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Plancherel measure for $\mathrm{GL}(n, F)$ and $\mathrm{GL}(m, D)$: Explicit formulas and Bernstein decomposition

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Abstract

Let F be a nonarchimedean local field, let D be a division algebra over F, let $\mathrm{GL}(n) = \mathrm{GL}(n, F)$. Let ν denote Plancherel measure for $\mathrm{GL}(n)$. Let Ω be a component in the Bernstein variety $\Omega(\mathrm{GL}(n))$. Then Ω yields its fundamental invariants: the cardinality q of the residue field of F, the sizes $m_1, \ldots, m_t$, exponents $e_1, \ldots, e_t$, torsion numbers $r_1, \ldots, r_t$, formal degrees $d_1, \ldots, d_t$ and conductors $f_{11}, \ldots, f_{tt}$. We provide explicit formulas for the Bernstein component ν_Ω of Plancherel measure in terms of the fundamental invariants. We prove a transfer-of-measure formula for $\mathrm{GL}(n)$ and establish some new formal degree formulas. We derive, via the Jacquet–Langlands correspondence, the explicit Plancherel formula for $\mathrm{GL}(m, D)$.

MSC: primary 22E50; secondary 11F70; 11S40

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1. Introduction

In this article we provide an explicit Plancherel formula for the p-adic group $\mathrm{GL}(n)$. Moreover, we determine explicitly the Bernstein decomposition of Plancherel measure, including all numerical constants.

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Let F be a nonarchimedean local field with ring of integers $\mathfrak{o}_F$, let $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$. We will use the standard normalization of Haar measure on $\mathrm{GL}(n)$ for which the volume of $\mathrm{GL}(n, \mathfrak{o}_F)$ is 1. Plancherel measure ν is then uniquely determined by the equation

$$f(g) = \int \operatorname{trace} \pi(\lambda(g) f^\vee)\, \mathrm{d}\nu(\pi)$$

for all $g \in G$, $f \in \mathcal{C}(G)$, where $f^\vee(g) = f(g^{-1})$.

The Harish-Chandra Plancherel Theorem expresses the Plancherel measure in the following form:

$$\mathrm{d}\nu(\omega) \;=\; c(G|M)^{-2}\, \gamma(G|M)^{-1}\, \mu_{G|M}(\omega)\, d(\omega)\, \mathrm{d}\omega,$$

where M is a Levi subgroup of G, $\omega \in E_2(M)$ the discrete series of M, $c(G|M)$ and $\gamma(G|M)$ are certain constants, $\mu_{G|M}$ is a certain rational function, $d(\omega)$ is the formal degree of ω, and $\mathrm{d}\omega$ is the Harish-Chandra canonical measure.

In this article we determine explicitly

$$c(G|M)^{-2}\gamma(G|M)^{-1}\mu_{G|M}(\omega)\, \mathrm{d}(\omega)\, \mathrm{d}\omega$$

for $\mathrm{GL}(n)$.

The support of Plancherel measure ν admits a Bernstein decomposition [23] and therefore ν admits a canonical decomposition

$$\nu = \bigsqcup \nu_\Omega,$$

where Ω is a component in the Bernstein variety $\Omega(G)$. We determine explicitly the Bernstein component ν_Ω for $\mathrm{GL}(n)$.

We can think of Ω as a vector of irreducible supercuspidal representations of smaller general linear groups. If the vector is

$$(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_t)$$

with σ_i repeated e_i times, $1 \leqslant i \leqslant t$, and $\sigma_1, \ldots, \sigma_t$ pairwise distinct (after unramified twist) then we say that Ω has *exponents* $e_1, \ldots, e_t$.

Each representation σ_i of $\mathrm{GL}(m_i)$ has a *torsion number*: the order of the cyclic group of all those unramified characters η for which $\sigma_i \otimes \eta \cong \sigma_i$. The torsion number of σ_i will be denoted r_i.

We may choose each representation σ_i of $\mathrm{GL}(m_i)$ to be unitary: in that case σ_i has a *formal degree* $d_i = d(\sigma_i)$. We have $0 < d_i < \infty$.

We will denote by $f_{ij} = f(\sigma_i^\vee \times \sigma_j)$ the conductor of the pair $\sigma_i^\vee \times \sigma_j$. An explicit conductor formula is obtained in the article by Bushnell et al. [9].

In this way, the Bernstein component Ω yields up the following *fundamental invariants*:

- the cardinality q of the residue field of F;
- the sizes $m_1, m_2, \ldots, m_t$ of the smaller general linear groups;
- the exponents $e_1, e_2, \ldots, e_t$;
- the torsion numbers $r_1, r_2, \ldots, r_t$;
- the formal degrees $d_1, d_2, \ldots, d_t$;
- the conductors for pairs $f_{11}, f_{12}, \ldots, f_{tt}$.

Our Plancherel formulas are built from precisely these numerical invariants.

If Ω has the single exponent e, then the fundamental invariants yielded up by Ω are q, m, e, r, d, f. The component Ω determines a representation in the discrete series of $\mathrm{GL}(n)$, namely the generalized Steinberg representation $\mathrm{St}(\sigma, e)$. The formal degree of $\pi = \mathrm{St}(\sigma, e)$ is given by the following new formula, which is intricate, but depends only on the fundamental invariants of Ω, in line with our general philosophy:

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}e} \cdot q^{(e^2-e)(f(\sigma^\vee \times \sigma)+r-2m^2)/2} \cdot \frac{(q^r-1)^e}{q^{er}-1} \cdot \frac{|\mathrm{GL}(em, q)|}{|\mathrm{GL}(m, q)|^e}.$$

In Section 2, we give a précis of the background material which we need, following the recent article of Waldspurger [34].

The Langlands–Shahidi formula gives the rational function $\mu_{G|M}$ as a ratio of certain L-factors and ε-factors [26]. In Sections 3 and 4, we compute explicitly the expression

$$c(G|M)^{-2}\gamma(G|M)^{-1}\mu_{G|M}(\omega)\,\mathrm{d}\omega$$

when M is a maximal parabolic. The resulting formula is stated in Theorem 4.4: in this formula we correct certain misprints in [27, pp. 292–293].

In Section 5, we compute the Plancherel density $\mu_{G|M}$ in the general case by using the Harish-Chandra product formula and we give the explicit Bernstein decomposition of Plancherel measure.

As a special case, we derive the explicit Plancherel formula for the (extended) affine Hecke algebra $\mathcal{H}(n, q)$.

We have, in effect, extended the classical formula of Macdonald [19,20, Theorem 5.1.2] from the spherical component of $\mathrm{GL}(n)$ to the whole of the tempered dual.

The Plancherel formulas for $\mathrm{GL}(n, F)$ and $\mathrm{GL}(m, D)$ are dominated by *repeating patterns*, which we now attempt to explain. The repeating patterns are expressed by transfer-of-measure theorems, of which the first is as follows. With $j = 1, 2$, let F_j be a nonarchimedean local field and let Ω_j be a component in the Bernstein variety of $\mathrm{GL}(n_j, F_j)$. Let $\nu^{(j)}$ denote the Plancherel measure of $\mathrm{GL}(n_j, F_j)$. If Ω_1, Ω_2 share the same fundamental invariants, then

$$\nu^{(1)}_{\Omega_1} = \nu^{(2)}_{\Omega_2}.$$

The next transfer-of-measure theorem is more surprising. Let Ω be a component in the Bernstein variety of $\mathrm{GL}(n, F)$, and let ν be Plancherel measure. Let Ω have the fundamental invariants (q, m, e, r, d, f). Let K/F be an extension field with $q_K = q^r$. Let $G_0 := \mathrm{GL}(e, K)$, let Ω_0 be a component in the Bernstein variety of G_0, and let $\nu^{(0)}$ be Plancherel measure. If Ω_0 has fundamental invariants $(q^r, 1, e, 1, 1, 1)$ then ν_Ω and $\nu^{(0)}_{\Omega_0}$ are *proportional*, i.e.,

$$\nu_\Omega = \kappa \cdot \nu^{(0)}_{\Omega_0},$$

where $\kappa = \kappa(q, m, e, r, d, f)$. This phenomenon was first noted by Bushnell et al. [10, Theorem 4.1] working in the context of types and Hilbert algebras. We reconcile our result for $\mathrm{GL}(n)$ with (a special case of) their result by proving that

$$\kappa(q, m, e, r, d, f) = \mathrm{vol}(J)^{-1} \cdot \mathrm{vol}(I_0) \cdot \dim(\lambda),$$

where (J, λ) is an Ω-type, I_0 is an Iwahori subgroup of G_0: for this result, see Theorem 6.12. Theorem 5.7, which in essence is the Harish-Chandra product formula, then allows one to compute the Plancherel measure ν_Ω for any component Ω.

Using the explicit value for the formal degree of any representation in the discrete series of G previously obtained by Silberger and Zink, we show that the comparison formula between formal degrees, proved by Corwin, Moy and Sally in the tame case [14], is valid in general.

In the last section of the paper we consider the case of a group $\mathrm{GL}(n', D)$, where D is a central division algebra of index d over F. We extend the transfer-of-measure result of Arthur and Clozel [1, pp. 88–90] to the case when F is of positive characteristic, by using results of Badulescu.

Let $G' = \mathrm{GL}(n', D)$, $G = \mathrm{GL}(n, F)$ with $n = \mathrm{d}n'$. Let ν', ν denote the Plancherel measure for G', G, each with the standard normalization of Haar measure on G', G. Let $\mathrm{JL} : E_2(G') \to E_2(G)$ denote the Jacquet–Langlands correspondence. Then we have

$$\mathrm{d}\nu'(\omega') = \lambda(D/F) \cdot \mathrm{d}\nu(\mathrm{JL}(\omega')),$$

where

$$\lambda(D/F) = \prod (q^m - 1)^{-1}$$

the product taken over all m such that $1 \leqslant m \leqslant n - 1, m \neq 0 \bmod d$.

For example, let $G' = \mathrm{GL}(3, D), G = \mathrm{GL}(6, F)$ with D of index 2. Then we have

$$\mathrm{d}\nu'(\omega') = (q - 1)^{-1}(q^3 - 1)^{-1}(q^5 - 1)^{-1} \cdot \mathrm{d}\nu(\mathrm{JL}(\omega')).$$

Our proof of this is in local harmonic analysis, cf. [1, pp. 88–90].

Historical note. The Harish-Chandra Plancherel Theorem, and the Product Theorem for Plancherel Measure, were published posthumously in his collected papers in 1984, see [16]. The theorems were stated without proof (although Harish-Chandra had apparently written out the proofs). At this point, we quote from Silberger's article [30], published in 1996:

> In [16] Harish-Chandra has summarized the theory underlying the Plancherel formula for G and sketched a proof of the Plancherel theorem. To complete this sketch it seems to this writer that details need to be supplied justifying only one assertion of [16], namely Theorem 11. Every other assertion in this paper can be readily proved either by using prior published work of Harish-Chandra or the present author's notes on Harish-Chandra's lectures.

For Silberger's Notes, published in 1979, see [29]. Complete and detailed proofs were finally published by Waldspurger in 2003, see [34, V.2.1, VIII.1.1]. None of these sources contains any explicit computations for $\mathrm{GL}(n)$.

Some of the results in this article have been announced in [2].

2. The Plancherel formula after Harish-Chandra

We shall follow very closely the notation and terminology in [34].

Let $\mathcal{K} = \mathrm{GL}(n, \mathfrak{o}_F)$. Let H be a closed subgroup of $G = \mathrm{GL}(n, F)$. We use the *standard* normalization of Haar measures, following [34, I.1, p. 240]. Then Haar measure μ_H on H is chosen so that $\mu_H(H \cap \mathcal{K}) = 1$. If $Z = A_G$ is the centre of G then we have $\mu_Z(Z \cap \mathcal{K}) = 1$. If $H = G$ then Haar measure $\mu = \mu_G$ is normalized so that the volume of $\mathcal{K}$ is 1.

Denote by Θ the set of pairs $(\mathcal{O}, P = MU)$ where P is a semi-standard parabolic subgroup of G and $\mathcal{O} \subset E_2(M)$ is an orbit under the action of $\mathrm{Im}\, X(M)$. (Here $E_2(M)$ is the set of equivalence classes of the discrete series of the Levi subgroup M, and $\mathrm{Im}\, X(M)$ is the group of the unitary unramified characters of M.)

Two elements $(\mathcal{O}, P = MU)$ and $(\mathcal{O}', P' = M'U')$ are *associated* if there exists $s \in W^G$ such that $s \cdot M = M'$, $s\mathcal{O} = \mathcal{O}'$. We fix a set Θ/assoc of representatives in Θ for the classes of association. For $(\mathcal{O}, P = MU) \in \Theta$, we set $W(G|M) = \{s \in W^G : s \cdot M = M\}/W^M$, and

$$\mathrm{Stab}(\mathcal{O}, M) = \{s \in W(G|M) : s\mathcal{O} = \mathcal{O}\}.$$

Let $\mathcal{C}(G)$ denote the Harish-Chandra Schwartz space of G and let $I_P^G\omega$ denote the normalized induced representation from ω. Let $f \in \mathcal{C}(G)$, $\omega \in E_2(M)$. We will write

$$\pi = I_P^G\omega, \quad \pi(f) = \int f(g)\pi(g)\,\mathrm{d}g, \quad \theta_\omega^G(f) = \mathrm{trace}\,\pi(f).$$

Theorem 2.1 (*The Plancherel formula [34, VIII.1.1]*). *For each $f \in \mathcal{C}(G)$ and each $g \in G$ we have*

$$f(g) = \sum c(G|M)^{-2}\gamma(G|M)^{-1}|\mathrm{Stab}(\mathcal{O}, M)|^{-1} \int_{\mathcal{O}} \mu_{G|M}(\omega)d(\omega)\theta_\omega^G(\lambda(g)f^\vee)\,\mathrm{d}\omega$$

where the sum is over all the pairs $(\mathcal{O}, P = MU) \in \Theta/\mathrm{assoc}$.

Note that

$$\mu_{G|M}(\omega)\cdot c(G|M)^{-2}\cdot\gamma(G|M)^{-1} = \gamma(G|M)\,\cdot\, j(\omega)^{-1}, \tag{1}$$

where j denotes the composition of intertwining operators defined in [34, IV.3 (2)].

The map

$$(\mathcal{O}, P = MU) \to \mathrm{Irr}^{\mathrm{t}}(G),\ \omega \mapsto I_P^G\omega$$

determines a *bijection*

$$\bigsqcup(\mathcal{O}, P = MU)/\mathrm{Stab}(\mathcal{O}, M) \longrightarrow \mathrm{Irr}^{\mathrm{t}}(G).$$

The tempered dual $\mathrm{Irr}^{\mathrm{t}}(G)$ acquires, by transport of structure, the structure of *disjoint union of countably many compact orbifolds*.

According to [34, V.2.1], the function $\mu_{G|M}$ is a rational function on $\mathcal{O}$. We have $\mu_{G|M}(\omega)\geqslant 0$ and $\mu_{G|M}(s\omega) = \mu(\omega)$ for each $s \in W^G, \omega \in \mathcal{O}$. This invariance property implies that μ *descends* to a function on the orbifold $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$. We can view μ either as an *invariant* function on the orbit $\mathcal{O}$ or as a function on the orbifold $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$.

We now define the *canonical measure* $\mathrm{d}\omega$. The map $\mathrm{Im}\,X(M) \to \mathcal{O}$ sends $\chi \mapsto \omega\otimes\chi$; the map $\mathrm{Im}\,X(M) \to \mathrm{Im}\,X(A_M)$ is determined by restriction. Let $(Y_i, \mathcal{B}_i, \mu_i)$ be finite measure spaces with $i = 1, 2$ and let $f : Y_1 \to Y_2$ be a measurable map. Then μ_1 is the *pull-back* of μ_2 if $\mu_1(f^{-1}E) = \mu_2(E)$ for all $E \in \mathcal{B}_2$. This surely is the meaning of *préserve localement les mesures* in [34, pp. 239, 302].

The compact group $\mathrm{Im}\,X(A_M)$ is assigned the Haar measure of total mass 1. Choose Haar measure on the compact orbit $\mathcal{O}$. Now $\mathrm{Im}\,X(M)$ admits two pull-back measures:

$$\mathrm{Im}\,X(A_M) \leftarrow \mathrm{Im}\,X(M) \to \mathcal{O}.$$

These must coincide: this fixes the Haar measure $\mathrm{d}\omega$ on $\mathcal{O}$, see [34, pp. 239, 302].

Let E be a Borel set in $\mathcal{O}$ which is also a fundamental domain for the action of $\mathrm{Stab}(\mathcal{O}, M)$ on $\mathcal{O}$. Since $F(\omega) := \mu_{G|M}(\omega)d(\omega)\,\theta_\omega^G(\lambda(g)f^\vee)$ is $\mathrm{Stab}(\mathcal{O}, M)$-invariant, we have

$$|\mathrm{Stab}(\mathcal{O}, M)|^{-1} \cdot \int_{\mathcal{O}} F(\omega)\, \mathrm{d}\omega = \int_E F(\omega)\, \mathrm{d}\omega.$$

The *Plancherel density*, with respect to the canonical measure $\mathrm{d}\omega$, is therefore

$$c(G|M)^2 \cdot \gamma(G|M)^{-1} \cdot \mu_{G|M}(\omega)\, d(\omega),$$

where $d(\omega)$ is the formal degree of ω. It is precisely this expression which we will compute explicitly for $\mathrm{GL}(n)$. To this end, we will use the following result.

Theorem 2.2 (*The Product formula [34, V.2.1]*). *With* $M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k) \subset \mathrm{GL}(n)$ *and* $\omega = \omega_1 \otimes \cdots \otimes \omega_k$ *we have*

$$\mu_{G|M}(\omega) = \prod_{1 \leqslant j < i \leqslant k} \mu_{\mathrm{GL}(n_i+n_j)|\mathrm{GL}(n_i)\times\mathrm{GL}(n_j)}(\omega_i \otimes \omega_j).$$

The Plancherel measure v is determined by the equation

$$f(g) = \int \mathrm{trace}\, \pi(\lambda(g) f^{\vee})\, \mathrm{d}v(\pi)$$

for all $f \in \mathcal{C}(G)$.

Theorem 2.3 (*The Bernstein decomposition [23]*). *The Plancherel measure v admits a canonical Bernstein decomposition*

$$v = \bigsqcup v_\Omega,$$

where Ω is a component in the Bernstein variety $\Omega(G)$. The domain of each v_Ω is a finite union of orbifolds of the form $\mathcal{O}/\mathrm{Stab}(\mathcal{O}, M)$ and is precisely a single extended quotient.

We will use Theorem 2.3 to compute the Plancherel measure of the (extended) affine Hecke algebra $\mathcal{H}(n, q)$ (see Remark 5.6).

3. Calculation of the γ factors

Theorem 3.1. *We have*

$$\gamma(G|M) = q^{-2\sum_{1\leqslant i<j\leqslant k} n_i n_j}\, \frac{|\mathrm{GL}(n, q)|}{|\mathrm{GL}(n_1, q)| \times \cdots \times |\mathrm{GL}(n_k, q)|}.$$

Proof. By applying the formula given in [34, p. 241, l.7] to the group $H = I_n + \varpi M(n, \mathfrak{o}_F)$, we obtain

$$\gamma(G|M) = q^{-2R} \frac{\mu(M \cap H)}{\mu(H)},$$

with $R = \Sigma(G)^+ - \Sigma(M)^+$, where $\Sigma(G)^+$ (resp. $\Sigma(M)^+$) denotes the set of positive roots in G (resp. M). We have

$$R = \sum_{1 \leqslant i < j \leqslant k} n_i n_j.$$

On the other hand, since the Haar measure on G is normalized so that the volume of $\mathcal{K}$ is 1, it follows from the exact sequence

$$1 \to H \to \mathcal{K} \to \mathrm{GL}(n, q),$$

that

$$\mu(H) = |\mathrm{GL}(n, q)|^{-1} \quad \text{and} \quad \mu(H \cap M) = |\mathrm{GL}(n_1, q)|^{-1} \times \cdots \times |\mathrm{GL}(n_k, q)|^{-1}. \qquad \square$$

Remark 3.2. Observe that $2 \sum_{1 \leqslant i < j \leqslant k} n_i n_j$ equals the length of the element $w = w_M w_{\mathrm{GL}(n)}$, where w_M (resp. $w_{\mathrm{GL}(n)}$) denotes the longest element in the Weyl group of M (resp. $\mathrm{GL}(n)$). Let $P_{S_n}(X)$ denote the Poincaré polynomial of the Coxeter group S_n. Then, using the fact that (see for instance [21, (2.6)])

$$P_{S_n}(q^{-1}) = \frac{|\mathrm{GL}(n, q)|}{q^{n^2-n}(q-1)^n}, \tag{2}$$

we obtain from Theorem 3.1

$$\gamma(G|M) = \frac{P_{S_n}(q^{-1})}{P_{S_{n_1}}(q^{-1}) \times \cdots \times P_{S_{n_k}}(q^{-1})}. \tag{3}$$

This gives the following expression for the c-function defined in [34, I.1]:

$$c(G|M) = \frac{\prod_{1 \leqslant i < j \leqslant k} P_{S_{n_i+n_j}}(q^{-1})}{P_{S_n}(q^{-1}) \cdot \prod_{i=1}^{k} (P_{S_{n_i}}(q^{-1}))^{k-2}}. \tag{4}$$

4. The Langlands–Shahidi formula

Let ϖ denote a fixed uniformizer. We will choose a continuous additive character Ψ such that the conductor of Ψ is $\mathfrak{o}_F$. Note that Shahidi uses precisely this normalization in [25]. We shall need the L-factor $L(s, \pi_1 \times \pi_2)$ and the ε-factor $\varepsilon(s, \pi_1 \times \pi_2, \Psi)$ for pairs, where s denotes a complex variable (see [18,26]). We define the conductor $f(\pi_1 \times \pi_2)$ (see [9]) and the γ-factor $\gamma(s, \pi_1 \times \pi_2, \Psi)$ (see [18, p. 374]) for pairs as

$$\varepsilon(0, \pi_1 \times \pi_2, \Psi) = q^{f(\pi_1 \times \pi_2)} \cdot \varepsilon(1, \pi_1 \times \pi_2, \Psi), \tag{5}$$

$$\gamma(s, \pi_1 \times \pi_2, \Psi) = \varepsilon(s, \pi_1 \times \pi_2, \Psi) \cdot L(1 - s, \pi_1^\vee \times \pi_2^\vee)/L(s, \pi_1 \times \pi_2). \tag{6}$$

We assume in this section that P is the upper block triangular maximal parabolic subgroup of G with Levi subgroup $M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$. We have the Langlands–Shahidi formula for the Harish-Chandra μ-function, see [25, §6] or [27, §7]:

$$\mu_{G|M}(\omega_1 \otimes \omega_2) = \gamma(G|M)^2 \cdot \frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)}. \tag{7}$$

It is useful to note that

$$\frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)} = q^{f(\omega_1^\vee \times \omega_2)} \cdot L'', \tag{8}$$

where

$$L'' = \frac{L(1, \omega_1 \times \omega_2^\vee) L(1, \omega_1^\vee \times \omega_2)}{L(0, \omega_1 \times \omega_2^\vee) L(0, \omega_1^\vee \times \omega_2)}. \tag{9}$$

For any smooth representation π of G and any quasi-character χ, we denote by $\chi\pi$ the twist of π by χ:

$$\chi\pi := (\chi \circ \det) \otimes \pi.$$

If σ_1 (resp. σ_2) is an irreducible supercuspidal representation of $\mathrm{GL}(m_1)$ (resp. $\mathrm{GL}(m_2)$), then we have $L(s, \sigma_1 \times \sigma_2^\vee) = 1$ unless $\sigma_1 \cong \chi\sigma_2$ with χ an unramified quasi-character of $F^\times$.

The next formula is from [18, Proposition 8.1] or [27, p. 292].

Lemma 4.1. Let σ_2 *have torsion number* r *and let* $\sigma_1 \cong \chi\sigma_2$ *with* χ *an unramified quasi-character such that* $\chi(\varpi) = \zeta$. *Then we have*

$$L(s, \sigma_1 \times \sigma_2^\vee) = (1 - \zeta^{-r} q^{-rs})^{-1}.$$

Let χ_1, χ_2 be unramified (unitary) characters of $F^\times$. The group of unramified (unitary) characters $\operatorname{Im} X(M)$ of M has, via the map

$$(\chi_1 \circ \det) \otimes (\chi_2 \circ \det) \mapsto (\chi_1(\varpi), \chi_2(\varpi))$$

the structure of the compact torus $\mathbb{T}^2$.

Let π_i be in the discrete series of $\mathrm{GL}(n_i)$ with $i = 1, 2$, and let π_i have torsion number r. Consider now the *orbit* $\operatorname{Im} X(M) \cdot (\pi_1 \otimes \pi_2)$ in the Harish-Chandra parameter space $\Omega^{\mathrm{t}}(G)$. The action of $\operatorname{Im} X(M)$ creates a short exact sequence

$$1 \to \mathcal{G} \to \mathbb{T}^2 \to \mathbb{T}^2 \to 1$$

with

$$\mathbb{T}^2 \to \mathbb{T}^2, \ (\zeta_1, \zeta_2) \mapsto (\zeta_1^r, \zeta_2^r).$$

The finite group $\mathcal{G}$ is precisely the finite group in [5, Lemma 25] and is the product of cyclic groups:

$$\mathcal{G} = \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}.$$

We will write $z_1 = \zeta_1^r, z_2 = \zeta_2^r$ so that z_1, z_2 are precisely the co-ordinates of a point in the orbit.

Remark 4.2. We recall the following facts about the discrete series of $\mathrm{GL}(n)$. Let π_1 and π_2 be two discrete series representations of $\mathrm{GL}(n_1)$ and $\mathrm{GL}(n_2)$, respectively. By Zelevinsky [35], there exist two pairs of integers (m_1, l_1) and (m_2, l_2) and two irreducible unitary supercuspidal representations σ_1 and σ_2 of $\mathrm{GL}(m_1)$ and $\mathrm{GL}(m_2)$, respectively, such that, for $i = 1, 2$, we have $l_i m_i = n_i$ and the representation π_i is the unique irreducible quotient associated to the Zelevinsky segment

$$\{|\det|^{-g_i}\sigma_i, |\det|^{-g_i+1}\sigma_i, \ldots, |\det|^{g_i-1}\sigma_i, |\det|^{g_i}\sigma_i\},$$

where $2g_i + 1 = l_i$. We will follow the notation in [1, p. 61] and write

$$\pi_i = \mathrm{St}(\sigma_i, l_i).$$

So π_i is a *generalized Steinberg representation*. We observe that

$$\chi\pi_i = \mathrm{St}(\chi\sigma_i, l_i).$$

It follows that the torsion numbers of π_i and σ_i are equal.

Theorem 4.3. *Let σ_1, σ_2 be irreducible unitary supercuspidal representations of* $\mathrm{GL}(m_1), \mathrm{GL}(m_2)$. *Let π_1, π_2 be discrete series representations of* $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$ *such that $\pi_i = \mathrm{St}(\sigma_i, l_i)$. Let χ_1, χ_2 be unramified characters. If $\sigma_1 \neq \chi\sigma_2$ for any unramified quasi-character χ of $F^\times$ then, as a function on the compact torus $\mathbb{T}^2$, $\mu_{G|M}(\chi_1\pi_1 \otimes \chi_2\pi_2)$ is constant: we have*

$$\mu_{G|M}(\chi_1\pi_1 \otimes \chi_2\pi_2) = \gamma(G|M)^2 \cdot q^{l_1 l_2 f(\sigma_1^\vee \times \sigma_2)}$$

We also have

$$f(\pi_1^\vee \times \pi_2) = l_1 l_2 f(\sigma_1^\vee \times \sigma_2).$$

Proof. Let $\omega_i = \chi_i\pi_i$ and $\tau_i = \chi_i\sigma_i$ for $i = 1, 2$. We will use the multiplicative property of the γ-factors. From [17, p. 254] or [18, Theorem 3.1], we have, with $b = g_1 + g_2$,

$$\gamma(s, \omega_1^\vee \times \omega_2, \Psi) = \prod_{i=0}^{l_1-1} \prod_{j=0}^{l_2-1} \gamma(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi).$$

On the other hand, $\gamma(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)$ equals

$$\varepsilon(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi) \cdot \frac{L(1-s, |\ |^{-i-j+b}\tau_1 \times \tau_2^\vee)}{L(s, |\ |^{i+j-b}\tau_1^\vee \times \tau_2)}.$$

Since

$$\frac{\varepsilon(0, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)}{\varepsilon(1, |\ |^{i+j-b}\tau_1^\vee \times \tau_2, \Psi)} = q^{f(|\ |^{i+j-b}\tau_1^\vee \times \tau_2)} = q^{f(\tau_1^\vee \times \tau_2)} = q^{f(\sigma_1^\vee \times \sigma_2)},$$

it follows that

$$\frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)} = q^{l_1 l_2 \cdot f(\sigma_1^\vee \times \sigma_2)} \cdot L', \tag{10}$$

with

$$L' = \prod_{i=0}^{l_1-1} \prod_{j=0}^{l_2-1} \frac{L(1, |\ |^{-i-j+b}\tau_1 \times \tau_2^\vee)}{L(0, |\ |^{i+j-b}\tau_1^\vee \times \tau_2)} \cdot \frac{L(1, |\ |^{i+j-b}\tau_1^\vee \times \tau_2)}{L(0, |\ |^{-i-j+b}\tau_1 \times \tau_2^\vee)}. \tag{11}$$

Since $\sigma_1 \neq \chi\sigma_2$, then $\tau_1 \neq \chi\tau_2$ for any unramified quasi-character χ, and $L' = 1$.

The multiplicative property of the L-factors [18, Theorem 8.2] implies that $L'' = 1$. Therefore, by (8) we have

$$\frac{\gamma(0, \omega_1^\vee \times \omega_2, \Psi)}{\gamma(1, \omega_1^\vee \times \omega_2, \Psi)} = q^{f(\omega_1^\vee \times \omega_2)} \tag{12}$$

Then the results follow from the Langlands–Shahidi formula (7), and from (10) and (12). $\square$

Theorem 4.4. *Let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ with torsion number r. Let π_1, π_2 be discrete series representations of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$, with $n_i = l_i m$, such that $\pi_i = \mathrm{St}(\sigma, l_i)$. Let χ_1, χ_2 be unramified characters. Let $\chi_i(\varpi) = \zeta_i$, $z_i = \zeta_i^r$, $i = 1, 2$. Then, as a function on the compact torus $\mathbb{T}^2$ with co-ordinates (z_1, z_2), we have*

$$\mu_{G|M}(\chi_1\pi_1 \otimes \chi_2\pi_2) = \gamma(G|M)^2 \cdot q^{l_1 l_2 f(\sigma^\vee \times \sigma)} \cdot \prod \left| \frac{1 - z_2 z_1^{-1} q^{gr}}{1 - z_2 z_1^{-1} q^{-(g+1)r}} \right|^2,$$

where the product is over those g for which $|g_1 - g_2| \leqslant g \leqslant g_1 + g_2$. Note that $g_1 - g_2$ and $g_1 + g_2$ can both be half integers.

We also have

$$f(\pi_1^\vee \times \pi_2) = l_1 l_2 f(\sigma^\vee \times \sigma) + r(l_1 l_2 - \min(l_1, l_2)).$$

Proof. Let $\tau_i = \chi_i \sigma$. We have

$$L' = \prod_{i=0}^{l_1-1} \prod_{j=0}^{l_2-1} \frac{L(1 - i - j + b, \tau_1 \times \tau_2^\vee)}{L(i + j - b, \tau_1^\vee \times \tau_2)} \cdot \frac{L(i + j + 1 - b, \tau_1^\vee \times \tau_2)}{L(-i - j + b, \tau_1 \times \tau_2^\vee)},$$

where L' is defined by (11).

Now we delve into the combinatorics. To this end, we make a change of variable, and a change of notation.

Let $\lambda(s) = L(s, \tau_1^\vee \times \tau_2)$, $\lambda^*(s) = L(s, \tau_1 \times \tau_2^\vee)$. Note that, for all $s \in \mathbb{R}$, $\lambda^*(s)$ is the complex conjugate of $\lambda(s)$. Let now $k = i + j - b$. We have

$$L' = \prod_{i=0}^{l_1-1} \prod_{j=0}^{l_2-1} \frac{\lambda^*(1 - k)}{\lambda(k)} \cdot \frac{\lambda(1 + k)}{\lambda^*(-k)}.$$

We now define the function

$$a : \{-b, -b + 1, \ldots, b - 1, b\} \longrightarrow \{1, 2, 3, \ldots, \min(l_1, l_2)\}$$

as follows:

$$a(k) = \sharp\{(i, j) : k = i + j - b, 0 \leqslant i \leqslant l_1 - 1, 0 \leqslant j \leqslant l_2 - 1\}.$$

Note that the function a is even: $a(-k) = a(k)$. It first increases, then is constant with its maximum value $\min(l_1, l_2)$, then decreases. Quite specifically, we have

- $a(-b) = 1$,
- $-b \leqslant k < -|g_1 - g_2| \Rightarrow a(k+1) - a(k) = 1$,
- $a(-|g_1 - g_2|) = \min(l_1, l_2)$,
- $-|g_1 - g_2| \leqslant k < |g_1 - g_2| \Rightarrow a(k+1) = a(k)$,
- $a(|g_1 - g_2|) = \min(l_1, l_2)$,
- $|g_1 - g_2| \leqslant k < b \Rightarrow a(k+1) - a(k) = -1$,
- $a(b) = 1$.

We have

$$\begin{aligned} L' &= \prod_{k=-b}^{b} \frac{\lambda^*(1-k)^{a(k)}}{\lambda(k)^{a(k)}} \cdot \frac{\lambda(1+k)^{a(k)}}{\lambda^*(-k)^{a(k)}} \\ &= \prod_{k=-b}^{b} \frac{\lambda^*(1+k)^{a(k)}}{\lambda(k)^{a(k)}} \cdot \frac{\lambda(1+k)^{a(k)}}{\lambda^*(k)^{a(k)}} \\ &= \prod_{k=-b}^{b} \left| \frac{\lambda(1+k)^{a(k)}}{\lambda(k)^{a(k)}} \right|^2 . \end{aligned} \tag{13}$$

We also have, setting $a(1+b) = 0$,

$$\begin{aligned} \prod_{k=-b}^{b} \frac{\lambda(1+k)^{a(k)}}{\lambda(k)^{a(k)}} &= \frac{1}{\lambda(-b)} \cdot \prod_{k=-b}^{b} \frac{\lambda(1+k)^{a(k)}}{\lambda(1+k)^{a(1+k)}} \\ &= \frac{1}{\lambda(-b)} \prod_{k=-b}^{-|g_1-g_2|-1} \frac{1}{\lambda(k+1)} \cdot \prod_{k=|g_1-g_2|}^{b} \lambda(1+k) \\ &= \frac{\lambda(1+b)}{\lambda(-b)} \cdots \frac{\lambda(1+|g_1-g_2|)}{\lambda(-|g_1-g_2|)} \\ &= \prod_{g=|g_1-g_2|}^{g_1+g_2} \frac{\lambda(1+g)}{\lambda(-g)}. \end{aligned} \tag{14}$$

Note that $\tau_2 = \chi\tau_1$ where $\chi(\varpi) = \zeta_2\zeta_1^{-1}$. Therefore $\chi(\varpi)^{-r} = z_1 z_2^{-1}$. The first result now follows immediately from Lemma 4.1, since

$$\lambda(g) = L(g, \tau_1^\vee \times \tau_2) = L(g, \tau_2 \times \tau_1^\vee) = (1 - z_1 z_2^{-1} q^{-gr})^{-1}.$$

Note also that $|1 - z_2 z_1^{-1} q^{-gr}| = |1 - z_1 z_2^{-1} q^{-gr}|$ since $z_2 z_1^{-1}, z_1 z_2^{-1}$ are complex conjugates.

In addition we have

$$|1 - z_2 z_1^{-1} q^{gr}|^2 = |q^{gr} - z_2 z_1^{-1}|^2 = q^{2gr} |1 - z_2 z_1^{-1} q^{-gr}|^2$$

and so we have

$$\left| \frac{\lambda(g)}{\lambda(-g)} \right|^2 = q^{2gr}.$$

The multiplicative property of the L-factors [18, Theorem 8.2] leads to the equation

$$L'' = \prod_{g=|g_1-g_2|}^{g_1+g_2} \left| \frac{\lambda(1+g)}{\lambda(g)} \right|^2 .$$

Therefore, we have

$$\begin{aligned} L'/L'' &= \prod_{g=|g_1-g_2|}^{g_1+g_2} \left| \frac{\lambda(g)}{\lambda(-g)} \right|^2 \\ &= \prod_{g=|g_1-g_2|}^{g_1+g_2} q^{2rg} \\ &= q^{r(l_1 l_2 - \min(l_1, l_2))} \end{aligned} \tag{15}$$

thanks to the identity

$$2|g_1 - g_2| + \cdots + 2(g_1 + g_2) = l_1 l_2 - \min(l_1, l_2)$$

which follows from the classic identity

$$2|g_1 - g_2| + 1 + \cdots + 2(g_1 + g_2) + 1 = l_1 l_2.$$

Since

$$\frac{\gamma(1,\omega_1^\vee\times\omega_2,\psi_F)}{\gamma(0,\omega_1^\vee\times\omega_2,\psi_F)}=q^{f(\omega_1^\vee\times\omega_2)}\cdot L''=q^{l_1l_2f(\sigma^\vee\times\sigma)}\cdot L'$$

we have

$$q^{f(\omega_1^\vee\times\omega_2)}=q^{l_1l_2f(\sigma^\vee\times\sigma)}\cdot L'/L''=q^{l_1l_2f(\sigma^\vee\times\sigma)}q^{r(l_1l_2-\min(l_1,l_2))}$$

and we conclude that

$$f(\pi_1^\vee\times\pi_2)=l_1l_2f(\sigma^\vee\times\sigma)+r(l_1l_2-\min(l_1,l_2)). \qquad \square$$

The above formulas are invariant under the map $(z_1,z_2)\mapsto(\lambda z_1,\lambda z_2)$ with λ a complex number of modulus 1, and under the map $(z_1,z_2)\mapsto(z_2,z_1)$. In Section 6 of the paper we shall interpret q^r as the cardinality q_K of the residue field of a canonical extension field K/F.

For example, let $M=\mathrm{GL}(1)\times\mathrm{GL}(2)\subset\mathrm{GL}(3)$, $\omega_1=1$, $\omega_2=\mathrm{St}(2)=\mathrm{St}(1,2)$. We have $l_1=1$, $l_2=2$, $g_1=0$, $g_2=1/2$, $r=1$. This gives the following (rational) function on the 2-torus:

$$\mu(\chi_1\otimes\chi_2\,\mathrm{St}(2))=\gamma(\mathrm{GL}(3)|M)^2\cdot q\cdot\left|\frac{1-z_2z_1^{-1}q^{-1/2}}{1-z_2z_1^{-1}q^{-3/2}}\right|^2.$$

Theorem 4.5. *Let* $G=\mathrm{GL}(2m)$, $M=\mathrm{GL}(m)\times\mathrm{GL}(m)$ *and let* σ *be an irreducible unitary supercuspidal representation of* $\mathrm{GL}(m)$ *with torsion number r. Then we have*

$$\mu_{G|M}(\chi_1\sigma\otimes\chi_2\sigma)=\gamma(G|M)^2\cdot q^{f(\sigma^\vee\times\sigma)}\cdot\left|\frac{1-z_2z_1^{-1}}{1-z_2z_1^{-1}q^{-r}}\right|^2$$

Proof. This follows from Theorem 4.4 by taking $l_1=l_2=1$, so that $g_1=g_2=g=0$. $\square$

5. The Bernstein decomposition of Plancherel measure

5.1. The one exponent case

Let X be a space on which the finite group Γ acts. The extended quotient associated to this action is the quotient space $\tilde{X}/\Gamma$ where

$$\tilde{X}=\{(\gamma,x)\in\Gamma\times X:\gamma x=x\}.$$

The group action on $\tilde{X}$ is $g.(\gamma, x) = (g\gamma g^{-1}, gx)$. Let $X^\gamma = \{x \in X : \gamma x = x\}$ and let $Z(\gamma)$ be the Γ-centralizer of γ. Then the extended quotient is given by:

$$\tilde{X}/\Gamma = \bigsqcup_\gamma X^\gamma / Z(\gamma),$$

where one γ is chosen in each Γ-conjugacy class. If $\gamma = 1$ then $X^\gamma/Z(\gamma) = X/\Gamma$ so *the extended quotient always contains the ordinary quotient*:

$$\tilde{X}/\Gamma = X/\Gamma \sqcup \cdots \ .$$

We shall need only the special case in which X is the compact torus $\mathbb{T}^n$ of dimension n and Γ is the symmetric group S_n acting on $\mathbb{T}^n$ by permuting co-ordinates.

Let β be a partition of n, and let γ have cycle type β. Each cycle provides us with one circle, and cycles of equal length provide us with a symmetric product of circles. For example, the extended quotient $\widetilde{\mathbb{T}^5}/S_5$ is the following disjoint union of compact orbifolds (one for each partition of 5):

$$\mathbb{T} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^3\mathbb{T}) \sqcup \mathrm{Sym}^5\mathbb{T},$$

where $\mathrm{Sym}^n\mathbb{T}$ is the n-fold symmetric product of the circle $\mathbb{T}$. This extended quotient is a model of the arithmetically unramified tempered dual of $\mathrm{GL}(5)$.

Let $\Omega \subset \Omega(\mathrm{GL}(n))$ have one exponent e. Then we have $e|n$ and so $em = n$.

There exists an irreducible unitary supercuspidal representation σ of $\mathrm{GL}(m)$ such that the conjugacy class of the cuspidal pair $(\mathrm{GL}(m) \times \cdots \times \mathrm{GL}(m), \sigma \otimes \cdots \otimes \sigma)$ is an element in Ω. We have $\Omega \cong \mathrm{Sym}^e\,\mathbb{C}^\times$ as complex affine algebraic varieties. Consider now a partition $p = (l_1, \ldots, l_k)$ of e into k parts, and write $2g_1 + 1 = l_1, \ldots, 2g_k + 1 = l_k$. Let

$$\pi_i = \mathrm{St}(\sigma, l_i)$$

as in Remark 3.2. Then $\pi_1 \in E_2(\mathrm{GL}(ml_1)), \ldots, \pi_k \in E_2(\mathrm{GL}(ml_k))$. Note that $ml_1 + \cdots + ml_k = n$ so that $\mathrm{GL}(ml_1) \times \cdots \times \mathrm{GL}(ml_k)$ is a standard Levi subgroup M of $\mathrm{GL}(n)$. Now consider

$$\pi = \chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k$$

with $\chi_1, \ldots, \chi_k$ unramified (unitary) characters. Then $\pi \in E_2(M)$. We have

$$\omega = I_{MN}^{\mathrm{GL}(n)}(\pi \otimes 1) \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$$

and each element $\omega \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$ for which $\mathrm{inf.ch.}\omega \in \Omega$ is accounted for in this way. As explained in detail in [23], we have

$$\widetilde{X}/\Gamma \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_{\Omega}, \tag{16}$$

where $X = \mathbb{T}^e$, $\Gamma = S_e$, i.e.,

$$\bigsqcup_{\gamma} X^{\gamma}/Z(\gamma) \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_{\Omega}.$$

The partition $p = (l_1, \ldots, l_k)$ of e determines a permutation γ of the set $\{1, 2, \ldots, e\}$: γ is the product of the cycles $(1, \ldots, l_1) \cdots (1, \ldots, l_k)$. Then the fixed set X^{γ} is

$$\{(z_1, \ldots, z_1, \ldots, z_k, \ldots, z_k) \in \mathbb{T}^e : z_1, \ldots, z_k \in \mathbb{T}\}$$

and so $X^{\gamma} \cong \mathbb{T}^k$.

Explicitly, we have

$$X^{\gamma} \longrightarrow \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_{\Omega},$$

$$(z_1, \ldots, z_k) \mapsto I_{MN}^{\mathrm{GL}(n)}(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k)$$

with $\chi_1(\varpi) = \zeta_1, \ldots, \chi_k(\varpi) = \zeta_k, z_1 = \zeta_1^r, \ldots, z_k = \zeta_k^r$ exactly as in Remark 3.2. This map is constant on each $Z(\gamma)$-orbit and descends to an *injective* map

$$X^{\gamma}/Z(\gamma) \to \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_{\Omega}.$$

Taking one γ in each Γ-conjugacy class we have the bijective map

$$\bigsqcup_{\gamma} X^{\gamma}/Z(\gamma) \cong \mathrm{Irr}\,\mathrm{GL}(n)_{\Omega}.$$

This bijection, by transport of structure, equips $\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_{\Omega}$ with the structure of disjoint union of finitely many compact orbifolds.

We now describe the restriction μ_{Ω} of Plancherel density to the compact orbifold $X^{\gamma}/Z(\gamma)$.

Theorem 5.1. *Let σ be an irreducible unitary supercuspidal representation of* $\mathrm{GL}(m)$ *with torsion number r. For $i = 1, \ldots, k$, let*

$$\pi_i = \mathrm{St}(\sigma, l_i),$$

let χ_i be an unramified character with $\chi_i(\varpi) = \zeta_i$, and let $z_i = \zeta_i^r$.

Then, as a function on the compact torus $\mathbb{T}^k$ *with co-ordinates* $(z_1, \ldots, z_k)$ *we have*

$$\mu(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k) = \text{const.} \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2,$$

where the product is taken over those i, j, g *for which the following inequalities hold*: $1 \leqslant i < j \leqslant k$, $|g_i - g_j| \leqslant g \leqslant g_i + g_j$, $2g_i + 1 = l_i$.

Proof. Apply Theorem 4.4 and the Harish-Chandra product formula, Theorem 2.2. Note that the function

$$(z_1, \ldots, z_k) \mapsto \text{const.} \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2$$

is a $Z(\gamma)$-invariant function on the γ-fixed set $X^\gamma = \mathbb{T}^k$, and descends to a nonnegative function on the orbifold $X^\gamma / Z(\gamma)$:

$$X^\gamma / Z(\gamma) \longrightarrow \mathbb{R}_+. \qquad \square$$

In the above theorem, the co-ordinates $z_1, \ldots, z_k$ should be thought of as *generalized Satake parameters*. The k-tuple $t = (z_1, \ldots, z_k)$ is a point in the standard maximal torus T of the unitary group $U(k, \mathbb{C})$. In that case, the roots of the unitary group are given by

$$\alpha_{ij}(t) = z_i / z_j.$$

The μ-function may now be written in the more invariant form

$$\mu(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k) = \text{const.} \prod (1 - \alpha(t) q^{gr})(1 - \alpha(t) q^{-(g+1)r})^{-1},$$

where the product is taken over all roots $\alpha = \alpha_{ij}$ of $U(k, \mathbb{C})$ and all g for which the following inequalities hold: $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$, $i \neq j$, $|g_i - g_j| \leqslant g \leqslant g_i + g_j$, $2g_i + 1 = l_i$.

Theorem 5.2. *We have the following numerical formula for* const.

$$\text{const.} = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \gamma(G|M)^2 \cdot c(G|M)^2,$$

where $\ell(\gamma) = \sum_{1 \leqslant i < j \leqslant k} l_i l_j$.

Proof. The numerical constant is determined by Theorems 4.4 and 2.2. Explicitly, for $i, j \in \{1, \ldots, k\}$, setting

$$\gamma_{i,j} := \gamma(\mathrm{GL}(n_i + n_j) | \mathrm{GL}(n_i) \times \mathrm{GL}(n_j)),$$

for the γ-factor of the Levi subgroup $\mathrm{GL}(n_i)\times\mathrm{GL}(n_j)$ of the maximal standard parabolic subgroup in $\mathrm{GL}(n_i+n_j)$,

$$\begin{aligned}\text{const.} &= q^{\sum_{1\leqslant i<j\leqslant k} l_i l_j f(\sigma^\vee\times\sigma)} \cdot \prod_{1\leqslant i<j\leqslant k}\gamma_{i,j}^2\\ &= q^{\ell(\gamma) f(\sigma^\vee\times\sigma)} \cdot \gamma(G|M)^2\cdot c(G|M)^2. \qquad \square\end{aligned}$$

Corollary 5.3. *We have*

$$j(\omega) = q^{\ell(\gamma) f(\sigma^\vee\times\sigma)} \cdot \prod \left|\frac{1-z_j z_i^{-1} q^{-(g+1)r}}{1-z_j z_i^{-1} q^{gr}}\right|^2$$

Proof. This follows immediately from Theorems 5.1, 5.2 and the fact that

$$c(G|M)^{-2}\,\gamma(G|M)^{-1}\,\mu_{G|M}(\omega) = \gamma(G|M)\,j(\omega)^{-1}. \qquad \square$$

Given $G=\mathrm{GL}(n)=\mathrm{GL}(n,F)$ choose $e|n$ and let $m=n/e$. Let Ω be a Bernstein component in $\Omega(\mathrm{GL}(n))$ with one exponent e. The compact extended quotient attached to Ω has finitely many components, each component is a compact orbifold. We now have enough results to write down explicitly the component μ_Ω. Let $l_1+\cdots+l_k=e$ be a partition of e, let $\gamma=(1,\ldots,l_1)\cdots(1,\ldots,l_k)\in S_e=\Gamma$, $g_1=(l_1-1)/2,\ldots,g_k=(l_k-1)/2$. Then we have the fixed set $X^\gamma=\mathbb{T}^k$. Let σ be an irreducible unitary supercuspidal representation of the group $\mathrm{GL}(m)$ and let the conjugacy class of the cuspidal pair $(\mathrm{GL}(m)^e,\sigma^{\otimes e})$ be a point in the Bernstein component Ω. Let r be the torsion number of σ and choose a field K such that $q_K=q_F^r$.

We have (16):

$$\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n,F)_\Omega \cong \widetilde{X}/\Gamma.$$

This compact Hausdorff space admits the Harish-Chandra *canonical measure* $\mathrm{d}\omega$: on each connected component in the extended quotient $\widetilde{X}/\Gamma$, $\mathrm{d}\omega$ restricts to the quotient by the centralizer $Z(\gamma)$ of the normalized Haar measure on the compact torus X^γ.

Let $\mathrm{d}\nu$ denote Plancherel measure on the tempered dual of $\mathrm{GL}(n,F)$.

Theorem 5.4. *On the component $X^\gamma/Z(\gamma)$ of the extended quotient $\widetilde{X}/\Gamma$ we have*:

$$\mathrm{d}\nu(\omega) = q^{\ell(\gamma) f(\sigma^\vee\times\sigma)} \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left|\frac{1-z_j z_i^{-1} q^{gr}}{1-z_j z_i^{-1} q^{-(g+1)r}}\right|^2 \cdot \mathrm{d}\omega.$$

Proof. By 2.1, the Plancherel measure on $\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n, F)_{\Omega}$ is given by

$$\mathrm{d}\nu(\omega) = c(G|M)^{-2}\,\gamma(G|M)^{-1}\,\mu(\omega)\,d(\omega)\ \mathrm{d}\omega.$$

Then, the result follows from Theorems 5.1 and 5.2. $\square$

Let T be the diagonal subgroup of G and take for Ω the Bernstein component in $\Omega(G)$ which contains the cuspidal pair $(T, 1)$. Then Ω has the single exponent n and parametrizes those irreducible smooth representations of $\mathrm{GL}(n, F)$ which admit nonzero Iwahori fixed vectors.

Now let $l_1 + \cdots + l_k$ be a partition of n, and let

$$M = \mathrm{GL}(l_1, F) \times \cdots \times \mathrm{GL}(l_k, F) \subset \mathrm{GL}(n, F).$$

The formal degree of the Steinberg representation $\mathrm{St}(l_i)$ is given by

$$d(\mathrm{St}(l_i)) = \frac{q^{(l_i - l_i^2)/2}}{l_i} \cdot \frac{|\mathrm{GL}(l_i, q)|}{q^{l_i} - 1} = \frac{1}{l_i} \cdot \prod_{j=1}^{l_i - 1}(q^j - 1). \tag{17}$$

We also have the inner product identity in pre-Hilbert space:

$$\langle(\sigma_1 \otimes \cdots \otimes \sigma_k)(g)\xi_1 \otimes \cdots \xi_k, \xi_1 \otimes \cdots \otimes \xi_k\rangle = \prod \langle\sigma_j(g)\xi_j, \xi_j\rangle.$$

Let each $\xi_j \in V_j$ be a unit vector. With respect to the standard normalization of all Haar measures we then have (cf. [11, (7.7.9)])

$$1/d_{\sigma_1 \otimes \cdots \otimes \sigma_k} = \prod \int | < \sigma_j(g)\xi_j, \xi_j > |^2 d\dot{\mu}_j = \prod 1/d_{\sigma_j}$$

and so

$$d_{\sigma_1 \otimes \cdots \otimes \sigma_k} = \prod d_{\sigma_j}. \tag{18}$$

Using (18) and Theorem 3, we obtain the following result.

Corollary 5.5. *On the orbifold $X^{\gamma}/Z(\gamma)$ we have*

$$\mathrm{d}\nu(\omega) = \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot \mathrm{d}\omega$$

where

$$d(\omega) = \prod d(\mathrm{St}(l_i)).$$

So we have

$$\begin{aligned}\mathrm{d}\nu(\omega) &= \gamma(G|M) \cdot \prod_{i=1}^{k} \frac{1}{l_i} \prod_{j=1}^{l_i-1} (q^j - 1) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot \mathrm{d}\omega \\ &= \prod_{i=1}^{k} \frac{q^{\frac{l_i^2 - l_i}{2}} (q-1)^{l_i}}{l_i (q^{l_i} - 1)} \cdot P_{S_n}(q^{-1}) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot \mathrm{d}\omega. \quad (19)\end{aligned}$$

Remark 5.6. Using [10, Theorem 3.3], we obtain that the Plancherel measure of the (extended) affine Hecke algebra $\mathcal{H}(n, q)$ is given on $X^{\gamma}/Z(\gamma)$ by

$$\mu(I) \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot \mathrm{d}\omega.$$

Concerning the volume $\mu(I)$: by Bushnell and Kutzko [11, 5.4.3] we have

$$\mu(\mathrm{GL}(n, \mathfrak{o}_F)) = \sum_{w \in W_0} \mu(IwI) = \sum_{w \in W_0} \mu(I) \cdot q^{\ell(w)} = P_{S_n}(q) \cdot \mu(I).$$

The explicit formula is then (using (2)):

$$\mathrm{d}\nu_{\mathcal{H}(n,q)}(\omega) = \prod_{i=1}^{k} \frac{q^{\frac{l_i^2 - l_i}{2}} (q-1)^{l_i}}{l_i (q^{l_i} - 1)} \cdot q^{\frac{n-n^2}{2}} \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^g}{1 - z_j z_i^{-1} q^{-(g+1)}} \right|^2 \cdot \mathrm{d}\omega,$$

where the second product is taken over those i, j, g for which the following inequalities hold: $1 \leqslant i < j \leqslant k$, $|g_i - g_j| \leqslant g \leqslant g_i + g_j$, $2g_i + 1 = l_i$. Note that Plancherel measure for Iwahori Hecke algebras has been already calculated by Opdam (see [22, 2.8.3]).

We will now consider a special case. The p-adic gamma function attached to the local field K (see [32, p. 51]) is the following meromorphic function of a single complex variable:

$$\Gamma_1(\zeta) = \frac{1 - q_K^{\zeta}/q_K}{1 - q_K^{-\zeta}}.$$

We will change the variable via $s = q_K^{\zeta}$ and write

$$\Gamma_K(s) = \frac{1 - s/q_K}{1 - s^{-1}},$$

a rational function of s. Let $s \in i\mathbb{R}$ so that s has modulus 1. Then we have

$$1/|\Gamma_K(s)|^2 = \left| \frac{1 - s}{1 - q_K^{-1} s} \right|^2 .$$

Let T be the standard maximal torus in $\mathrm{GL}(n)$ and let $\widehat{T}$ denote the unitary dual of T. Then $\widehat{T}$ has the structure of a compact torus $\mathbb{T}^n$ (the space of Satake parameters) and the unramified unitary principal series of $\mathrm{GL}(n)$ is parametrized by the quotient $\mathbb{T}^n/S_n$. Let now $t = (z_1, \ldots, z_n) \in \mathbb{T}^n$. Applying the above formulas the Plancherel density $\mu_{G|T}$ is given by

$$\mu_{G|T} = \mathrm{const.} \cdot \prod_{i<j} \left| \frac{1 - z_j z_i^{-1}}{1 - z_j z_i^{-1}/q} \right|^2, \tag{20}$$

$$\mu_{G|T} = \mathrm{const.} \cdot \prod_{0<\alpha} \left| \frac{1 - \alpha(t)}{1 - \alpha(t)/q} \right|^2, \tag{21}$$

$$\mu_{G|T} = \mathrm{const.} \cdot \prod_{\alpha} 1/\Gamma(\alpha(t)), \tag{22}$$

where α is a root of the Langlands dual group $\mathrm{GL}(n, \mathbb{C})$ so that $\alpha_{ij}(t) = z_i/z_j$.

For $\mathrm{GL}(n)$, one connected component in the tempered dual is the compact orbifold $\mathbb{T}^n/S_n$, the symmetric product of n circles. On this component we have the Macdonald formula [19]:

$$\mathrm{d}\mu(\omega_\lambda) = \mathrm{const.} \cdot \mathrm{d}\lambda / \prod_{\alpha} \Gamma(i\lambda(\alpha^\vee))$$

the product over all roots α where $\alpha^\vee$ is the coroot. This formula is a very special case of our formula for $\mathrm{GL}(n)$.

5.2. General case

We now pass to the general case of a component $\Omega \subset \Omega(\mathrm{GL}(n))$ with exponents $e_1, \ldots, e_t$. We first note that each component $\Omega \subset \Omega(\mathrm{GL}(n))$ yields up its fundamental invariants:

- the cardinality q of the residue field of F;
- the sizes m_i of the small general linear groups;

- the exponents e_i;
- the torsion numbers r_i;
- the formal degrees d_i;
- the conductors $f_{ij} = f(\sigma_i^\vee \times \sigma_j)$;

with $1 \leqslant i \leqslant t$.

We now construct the disjoint union

$$E = \Omega(\mathrm{GL}(\infty)) = \left\{ \bigsqcup \Omega(\mathrm{GL}(n)) : n = 0, 1, 2, 3, \ldots \right\}$$

with the convention that $\Omega(\mathrm{GL}(0)) = \mathbb{C}$.

We will say that two components $\Omega_1, \Omega_2 \in E$ are *disjoint* if none of the irreducible supercuspidals which occur in Ω_1 is equivalent (after unramified twist) to any of the supercuspidals which occur in Ω_2. We now define a law of composition on *disjoint components* in E. With the cuspidal pair $(M_1, \sigma_1) \in \Omega_1$ and the cuspidal pair $(M_2, \sigma_2) \in \Omega_2$ we define $\Omega_1 \times \Omega_2$ as the unique component determined by

$$(M_1 \times M_2, \sigma_1 \otimes \sigma_2).$$

The set E admits a law of composition *not everywhere defined* such that E is unital, commutative and associative. Rather surprisingly, E admits prime elements: the prime elements are precisely the components with a single exponent. Each element in E admits a unique factorization into prime elements:

$$\Omega = \Omega_1 \times \cdots \times \Omega_t.$$

Plancherel measure respects the unique factorization into prime elements, modulo constants. Quite specifically, we have

Theorem 5.7. *Let Ω have the unique factorization*

$$\Omega = \Omega_1 \times \cdots \times \Omega_t$$

so that Ω has exponents $e_1, \ldots, e_t$ and $\Omega_1, \ldots, \Omega_t$ are pairwise disjoint prime elements with the individual exponents $e_1, \ldots, e_t$. Let

$$\nu = \bigsqcup \nu_\Omega$$

denote the Bernstein decomposition of Plancherel measure. Then we have

$$\nu_\Omega = const. \, \nu_{\Omega_1} \cdots \nu_{\Omega_t},$$

where $v_{\Omega_1}, \ldots, v_{\Omega_t}$ *are given by Theorem* 5.1 *and the constant is given, in terms of the fundamental invariants, by Theorem* 5.2.

Proof. In the Harish-Chandra product formula, all the cross-terms are constant, by Theorem 4.3. □

6. Transfer-of-measure, conductor, and the formal degree formulas

6.1. Torsion number

The theory of types of [11] produces a canonical extension K of F such that $q_K = q^r$. Indeed, let σ be an irreducible supercuspidal representation of $\mathrm{GL}(m)$, and let (J, λ) be a maximal simple type occurring in it. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in $A = M(m, F)$ and let $E = F[\beta]$ be the field extension of F attached to the stratum (see [11, Definition 5.5.10 (iii)]). It is proved in [11, Lemma 6.2.5] that

$$r = \frac{m}{e(E|F)}, \tag{23}$$

where $e(E|F)$ denotes the ramification index of E with respect to F. Let B denote the centralizer of E in A. We set $\mathfrak{B} := \mathfrak{A} \cap B$. Then $\mathfrak{B}$ is a maximal hereditary order in B, see [11, Theorem 6.2.1]. Let K be an unramified extension of E which normalizes $\mathfrak{B}$ and is maximal with respect to that property, as in [11, Proposition 5.5.14]. Then $[K : F] = m$, and (23) gives that r is equal to the residue index $f(K|F)$ of K with respect to F. Thus $Q = q^r$ is equal to the order q_K of the residue field of K.

Also the number Q is the one which occurs for the Hecke algebra $\mathcal{H}(\mathrm{GL}(m), \lambda)$ associated to (J, λ), see [11, Theorem 5.6.6]. Indeed, since the order of the residue field of E is equal to $q^{f(E|F)}$, that number is $(q^{f(E|F)})^f$, with

$$f = \frac{m}{[E : F]\, e(\mathfrak{B})},$$

where $e(\mathfrak{B})$ denotes the period of a lattice chain attached to $\mathfrak{B}$ as in [11, (1.1)]. Since σ is supercuspidal, $e(\mathfrak{B}) = 1$ (see [11, Corollary 6.2.3]). It follows that

$$f \cdot f(E|F) = \frac{m \cdot f(E|F)}{[E : F]} = \frac{m}{e(E|F)} = r. \tag{24}$$

6.2. Normalization of measures

We will relate our normalization of measures to the measures used in [11, (7.7)]. Bushnell and Kutzko work with a quotient measure $\dot{\mu}$, the quotient of μ_G by μ_Z.

Let Z denote the centre of $\mathrm{GL}(n)$. The second isomorphism theorem in group theory gives:

$$JZ/Z \cong J/J \cap Z.$$

We have

$$J \cap Z = \mathfrak{o}_F^\times.$$

One way to see this would be: J contains $\mathfrak{A}^\times \cap B$, where B is the centralizer in $M(n, F)$ of the extension E. Now certainly Z is contained in B. On the other hand, $\mathfrak{A}$ is an $\mathfrak{o}_F$-order so $\mathfrak{A}$ certainly contains $\mathfrak{o}_F$. Thanks to Shaun Stevens for this remark.

Then we have

$$JZ/Z \cong J/\mathfrak{o}_F^\times.$$

Now J is a principal $\mathfrak{o}_F^\times$-bundle over $J/\mathfrak{o}_F^\times$. Each fibre over the base $J/\mathfrak{o}_F^\times$ has volume 1. The quotient measure of the base space is then given by

$$\dot{\mu}(JZ/Z) = \mu(J). \tag{25}$$

Similar normalizations are done with $G_0 = \mathrm{GL}(e, K)$. We also need the corresponding quotient measure $\ddot{\mu}$ (see [11, (7.7.8)]). We have

$$\ddot{\mu}(IK^\times/K^\times) = \mu_{G_0}(I).$$

Let $M = \prod \mathrm{GL}(n_j)$. We have $Z_M = \prod Z_j$, $\mathcal{K} = \prod \mathcal{K}_j$, with $Z_j = Z_{\mathrm{GL}(n_j,F)}$ and $\mathcal{K}_j = \mathrm{GL}(n_j, \mathfrak{o}_F)$. With respect to the standard normalization of all Haar measures, we have $\mu_M = \prod \mu_j$ (where μ_j denotes $\mu_{\mathrm{GL}(n_i,F)}$) and $\mu_{Z_M} = \prod \mu_{Z_j}$. This then guarantees that

$$\dot{\mu}_M = \prod \dot{\mu}_j. \tag{26}$$

6.3. Conductor formulas (the supercuspidal case)

We will first recall results from [9] in a suitable way for our purpose.

Let $(J^{\mathrm{s}}, \lambda^{\mathrm{s}})$ be a simple type in $\mathrm{GL}(2m)$ with associated maximal simple type (J, λ) (in the terminology of [11, (7.2.18) (iii)]). When (J, λ) is of positive level, we set $J_P = (J^{\mathrm{s}} \cap P)H^1(\beta, \mathfrak{A}) \subset J^{\mathrm{s}}$ (in notation [11, (3.1.4)]), where P denotes the upper-triangular parabolic subgroup of $\mathrm{GL}(2m)$ with Levi component $M = \mathrm{GL}(m) \times \mathrm{GL}(m)$, and unipotent radical denoted by N. Following [11, Theorem 7.2.17], we define λ_P as the natural representation of J_P on the space of $(J \cap N)$-fixed vectors in λ^{s}. The representation λ_P is irreducible and $\lambda_P \simeq \text{c–Ind}_{J_P}^{J^{\mathrm{s}}}(\lambda^{\mathrm{s}})$.

The pair $(J \times J, \lambda \otimes \lambda)$ is a type in M which occurs in $\sigma \otimes \sigma$, and, as shown in [13, Proposition 1.4], (J_P, λ_P) is a $\mathrm{GL}(2m)$-cover of $(J \times J, \lambda \otimes \lambda)$.

Theorem 6.1 (*Conductor formulas [9]*). *Let $G_0 = \mathrm{GL}(2, K)$, let N_0 denote the unipotent radical of the standard Borel subgroup of G_0 and let I denote the standard Iwahori subgroup of G_0. We will denote by μ_0 the Haar measure on G_0 normalized as in Section* 6.2.

Let $(J^{\mathrm{GL}(2m)}, \lambda^G)$ be any $\mathrm{GL}(2m)$-cover of $(J \times J, \lambda \otimes \lambda)$.

Then

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-f(\sigma^\vee \times \sigma)} = \frac{j(\sigma \otimes \sigma)}{j_0(1)},$$

where j, j_0 *denote the j-functions for the group* G, G_0 *respectively.*

Proof. The first equality is [9, Theorem in §5.4], using the fact that $\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0) = q_K^{-1}$. The second equality is [9, Theorem in §5.4] (note that in [9] the normalizations have been taken so that $\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N}) = \mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)$). It also follows directly from our Corollary 5.3. $\square$

We will now extend the above theorem to the case of $M = \mathrm{GL}(m)^{\times e}$, with e arbitrary.

Corollary 6.2. *Let* $M = \mathrm{GL}(m)^{\times e}$ *with* $n = em$, *and* $G_0 = \mathrm{GL}(e, K)$, *let* N_0 *denote the unipotent radical of the standard Borel subgroup of* G_0 *and let* I *denote the standard Iwahori subgroup of* G_0.

Let (J^G, λ^G) *be a cover in* $G = \mathrm{GL}(n)$ *of* $(J^{\times e}, \lambda^{\otimes e})$ (*the existence of which is guaranteed by Bushnell and Kutzko* [13]).

Then

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-\frac{e(e-1)}{2} f(\sigma^\vee \times \sigma)} = \frac{j(\sigma^{\otimes e})}{j_0(1)}.$$

Proof. Let M' be a Levi subgroup of a parabolic subgroup in G such that P is a maximal parabolic subgroup of M'. Then, $M'/M \simeq \mathrm{GL}(2m)/\mathrm{GL}(m) \times \mathrm{GL}(m)$ and

$$\mu(J^G \cap M' \cap N) = \mu(J^{\mathrm{GL}(2m)} \cap \mathrm{GL}(2m) \cap N).$$

It follows from [12, Proposition 8.5 (ii)] that $(J^G \cap M', \lambda^G | J^G \cap M')$ is an M'-cover of $(J^{\times e}, \lambda^{\otimes e})$.

Because of the unipotency of N, we have

$$\mu(J^G \cap N) = (\mu(J^{\mathrm{GL}(2m)} \cap \mathrm{GL}(2m) \cap N))^{\frac{e(e-1)}{2}} \tag{27}$$

and similar equalities for the three others terms. Since $\mathrm{GL}(2m) \cap N$ is the unipotent radical of the parabolic subgroup of $\mathrm{GL}(2m)$ with Levi $\mathrm{GL}(m) \times \mathrm{GL}(m)$, the first equality in the corollary follows from Theorem 6.1.

The second equality follows from our Corollary 5.3. It is also a direct consequence of Theorem 6.1, using the product formula for j and for j_0 from [34, IV.3. (5)]. $\square$

6.4. Formal degree formulas

Using Corollary 6.2, we will deduce from [11, (7.7.11)] a formula relating the formal degree of any discrete series of $\mathrm{GL}(n)$ and the formal degree of a supercuspidal representation in its inertial support.

Given $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$ choose $e|n$ and let $m = n/e$. Let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ and let (J, λ) be a maximal simple type occurring in it. Let $g = (e-1)/2$. We consider the standard Levi subgroup $M = \mathrm{GL}(m)^{\times e}$ of $\mathrm{GL}(n, F)$ and the supercuspidal representation

$$\sigma_M = |\det(\ )|^{-g}\sigma \otimes \cdots \otimes |\det(\ )|^{g}\sigma$$

of it. Then $(J_M, \lambda_M) = (J^{\times e}, \lambda^{\otimes e})$ is a type in M occurring in σ_M.

Let $\pi = \mathrm{St}(\sigma, e)$ and let $(J^{\mathrm{s}}, \lambda^{\mathrm{s}})$ be a simple type in $\mathrm{GL}(n)$ occurring in π (it has associated maximal simple type (J, λ)).

The following result is rather intricate, but note that only the *fundamental invariants* $m, e, r, d, f(\sigma^\vee \times \sigma)$ occur in it, in line with our general philosophy.

Theorem 6.3. *We have*

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}e} \cdot q^{\frac{e^2-e}{2}(f(\sigma^\vee\times\sigma)+r-2m^2)} \cdot \frac{(q^r-1)^e}{q^{er}-1} \cdot \frac{|\mathrm{GL}(em,q)|}{|\mathrm{GL}(m,q)|^e}.$$

Remark 6.4. The right-hand side in the above equality can be rewritten, by using (17), as

$$r^{1-e} \cdot \frac{(q^{em}-1)(q^r-1)^e}{(q^m-1)^e(q^{er}-1)} \cdot q^{\frac{e^2-e}{2}(f(\sigma^\vee\times\sigma)+r-m^2)} \cdot \frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^e}.$$

Proof. Let T denote the diagonal torus in $\mathrm{GL}(e, K)$ and let I denote the Iwahori subgroup of $G_0 = \mathrm{GL}(e, K)$ attached to the Bernstein component in $\Omega(\mathrm{GL}(e, K))$ which contains the cuspidal pair $(T, 1)$. Note that $I \cap T = \mathrm{GL}(1, \mathfrak{o}_K)^{\times e}$. From [11, (7.7.11)], applied to the representations π and σ, we have

$$d(\pi) = \frac{\mu_0(I)}{\mu(J^{\mathrm{s}})} \cdot \frac{\dim(\lambda^{\mathrm{s}})}{e(E|F)} \cdot d(\pi)_0, \tag{28}$$

where $d(\pi)_0$ denotes the formal degree of $\pi \in E_2(G_0)$, and

$$d(\sigma) = \frac{\mu(\mathrm{GL}(1, \mathfrak{o}_K))}{\mu(J)} \cdot \frac{\dim(\lambda)}{e(E|F)}. \tag{29}$$

Using (28), (29) and (24), we obtain

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot \frac{\mu_0(I)}{\mu(J^{\mathrm{s}})} \cdot \frac{\mu(J^{\times e})}{\mu(\mathrm{GL}(1, \mathfrak{o}_K)^{\times e})} \cdot \frac{\dim(\lambda^{\mathrm{s}})}{(\dim(\lambda))^e} \cdot d(\pi)_0. \tag{30}$$

We set $J_P = (J^{\mathrm{s}} \cap P)H^1(\beta, \mathfrak{A}) \subset J^{\mathrm{s}}$, where P is the upper-triangular parabolic subgroup of G with Levi component M, and unipotent radical N. We define λ_P as the natural representation of J_P on the space of $(J \cap N)$-fixed vectors in λ^{s}. The representation λ_P is irreducible and $\lambda_P \simeq \text{c–Ind}_{J_P}^{J^{\mathrm{s}}}(\lambda^{\mathrm{s}})$. Then (J_P, λ_P) is a G-cover of (J_M, λ_M). In the case where (J, λ) is of zero level, we denote by $(J^{\mathrm{s}}, \lambda^{\mathrm{s}}) = (J_P, \lambda_P)$ an arbitrary G-cover of (J_M, λ_M).

Since $J^{\mathrm{s}} \cap M = J^{\times e} = J_M = J_P \cap M$, and

$$\dim(\lambda)^e = \dim(\lambda_M) = \dim(\lambda_P) = [J^{\mathrm{s}} : J_P]^{-1} \dim(\lambda^{\mathrm{s}}),$$

(30) gives

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot \frac{\mu_0(I)}{\mu(J_P)} \cdot \frac{\mu(J_M)}{\mu_0(I \cap T)} \cdot d(\pi)_0.$$

On the other hand, by applying formula [34, p. 241, l.7] to the group J, we obtain

$$\gamma(G|M) = \frac{\mu(J_P \cap N) \cdot \mu(J_P \cap M) \cdot \mu(J_P \cap \overline{N})}{\mu(J_P)}. \tag{31}$$

Similarly, we have

$$\gamma(G_0|T) = \frac{\mu_0(I \cap N_0) \cdot \mu_0(I \cap T) \cdot \mu_0(I \cap \overline{N}_0)}{\mu_0(I)}.$$

We then obtain

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{\gamma(G|M)}{\gamma(G_0|T)} \cdot \frac{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)}{\mu(J_P \cap N) \cdot \mu(J_P \cap \overline{N})} \cdot d(\pi)_0.$$

Applying Corollary 6.2, we get

$$\frac{d(\pi)}{d(\sigma)^e} = \frac{m^{e-1}}{r^{e-1}} \cdot q^{\frac{e(e-1)}{2} f(\sigma^\vee \times \sigma)} \cdot \frac{\gamma(G|M)}{\gamma(G_0|T)} \cdot d(\pi)_0.$$

Since Haar measure on $\mathrm{GL}(e, K)$ has been normalized so that the volume of $\mathrm{GL}(e, \mathfrak{o}_K)$ is equal to one, the formal degree of the Steinberg representation of $\mathrm{GL}(e, K)$ is given as in (17) by

$$d(\pi)_0 = \frac{q_K^{(e-e^2)/2}}{e} \cdot \frac{|\mathrm{GL}(e, q_K)|}{q_K^e - 1}.$$

On the other hand, Theorem 3.1 gives

$$\gamma(G|M) = q^{mn-n^2} \cdot \frac{|\mathrm{GL}(n, q)|}{|\mathrm{GL}(m, q)|^e} \quad \text{and} \quad \gamma(G_0|T) = q^{e-e^2} \cdot \frac{|\mathrm{GL}(e, q_K)|}{(q_K - 1)^e}.$$

The result follows. $\square$

We will now recall the explicit formulas for $d(\pi)$ and $d(\sigma)$ from [31], using also [36]. We would like to thank Wilhelm Zink for explaining these works to us.

Let η be the Heisenberg representation of $J^1(\beta, \mathfrak{A})$ attached to a maximal simple type $(J(\beta, \mathfrak{A}), \lambda)$ occurring in the supercuspidal representation σ of $\mathrm{GL}(m)$ (see [11, (5.1.1), (5.5.10)]). Let $\mathfrak{P}$ denote the Jacobson radical of $\mathfrak{A}$ and let $U^i(\mathfrak{A}) = 1+\mathfrak{P}^i$. Let π_β^1 be the compactly induced representation $\text{c–Ind}_{J^1(\beta,\mathfrak{A})}^{U^1(\mathfrak{A})}(\eta)$. Then π_β^1 is irreducible, see [11, (5.2.3)]. More generally the restriction of η to $J^i(\beta, \mathfrak{A}) = J^1(\beta, \mathfrak{A}) \cap (1+\mathfrak{P}^i)$ is a multiple of an irreducible representation η^i which induces irreducibly to a representation π_β^i of $U^i(\mathfrak{A})$ (see [36, 2.2]). Let E_{-i} be any field such that

$$U^1(\mathfrak{A}) \cdot I_{\mathrm{GL}(m)}(\pi_\beta^{i+1}) \cdot U^1(\mathfrak{A}) \;=\; U^1(\mathfrak{A}) \cdot \mathrm{GL}(m/[E_{-i} : F], E_{-i}) \cdot U^1(\mathfrak{A}),$$

where $I_{\mathrm{GL}(m)}(\pi_\beta^{i+1})$ denotes the intertwining of π_β^{i+1} in $\mathrm{GL}(m, F)$. In particular, we have $E_0 = E$.

Theorem 6.5 (*Explicit formal degrees formulas [31,36]*). *The formal degrees of σ and π are, respectively, given by*

$$d(\sigma) \;=\; r \cdot \frac{q^m - 1}{q^r - 1} \cdot q^{(r-m+\delta)/2} \cdot \deg(\mathrm{St}(m)),$$

$$d(\pi) \;=\; r \cdot \frac{q^{em} - 1}{q^{er} - 1} \cdot q^{(er-em+e^2\delta)/2} \cdot \deg(\mathrm{St}(em)),$$

where

$$\delta = rm \cdot \sum_{i \geqslant 0} (1 - [E_{-i} : F]^{-1}).$$

Proof. It follows directly from [31, Theorem 1.1] and [36, Corollary 6.7], using the fact that $r = f(K|F)$ and $m/e(E|F) = r$. $\square$

As immediate consequences, we obtain the following results.

Corollary 6.6.

$$\frac{d(\pi)}{d(\sigma)^{e^2}} = r^{1-e^2} \cdot \frac{(q^{em}-1)(q^r-1)^{e^2}}{(q^{er}-1)(q^m-1)^{e^2}} \cdot q^{(e^2-e)(m-r)/2} \cdot \frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^{e^2}}.$$

Remark 6.7. We observe that the above formula extends to the general case the formula obtained in [14, Theorem 4.6] in the case where $(n, p) = 1$ and F has characteristic zero. The existence of such a formula was expected in [14, Remark 4.7]. Our formula also extends [33, Theorem VII.3.2].

Corollary 6.8.

$$\frac{d(\pi)}{d(\sigma)^{e}} = r^{1-e} \cdot \frac{(q^{em}-1)(q^r-1)^{e}}{(q^{er}-1)(q^m-1)^{e}} \cdot q^{(e^2-e)\delta/2} \cdot \frac{\deg(\mathrm{St}(em))}{(\deg(\mathrm{St}(m)))^{e}}.$$

The comparison of Corollary 6.8 with Remark 6.4 gives the following expression for the conductor for pairs $f(\sigma^\vee \times \sigma)$.

Theorem 6.9. *We have*

$$f(\sigma^\vee \times \sigma) = \delta + m^2 - r.$$

Remark 6.10. In [10, §6.4] (see also [10, 6.13]) is introduced a certain discriminant function $C(\beta)$ and an integer $\mathfrak{c}(\beta)$ such that $C(\beta) = q^{\mathfrak{c}(\beta)}$. It follows from our Theorem 5.1 and [10, Theorem 6.5 (i)] that

$$\mathfrak{c}(\beta) = \frac{[E:F]^2}{m^2} \cdot \delta.$$

6.5. Conductor formulas (the discrete series case)

Let σ be an irreducible supercuspidal representation of $\mathrm{GL}(m)$, and let (J, λ) be a maximal simple type occurring in it. Let $e|n$, and let $l_1 + \cdots + l_k = e$ be a partition of e. It determines the standard Levi subgroup

$$M = \mathrm{GL}(l_1 m) \times \cdots \times \mathrm{GL}(l_k m) \subset \mathrm{GL}(n, F). \tag{32}$$

Let $g_1 = (l_1 - 1)/2, \ldots, g_k = (l_k - 1)/2$, and let $\pi_1, \ldots, \pi_k$ be discrete series representations of $\mathrm{GL}(l_1 m), \ldots, \mathrm{GL}(l_k m)$ such that $\pi_i = \mathrm{St}(\sigma, l_i)$. Let $\pi = \pi_1 \otimes$

$\cdots \otimes \pi_k$ be the corresponding discrete series representation of M. For each $i \in \{1, \ldots, k\}$, we fix a $\mathrm{GL}(l_i m)$-cover $(J^{\mathrm{GL}(l_i m)}, \lambda^{\mathrm{GL}(l_i m)})$ of $(J^{\times l_i}, \lambda^{\otimes l_i})$ (as in the proof of Theorem 6.3). Then

$$(J_M, \lambda_M) = (J^{\mathrm{GL}(l_1 m)} \times \cdots \times J^{\mathrm{GL}(l_k m)}, \lambda^{\mathrm{GL}(l_1 m)} \otimes \cdots \otimes \lambda^{\mathrm{GL}(l_k m)}) \tag{33}$$

is an M-cover of $(J^{\times e}, \lambda^{\otimes e})$. Then let (J^G, λ^G) denote a G-cover of (J_M, λ_M) (the existence of which is guaranteed by Bushnell and Kutzko [13, Main Theorem (second version)]).

At the same time the partition $(l_1, \ldots, l_k)$ determines the standard Levi subgroup

$$M_0 = \mathrm{GL}(l_1) \times \cdots \times \mathrm{GL}(l_k) \subset \mathrm{GL}(e, K) = G_0. \tag{34}$$

Let P (resp. P_0) be the upper-triangular parabolic subgroup of G (resp. G_0) with Levi component M (resp. M_0), and unipotent radical denoted by N (resp. N_0). Let I denote the standard Iwahori subgroup of G_0.

Theorem 6.11. *We have*

$$\frac{\mu(J^G \cap N) \cdot \mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0) \cdot \mu_0(I \cap \overline{N}_0)} = q^{-\ell(\gamma) f(\sigma^\vee \times \sigma)} = \frac{j(\sigma^{\otimes e})}{j_0(1)}.$$

Proof. The second equality follows from our Corollary 5.3.

We will prove the first equality. Let U denote the unipotent radical of the upper-triangular parabolic subgroup of G with Levi component $\mathrm{GL}(m)^{\times e}$, and, for $i = 1, \ldots, k$, let U_i denote the unipotent radical of the upper-triangular parabolic subgroup of $\mathrm{GL}(l_i m)$ with Levi component $\mathrm{GL}(m)^{\times l_i}$. We observe that

$$U = N \times (U \cap M) = N \times \prod_{i=1}^{k} U_i.$$

Similarly, let U_0 be the unipotent radical of the standard Borel subgroup of G_0, and, for $i = 1, \ldots, k$, let $U_{0,i}$ be the unipotent radical of the standard Borel subgroup of $\mathrm{GL}(l_i, K)$. We have

$$U_0 = N_0 \times (U_0 \cap M_0) = N_0 \times \prod_{i=1}^{k} U_{0,i}.$$

It follows from [12, Proposition 8.5 (i)] that (J^G, λ^G) is also a G-cover of $(J^{\times e}, \lambda^{\otimes e})$. Applying Corollary 6.2 to (J^G, U) and to $(J^{\mathrm{GL}(l_i m)}, U_i)$ for each $i \in \{1, \ldots, k\}$, we obtain

$$\frac{\mu(J^G \cap U)\cdot\mu(J^G\cap\overline{U})}{\mu_0(I\cap U_0)\cdot\mu_0(I\cap\overline{U}_0)} = q^{-\frac{e(e-1)}{2}f(\sigma^\vee\times\sigma)}$$

$$\frac{\mu(J^{\mathrm{GL}(l_i m)} \cap U_i)\cdot\mu(J^{\mathrm{GL}(l_i m)}\cap\overline{U}_i)}{\mu_0(I\cap U_{0,i})\cdot\mu_0(I\cap\overline{U}_{0,i})} = q^{-\frac{l_i(l_i-1)}{2}f(\sigma^\vee\times\sigma)}.$$

Since $J^G\cap M = J_M$ (by definition of covers), it follows from (33) that $J^G\cap\mathrm{GL}(l_i m) = J^{\mathrm{GL}(l_i m)}$. Then using the fact that

$$\mu(J^G\cap N) = \mu(J^G\cap U)\ \times\ \prod_{i=1}^{k}\mu(J^{\mathrm{GL}(l_i m)}\cap U_i)$$

and the analogous equalities for the others terms, we obtain

$$\frac{\mu(J^G \cap N)\cdot\mu(J^G\cap\overline{N})}{\mu_0(I\cap N_0)\cdot\mu_0(I\cap\overline{N}_0)} = q^{(-\frac{e(e-1)}{2}+\sum_{i=1}^{k}\frac{l_i(l_i-1)}{2})f(\sigma^\vee\times\sigma)}, \tag{35}$$

$$\frac{\mu(J^G \cap N)\cdot\mu(J^G\cap\overline{N})}{\mu_0(I\cap N_0)\cdot\mu_0(I\cap\overline{N}_0)} = q^{-\ell(\gamma)f(\sigma^\vee\times\sigma)}. \qquad \square \tag{36}$$

6.6. *Transfer-of-measure*

The following result reduces the case of an arbitrary component Ω to the one (studied in Corollary 5.5) of a component (of a possibly different group G_0) which contains the cuspidal pair $(T, 1)$. We give a direct proof which is based on our previous calculations. It is worth noting that it is also a direct application of [10, Theorem 4.1].

Let $\Omega = \sigma^e$ be a Bernstein component in $\Omega(\mathrm{GL}(n))$ with single exponent e. Let T be the diagonal subgroup of $G_0 = \mathrm{GL}(e, K)$, and let Ω_0 be the Bernstein component in $\Omega(\mathrm{GL}(e, K))$ which contains the cuspidal pair $(T, 1)$. The components Ω, Ω_0 each have the single exponent e, and we have a homeomorphism of compact Hausdorff spaces

$$\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n, F)_\Omega \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(e, K)_{\Omega_0}. \tag{37}$$

This homeomorphism is determined by the map

$$\bigotimes_{i=1}^{k}\zeta_i^{\mathrm{val}_F\circ\det_F}\otimes\pi_i \mapsto \bigotimes_{i=1}^{k}(\zeta_i^r)^{\mathrm{val}_K\circ\det_K}\otimes\mathrm{St}(l_i).$$

This formula precisely allows for the fact that π_i has torsion number r and that $\mathrm{St}(l_i)$ has torsion number 1. Note that when ζ is replaced by $\omega\zeta$, where ω is an rth root of unity, each term remains unaltered.

The equation $r = f(K|F)$ and the standard formula

$$\mathrm{val}_K(y) = f(K|F)^{-1}\,\mathrm{val}_F(N_{K|F}(y))$$

lead to the more invariant formula:

$$\bigotimes_{i=1}^{k} (\chi_i \circ \det{}_F) \otimes \pi_i \mapsto \bigotimes_{i=1}^{k} (\chi_i \circ N_{K|F} \circ \det{}_K) \otimes \mathrm{St}(l_i),$$

where χ_i is an unramified character of $F^\times$.

Let (J^G, λ^G) be defined as in the previous subsection. It is a type in G attached to Ω. Recall that I denotes the standard Iwahori subgroup of G_0.

Theorem 6.12. *Let* $\mathrm{d}\nu$, $\mathrm{d}\nu_0$, *respectively, denote Plancherel measure on* $\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n, F)_\Omega$, $\mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(e, K)_{\Omega_0}$. *We have*

$$\frac{\mu(J^G)}{\dim(\lambda^G)} \cdot \mathrm{d}\nu(\omega) = \mu_0(I) \cdot \mathrm{d}\nu_0(\omega_0),$$

where

$$\omega = \chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k$$

and

$$\omega_0 = (\chi_1 \circ N_{K|F})\mathrm{St}(l_1) \otimes \cdots \otimes (\chi_k \circ N_{K|F})\mathrm{St}(l_k).$$

Proof. We first have to elucidate the canonical measures $\mathrm{d}\omega, \mathrm{d}\omega_0$. First, let $M = \mathrm{GL}(n)$, and let ω have torsion number r. Then the map $\mathrm{Im}\,X(M) \to \mathcal{O}$ is the r-fold covering map: $\mathbb{T} \to \mathbb{T}, z \mapsto z^r$. The map $\mathrm{Im}\,X(M) \to \mathrm{Im}\,X(A_M)$ sends the map $T \mapsto z^{\mathrm{val}(\det(T))}$ to the map $x \mapsto z^{\mathrm{val}(\det(xI_n))} = (z^n)^{\mathrm{val}(\det(x))}$ and so induces the n-fold covering map $\mathbb{T} \to \mathbb{T}$. The canonical measure $\mathrm{d}\omega$ on the orbit $\mathcal{O}$ is the Haar measure of total mass n/r. If $M = \mathrm{GL}(l_1) \times \cdots \times \mathrm{GL}(l_k)$ and ω_j has torsion number r_j then the canonical measure $\mathrm{d}\omega$ on the orbit $\mathcal{O}$ of $\omega_1 \otimes \cdots \otimes \omega_k$ is the Haar measure of total mass $l_1 \cdots l_k / r_1 \cdots r_k$. For the canonical measures $\mathrm{d}\omega$, $\mathrm{d}\omega_0$ we therefore have

$$\mathrm{d}\omega = (ml_1 \cdots ml_k/r^k) \cdot \mathrm{d}\tau = l_1 \cdots l_k \cdot (m^k/r^k) \cdot \mathrm{d}\tau,$$

$$\mathrm{d}\omega_0 = l_1 \cdots l_k \cdot \mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is the Haar measure on $\mathbb{T}^k$ of total mass 1. So, we have

$$\mathrm{d}\omega = (m^k/r^k) \cdot \mathrm{d}\omega_0. \tag{38}$$

By Theorem 5.4,

$$\mathrm{d}\nu(\omega) = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \gamma(G|M) \cdot d(\omega) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2 \cdot \mathrm{d}\omega$$

and

$$\mathrm{d}\nu_0(\omega_0) = \gamma(G_0|M_0) \cdot d(\omega_0) \cdot \prod \left| \frac{1 - z_j z_i^{-1} q^{gr}}{1 - z_j z_i^{-1} q^{-(g+1)r}} \right|^2 \cdot \mathrm{d}\omega_0.$$

Hence

$$\frac{\mathrm{d}\nu(\omega)}{\mathrm{d}\nu_0(\omega_0)} = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \frac{\gamma(G|M)}{\gamma(G_0|M_0)} \cdot \frac{d(\omega)}{d(\omega_0)} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}\omega_0}. \tag{39}$$

We keep the notation of Section 6.5. It follows from (26), (25) that

$$\mu(J_M) = \mu(J^{\mathrm{GL}(l_1 m)}) \times \cdots \times \mu(J^{\mathrm{GL}(l_k m)}), \tag{40}$$

since $J_M = J^{\mathrm{GL}(l_1 m)} \times \cdots \times J^{\mathrm{GL}(l_k m)}$. In the same way, we have

$$\mu_0(I \cap M_0) = \mu_0(I \cap \mathrm{GL}(l_1 m)) \times \cdots \times \mu_0(I \cap \mathrm{GL}(l_k m)), \tag{41}$$

On the other hand, formula [11, (7.7.11)] gives

$$\mu(J^{\mathrm{GL}(l_i m)}) \cdot d(\pi_i) = \mu_0(I \cap \mathrm{GL}(l_i, K)) \cdot \frac{\dim(\lambda^{\mathrm{GL}(l_i m)})}{e(E|F)} \cdot d(\mathrm{St}(l_i)).$$

Then (40), (41), (33), and (18) imply

$$\mu(J_M) \cdot d(\omega) = \mu_0(I \cap M_0) \cdot \frac{\dim(\lambda_M)}{e(E|F)^k} \cdot d(\omega_0). \tag{42}$$

Applying (31) to both $\gamma(G|M)$ and $\gamma(G_0|M_0)$, we obtain

$$\frac{\gamma(G|M)}{\gamma(G_0|M_0)} = \frac{\mu(J^G \cap N)\mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0)\mu_0(I \cap \overline{N}_0)} \cdot \frac{\mu_0(I)}{\mu(J^G)} \cdot \frac{\mu(J_M)}{\mu_0(I \cap M_0)}. \tag{43}$$

It then follows from (39), (42) and (43) that

$$\frac{\mathrm{d}\nu(\omega)}{\mathrm{d}\nu_0(\omega_0)} = q^{\ell(\gamma) f(\sigma^\vee \times \sigma)} \cdot \frac{\mu(J^G \cap N)\mu(J^G \cap \overline{N})}{\mu_0(I \cap N_0)\mu_0(I \cap \overline{N}_0)} \cdot \frac{\mu_0(I)}{\mu(J^G)} \cdot \frac{\dim(\lambda_M)}{e(E|F)^k} \frac{\mathrm{d}\omega}{\mathrm{d}\omega_0}.$$

Noting that $\dim(\lambda^G) = \dim(\lambda_M)$, and using Eq. (24) and Theorem 6.11, we have

$$\frac{\mathrm{d}\nu(\omega)}{\mathrm{d}\nu_0(\omega_0)} = \frac{\mu_0(I)}{\mu(J^G)} \cdot \dim(\lambda^G) \cdot \frac{r^k}{m^k} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}\omega_0} = \frac{\mu_0(I)}{\mu(J^G)} \cdot \dim(\lambda^G),$$

using (38). □

7. The central simple algebras case

Let D be a central division algebra of index d over F and ring of integers $\mathfrak{o}_D$, and let $A = A(n')$ denote the algebra of $n' \times n'$ matrices with coefficients in D. Then A is a central simple algebra with centre F of reduced degree $n = \mathrm{d}n'$ and the group of units of A is the group $G' = \mathrm{GL}(n', D)$. In Theorem 7.2 we will prove a transfer of Plancherel measure formula for G': this will be deduced from properties of the Jacquet–Langlands correspondence. In order to do this, we will adapt the proof of [1, (2.5), p. 88] to the case when F is of positive characteristic by using results of Badulescu.

We use the *standard* normalization of Haar measures, in particular $\mu_{G'}$ is normalized so that the volume of $\mathcal{K}' = \mathrm{GL}(n', \mathfrak{o}_D)$ is 1.

7.1. A transfer-of-measure formula

The aim of this subsection is to prove the transfer-of-measure formula stated in Theorem 7.2.

An element x' in G' will be called *semisimple* (resp. *regular semisimple*) if its orbit $O_{G'}(x') = \{yx'y^{-1} \ : \ y \in G'\}$ is a closed subset of G' (resp. if its characteristic polynomial admits only simple roots in an algebraic closure of F). Let G'_{rs} denote the set of regular semisimple elements in G'.

Let $G'_{x'}$ denote the centralizer in G' of x'. Then the group $G'_{x'}$ is unimodular, and the choice of Haar measures on G' and $G'_{x'}$ induces an invariant measure $\mathrm{d}x$ on $G'/G'_{x'}$. The orbital integral of $f' \in C_c(G')$ at x' is defined as

$$\Phi(f', x') = \int_{G'/G'_{x'}} f'(y^{-1}x'y)\,\mathrm{d}y. \tag{44}$$

Since the orbit $O_{G'}(x')$ is closed in G', the integral is absolutely convergent. Indeed, it is a finite sum, since the restriction of f' to $O_{G'}(x')$ is locally constant with compact support. Note that, if $x' \in G'_{\mathrm{rs}}$, then $G'_{x'}$ is a maximal torus in G'.

Orbital integrals have a local expansion, due to Shalika [28], which we will now recall. If O' is a unipotent orbit in G', let $\Lambda_{O'}$ denote the distribution given by integration over the orbit O'. There exist functions $\Gamma_{O'}^{G'}: G'_{\mathrm{rs}} \to \mathbb{R}$ (the *Shalika germs*) indexed by unipotent orbits of G' with the following property:

$$\Phi(f', x') = \sum_{O'} \Gamma_{O'}^{G'}(x') \cdot \Lambda_{O'}(f'), \tag{45}$$

for $x' \in G'_{\mathrm{rs}}$ sufficiently close to the identity. Observe that $\Lambda_1 = f'(1)$.

Harish-Chandra proved that the germ $\Gamma_1^{G'}$ associated to the trivial unipotent orbit is constant, and Rogawski [24] has determined its value assuming the characteristic of F to be zero:

$$\Gamma_1^{G'} = \frac{(-1)^{n-n'}}{d(\mathrm{St}_{G'})}. \tag{46}$$

Equality (46) is still valid in the case when F is of positive characteristic. Indeed, let F be of positive characteristic and let E be a field of zero characteristic sufficiently close to F, that is, such that there exists a ring isomorphism from $\mathfrak{o}_F/\varpi^l\mathfrak{o}_F$ to $\mathfrak{o}_E/\varpi^l\mathfrak{o}_E$, for some sufficiently big integer $l \geqslant 1$. Let D_E be a central division algebra over E with the same index d. Then by Badulescu [4, Lemma 3.8] the lifts f'_E of f' to $G'_E = \mathrm{GL}(m, D_E)$ (resp. f_E of f to $G_E = \mathrm{GL}(n, E)$) also satisfy $f_E \leftrightarrow (-1)^{n-n'} f'_E$. On the other hand, $f'_E(1) = f'(1)$, independently of m: since the way to lift f' to f'_E consists in cutting the group G' into compact open subsets on which f' is constant, in associating to these subsets compact open subsets in G'_E, and assigning to *these* subsets the same constants in order to define f'_E; but the compact open subset of G' containing 1 corresponds to the compact open subset in G'_E containing 1.

If π is a smooth representation of G or G' with finite length, we will denote by θ_π its character.

Theorem 7.1 (*The Jacquet–Langlands correspondence [3,15]*). *There exists a bijection*

$$\mathrm{JL}: E_2(G') \to E_2(G)$$

such that for each $\pi' \in E_2(G')$:

$$\theta_{\pi'}(x') = (-1)^{n-n'}\,\theta_{\mathrm{JL}(\pi')}(x), \tag{47}$$

for any $(x, x') \in G \times G'$ *such that* $x \leftrightarrow x'$.

Recall that $A = A(n')$ denotes the algebra of $n' \times n'$ matrices with coefficients in D. Let $\mathrm{Nrd}_{A|F}: A \to F$ denote the reduced norm of A over F as defined in [7, §12.3, p. 142]. We shall view the reduced norm $\mathrm{Nrd}_{A|F}$ as a homomorphism from G' to $F^\times$.

If η is a quasi-character of $F^\times$ then we will write

$$\eta\pi' = (\eta \circ \mathrm{Nrd}_{A|F}) \otimes \pi'.$$

If η is an unramified quasi-character then we will refer to $\eta\pi'$ as an *unramified twist* of π'.

Each representation π' of G' has a *torsion number*: the order of the cyclic group of all those unramified characters η of $F^\times$ for which

$$\eta\pi' \cong \pi'.$$

The Jacquet–Langlands correspondence has the property that

$$\eta(\mathrm{JL}(\pi')) = \mathrm{JL}(\eta\pi'), \tag{48}$$

for any square integrable representation π' of G' and any (unitary) character η of $F^\times$ (see [15, (4), p. 35]). It follows that the torsion number of π' is equal to that of $\mathrm{JL}(\pi')$.

For each Levi subgroup $M = \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$ of G such that d does not divide n_i for some $i \in \{1, \ldots, k\}$, we have

$$\theta^G_\omega(f) = 0, \quad \text{for any } \omega \in E_2(M)$$

(see the beginning of [4, §3] and the proof of [4, Lemma 3.3]).

We consider now a Levi subgroup M of the form $M = \mathrm{GL}(\mathrm{d}n'_1, F) \times \cdots \times \mathrm{GL}(\mathrm{d}n'_k, F)$, and define $M' = \mathrm{GL}(n'_1, D) \times \cdots \times \mathrm{GL}(n'_k, D)$ (a Levi subgroup of G'): M is the *transfer* of M'. The Jacquet–Langlands correspondence induces a bijection $\mathrm{JL}\colon E_2(M') \to E_2(M)$, by setting

$$\mathrm{JL}(\omega'_1 \otimes \cdots \otimes \omega'_k) = \mathrm{JL}(\omega'_1) \otimes \cdots \otimes \mathrm{JL}(\omega'_k).$$

For any $\omega \in E_2(M)$, there exists $\omega' \in E_2(M')$ such that $\omega = \mathrm{JL}(\omega')$.

Let $\Omega^{\mathrm{t}}(G')$, $\Omega^{\mathrm{t}}(G)$ denote the Harish-Chandra parameter space of G', G. Each point in $\Omega^{\mathrm{t}}(G')$ is a G'-conjugacy class of discrete-series pairs (M', ω') with $\omega' \in E_2(M')$. The topology on $\Omega^{\mathrm{t}}(G')$ is determined by the unramified unitary twists: then $\Omega^{\mathrm{t}}(G')$ is a locally compact Hausdorff space. The map

$$(M', \omega') \mapsto (M, \mathrm{JL}(\omega')),$$

where M is the transfer of M', secures an *injective* map

$$\mathrm{JL}\colon \Omega^{\mathrm{t}}(G') \to \Omega^{\mathrm{t}}(G).$$

We will write $Y = \mathrm{JL}(\Omega^{\mathrm{t}}(G'))$. Since the JL-map respects unramified unitary twists, we obtain a homeomorphism of $\Omega^{\mathrm{t}}(G')$ onto its image:

$$\mathrm{JL} : \Omega^{\mathrm{t}}(G') \cong Y \subset \Omega^{\mathrm{t}}(G).$$

Theorem 7.2 (*Transfer of Plancherel measure*). *Let* $G' = \mathrm{GL}(n', D), G = \mathrm{GL}(n, F)$ *with* $n = \mathrm{d}n'$. *Let* ν', ν *denote the Plancherel measure for* G', G, *each with the standard normalization of Haar measure on* G', G. *Then we have*

$$\mathrm{d}\nu'(\omega') = \lambda(D/F) \cdot \mathrm{d}\nu(\mathrm{JL}(\omega')),$$

where

$$\lambda(D/F) = \prod (q^m - 1)^{-1}$$

the product taken over all m *such that* $1 \leqslant m \leqslant n-1, m \neq 0 \bmod d$.

Proof. If $x \in G$ and $x' \in G'$, we will write $x \leftrightarrow x'$ if x, x' are regular semisimple and have the same characteristic polynomial. If $x \in G$, we will say that x *can be transferred* if there exists $x' \in G'$ such that $x \leftrightarrow x'$.

Let $f' \in C_c(G')$. Then, by [4, Theorem 3.2.], there exists $f \in C_c(G)$ such that

$$\Phi(f, x) = \begin{cases} (-1)^{n-n'} \cdot \Phi(f', x') & \text{for each } x' \in G' \text{ such that } x \leftrightarrow x', \\ 0 & \text{if } x \text{ cannot be transferred}, \end{cases}$$

for any $x \in G_{\mathrm{rs}}$.

It then follows from the germ expansion (45) that

$$f'(1) \cdot \Gamma_1^{G'} = (-1)^{n-n'} \cdot f(1) \cdot \Gamma_1^{G},$$

that is, using (46),

$$\frac{f'(1)}{d(\mathrm{St}_{G'})} = \frac{f(1)}{d(\mathrm{St}_G)}. \tag{49}$$

We recall that $\theta_\omega^G(f) = 0$ on the complement of Y in $\Omega^{\mathrm{t}}(G)$. Next, we use Eq. (49), and apply twice the Harish-Chandra Plancherel theorem, first for G', then for G. We obtain

$$\begin{aligned} \int \theta_{\omega'}^{G'}(f')\, \mathrm{d}\nu'(\omega') &= f'(1) \\ &= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot f(1) \end{aligned}$$

$$= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot \int \theta_{\omega}^{G}(f)\,\mathrm{d}\nu(\omega)$$

$$= d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot \int \theta_{\omega}^{G}(f)\,\mathrm{d}\nu|_Y(\omega) \tag{50}$$

for all $f' \in C_c(G')$.

We recall that the parameter space $\Omega^{\mathrm{t}}(G')$ is the *domain* of the Plancherel measure ν'.

By the refinement of the trace Paley–Wiener theorem due to Badulescu [4, Lemma 3.4] we have

$$\{\omega' \mapsto \theta_{\omega'}^{G'}(f'^{\vee}) : f' \in C_c(G'), \omega' \in \Omega^{\mathrm{t}}(G')\} = L(\Omega^{\mathrm{t}}(G')),$$

where $L(\Omega^{\mathrm{t}}(G'))$ is the space of compactly supported functions on $\Omega^{\mathrm{t}}(G')$ which, upon restriction to each connected component (a quotient of a compact torus $\mathbb{T}'^k$ by a product of symmetric groups), are Laurent polynomials in the co-ordinates $(z_1, z_2, \ldots, z_k)$.

Now $L(\Omega^{\mathrm{t}}(G'))$ is a dense subspace of $C_0(\Omega^{\mathrm{t}}(G'))$, the continuous complex-valued functions on $\Omega^{\mathrm{t}}(G')$ which vanish at infinity. On the other hand, it follows from [4, Proposition 3.6] that

$$\theta_{\omega'}^{G'}(f') = \theta_{\mathrm{JL}(\omega')}^{G}(f) \quad \text{for any } \omega' \in E_2(M'). \tag{51}$$

Eq. (50) therefore provides us with two Radon measures (continuous linear functionals) which agree on a *dense subspace* of $C_0(\Omega^{\mathrm{t}}(G'))$. Therefore the measures are equal:

$$\mathrm{d}\nu'(\omega') = d(\mathrm{St}_{G'}) \cdot d(\mathrm{St}_G)^{-1} \cdot \mathrm{d}\nu|_Y(\omega). \tag{52}$$

At this point, we have to elucidate a normalization issue. Let $K' = \mathrm{GL}(n', \mathfrak{o}_D)$. The group $A_{G'}$ by definition is the F-split component of the centre of G' and can be identified with $F^{\times}$. As in Section 6.2, we have $F^{\times}K'/F^{\times} = K'/K' \cap F^{\times} = K'/\mathfrak{o}_F^{\times}$. But the Haar measure on $A_{G'}$ has, as in [34, p. 240], the standard normalization $\mathrm{mes}(K' \cap A_{G'}) = 1$, i.e., $\mathrm{mes}(\mathfrak{o}_F^{\times}) = 1$. Since $\mathrm{mes}(K') = 1$, we have $\mathrm{mes}(F^{\times}K'/F^{\times}) = 1$. It follows (see for instance [31, 3.7]) that the formal degree of the Steinberg representation $\mathrm{St}_{G'}$ is given by

$$d(\mathrm{St}_{G'}) = \frac{1}{n} \prod_{j=1}^{n'-1} (q^{dj} - 1).$$

We then have

$$\mathrm{d}\nu'(\omega') = \lambda(D/F) \cdot \mathrm{d}\nu(\omega), \tag{53}$$

where

$$\lambda(D/F) = (q^d - 1)(q^{2d} - 1)\cdots(q^{(n'-1)d} - 1)(q-1)^{-1}(q^2-1)^{-1}\cdots(q^{n-1}-1)^{-1},$$

so that

$$\lambda(D/F) = \prod (q^m - 1)^{-1} \tag{54}$$

the product taken over all m such that $1 \leqslant m \leqslant n-1$, $m \neq 0 \bmod d$. □

This result may be expressed as follows:

Theorem 7.3. *Let* $(\Omega^{\mathrm{t}} G', \mathcal{B}', \nu')$ *be the measure space determined by the Plancherel measure* ν', *let* $(Y, \mathcal{B}, \lambda(D/F)\cdot\nu|_Y)$ *be the measure space determined by the restriction of* $\lambda(D/F)\cdot\nu$ *to* $Y = \mathrm{JL}(\Omega^{\mathrm{t}}(G') \subset \Omega^{\mathrm{t}}(G)$. *Then these two measure spaces are isomorphic*:

$$(\Omega^{\mathrm{t}} G', \mathcal{B}', \nu') \cong (Y, \mathcal{B}, \lambda(D/F)\cdot\nu|_Y).$$

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