^{2ir}| > 1$ on C_2 and $h(r) = h(-r) (r \in \mathbb{C})$. Therefore we have

$$
I(n) = \Phi\left(\frac{1}{2}\right) \left(\frac{1}{2} c(0) + \sum_{m \geqslant 1} c(2m) \right) + o(1).
$$

Combining the above results, the expression $E(n)$ is computed as

$$
\begin{aligned}
E(n) &= \frac{\log q}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r) \left((2n+1)\mu - \frac{1}{\log q} \frac{\varphi'}{\varphi} \left(\frac{1}{2} + ir \right) \right) dr \\
&\quad + Tr\, \Phi\left(\frac{1}{2}\right) \left(\frac{1}{2} c(0) + \sum_{m \geqslant 1} c(2m) \right) + o(1) \\
&= \mu(2n+1) \frac{1}{2} c(0) - \frac{1}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r) \frac{\varphi'}{\varphi} \left(\frac{1}{2} + ir \right) dr \\
&\quad + Tr\, \Phi\left(\frac{1}{2}\right) \left(\frac{1}{2} c(0) + \sum_{m \geqslant 1} c(2m) \right) + o(1).
\end{aligned}
$$

Hence we finally obtain (42). From (35) and (42), we also obtain (43). ∎

From (32), Proposition 3.1 and Proposition 3.2, we finally obtain explicitly the Selberg trace formula for $\Gamma(A)$.

THEOREM 3.2. *Let q be an odd prime power and $\Gamma = \Gamma(A)$($A \in \mathbb{F}_q[t]$, $\deg(A) = a \geqslant 1$). Assume that a sequence $c(n) \in \mathbb{C}$ ($n \in \mathbb{Z}$) satisfies $c(n) = c(-n)$ and $\sum_{n \geqslant m} q^{n/2} |c(n)| = O(q^{-m})$. Then we have the following formula:*

$$\sum_{n=1}^{M} h(r_n) = C(I) + C(H) + C(P_1) + C(P_2) + C(P_3), \tag{45}$$

where

$$C(I) = vol(\Gamma \backslash X)\, k(0),$$

$$C(H) = \sum_{\{P\}_\Gamma \in \mathfrak{P}_\Gamma} \sum_{l=1}^{\infty} \frac{\deg P}{q^{l \deg P/2}}\, c(l \deg P),$$

$$C(P_1) = \left(\mu - Tr\, \Phi\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\, c(0) + \sum_{m=1}^{\infty} c(2m)\right),$$

$$C(P_2) = \frac{1}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r)\, \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr,$$

$$C(P_3) = -\mu\left(a + \frac{1}{q-1}\right) c(0).$$

Furthermore we will investigate the following integral, which is the contribution of the continuous spectra in the trace formula:

$$\frac{1}{4\pi} \int_{-(\pi/\log q)}^{\pi/\log q} h(r)\, \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr.$$

Since from Theorem 2.2 $\varphi(s)$ is a rational function in q^{2s}, we now put

$$\varphi(s) = c\, \frac{(q^{2s} - qa_1)(q^{2s} - qa_2) \cdots (q^{2s} - qa_m)}{(q^{2s} - qb_1)(q^{2s} - qb_2) \cdots (q^{2s} - qb_m)}, \tag{51}$$

where c is a constant and assume the right hand side is written to be irreducible. Then we have

LEMMA 3.5. *The moduli of a_i, b_j ($i = 1, ..., m; j = 1, ..., n$) are not equal to* 1.

Proof. Recall that $\varphi(s)$ is holomorphic on the line $Re(s) = \frac{1}{2}$. The functional equation $\Phi(\frac{1}{2} + ir)\, \Phi(\frac{1}{2} - ir) = I (r \in \mathbb{R})$ implies that $\varphi(s)$ is non-zero on $Re(s) = 1/2$. Hence we have the assertion. ∎

Lemma 3.6. *Let $\varphi(s)$ be written as* (51). *Then we have*

$$\frac{1}{4\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr$$
$$=\left(\sum_{|a_i|<1}\sum_{l=0}^{\infty} c(2l)\,a_i^l-\sum_{|a_i|>1}\sum_{l=1}^{\infty}\frac{c(2l)}{a_i^l}\right)$$
$$-\left(\sum_{|b_j|<1}\sum_{l=0}^{\infty} c(2l)\,b_j^l-\sum_{|b_j|>1}\sum_{l=1}^{\infty}\frac{c(2l)}{b_j^l}\right),$$

where a_i, b_j are as in (51).

Proof. The equation (51) implies

$$\frac{\varphi'}{\varphi}(s)=\sum_{i=1}^{m}\frac{2(\log q)\,q^{2s}}{q^{2s}-qa_i}-\sum_{j=1}^{n}\frac{2(\log q)\,q^{2s}}{q^{2s}-qb_j},$$

so it suffices to consider the integral

$$\frac{\log q}{2\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{q^{2s}}{q^{2s}-qa}\,dr \qquad \left(s=\frac{1}{2}+ir\right), \tag{52}$$

where $a\in\mathbb{C}$ satisfies $|a|\neq 1$. Change the variable r here to $z=q^{ir}$, then the above integral (52) is given by

$$\frac{1}{2\pi i}\int_{S^1} h(z)\,\frac{z}{z^2-a}\,dz,$$

where $h(r)=\sum_{n=-\infty}^{\infty} c(n)\,z^n$ and $S^1=\{z\in\mathbb{C}\mid |z|=1\}$. By simple calculation it is computed as

$$\begin{cases}\displaystyle\sum_{l=0}^{\infty} c(2l)\,a^l & (|a|<1)\\[2ex] \displaystyle -\sum_{l=1}^{\infty}\frac{c(2l)}{a^l} & (|a|>1).\end{cases}$$

Therefore we have the assertion. ∎

By using the functional equation $\Phi(s)\,\Phi(1-s)=I$, we see easily the following lemma:

Lemma 3.7. *Assume that $\varphi(s)$ is written as* (51). *Then $\{q^{2s}-qa_i \mid a_i\neq 0, i=1, 2, ..., m\}$ and $\{q^{2s}-qb_j \mid b_j\neq 0, j=1, 2, ..., n\}$ are one-to-one correspondent in such a way that the term $q^{2s}-qa(a\neq 0)$ in the numerator corresponds to the term $q^{2s}-qb(b=1/a)$ in the denominator.*

4. SELBERG ZETA FUNCTION

As before let Γ be a principal congruence subgroup $\Gamma(A)(A \in \mathbb{F}_q[t]$, $\deg(A) \geqslant 1)$. In this section we define the Selberg zeta function attached to Γ and obtain its determinant expression in terms of T_Γ. As before denote the set of primitive hyperbolic conjugacy classes of Γ by $\mathfrak{P}_\Gamma$. For $\{P\}_\Gamma \in \mathfrak{P}_\Gamma$, let us put $N(P) = \sup\{|\lambda_i|_\infty^2 \mid \lambda_i$ is an eigenvalue of the matrix $P\}$. In the present case since $\Gamma \subset PGL(2, \mathbb{F}_q[t])$, we see that $N(P) = q^{\deg P}$, where $\deg P$ is as in Lemma 2.1. Then the Selberg zeta function $Z_\Gamma(s)(s \in \mathbb{C})$ attached to Γ is defined by

$$Z_\Gamma(s) := \prod_{\{P\}_\Gamma \in \mathfrak{P}_\Gamma} (1 - N(P)^{-s})^{-1}. \tag{53}$$

We will apply the trace formula in Theorem 3.2 for the study of $Z_\Gamma(s)$. We take the following function $c(n, s)(n \in \mathbb{Z}, s \in \mathbb{C})$ as the test function $c(n)$ in Theorem 3.2:

$$c(n, s) = \begin{cases} -(\log q)\, q^{-|n|\,(s - 1/2)} & (n \neq 0) \\ 0 & (n = 0), \end{cases}$$

where s is fixed with $Re(s) \geqslant 2$. Then for $Re(s) \geqslant 2$ the function $c(n, s)$ satisfies the required conditions of Theorem 3.2. By direct computation its Fourier transform $h(z, s)$ is given by

$$h(z, s) := \sum_{n = -\infty}^{\infty} c(n, s)\, z^n = \frac{d}{ds} \log \frac{1}{1 - \sqrt{q}\,(z + z^{-1})\, q^{-s} + q^{-2s+1}}.$$

In the trace formula (45), $C(I)$ is computed as

$$vol(\Gamma\backslash X)\, k(0) = vol(\Gamma\backslash X)\, \frac{q-1}{2}\, \frac{d}{ds} \log(1 - q^{-2s}).$$

The contribution of the hyperbolic classes $C(H)$ is precisely equal to the logarithmic derivative of the Selberg zeta function $Z_\Gamma(s)$:

$$-\log q \sum_{\{P\}_\Gamma \in \mathfrak{P}_\Gamma} \deg P \sum_{l=1}^{\infty} q^{-sl \deg P} = \frac{d}{ds} \log Z_\Gamma(s).$$

The expression $C(P_1)$ becomes

$$\left(\mu - Tr\,\Phi\left(\frac{1}{2}\right)\right) \sum_{m=1}^{\infty} c(2m, s) = -\frac{\mu - Tr\,\Phi\,(\frac{1}{2})}{2}\, \frac{d}{ds} \log(1 - q^{-2s+1}).$$

As for $C(P_2)$, we have from Lemma 3.6 and Lemma 3.7 that

$$\frac{1}{4\pi}\int_{-(\pi/\log q)}^{\pi/\log q} h(r)\,\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr$$

$$= -2\sum_{|b_j|<1}\sum_{l=1}^{\infty} c(2l,s)\,b_j^l + 2\sum_{|b_j|>1}\sum_{l=1}^{\infty}\frac{c(2l,s)}{b_j^l}$$

$$= \sum_{|b_j|<1}\frac{d}{ds}\log(1-q^{-2s+1}b_j) - \sum_{|b_j|>1}\frac{d}{ds}\log(1-q^{-2s+1}b_j^{-1}),$$

where $\varphi(s)$ is as in (51). Combining the above results, the Selberg trace formula under the function $c(n,s)$ defined by (54) yields the following:

$$\sum_{n=1}^{M}\frac{d}{ds}\log(1-\lambda_n q^{-s}+q^{1-2s})^{-1}$$

$$= vol(\Gamma\backslash X)\,\frac{q-1}{2}\,\frac{d}{ds}\log(1-q^{-2s})$$

$$+\frac{d}{ds}\log Z_\Gamma(s)$$

$$-\frac{\mu - Tr\,\Phi\left(\frac{1}{2}\right)}{2}\,\frac{d}{ds}\log(1-q^{-2s+1})$$

$$+\sum_{|b_j|<1}\frac{d}{ds}\log(1-q^{-2s+1}b_j) - \sum_{|b_j|>1}\frac{d}{ds}\log(1-q^{-2s+1}b_j^{-1}).$$

Since $Z_\Gamma(s)\to 1$ as $Re(s)\to\infty$, this equation implies

$$\prod_{n=1}^{M}(1-\lambda_n q^{-s}+q^{1-2s})^{-1}$$

$$=(1-q^{-2s})^{vol(\Gamma\backslash X)(q-1)/2}$$

$$\times Z_\Gamma(s)$$

$$\times(1-q^{-2s+1})^{-((\mu - Tr\,\Phi(1/2))/2)}$$

$$\times\prod_{|b_j|<1}(1-q^{-2s+1}b_j)\prod_{|b_j|>1}(1-q^{-2s+1}b_j^{-1})^{-1}.$$

Now we define the determinant function for T_Γ by

$$\det(T_\Gamma, s) := \det_D(T_\Gamma, s) \cdot \det_C(T_\Gamma, s), \tag{55}$$

where

$$\det_D(T_\Gamma, s) := \det_D(1 - T_\Gamma\, q^{-s} + q^{1-2s}) = \prod_{n=1}^{M} (1 - \lambda_n q^{-s} + q^{1-2s}),$$

$$\det_C(T_\Gamma, s) := \prod_{|b_j| < 1} (1 - q^{-2s+1} b_j) \prod_{|b_j| > 1} (1 - q^{-2s+1} b_j^{-1})^{-1},$$

and b_j is as in (51). We finally obtain the following:

THEOREM 4.1. *Let q be an odd prime power and $\Gamma = \Gamma(A)$ with* $\deg(A) \geqslant 1$. *Then the Selberg zeta function $Z_\Gamma(s)$ attached to Γ has the following determinant expression*;

$$Z_\Gamma(s)^{-1} = (1 - q^{-2s})^{\chi}\, (1 - q^{-2s+1})^{-\rho} \det(T_\Gamma, s), \tag{56}$$

where $\chi := vol(\Gamma \backslash X)\, \frac{q-1}{2}$, $\rho := \frac{1}{2}\, Tr(I_\mu - \Phi\,(\frac{1}{2}))$ and I_μ is the $\mu \times \mu$-identity matrix.

Remark. If $\varphi(s)$ is written as (51), the factor $q^{2s} - qb_i$ with $|b_i| > 1$ (resp. $|b_i| < 1$) corresponds to the pole of $\varphi(s)$ on $Re(s) > 1/2$ (resp. $Re(s) < 1/2$). Hence the definition of $\det_C(T_\Gamma, s)$ denotes the product over the poles of $\varphi(s)$ (or the zeros of $\varphi(s)$ if we use Lemma 3.7). See also the determinant of the Laplacian with respect to a Riemann surface of finite volume in [E1].

In the present case $\Gamma = \Gamma(A)$, we see ([Na]) that $\varphi(s)$ has no exceptional poles, i.e., $\varphi(s)$ is holomorphic on $Re(s) > 1/2$ except for simple poles at $s = 1 + n\pi i/\log q$ ($n \in \mathbb{Z}$). Namely $\varphi(s)$ has a factor $q^{2s} - q^2$ with multiplicity one. Moreover by using the results in [L1] [M2], $Z_\Gamma(s)$ is computed as follows:

COROLLARY 4.1. *Let q be an odd prime power and $\Gamma = \Gamma(A)(A \in \mathbb{F}_q[t])$ with* $\deg(A) = a \geqslant 1$. *We let $A = A_1^{e_1} A_2^{e_2} \cdots A_2^{e_l}$ be the decomposition of A into distinct irreducible polynomials, where* $\deg(A_i) = a_i$, $\sum_{i=1}^{l} e_i\, a_i = \deg(A)$. *Moreover let* $\det_D^{(1)}(T_\Gamma, s) = \prod_{\lambda_n} (1 - \lambda_n q^{-s} + q^{1-2s})$, *where the product is* *taken over discrete spectra of* T_Γ *except for two trivial eigenvalues* $\pm(q+1)$. *Then we have*

$$Z_\Gamma(s)^{-s} = (1-q^{-2s})^{\chi}\,(1-q^{-2s+1})^{-\rho}$$

$$\times(1-q^{2-2s})\,\det{}^{(1)}_D(T_\Gamma, s)\prod_{|b_j|<1}(1-q^{-2s+1}b_j),$$

where b_j *is as in* (51) *and*

$$\chi = \frac{q^{3a}}{(q+1)(q-1)}\prod_{i=1}^{l}(1-q^{-2a_i}),$$

$$\rho = \frac{1}{2}\left(\frac{q^{2a}}{q-1}\prod_{i=1}^{l}(1-q^{-2a_i}) + q^a\prod_{i=1}^{l}(1+q^{-a_i})\right).$$

In particular, $Z_\Gamma(s)$ *is a rational function in* q^{-s}.

Proof. By the results in [M2, p. 117], the number of cusps $\mu=\mu(\Gamma)$ and the total volume of $\Gamma\backslash X$ are computed as follows:

$$\mu = \frac{q^{2a}}{q-1}\prod_{i=1}^{l}(1-q^{-2a_i}),$$

$$vol(\Gamma\backslash X) = \frac{2q^{3a}}{(q+1)(q-1)^2}\prod_{i=1}^{l}(1-q^{-2a_i}).$$

The function $\varphi_{ij}(s)$ in Fourier series expansion of $E_i(g,s)$ at a cusp κ_j is computed in [L1, p. 240 (2.8)]. Here $\Gamma(A)$ is a normal subgroup of $\Gamma(1)$, so we see that $Tr\,\Phi(1/2)$ is equal to $\mu\cdot\varphi_{11}(1/2)$. Since

$$\varphi_{11}(s) = q^{1-a}\,(q-1)\,q^{a(1-2s)}\,\frac{1-q^{-2s}}{1-q^{2-2s}}\prod_{i=1}^{l}(1-q^{-2a_i s})^{-1},$$

we have

$$Tr\,\Phi\left(\frac{1}{2}\right) = -q^a\prod_{i=1}^{l}(1+q^{-a_i}).$$

The two trivial discrete spectra $\pm(q+1)$ correspond to the pole of $\varphi(s)$ coming from the factor $q^{2s}-q^2$. Combining the above, we have this corollary, since we can check that χ and ρ are integers. ∎

For example we will compute the Selberg zeta function for $\Gamma(t)$. Since the quotient graph $\Gamma(t)\backslash X$ can be taken as the union of the vertex σ_0 together with $q+1$ ends, we see that there are no eigenvalues of

T_Γ except for $\pm(q+1)$ (see also [L1, p. 256]). By using [L1, p. 240 (2.8)], it is computed that the scattering matrix $\Phi_{\Gamma(t)}(s)$ is $(q+1)\times(q+1)$-matrix which entries are $\varphi_{11}(s)=\frac{q(q-1)}{q^{2s}-q^2}$ in the diagonal part and $\varphi_{11}(s)+1$ in the other part. So the scattering determinant $\varphi_{\Gamma(t)}(s) := \det\ \Phi_{\Gamma(t)}(s)$ becomes

$$\varphi_{\Gamma(t)}(s) = -((q+1)\,\varphi_{11}(s)+q) = -\frac{q(q^{2s}-1)}{q^{2s}-q^2}.$$

In Corollary 4.1 we have $\chi=q$ and $\rho=q+1$ for $\Gamma(t)$. Hence we obtain

$$Z_{\Gamma(t)}(s)^{-1} = \frac{(1-q^{-2s})^q\,(1-q^{2-2s})}{(1-q^{1-2s})^{q+1}}.$$

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