Journal of Number Theory 133 (2013) 3007–3032

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On higher order Stickelberger-type theorems

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ARTICLE INFO

Article history:
Received 22 November 2012
Revised 8 January 2013
Accepted 28 February 2013
Available online 14 May 2013
Communicated by D. Burns

Keywords:
Stickelberger's theorem
Stark's conjecture
L-functions
Annihilation
Class groups

ABSTRACT

We discuss an explicit refinement of Rubin's integral version of Stark's conjecture. We prove that this refinement is a consequence of the relevant case of the Equivariant Tamagawa Number Conjecture of Burns and Flach, hence obtaining a full proof in several important cases. We also derive several explicit consequences of this refinement concerning the annihilation as Galois modules of ideal class groups by explicit elements constructed from the values of higher order derivatives of Dirichlet L-functions. We finally describe the relation between our approach and those found in recent work of Emmons and Popescu and of Buckingham.

1. Introduction

1.1. Stark's conjecture and Rubin's integral refinement

Let K/k be a finite abelian extension of global fields with Galois group G and set $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$. Let S be a finite non-empty set of places of k containing all archimedean places (if any) and all those that ramify in the extension K/k.

We define the S-truncated $\mathbb{C}[G]$-valued L-function of K/k by setting

$$\theta_{K/k,S}(s) := \sum_{\chi\in\widehat{G}} L_{K/k,S}\left(s, \chi^{-1}\right) e_\chi,$$

where $L_{K/k,S}(s, \chi)$ denotes the S-truncated Dirichlet L-function at χ and $e_\chi := (1/|G|)\sum_{g\in G}\chi(g)g^{-1}$ is the idempotent at χ. Its leading term at $s = 0$ is then equal to

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.02.015

$$\theta^*_{K/k,S}(0) := \sum_{\chi \in \widehat{G}} L^*_{K/k,S}\big(0, \chi^{-1}\big) e_\chi,$$

where $L^*_{K/k,S}(0,\chi)$ is the leading coefficient in the Taylor expansion of $L_{K/k,S}(s,\chi)$ at 0. It is easily shown that $\theta^*_{K/k,S}(0)$ belongs to $\mathbb{R}[G]^\times$.

To remove transcendence from this element however requires an appropriate regulator. To do this we write $Y_{K,S}$ for the free abelian group on the set $S(K)$ of places of K which lie above those in S and $X_{K,S}$ for the kernel of the homomorphism $Y_{K,S} \to \mathbb{Z}$ that sends each element of $S(K)$ to 1 (both with natural actions of G). We write $\mathcal{O}_{K,S}$ for the ring of $S(K)$-integers in K and $\mathcal{O}^\times_{K,S}$ for its group of units. Let also $R_{K,S} : \mathcal{O}^\times_{K,S} \otimes_{\mathbb{Z}} \mathbb{R} \to X_{K,S} \otimes_{\mathbb{Z}} \mathbb{R}$ denote the Dirichlet regulator isomorphism, which at each element u of $\mathcal{O}^\times_{K,S}$ satisfies

$$R_{K,S}(u) = -\sum_{w \in S(K)} \log |u|_w \cdot w,$$

with $|\cdot|_w$ denoting the normalised absolute value of the place w.

Then Stark's conjecture (as reformulated by Tate in [39, Chapter 1, Conjecture 5.1]) asserts that for each $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S}, X_{K,S})$ we have

$$\theta^*_{K/k,S}(0) \cdot \det_{\mathbb{R}[G]}\big(R^{-1}_{K,S} \circ (\phi \otimes_{\mathbb{Z}} \mathbb{R})\big) \in \mathbb{Q}[G].$$

In order to study integral properties of the above conjectural elements of $\mathbb{Q}[G]$ it is convenient to make certain technical assumptions and modifications. Namely, we follow Rubin [35] in fixing a second finite non-empty set T of places of k that is disjoint from S and we write $\mathcal{O}^\times_{K,S,T}$ for the (finite index) subgroup of $\mathcal{O}^\times_{K,S}$ consisting of those elements that are congruent to 1 modulo all places in $T(K)$. Since each place in T is then in particular both non-archimedean and unramified in K/k, we may define an element

$$\delta_T := \prod_{v \in T} \big(1 - \mathrm{N}v \cdot \mathrm{Fr}_v^{-1}\big)$$

of $\mathbb{Z}[G]$ where $\mathrm{N}v$ denotes the absolute norm of v and Fr_v denotes its Frobenius automorphism in G. We also note that, since $\mathcal{O}^\times_{K,S} \otimes_{\mathbb{Z}} \mathbb{R}$ is canonically isomorphic to $\mathcal{O}^\times_{K,S,T} \otimes_{\mathbb{Z}} \mathbb{R}$, we may and will by abuse of notation consider $R_{K,S}$ as a map from $\mathcal{O}^\times_{K,S,T} \otimes_{\mathbb{Z}} \mathbb{R}$ to $X_{K,S} \otimes_{\mathbb{Z}} \mathbb{R}$. In order to simplify matters, we elect to focus separately on the leading terms of each of the classes of Wedderburn components of the equivariant meromorphic function $\theta_{K/k,S}(s)$ which are characterised by having the same order of vanishing at $s = 0$, and for this purpose we once again follow Rubin in fixing a non-negative integer r and then, with respect to r, making the following assumptions on the sets S and T: we assume that S contains at least r places which split completely in K/k and has cardinality strictly greater than r, and that $\mathcal{O}^\times_{K,S,T}$ is torsion-free. We then know by [39, Chapter I, Proposition 3.4] that $s^{-r}\theta_{K/k,S}(s)$ is holomorphic at $s = 0$ and we proceed to isolate the relevant leading terms 'of order r' by defining an element

$$\theta^{(r)}_{K/k,S}(0) := \frac{1}{r!}\left(\frac{d}{ds}\right)^r \theta_{K/k,S}(s)\bigg|_{s=0}$$

of $\mathbb{C}[G]$ and then also, in order to avoid certain technical difficulties related to the roots of unity of K, a 'T-modified' version

$$\theta^{(r)}_{K/k,S,T}(0) := \delta_T \cdot \theta^{(r)}_{K/k,S}(0).$$

Rubin's conjecture [35, Conjecture B'] for the set of data $(K/k, S, T, r)$ satisfying the above hypotheses can then be reformulated as the assertion that, for each $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,S,T}, X_{K,S})$, one has that

$$\theta^{(r)}_{K/k,S,T}(0) \cdot \det_{\mathbb{R}[G]}\bigl(R^{-1}_{K,S} \circ (\phi \otimes_{\mathbb{Z}} \mathbb{R})\bigr) \in \mathbb{Z}[G]. \tag{1}$$

We refer the reader to Theorem 6.4 (i) and Remark 6.3 below for a proof of the fact that this restatement is indeed equivalent to Rubin's original formulation. We recall that Rubin shows in [35, Proposition 2.5] that this conjecture specialises to recover the conjecture 'over $\mathbb{Z}$' formulated by Stark in [38] and that results of Rubin [35, Proposition 2.4] and Popescu [32, Theorem 5.5.1] also combine to imply that Rubin's conjecture implies the validity of the conjecture formulated by Popescu in [32].

Our attempts in this article to construct integral annihilators of class groups from elements of the form $\theta^{(r)}_{K/k,S,T}(0)$ will lead us to conjecture (see Theorem 2.1 below) that a statement stronger than (1) should hold: namely, we explicitly define a natural arithmetic $\mathbb{Z}[G]$-module whose Fitting ideal, which is often strictly contained in $\mathbb{Z}[G]$, we expect to contain all elements of the form $\theta^{(r)}_{K/k,S,T}(0) \cdot \det_{\mathbb{R}[G]}(R^{-1}_{K,S} \circ (\phi \otimes_{\mathbb{Z}} \mathbb{R}))$, and furthermore we work with more general regulators than those which occur in (1). We obtain a full proof of our stronger statement in several important cases (see Corollary 2.4 below). It is also worth pointing out that, although Rubin formulates his conjecture specifically in the number field setting, we make no distinction with the global function field setting in which, in particular, our refinement of the natural analogue of Rubin's conjecture given by (1) is proved to hold unconditionally.

1.2. Stickelberger's theorem, Brumer's conjecture and annihilation of class groups

For a finite abelian extension K with group of roots of unity μ_K of a number field k and any finite non-empty set S of places of k containing all archimedean places and all those that ramify in the extension K/k, Brumer's conjecture asserts that any element of $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\theta_{K/k,S}(0)$ belongs to $\mathbb{Z}[G]$ and furthermore annihilates the ideal class group of K. We note first that the former assertion has been known to hold for some time now by independent results of Cassou-Noguès [17] and of Deligne and Ribet [19]. This conjecture has been studied extensively in the last decades by, amongst others, Burns, Greither, Popescu and Wiles. There is in particular a large body of supporting evidence (see, for example, the expository article [23]). Furthermore, Burns [10] and Nickel [31] have recently independently formulated a non-abelian generalisation of Brumer's conjecture. In addition, there has been recent extensive study of a refinement of Brumer's conjecture, the 'Brumer–Stark conjecture' (see the work of Popescu in [34] for a statement of this conjecture and a discussion of its connection to Brumer's). In particular, in [25], Greither and Popescu prove the validity of a refinement of the Brumer–Stark conjecture (away from its 2-primary part and under the assumed vanishing of the relevant classical μ-invariants) in the number field setting while, in [24], motivated by the proof by Deligne and Tate of a natural function field analogue of the Brumer–Stark conjecture (see [39, Chapter V]), they also prove the validity of the natural function field analogue of their refined conjecture for number fields.

We also recall that, if $k = \mathbb{Q}$ and S contains only the archimedean place of $\mathbb{Q}$ and those places that ramify in $K/\mathbb{Q}$, then Brumer's conjecture specialises to recover the classical theorem of Stickelberger. However, in many situations of interest, the element $\theta_{K/k,S}(0)$ vanishes and therefore neither Brumer's conjecture nor Stickelberger's theorem can provide any information whatsoever on the object of interest, namely the class group of K. This is for instance the case whenever K is a totally real number field (because then all archimedean places split completely in K/k).

One of the main aims of this article is to construct integral annihilators of class groups which both apply in more general settings, namely to arbitrary finite abelian extension of global fields, and are non-trivial much more often than those predicted by Brumer's conjecture. The approach we follow is to consider the values at $s = 0$ of higher order derivatives of L-functions, normalised by appropriate regulators (as discussed in Section 1.1), rather than simply the values of the L-functions themselves.

The link between elements of $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\theta_{K/k,S}(0)$ and those elements occurring in (1) is given by the following facts: $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated over $\mathbb{Z}[G]$ by $\{\delta_T : T\}$, where T runs over those sets of places of k satisfying the hypotheses required in Section 1.1 (see, for instance, [33, Lemma 1.2.2]), while one easily shows that, for any ϕ as above,

$$\theta^{(0)}_{K/k,S,T}(0)\cdot \det_{\mathbb{R}[G]}\bigl(R^{-1}_{K,S}\circ(\phi\otimes_{\mathbb{Z}}\mathbb{R})\bigr)=\theta^{(0)}_{K/k,S,T}(0)=\delta_T\cdot\theta^{(0)}_{K/k,S}(0)=\delta_T\cdot\theta_{K/k,S}(0).$$

Since, for $r=0$, S automatically contains at least r places which split completely in K/k and has cardinality strictly greater than r, Brumer's conjecture is equivalent to the $r=0$ case of the assertion that for sets S and T satisfying Rubin's hypotheses the element $\theta^{(r)}_{K/k,S,T}(0)\cdot\det_{\mathbb{R}[G]}(R^{-1}_{K,S}\circ(\phi\otimes_{\mathbb{Z}}\mathbb{R}))$ belongs to $\mathbb{Z}[G]$ and annihilates the ideal class group of K for each $\phi\in\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,S,T},X_{K,S})$.

Our work will lead us to obtain strong evidence, including a full proof in several important cases, in support of statements about the annihilation of certain modifications of the ideal class group of K, such as the ideal class group of $\mathcal{O}_{K,S}$ and certain natural quotients of the class group of K itself, by elements of the form $\theta^{(r)}_{K/k,S,T}(0)\cdot\det_{\mathbb{R}[G]}(R^{-1}_{K,S}\circ(\phi\otimes_{\mathbb{Z}}\mathbb{R}))$ (see Corollaries 2.3 and 2.4 below). It is in particular motivated by the following question.

Question 1.1 *(Burns).* Let K/k be a finite abelian extension of global fields with Galois group G and let S be a finite non-empty set of places of k containing all archimedean places (if any) and all those that ramify in the extension K/k. Define $r:=\min\{\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi)\colon\ \chi\in\widehat{G}\}$. Then, for any $\phi\in\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,S},X_{K,S})$, does every element of

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\cdot\theta^{(r)}_{K/k,S}(0)\cdot\det_{\mathbb{R}[G]}\bigl(R^{-1}_{K,S}\circ(\phi\otimes_{\mathbb{Z}}\mathbb{R})\bigr)$$

belong to $\mathbb{Z}[G]$ and annihilate the ideal class group of $\mathcal{O}_{K,S}$?

The reader can find more details about similar questions in [10].

In [30], the present author already refines work of Rubin and Sands in [35] and [36] respectively to prove similar annihilation statements for the ideal class group of $\mathcal{O}_{K,S}$ for a wide range of multi-quadratic extensions.

2. Statement of the main results

We use the following notational conventions: Given a commutative ring R, an R-module M and a homomorphism of R-modules λ, we set $\wedge^0_R M:=R$ and let $\wedge^0_R\lambda$ denote the identity automorphism of R.

As mentioned in the introduction, we aim to be able to use regulators which are more general than those in terms of which Stark's and Rubin's conjectures can be formulated. For this purpose, and for any finite non-empty set T of places of k that is disjoint from S, any non-negative integer t and any $\Phi\in\mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^t_{\mathbb{Z}[G]}\mathcal{O}^{\times}_{K,S,T},\wedge^t_{\mathbb{Z}[G]}X_{K,S})$, we set

$$R(\Phi):=\det_{\mathbb{R}[G]}\bigl(\wedge^t_{\mathbb{R}[G]}R^{-1}_{K,S}\circ(\Phi\otimes_{\mathbb{Z}}\mathbb{R})\bigr).$$

Note that our notation differs from the similar one used by Tate in [39] in that, if $\phi\in\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,S},X_{K,S})$ has finite kernel and cokernel, which is necessary to define Tate's $R(\phi,\chi)$ for $\chi\in\widehat{G}$, and ϕ' denotes the restriction of ϕ to $\mathcal{O}^{\times}_{K,S,T}$, then our $R(\phi')$ (taking $t=1$, obviously) is the inverse of the element $\sum_{\chi\in\widehat{G}}R(\phi,\chi)e_\chi$ of $\mathbb{R}[G]^{\times}$ in Tate's notation.

We now fix a non-negative integer r and, following Rubin [35], assume throughout that there is a subset S_1 strictly contained in S of cardinality r comprising places which split completely in K/k. Note that in particular, by [39, Chapter I, Proposition 3.4], this assumption ensures that r is less than or equal to $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi)$ for every character $\chi \in \widehat{G}$. We further assume that the auxiliary set T is chosen so that $\mathcal{O}_{K,S,T}^{\times}$ is torsion-free, and note that in the function field case this condition is automatically satisfied while in the number field case it is satisfied whenever, for example, not all the places in T have the same residue characteristic.

Given a Dedekind Domain $\mathcal{O}$, we write $\mathrm{Cl}(\mathcal{O})$ for its ideal class group. We also write $\mathrm{Cl}^T(\mathcal{O}_{K,S})$ for the quotient of the group of fractional ideals of $\mathcal{O}_{K,S}$ that are coprime to all places in $T(K)$ by the subgroup of principal ideals with a generator congruent to 1 modulo all places in $T(K)$.

Theorem 2.1. *Assume that the central conjecture (Conjecture* $\mathrm{C}(K/k)$*) of [16] is valid for K/k. Then there exists a natural arithmetic G-module $H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'$ which contains a submodule isomorphic to* $\mathrm{Cl}^T(\mathcal{O}_{K,S})$ *and has the property that, for each element Φ of* $\mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r_{\mathbb{Z}[G]}\mathcal{O}^{\times}_{K,S,T}, \wedge^r_{\mathbb{Z}[G]}X_{K,S})$*, we have*

$$\theta^{(r)}_{K/k,S,T}(0)R(\Phi) \in \mathrm{Fitt}_{\mathbb{Z}[G]}\bigl(H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'\bigr). \tag{2}$$

Remark 2.2.

(i) In the function field case [5, Remark 3] shows that Conjecture $\mathrm{C}(K/k)$ is equivalent to the conjecture formulated in [5, §2.2]. In the number field case, $\mathrm{C}(K/k)$ is equivalent to the Equivariant Tamagawa Number Conjecture ('ETNC' for brevity) of [13, Conjecture 4 (iv)] for the pair $(h^0(\mathrm{Spec}(K)),\mathbb{Z}[G])$. In particular, from these equivalences and [5, Remark 2], resp. [14, §2], one finds that in the function field, resp. number field case, $\mathrm{C}(K/k)$ is an equivariant version of the relevant case of Lichtenbaum's conjecture [29, Conjecture 8.1e)], resp. is a version without 'sign ambiguities' of the form discussed in [13, Remark 9] of the relevant case of Kato's 'generalised Iwasawa main conjecture' [26, Conjecture 3.2.2]. It is known to be valid in several important cases (see Corollary 2.4 below). For more details concerning explicit consequences of the validity of the ETNC for various motives, see [8].

(ii) Theorem 2.1 makes it reasonable to conjecture that (2) is valid for all such Φ. Given $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,S,T}, X_{K,S})$, we show in the proof of Corollary 4.2 below that $\theta^{(r)}_{K/k,S,T}(0)R(\phi) = \theta^{(r)}_{K/k,S,T}(0)R(\wedge^r_{\mathbb{Z}[G]}\phi)$. In particular, it follows that

$$\begin{aligned}&\bigl\{\theta^{(r)}_{K/k,S,T}(0)R(\phi)\colon\ \phi \in \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(\mathcal{O}^{\times}_{K,S,T}, X_{K,S}\bigr)\bigr\}\\ &\quad\subseteq \bigl\{\theta^{(r)}_{K/k,S,T}(0)R(\Phi)\colon\ \Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(\wedge^r_{\mathbb{Z}[G]}\mathcal{O}^{\times}_{K,S,T}, \wedge^r_{\mathbb{Z}[G]}X_{K,S}\bigr)\bigr\}.\end{aligned}$$

The reformulation of Rubin's integral refinement of Stark's conjecture given by Theorem 6.4 (i) and Remark 6.3 below, and already given in (1), makes it clear that this conjecture is stronger than Rubin's, and hence also than the conjecture formulated by Popescu in [32].

(iii) The G-module $H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'$ will be explicitly defined in Section 4. The fact that it contains a submodule isomorphic to $\mathrm{Cl}^T(\mathcal{O}_{K,S})$ will allow us to relate (2) to the desired statements about annihilation of class groups, as discussed in the introduction (see Corollary 2.3 below). In fact, it has a natural interpretation in terms of the Weil-étale cohomology theory of the sheaf $\mathbb{G}_m$ on $\mathrm{Spec}(\mathcal{O}_{K,S})$ (conjectural in the number field case to the existence of such a theory as predicted by Lichtenbaum). We will however not use this fact in the sequel.

To describe some explicit consequences of Theorem 2.1 we define a natural quotient of the ideal class group $\mathrm{Cl}(\mathcal{O}_K)$ of K. We label, and thereby order, the elements of S_1 as $\{v_i\colon\ 1 \leqslant i \leqslant r\}$, and for each i between 1 and r fix a place w_i of K above v_i. We set $S_0 := S \setminus S_1$, $S_{0,f} :=$ {non-archimedean places in S_0} and $c_0 := [\prod_{w \in S_{0,f}(K)} w] \in \mathrm{Cl}(\mathcal{O}_K)$ and then define a G-module by setting

$$\mathrm{Cl}(\mathcal{O}_K)_S := \mathrm{Cl}(\mathcal{O}_K)/\bigl(\bigl\langle\bigl[g(w_i)\bigr]\colon\ 1 \leqslant i \leqslant r,\ w_i \text{ non-arch.},\ g \in G\bigr\rangle\langle c_0\rangle\bigr).$$

We note in particular that if either $r = 0$ or all places v_i with $1 \leqslant i \leqslant r$ are archimedean, then $\mathrm{Cl}(\mathcal{O}_K)_S$ is simply equal to the quotient of $\mathrm{Cl}(\mathcal{O}_K)$ by the subgroup generated by the single element c_0.

In the following result we let $\mathbb{Z}_{(p)}$ denote the localisation of $\mathbb{Z}$ at p for any prime number p and, for any $\mathbb{Z}[G]$-module M, we set $M_{(p)} := M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Corollary 2.3. *Assume the notation and hypotheses of Theorem 2.1 and fix $\Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S,T}, \wedge^r_{\mathbb{Z}[G]} X_{K,S})$. Then all of the following claims are valid:*

(i) *The element $\theta^{(r)}_{K/k,S,T}(0)R(\Phi)$ of $\mathbb{Z}[G]$ annihilates the module $\mathrm{Cl}^T(\mathcal{O}_{K,S})$ and therefore also the module $\mathrm{Cl}(\mathcal{O}_{K,S})$.*
(ii) *The element $|G|\theta^{(r)}_{K/k,S,T}(0)R(\Phi)$ of $\mathbb{Z}[G]$ annihilates the module $\mathrm{Cl}(\mathcal{O}_K)_S$.*
(iii) *If p is any prime that does not divide $|G|$, then*

$$\theta^{(r)}_{K/k,S,T}(0)R(\Phi) \in \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathrm{Cl}(\mathcal{O}_K)_S\big)_{(p)}.$$

We also record that in several important cases the results of Theorem 2.1 and Corollary 2.3 are unconditional.

Corollary 2.4. *The results of Theorem 2.1 and Corollary 2.3 are unconditional in each of the following cases:*

(i) *K is a global function field.*
(ii) *K is a finite abelian extension of $\mathbb{Q}$.*

Furthermore, the results of Theorem 2.1 and Corollary 2.3 become unconditional after applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ to them in the following case:

(iii) *There exists an imaginary quadratic field F of class number one such that $F \subseteq k$, K/F is finite abelian and the degree of K/k is both odd and divisible only by primes which split in $F/\mathbb{Q}$.*

Proof. The key point here is that the central conjecture of [16] is known to be valid in each of the above cases. Indeed, if K is a global function field, it is proved by Burns (see [9, Theorem 1.1]). If K is a finite abelian extension of $\mathbb{Q}$, it follows from results of Burns and Greither (see [15, Theorem 8.1, Remark 8.1]) and Flach (see [21]). If there exists an imaginary quadratic field F of class number one such that $F \subseteq k$, K/F is finite abelian and the degree of K/k is both odd and divisible only by primes which split in $F/\mathbb{Q}$, then a result of Bley (see [2, Theorem 4.2]) implies the validity of the p-primary part of Conjecture $\mathrm{C}(K/k)$ for every prime $p \neq 2, 3$ (Bley works with odd primes but the prime 3 also has to be excluded because Bley's theorem depends on a result of Gillard in [22] on the vanishing of a certain μ-invariant which requires $p > 3$). But one can run through every single one of our arguments after applying the exact functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ and they remain valid. □

Remark 2.5. The results of Theorem 2.1 and Corollary 2.3 are also unconditional if K/k is a quadratic extension since the central conjecture of [16] is also known to be valid as a consequence of results of Kim in [27, (2.4) Proposition (a)], Burns in [4] and Tate in [39, II §6.8] (where the validity of the Strong Stark Conjecture for rational-valued characters is proved). For quadratic extensions, Theorem 2.1 and Corollary 2.3 therefore refine the result [30, Theorem 1.3] obtained by the present author concerning explicit annihilators of the ideal class group of $\mathcal{O}_{K,S}$.

Remark 2.6. If $k = \mathbb{Q}$ (so in particular K is a finite abelian extension of $\mathbb{Q}$ and the results of Corollary 2.3 are unconditional), we take $r = 0$, and S comprises the archimedean place of $\mathbb{Q}$ and all the places which ramify in $K/\mathbb{Q}$, then $\theta^{(r)}_{K/k,S}(0)$ is the classical Stickelberger element of $K/\mathbb{Q}$ and $\{R(\Phi)\colon\ \Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S,T}, \wedge^r_{\mathbb{Z}[G]} X_{K,S})\} = \mathbb{Z}[G]$, whilst $\mathrm{Cl}(\mathcal{O}_K)_S$ is just the quotient of $\mathrm{Cl}(\mathcal{O}_K)$

by the subgroup generated by the single element c_0 (which is in this case the class of the product of the primes of K that ramify in $K/\mathbb{Q}$). So in this case Corollary 2.3 recovers a weak version of the classical Stickelberger theorem.

In [20], Emmons and Popescu extend Rubin's conjecture to situations in which, although r is less than or equal to $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi)$ for every character $\chi \in \widehat{G}$, the set S need not contain r places which split completely, in a way which exactly recovers Rubin's original conjecture whenever it does. In Section 6.1, we explore the connection between the (conjectural) containment (2) and their conjecture under their more general hypotheses and, in the process, establish the fact that Rubin's conjecture can be reformulated in precisely the way claimed in Section 1.1.

In Section 6.2, we explore the connection between the set

$$\mathcal{I}_{S,T,r} := \bigl\{\theta^{(r)}_{K/k,S,T}(0)R(\phi)\colon \phi \in \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(\mathcal{O}^\times_{K,S,T}, X_{K,S}\bigr)\bigr\}$$

and the 'fractional Galois ideal' $\mathcal{J}(K/k,S,T)$ defined by Buckingham in [3]. This module was defined in a (somewhat analogous to ours) attempt to construct annihilators of class groups which are non-trivial more often than those given by Brumer's conjecture or Stickelberger's theorem. In particular, Buckingham proves that the validity of Rubin's conjecture would imply that, after inverting the order of G and restricting to e_r components, where $e_r := \sum_\chi e_\chi$ is the idempotent of $\mathbb{Q}[G]$ obtained by letting the sum run over all elements χ of $\widehat{G}$ such that $\dim_{\mathbb{C}}(e_\chi(\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{O}^\times_{K,S})) = r$, the fractional Galois ideal recovers the Fitting ideal of $\mathrm{Cl}^T(\mathcal{O}_{K,S})$ (see [3, Proposition 6.1]). In Theorem 6.5 below we prove that

$$\mathcal{I}_{S,T,r} \subseteq e_r\mathcal{J}(K/k,S,T). \tag{3}$$

This comparative statement is interesting because, since Buckingham required inverting the order of G in order to obtain conjectural annihilators of $\mathrm{Cl}^T(\mathcal{O}_{K,S})$, it combines with Corollary 2.3 to suggest that one may prefer to restrict attention to those elements which actually belong to $\mathcal{I}_{S,T,r}$. Furthermore, it combines with Buckingham's aforementioned [3, Proposition 6.1] to imply the following result which, although in the spirit of Corollary 2.3 and only dealing with primes not dividing the order of G, has the advantage of not being conditional to the validity of the whole of the central Conjecture $\mathrm{C}(K/k)$ of [16] but rather to the validity of the weaker conjecture of Rubin.

Proposition 2.7. *Assume that the set $\mathcal{I}_{S',T,r}$ is contained in $\mathbb{Z}[G]$ for every finite non-empty set S' of places of k containing all archimedean places (if any) and all those that ramify in the extension K/k which furthermore satisfies Rubin's hypotheses with respect to a fixed choice of T and r. Then, for every prime p which does not divide $|G|$, the set $\mathcal{I}_{S,T,r}$ is also contained in $e_r\,\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(\mathcal{O}_{K,S}))_{(p)}$.*

Proof. The claimed result follows directly from the following three facts: the fact that Rubin's conjecture [35, Conjecture B′] for the set of data (S',T,r) is equivalent to the containment $\mathcal{I}_{S',T,r} \subseteq \mathbb{Z}[G]$, as proved in Theorem 6.4 (i) and Remark 6.3 below; the containment (3) proved in Theorem 6.5 below; and Buckingham's result [3, Proposition 6.1]. □

3. The Conjecture $\mathrm{C}(K/k)$

In this section we first state explicitly the Conjecture $\mathrm{C}(K/k)$. We then construct an explicit representative of a 'sufficiently good approximation' to a perfect complex of $\mathbb{Z}[G]$-modules which occurs in $\mathrm{C}(K/k)$ and reinterpret $\mathrm{C}(K/k)$ in terms of this approximation. We will then in the next section combine this reinterpretation with a result of Burns in [6] to derive an explicit consequence of $\mathrm{C}(K/k)$ which will be crucial in our proof of Theorem 2.1.

In the sequel, unless explicitly indicated otherwise by context, unadorned tensor products are regarded as taken in the category of abelian groups. Unadorned exterior powers are regarded as taken in the category of either $\mathbb{Z}[G]$-modules, $\mathbb{Q}[G]$-modules, $\mathbb{R}[G]$-modules or $\mathbb{C}[G]$-modules, and the particular category will always be clear from the context. Given a commutative ring R and an R-module M, whenever we let an expression of the form $\wedge_{i=1}^{i=0} m_i$ denote an element of $\wedge_R^0 M = R$, we will always take it to be the element 1 in R. For any abelian group A we write A_{tor} for its torsion subgroup and set $A_{\mathrm{tf}} := A/A_{\mathrm{tor}}$ (regarded as a sublattice of the vector space $\mathbb{Q}\cdot A$ spanned by A).

3.1. Preliminaries

3.1.1. Étale cohomology

We write $\mathcal{D}(\mathbb{Z}[G])$ for the derived category of complexes of G-modules. For any object $C^\bullet$ of $\mathcal{D}(\mathbb{Z}[G])$ with differential d^i in each degree i and any integer m we write $C^\bullet[m]$ for the complex which is equal to C^{i+m} in each degree i and for which the differential in degree i is equal to $(-1)^m d^{i+m}$. For any G-module M and integer m we write $M[m]$ for the complex which is equal to M in degree $-m$ and is equal to 0 in all other degrees.

Let Σ be any finite non-empty set of places of k containing all archimedean places and all those that ramify in the extension K/k. If K is a global function field, with associated smooth projective curve C_K, then we set

$$R\Gamma_{\mathrm{c,\acute{e}t}}\big(\mathrm{Spec}(\mathcal{O}_{K,\Sigma}),\mathbb{Z}\big) := R\Gamma_{\mathrm{\acute{e}t}}(C_K, \iota_!\mathbb{Z})$$

where ι is the natural open immersion $\mathrm{Spec}(\mathcal{O}_{K,\Sigma}) \to C_K$ and $\mathbb{Z}$ is the constant étale sheaf on $\mathrm{Spec}(\mathcal{O}_{K,\Sigma})$. If K is a number field, then we follow [12, (3)] in defining the 'cohomology with compact support' of the constant étale sheaf $\mathbb{Z}$ on $\mathrm{Spec}(\mathcal{O}_{K,\Sigma})$ by setting

$$R\Gamma_{\mathrm{c,\acute{e}t}}\big(\mathrm{Spec}(\mathcal{O}_{K,\Sigma}),\mathbb{Z}\big) := \mathrm{cone}\Big(R\Gamma_{\mathrm{\acute{e}t}}\big(\mathrm{Spec}(\mathcal{O}_{K,\Sigma}),\mathbb{Z}\big) \to \bigoplus_{w\in\Sigma} R\Gamma_{\mathrm{\acute{e}t}}\big(\mathrm{Spec}(K_w), j_w^*(\mathbb{Z})\big)\Big)[-1]$$

where K_w is the completion of K at w, j_w the natural morphism $\mathrm{Spec}(K_w) \to \mathrm{Spec}(\mathcal{O}_{K,\Sigma})$ and the mapping cone is defined using Godement resolutions.

Lemma 3.1. *For any finite non-empty set Σ of places of k containing all archimedean places and all those that ramify in the extension K/k, there exists a complex $\widetilde{\Psi}_\Sigma^\bullet$ of $\mathbb{Z}[G]$-modules of the form*

$$\Psi_\Sigma^0 \xrightarrow{\delta_\Sigma} \Psi_\Sigma^1 \to X_{K,\Sigma}\otimes\mathbb{Q}$$

which has both of the following properties:

(i) *There exists a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form*

$$\widetilde{\Psi}_\Sigma^\bullet \to \mathrm{Hom}_{\mathbb{Z}}\big(R\Gamma_{\mathrm{c,\acute{e}t}}\big(\mathrm{Spec}(\mathcal{O}_{K,\Sigma}),\mathbb{Z}\big),\mathbb{Q}/\mathbb{Z}[-3]\big) \to \widehat{\mathcal{O}}_{K,\Sigma}^\times/\mathcal{O}_{K,\Sigma}^\times[0] \to \widetilde{\Psi}_\Sigma^\bullet[1]$$

where $\widehat{\mathcal{O}}_{K,\Sigma}^\times$ denotes the profinite completion of $\mathcal{O}_{K,\Sigma}^\times$ and the second arrow is the unique morphism in $\mathcal{D}(\mathbb{Z}[G])$ which induces upon cohomology (in degree 0) the composite of the canonical identification

$$H^0\big(\mathrm{Hom}_{\mathbb{Z}}\big(R\Gamma_{\mathrm{c,\acute{e}t}}\big(\mathrm{Spec}(\mathcal{O}_{K,\Sigma}),\mathbb{Z}\big),\mathbb{Q}/\mathbb{Z}[-3]\big)\big) \cong \widehat{\mathcal{O}}_{K,\Sigma}^\times$$

and the natural projection $\widehat{\mathcal{O}}_{K,\Sigma}^\times \to \widehat{\mathcal{O}}_{K,\Sigma}^\times/\mathcal{O}_{K,\Sigma}^\times$.

(ii) *Ψ^0_Σ is a finitely generated cohomologically-trivial G-module, Ψ^1_Σ is a finitely generated free $\mathbb{Z}[G]$-module, and the distinguished triangle in claim* (i) *induces exact sequences of the form*

$$0 \to \mathcal{O}^\times_{K,\Sigma} \to \Psi^0_\Sigma \xrightarrow{\delta_\Sigma} \Psi^1_\Sigma \to \mathrm{cok}(\delta_\Sigma) \to 0, \tag{4}$$

$$0 \to \mathrm{Cl}(\mathcal{O}_{K,\Sigma}) \to \mathrm{cok}(\delta_\Sigma) \to X_{K,\Sigma} \to 0. \tag{5}$$

Proof. This is just Lemma 2.1 in [16]. □

Remark 3.2. If K is a global function field, then it is known that $\mathrm{cok}(\delta_\Sigma)$ is canonically isomorphic to the Weil-étale cohomology group $H^1_W(\mathcal{O}_{K,\Sigma},\mathbb{G}_m)$ of the sheaf $\mathbb{G}_m$ on $\mathrm{Spec}(\mathcal{O}_{K,\Sigma})$. If K is a number field, then a Weil-étale topology has been conjectured to exist by Lichtenbaum. It has been shown by Burns (see [7, Remark 3.8]) that if such a topology exists with the expected properties, then $\mathrm{cok}(\delta_\Sigma)$ will be canonically isomorphic to the group $H^1_W(\mathcal{O}_{K,\Sigma},\mathbb{G}_m)$.

We fix henceforth a choice of perfect complex of $\mathbb{Z}[G]$-modules $\Psi^\bullet_S$ which, in terms of the notation of Lemma 3.1, is equal to $\Psi^0_S \xrightarrow{\delta_S} \Psi^1_S$ with the first term placed in degree 0 and the cohomology groups identified with $\mathcal{O}^\times_{K,S}$ and $\mathrm{cok}(\delta_S)$ by means of (4).

Corollary 3.3. *Let K/k and Σ be as above. Then for any finite set of places T of k that is disjoint from Σ there exists a perfect complex $\Psi^\bullet_{\Sigma,T}$ of $\mathbb{Z}[G]$-modules which is defined up to canonical isomorphism in $\mathcal{D}(\mathbb{Z}[G])$, is acyclic outside degrees 0 and 1 with canonical identifications $H^0(\Psi^\bullet_{\Sigma,T}) = \mathcal{O}^\times_{K,\Sigma,T}$, $H^1(\Psi^\bullet_{\Sigma,T})_{\mathrm{tor}} = \mathrm{Cl}^T(\mathcal{O}_{K,\Sigma})$ and $H^1(\Psi^\bullet_{\Sigma,T})_{\mathrm{tf}} = X_{K,\Sigma}$, and lies in a natural exact triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form*

$$\Psi^\bullet_{\Sigma,T} \to \Psi^\bullet_\Sigma \to \mathbb{F}^\times_T[0] \to \Psi^\bullet_{\Sigma,T}[1]. \tag{6}$$

In this triangle, $\mathbb{F}^\times_T$ denotes the direct sum of the multiplicative groups of the residue fields of all places in $T(K)$.

Proof. This is proved in claims (i) and (ii) of [11, Proposition 4.1]. Indeed, although [11, Proposition 4.1] is stated specifically in the number field case, all the relevant steps in the construction of the complex $\Psi^\bullet_{\Sigma,T}$ from the given complex $\Psi^\bullet_\Sigma$ remain valid in the global function field case. □

We fix henceforth a choice of finite set of places T of k that is disjoint from S with the property that $\mathcal{O}^\times_{K,S,T}$ is torsion-free.

3.1.2. Determinants

For any commutative ring R we write Det_R for the determinant functor of Knudsen and Mumford [28], valued in the category $\mathcal{P}(R)$ of graded invertible R-modules, and for any object $(\mathcal{L},r)$ of $\mathcal{P}(R)$ we set $(\mathcal{L},r)^{-1} := (\mathrm{Hom}_R(\mathcal{L},R),-r)$ (which is again an object of $\mathcal{P}(R)$).

We now assume to be given a perfect complex of $\mathbb{Z}[G]$-modules $C^\bullet$ that is concentrated in degrees 0 and 1 together with an isomorphism of $\mathbb{R}[G]$-modules $\lambda : \mathbb{R}\otimes H^0(C^\bullet) \to \mathbb{R}\otimes H^1(C^\bullet)$. For any such pair we write

$$\vartheta_{C^\bullet,\lambda} : \mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R}\otimes C^\bullet\big) \cong \big(\mathbb{R}[G],0\big)$$

for the isomorphism in $\mathcal{P}(\mathbb{R}[G])$ that is obtained by composing the isomorphism

$$\mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R}\otimes C^\bullet\big) \cong \mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R}\otimes H^0\big(C^\bullet\big)\big) \otimes_{\mathcal{P}(\mathbb{R}[G])} \mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R}\otimes H^1\big(C^\bullet\big)\big)^{-1}$$

induced by applying $\mathrm{Det}_{\mathbb{R}[G]}$ to the tautological short exact sequences

$$0 \to \mathbb{R} \otimes H^0(C^\bullet) \to \mathbb{R} \otimes C^0 \to \mathbb{R} \otimes \mathrm{im}(d) \to 0$$

and

$$0 \to \mathbb{R} \otimes \mathrm{im}(d) \to \mathbb{R} \otimes C^1 \to \mathbb{R} \otimes H^1(C^\bullet) \to 0$$

where d is the differential of $C^\bullet$, together with the isomorphism

$$\mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R} \otimes H^0(C^\bullet)\big) \otimes_{\mathcal{P}(\mathbb{R}[G])} \mathrm{Det}_{\mathbb{R}[G]}\big(\mathbb{R} \otimes H^1(C^\bullet)\big)^{-1} \cong \big(\mathbb{R}[G], 0\big)$$

obtained by composing $\mathrm{Det}_{\mathbb{R}[G]}(\lambda) \otimes_{\mathcal{P}(\mathbb{R}[G])} \mathrm{id}$ and the evaluation pairing on the space $\mathrm{Det}_{\mathbb{R}[G]}(\mathbb{R} \otimes H^1(C^\bullet))$.

3.2. *Statement of the conjecture*

We recall that the exact sequences (4) and (5) combine to identify $R_{K,S}$ with an isomorphism of the form $\mathbb{R} \otimes H^0(\Psi_S^\bullet) \to \mathbb{R} \otimes H^1(\Psi_S^\bullet)$.

Conjecture C(K/k) *(Burns). One has an equality in $\mathcal{P}(\mathbb{Z}[G])$ of the form*

$$\big(\theta^*_{K/k,S}(0) \cdot \mathbb{Z}[G], 0\big) = \vartheta_{\Psi_S^\bullet, R_{K,S}}\big(\mathrm{Det}_{\mathbb{Z}[G]}\big(\Psi_S^\bullet\big)\big).$$

Remark 3.4.

(i) Under the conditions of Lemma 3.1, the complex $\Psi_S^\bullet$ is unique to within an isomorphism in $\mathcal{D}(\mathbb{Z}[G])$ that induces the identity map on all (non-zero) degrees of cohomology, and this can be used to show that the (graded) lattice $\vartheta_{\Psi_S^\bullet, R_{K,S}}(\mathrm{Det}_{\mathbb{Z}[G]}(\Psi_S^\bullet))$ depends only upon the pair $(K/k, S)$.

(ii) The same argument as used to prove [4, Theorem 2.1.2 (i)] shows that the validity of Conjecture C(K/k) is independent of the chosen set S.

3.3. *An explicit reformulation of* C(K/k)

From [39, Chapter I, Proposition 3.4] one knows that for each $\chi \in \widehat{G}$ the order of vanishing of $L_{K/k,S}(s, \chi)$ at $s = 0$ is equal to

$$\dim_{\mathbb{C}}\big(e_\chi\big(\mathbb{C} \cdot \mathcal{O}^\times_{K,S}\big)\big) = \begin{cases} |\{v \in S\colon\ v \text{ splits completely in } K^{\ker(\chi)}/k\}|, & \text{if } \chi \text{ is non-trivial,} \\ |S| - 1, & \text{if } \chi \text{ is trivial.} \end{cases} \tag{7}$$

The equivariant function $\theta_{K/k,S}(s)$ therefore vanishes at $s = 0$ to order at least

$$r := \min\big\{|S| - 1, \big|\{v \in S\colon\ v \text{ splits completely in } K/k\}\big|\big\}.$$

We now set $n := |S| - 1$ and label, and thereby order, the elements of S as $\{v_i\colon\ 0 \leqslant i \leqslant n\}$. We always assume, as we may, that this numbering is chosen so that v_i splits completely in K/k for each i with $1 \leqslant i \leqslant r$. For each index $i \in \{0, \ldots, n\}$ we fix a place w_i of K lying over v_i. Set $S_1 := \{v_1, \ldots, v_r\}$.

We then define $\alpha : X_{K,S} \to Y_{K,S_1}$ by

$$\alpha\Big(\sum\nolimits_{w \in S(K)} n_w w\Big) := \sum\nolimits_{w \in S_1(K)} n_w w$$

and, for any complex $\Psi^{\bullet}_{S,T}$ as specified by Corollary 3.3, use the canonical identification $H^1(\Psi^{\bullet}_{S,T})_{\mathrm{tf}} = X_{K,S}$ to define a map $\alpha' : H^1(\Psi^{\bullet}_{S,T}) \to Y_{K,S_1}$ by the composition

$$H^1\big(\Psi^{\bullet}_{S,T}\big) \twoheadrightarrow X_{K,S} \xrightarrow{\alpha} Y_{K,S_1}.$$

To proceed, we now generalise constructions of Burns in [6, §7] to apply in our setting.

Lemma 3.5. *There exists a finitely generated free G-module F of rank d with $d \geqslant r$, a surjective homomorphism of G-modules $\pi : F \to H^1(\Psi^{\bullet}_{S,T})$ and an ordered $\mathbb{Z}[G]$-basis $\{b_i\colon 1 \leqslant i \leqslant d\}$ of F with the following property:*

$$\big(\alpha' \circ \pi\big)(b_i) = \begin{cases} w_i, & \text{if } 1 \leqslant i \leqslant r, \\ 0, & \text{if } r < i \leqslant d. \end{cases}$$

Proof. Fix a free G-module F_1 of rank r with basis $\{b_i\colon 1 \leqslant i \leqslant r\}$, define $\pi_1' : F_1 \to X_{K,S}$ by

$$\pi_1'(b_i) = w_i - w_0 \quad \text{for } i \in \{1, \ldots, r\}$$

and let $\pi_1 : F_1 \to H^1(\Psi^{\bullet}_{S,T})$ be a lift of π_1' through the natural projection $H^1(\Psi^{\bullet}_{S,T}) \twoheadrightarrow X_{K,S}$ (which clearly exists since F_1 is free).

Fix a free G-module F_2 of (large enough) rank c with basis $\{b_i\colon r < i \leqslant c+r\}$ such that there exists a surjective homomorphism $\pi_2' : F_2 \twoheadrightarrow \ker(\alpha')$ and let $\pi_2 : F_2 \to H^1(\Psi^{\bullet}_{S,T})$ denote the composite of π_2' and the tautological inclusion $\ker(\alpha') \subseteq H^1(\Psi^{\bullet}_{S,T})$.

Set now $F := F_1 \oplus F_2$, $\pi := \pi_1 \oplus \pi_2 : F \to H^1(\Psi^{\bullet}_{S,T})$, $d := r + c$. We have that $(\alpha' \circ \pi_1)(b_i) = \alpha(w_i - w_0) = w_i$ for $1 \leqslant i \leqslant r$, so $\alpha' \circ \pi_1 : F_1 \to Y_{K,S_1}$ is surjective (Y_{K,S_1} is clearly generated as a $\mathbb{Z}[G]$-module by $\{w_1, \ldots, w_r\}$). The tautological exact sequence

$$0 \to \ker\big(\alpha'\big) \to H^1\big(\Psi^{\bullet}_{S,T}\big) \xrightarrow{\alpha'} Y_{K,S_1} \to 0$$

therefore implies that π is surjective. $\square$

For any G-module M and homomorphism of G-modules λ, we set $M_{\mathbb{R}} := M \otimes \mathbb{R}$ and $\lambda_{\mathbb{R}} := \lambda \otimes \mathbb{R}$. The algebra $\mathbb{R}[G]$ is semisimple, and so for any $\mathbb{Z}[G]$-endomorphism φ of the finitely generated, free $\mathbb{Z}[G]$-module F, there exist $\mathbb{R}[G]$-equivariant sections ι_1 and ι_2 to the surjections $F_{\mathbb{R}} \to \mathrm{im}(\varphi)_{\mathbb{R}}$ and $F_{\mathbb{R}} \to \mathrm{cok}(\varphi)_{\mathbb{R}}$ that are induced by φ and by the tautological surjection respectively. This induces a direct sum decomposition of $\mathbb{R}[G]$-modules

$$F_{\mathbb{R}} = \ker(\varphi)_{\mathbb{R}} \oplus \iota_1\big(\mathrm{im}(\varphi)_{\mathbb{R}}\big)$$

and so for τ in $\mathrm{Hom}_{\mathbb{R}[G]}(\ker(\varphi)_{\mathbb{R}}, \mathrm{cok}(\varphi)_{\mathbb{R}})$ there is a unique $\langle \tau, \varphi, \iota_1, \iota_2 \rangle$ in $\mathrm{Hom}_{\mathbb{R}[G]}(F_{\mathbb{R}}, F_{\mathbb{R}})$ that is equal to $\iota_2 \circ \tau$ on $\ker(\varphi)_{\mathbb{R}}$ and to $\varphi_{\mathbb{R}}$ on $\iota_1(\mathrm{im}(\varphi)_{\mathbb{R}})$.

Proposition 3.6. *Let $\Psi^{\bullet}_{S,T}$ be a perfect complex of $\mathbb{Z}[G]$-modules as specified by Corollary 3.3 with respect to our fixed choices of S and T, and let $\pi : F \to H^1(\Psi^{\bullet}_{S,T})$ be a surjective homomorphism of $\mathbb{Z}[G]$-modules as specified by Lemma 3.5. Then there exists an exact sequence of $\mathbb{Z}[G]$-modules of the form*

$$0 \to \mathcal{O}^{\times}_{K,S,T} \xrightarrow{\iota} F \xrightarrow{\varphi} F \xrightarrow{\pi} H^1\big(\Psi^{\bullet}_{S,T}\big) \to 0$$

with the property that, for any choice of $\mathbb{R}[G]$-equivariant sections ι_1 and ι_2 to the surjective homomorphisms $F_{\mathbb{R}} \to \mathrm{im}(\varphi)_{\mathbb{R}}$ and $F_{\mathbb{R}} \to \mathrm{cok}(\varphi)_{\mathbb{R}} \xrightarrow{\sim} H^1(\Psi^{\bullet}_{S,T})_{\mathbb{R}}$ that are induced by φ and π respectively, the

$\mathbb{R}[G]$-endomorphism $\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle$ of $F_{\mathbb{R}}$ is invertible, and one has that Conjecture $\mathrm{C}(K/k)$ is valid if and only if the element

$$\theta^*_{K/k,S,T}(0) \cdot \det_{\mathbb{R}[G]}\big(\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle\big)^{-1}$$

belongs to $\mathbb{Z}[G]^\times$, where we write $\theta^*_{K/k,S,T}(0)$ for the product $\delta_T \cdot \theta^*_{K/k,S}(0)$.

Proof. Since $\Psi^\bullet_{S,T}$ is both perfect and acyclic outside degrees 0 and 1, a standard argument shows that it is isomorphic in $D(\mathbb{Z}[G])$ to a complex of the form $\Psi^0_{S,T} \xrightarrow{d} F$ where $\Psi^0_{S,T}$ occurs in degree 0 and is a finitely generated, cohomologically-trivial $\mathbb{Z}[G]$-module and furthermore with the property that $\mathrm{im}(d) = \ker(\pi)$. But both $H^0(\Psi^\bullet_{S,T}) = \mathcal{O}^\times_{K,S,T}$ and F are torsion-free and hence so is $\Psi^0_{S,T}$, which by [1, Theorem 8] is a projective $\mathbb{Z}[G]$-module. Also, there exist isomorphisms of $\mathbb{Q}[G]$-modules

$$\mathcal{O}^\times_{K,S,T} \otimes \mathbb{Q} \cong \mathcal{O}^\times_{K,S} \otimes \mathbb{Q} \cong X_{K,S} \otimes \mathbb{Q} \cong H^1\big(\Psi^\bullet_{S,T}\big) \otimes \mathbb{Q},$$

where the existence of the second isomorphism is implied from the fact that $\mathcal{O}^\times_{K,S} \otimes \mathbb{R}$ and $X_{K,S} \otimes \mathbb{R}$ are $\mathbb{R}[G]$-isomorphic by the Noether–Deuring theorem (see the remark after [37, Proposition 33]). Since $\mathbb{Q}[G]$ is semisimple, it is then easily seen that the $\mathbb{Q}[G]$-modules $\Psi^0_{S,T} \otimes \mathbb{Q}$ and $F \otimes \mathbb{Q}$ are isomorphic. Hence Swan's theorem [18, 32.1] implies that, for every prime number p, the $\mathbb{Z}_p[G]$-modules $\Psi^0_{S,T} \otimes \mathbb{Z}_p$ and $F \otimes \mathbb{Z}_p$ are isomorphic. Roiter's lemma [18, 31.6] finally implies that the $\mathbb{Z}[G]$-modules $\Psi^0_{S,T}$ and F are isomorphic, as required to complete the proof of the first claim of the proposition.

The exact triangle of perfect complexes (6) in $\mathcal{D}(\mathbb{Z}[G])$ now implies that

$$\begin{aligned}\mathrm{Det}_{\mathbb{Z}[G]}\big(\Psi^\bullet_S\big) &= \mathrm{Det}_{\mathbb{Z}[G]}\big(\Psi^\bullet_{S,T}\big) \cdot \mathrm{Det}_{\mathbb{Z}[G]}\big(\mathbb{F}^\times_T[0]\big)\\ &= \mathrm{Det}_{\mathbb{Z}[G]}\big(\Psi^\bullet_{S,T}\big) \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathbb{F}^\times_T\big)^{-1}\end{aligned}$$

while, since each place in T is unramified, the short exact sequence

$$0 \to \bigoplus_{v\in T} \mathbb{Z}[G] \xrightarrow{(1-\mathrm{N}v\cdot \mathrm{Fr}_v^{-1})_v} \bigoplus_{v\in T} \mathbb{Z}[G] \to \mathbb{F}^\times_T \to 0$$

implies that $\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathbb{F}^\times_T) = \delta_T \cdot \mathbb{Z}[G]$. It follows that Conjecture $\mathrm{C}(K/k)$ is valid if and only if

$$\vartheta_{\Psi^\bullet_{S,T}, R_{K,S}}\big(\mathrm{Det}_{\mathbb{Z}[G]}\big(\Psi^\bullet_{S,T}\big)\big) = \big(\theta^*_{K/k,S,T}(0) \cdot \mathbb{Z}[G], 0\big). \tag{8}$$

The definition of $\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle$ implies that both

$$\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle\big(\iota_1\big(\mathrm{im}(\varphi)_{\mathbb{R}}\big)\big) = \mathrm{im}(\varphi)_{\mathbb{R}}$$

and

$$\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle\big(\ker(\varphi)_{\mathbb{R}}\big) = \iota_2\big(H^1\big(\Psi^\bullet_{S,T}\big)_{\mathbb{R}}\big).$$

Since $F_{\mathbb{R}}$ is equal to the direct sum of $\mathrm{im}(\varphi)_{\mathbb{R}}$ and $\iota_2(H^1(\Psi^\bullet_{S,T})_{\mathbb{R}})$, one therefore has $\mathrm{im}(\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle) = F_{\mathbb{R}}$ and so $\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle$ is invertible.

Finally, by the explicit computation in [5, Lemma A1], and using (8), we have that Conjecture $\mathrm{C}(K/k)$ is valid if and only if

$$\big(\det_{\mathbb{R}[G]}\big(\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle\big) \cdot \mathbb{Z}[G], 0\big) = \big(\theta^*_{K/k,S,T}(0) \cdot \mathbb{Z}[G], 0\big),$$

as required. □

4. The proof of Theorem 2.1

We recall that, by the definition of r in Section 3.3, it is less than or equal to $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi)$ for every character $\chi \in \widehat{G}$, and that we have defined an element of $\mathbb{C}[G]$ by setting

$$\theta^{(r)}_{K/k,S}(0) := \frac{1}{r!}\left(\frac{d}{ds}\right)^r \theta_{K/k,S}(s)\bigg|_{s=0}.$$

In fact, it is easily seen that this element belongs to $\mathbb{R}[G]$. Set

$$H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)' := \ker\big(\alpha'\big).$$

Theorem 4.1. *Assume that* $\mathrm{C}(K/k)$ *is valid. Then for any element* Φ *of* $\mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r_{\mathbb{Z}[G]}\mathcal{O}^\times_{K,S,T}, \wedge^r_{\mathbb{Z}[G]}X_{K,S})$ *one has that the element* $\theta^{(r)}_{K/k,S,T}(0)R(\Phi)$ *belongs to* $\mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$.

Proof. Using the notation of Lemma 3.5, let first $f = (f_1, f_2) \in \ker(\pi)$ with $f_1 = \sum_{i=1}^{i=r} x_i b_i \in F_1$, $f_2 \in F_2$, $x_1, \dots, x_r \in \mathbb{Z}[G]$. Then $0 = (\alpha' \circ \pi)(f) = \sum_{i=1}^{i=r} x_i w_i + \alpha'(\pi_2(f_2)) = \sum_{i=1}^{i=r} x_i w_i$, and since Y_{K,S_1} is a free $\mathbb{Z}[G]$-module with basis $\{w_1, \dots, w_r\}$, we have that $x_1 = \cdots = x_r = 0$ and so $f_1 = 0$. This, combined with the exactness of $F \xrightarrow{\varphi} F \xrightarrow{\pi} H^1(\Psi^\bullet_{S,T})$ in Proposition 3.6, implies that $\mathrm{im}(\varphi) = \ker(\pi) = \ker(\pi_2) = \ker(\pi_2') \subseteq F_2$. We hence have an exact sequence

$$F \xrightarrow{\varphi} F_2 \xrightarrow{\pi_2'} H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)' \to 0. \tag{9}$$

Let now $N = (N_{st})$ be the matrix of $\varphi : F \to F_2$ with respect to the bases $\{b_1, \dots, b_d\}$ and $\{b_{r+1}, \dots, b_d\}$, i.e.,

$$\varphi(b_s) = \sum_{t=1}^{t=d-r} N_{st} b_{t+r} \quad \text{for } 1 \leqslant s \leqslant d.$$

Then, since the exact sequence (9) is a free resolution of $H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'$, the definition of Fitting ideals implies that

$$\det\big(N^\lambda\big) \in \mathrm{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\big) \tag{10}$$

for every $\lambda \in \Lambda_r(d)$ where we set $\Lambda_r(d) := \{(\lambda_1, \dots, \lambda_r)\colon\ 1 \leqslant \lambda_1 < \cdots < \lambda_r \leqslant d\}$ and for $\lambda = (\lambda_1, \dots, \lambda_r)$ we let N^λ denote the $(d-r) \times (d-r)$ matrix obtained by deleting the rows of N in positions $\lambda_1, \dots, \lambda_r$.

From Proposition 3.6, we have a short exact sequence

$$0 \to \mathcal{O}^\times_{K,S,T} \xrightarrow{\iota} F \xrightarrow{\varphi} \mathrm{im}(\varphi) \to 0$$

and, since F is $\mathbb{Z}$-free, so is $\mathrm{im}(\varphi) \subseteq F$. Hence $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{im}(\varphi), \mathbb{Z})$ vanishes and it follows that the map $\mathrm{Hom}_{\mathbb{Z}}(\iota, \mathbb{Z})$ is surjective. Now, the functors $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathbb{Z}[G])$ are naturally isomorphic, and hence we find that

$$\mathrm{Hom}_{\mathbb{Z}[G]}\big(\iota, \mathbb{Z}[G]\big) : \mathrm{Hom}_{\mathbb{Z}[G]}\big(F, \mathbb{Z}[G]\big) \to \mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G]\big)$$

is a surjective homomorphism of $\mathbb{Z}[G]$-modules. We fix an isomorphism of $\mathbb{Z}[G]$-modules $f : Y_{K,S_1} \cong \mathbb{Z}[G]^r$, where for any $\mathbb{Z}[G]$-module M, resp. homomorphism of $\mathbb{Z}[G]$-modules λ, we let M^r, resp. λ^r, denote the direct sum of r copies of M, resp. λ. It is then clear that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathcal{O}_{K,S,T}^{\times}, f^{-1}\big) \circ \mathrm{Hom}_{\mathbb{Z}[G]}\big(\iota, \mathbb{Z}[G]\big)^r \circ \mathrm{Hom}_{\mathbb{Z}[G]}(F, f) \\
&\quad = \mathrm{Hom}_{\mathbb{Z}[G]}(\iota, Y_{K,S_1}) : \mathrm{Hom}_{\mathbb{Z}[G]}(F, Y_{K,S_1}) \to \mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathcal{O}_{K,S,T}^{\times}, Y_{K,S_1}\big)
\end{aligned}$$

is also surjective.

We now fix $\Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r \mathcal{O}_{K,S,T}^{\times}, \wedge^r X_{K,S})$. For any finitely generated $\mathbb{Z}[G]$-modules M and N, we have a natural isomorphism

$$\wedge^r \mathrm{Hom}_{\mathbb{Z}[G]}(M, N) \cong \mathrm{Hom}_{\mathbb{Z}[G]}\big(\wedge^r M, \wedge^r N\big) \tag{11}$$

which is given by

$$\wedge_{i=1}^{i=r} \phi_i \mapsto \big[\wedge_{i=1}^{i=r} m_i \mapsto \wedge_{i=1}^{i=r} \phi_i(m_i)\big].$$

Hence, if $r > 0$, then for some $n \in \mathbb{N}$, we have that Φ corresponds under this isomorphism to $\sum_{j=1}^{j=n}(\wedge_{i=1}^{i=r}\phi_{i,j})$ for some $\phi_{i,j} \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^{\times}, X_{K,S})$. By the surjectivity of $\mathrm{Hom}_{\mathbb{Z}[G]}(\iota, Y_{K,S_1})$, we may then, for each $(i,j) \in \{1,\dots,r\} \times \{1,\dots,n\}$, choose $\phi'_{i,j} \in \mathrm{Hom}_{\mathbb{Z}[G]}(F, Y_{K,S_1})$ such that

$$\alpha \circ \phi_{i,j} = \phi'_{i,j} \circ \iota. \tag{12}$$

So, if $r > 0$, we let Φ' be the element of $\mathrm{Hom}_{\mathbb{Z}[G]}(\wedge^r F, \wedge^r Y_{K,S_1})$ that corresponds to $\sum_{j=1}^{j=n}(\wedge_{i=1}^{i=r}\phi'_{i,j})$ under the analogous isomorphism given by (11), and note that then (12) implies by naturality of the isomorphisms given by (11) that

$$\wedge^r \alpha \circ \Phi = \Phi' \circ \wedge^r \iota. \tag{13}$$

Now, for arbitrary r, $\wedge_{\mathbb{R}[G]}^r R_{K,S}$ gives an isomorphism of $\mathbb{R}[G]$-modules

$$\mathbb{R} \cdot \wedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^{\times} \xrightarrow{\sim} \mathbb{R} \cdot \wedge_{\mathbb{Z}[G]}^r X_{K,S}.$$

Let η denote the unique element of $\mathbb{R} \cdot \wedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^{\times}$ which satisfies

$$\big(\wedge_{\mathbb{R}[G]}^r R_{K,S}\big)(\eta) = \theta_{K/k,S,T}^{(r)}(0) \cdot \wedge_{i=1}^{i=r}(w_i - w_0) \tag{14}$$

in $\mathbb{R} \cdot \wedge_{\mathbb{Z}[G]}^r X_{K,S}$. By a theorem of Burns [6, Theorem 8.1], noting that all the arguments involved remain valid for our more general S, we deduce from Proposition 3.6 and the assumed validity of Conjecture $\mathrm{C}(K/k)$ that

$$\wedge^r \iota_{\mathbb{R}}(\eta) = x_T \sum_{\lambda \in \Lambda_r(d)} \mathrm{sgn}(\sigma_\lambda) \det\big(N^\lambda\big) \cdot \wedge_{i=1}^{i=r} b_{\lambda_i} \in \wedge^r \ker(\varphi)_{\mathbb{R}} \subseteq \wedge^r F_{\mathbb{R}} \tag{15}$$

with

$$x_T := \theta^*_{K/k,S,T}(0) \cdot \det_{\mathbb{R}[G]}\big(\langle R_{K,S}, \varphi, \iota_1, \iota_2\rangle\big)^{-1} \in \mathbb{Z}[G]^\times \tag{16}$$

and where σ_λ is just an element of a group of permutations and so $\operatorname{sgn}(\sigma_\lambda) \in \{1, -1\}$.

Hence, if $r > 0$, we have that

$$\begin{aligned}
\theta^{(r)}_{K/k,S,T}(0)R(\Phi) \cdot \wedge_{i=1}^{i=r} w_i &= \theta^{(r)}_{K/k,S,T}(0)\det_{\mathbb{R}[G]}\big(\wedge^r R^{-1}_{K,S} \circ \Phi_{\mathbb{R}}\big) \cdot \big(\wedge^r \alpha\big)\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big) \\
&= \big(\wedge^r \alpha_{\mathbb{R}}\big)\big(\theta^{(r)}_{K/k,S,T}(0)\det_{\mathbb{R}[G]}\big(\wedge^r R^{-1}_{K,S} \circ \Phi_{\mathbb{R}}\big) \cdot \big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big) \\
&= \big(\wedge^r \alpha_{\mathbb{R}}\big)\big(\theta^{(r)}_{K/k,S,T}(0)\big(\Phi_{\mathbb{R}} \circ \wedge^r R^{-1}_{K,S}\big)\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big) \\
&= \big(\wedge^r \alpha_{\mathbb{R}} \circ \Phi_{\mathbb{R}}\big)(\eta) \\
&= \big(\Phi'_{\mathbb{R}} \circ \wedge^r \iota_{\mathbb{R}}\big)(\eta) \\
&= \Phi'_{\mathbb{R}}\Big(x_T \sum\nolimits_{\lambda \in \Lambda_r(d)} \operatorname{sgn}(\sigma_\lambda)\det\big(N^\lambda\big) \cdot \wedge_{i=1}^{i=r} b_{\lambda_i}\Big) \\
&= x_T \sum\nolimits_{\lambda \in \Lambda_r(d)} \operatorname{sgn}(\sigma_\lambda)\det\big(N^\lambda\big) \cdot \Phi'\big(\wedge_{i=1}^{i=r} b_{\lambda_i}\big),
\end{aligned}$$

where the fourth, fifth and sixth equalities are implied by (14), (13) and (15) respectively. Now (10) combines with (16) to imply that the last displayed expression belongs to $\operatorname{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)') \cdot \wedge^r Y_{K,S_1}$. Hence, since $\{\wedge_{i=1}^{i=r} w_i\}$ is a $\mathbb{Z}[G]$-basis of $\wedge^r Y_{K,S_1} = \mathbb{Z}[G] \cdot \wedge_{i=1}^{i=r} w_i$, we finally deduce that $\theta^{(r)}_{K/k,S,T}(0)R(\Phi)$ belongs to $\operatorname{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$, as required.

If on the other hand $r = 0$, then we have that

$$\theta^{(r)}_{K/k,S,T}(0)R(\Phi) = \eta R(\Phi) = x_T \det(N)R(\Phi) \in \operatorname{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\big),$$

where the first and second equalities are implied by (14) and (15) respectively and the fact that $x_T \det(N)R(\Phi)$ belongs to $\operatorname{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$ follows from combining (10) with (16) and the fact that $R(\Phi) \in \mathbb{Z}[G]$ in this particular case. □

Corollary 4.2. *If* $\mathrm{C}(K/k)$ *is valid, then for any* $\phi \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,S,T}, X_{K,S})$ *one has* $\theta^{(r)}_{K/k,S,T}(0)R(\phi) \in \operatorname{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$.

Proof. Set $e_r := \sum_\chi e_\chi$ where the sum runs over all elements χ of $\widehat{G}$ such that $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes \mathcal{O}^\times_{K,S})) = r$. Then it follows from (7) that we have $\theta^{(r)}_{K/k,S,T}(0) = \theta^*_{K/k,S,T}(0)e_r = \theta^{(r)}_{K/k,S,T}(0)e_r$ and hence

$$\begin{aligned}
\theta^{(r)}_{K/k,S,T}(0)R(\phi) &= \theta^{(r)}_{K/k,S,T}(0)e_r \det_{\mathbb{R}[G]}\big(R^{-1}_{K,S} \circ \phi_{\mathbb{R}}\big) \\
&= \theta^{(r)}_{K/k,S,T}(0)\det_{\mathbb{R}[G]e_r}\big(R^{-1}_{K,S} \circ \phi_{\mathbb{R}} \mid e_r\big(\mathcal{O}^\times_{K,S}\big)_{\mathbb{R}}\big) \\
&= \theta^{(r)}_{K/k,S,T}(0)e_r \det_{\mathbb{R}[G]}\big(\wedge^r_{\mathbb{R}[G]} R^{-1}_{K,S} \circ \wedge^r_{\mathbb{R}[G]}\phi_{\mathbb{R}}\big) \\
&= \theta^{(r)}_{K/k,S,T}(0)R\big(\wedge^r_{\mathbb{Z}[G]}\phi\big) \\
&\in \operatorname{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\big),
\end{aligned}$$

where the third equality is a consequence of the fact that $e_r(\mathcal{O}^\times_{K,S} \otimes \mathbb{R})$ is a free $\mathbb{R}[G]e_r$-module of rank r and the last assertion is just the statement of Theorem 4.1 for $\Phi = \wedge^r_{\mathbb{Z}[G]}\phi$. □

5. The proof of Corollary 2.3

Recall the definitions of S_0 and $S_{0,f}$ from the Section 2. We now set

$$\mathrm{Cl}_S := \mathrm{Cl}(\mathcal{O}_{K,S}), \qquad \mathrm{Cl}_S^T := \mathrm{Cl}^T(\mathcal{O}_{K,S}),$$
$$\mathrm{Cl}'_S := \mathrm{Cl}(\mathcal{O}_K)/\bigl\langle\bigl[g(w_i)\bigr]\colon\ 1 \leqslant i \leqslant r,\ w_i \text{ non-arch.},\ g \in G\bigr\rangle,$$
$$c_0 := \Biggl[\prod_{w \in S_{0,f}(K)} w\Biggr] \in \mathrm{Cl}(\mathcal{O}_K),$$
$$\mathrm{Cl}(\mathcal{O}_K)_S := \mathrm{Cl}(\mathcal{O}_K)/\bigl(\bigl\langle\bigl[g(w_i)\bigr]\colon\ 1 \leqslant i \leqslant r,\ w_i \text{ non-arch.},\ g \in G\bigr\rangle\langle c_0\rangle\bigr).$$

We note first that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Cl}_S^T \arrow[r] \arrow[d] & H^1(\Psi^\bullet_{S,T}) \arrow[r] \arrow[d, "\alpha'"'] & X_{K,S} \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & 0 \arrow[r] & Y_{K,S_1} \arrow[r, "\mathrm{id}"] & Y_{K,S_1} \arrow[r] & 0
\end{tikzcd}$$

with the top row induced by the canonical identifications given in Corollary 3.3. Note also that, if we regard X_{K,S_0} as a submodule of $X_{K,S}$ in the obvious way, then it clearly coincides with $\ker(\alpha)$. Hence the Snake lemma induces a short exact sequence of G-modules

$$0 \to \mathrm{Cl}_S^T \to H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)' \to X_{K,S_0} \to 0. \tag{17}$$

5.1. *The proof of Corollary 2.3 (i)*

Combining Theorem 4.1 with (17), the assumed validity of Conjecture $\mathrm{C}(K/k)$ implies that

$$\begin{aligned}
\theta^{(r)}_{K/k,S,T}(0)R(\Phi) &\in \mathrm{Fitt}_{\mathbb{Z}[G]}\bigl(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\bigr) \\
&\subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\bigl(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\bigr) \\
&\subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\bigl(\mathrm{Cl}_S^T\bigr)
\end{aligned}$$

for any Φ. Furthermore, Cl_S^T naturally surjects onto Cl_S, so $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_S^T) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_S)$. This completes the proof of claim (i) in Corollary 2.3.

5.2. *The proofs of Corollary 2.3 (ii) and (iii)*

We have an exact sequence of G-modules

$$Y_{K,S_0} \to \mathrm{Cl}'_S \to \mathrm{Cl}_S \to 0 \tag{18}$$

where the first arrow is just the map which sends any place w of $S_0(K)$ to its class $[w]$ in $\mathrm{Cl}(\mathcal{O}_K)$ and then to its coset in the quotient Cl'_S.

For any G-module M, we set $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, and for any G-module homomorphism λ, we set $\lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\lambda, \mathbb{Z})$.

Lemma 5.1. *There is a canonical isomorphism of* $\mathbb{Z}[G]$*-modules*

$$X^*_{K,S_0} \cong Y_{K,S_0} / \left\langle \sum_{w \in S_0(K)} w \right\rangle.$$

Proof. The tautological short exact sequence

$$0 \to X_{K,S_0} \to Y_{K,S_0} \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ sends each place of $S_0(K)$ to 1, induces a short exact sequence

$$0 \to \mathbb{Z}^* \xrightarrow{\epsilon^*} Y^*_{K,S_0} \to X^*_{K,S_0} \to 0.$$

Set $J := \{0, r+1, \ldots, n\}$. We also have an isomorphism $\mathbb{Z}^* \cong \mathbb{Z}$ by sending each $\psi \in \mathbb{Z}^*$ to $\psi(1)$ and a composite canonical isomorphism

$$\begin{aligned} Y^*_{K,S_0} &= \mathrm{Hom}_{\mathbb{Z}}(Y_{K,S_0}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{j \in J} \mathbb{Z}[G/G_{w_j}], \mathbb{Z}\right) \\ &\cong \bigoplus_{j \in J} \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[G/G_{w_j}], \mathbb{Z}\big) \\ &\xrightarrow{\sim} \bigoplus_{j \in J} \mathrm{Hom}_{\mathbb{Z}[G/G_{w_j}]}\big(\mathbb{Z}[G/G_{w_j}], \mathbb{Z}[G/G_{w_j}]\big) \\ &\xrightarrow{\sim} \bigoplus_{j \in J} \mathbb{Z}[G/G_{w_j}] \\ &\xrightarrow{\sim} Y_{K,S_0}. \end{aligned}$$

Here G_{w_j} denotes the decomposition group of w_j, the first and last arrows are induced by the isomorphism

$$Y_{K,S_0} = \bigoplus_{j \in J} \mathbb{Z}[G] \cdot w_j \cong \bigoplus_{j \in J} \mathbb{Z}[G/G_{w_j}],$$

where each w_j is simply mapped to the identity element of G/G_{w_j}, the second arrow is induced by the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}[\Gamma]}\big(M, \mathbb{Z}[\Gamma]\big)$$

(with $\Gamma = G/G_{w_j}$ and $M = \mathbb{Z}[G/G_{w_j}]$) that is given for any finite group Γ and any $\mathbb{Z}[\Gamma]$-module M by

$$\phi \mapsto \left[m \mapsto \sum_{\gamma \in \Gamma} \phi\big(\gamma^{-1} \cdot m\big) \cdot \gamma\right],$$

and the third arrow is induced by the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[G/G_{w_j}]}\big(\mathbb{Z}[G/G_{w_j}], \mathbb{Z}[G/G_{w_j}]\big) \cong \mathbb{Z}[G/G_{w_j}]$$

given by sending each homomorphism ψ to $\psi(1)$.

It is now straightforward to check that the map from $\mathbb{Z}$ to Y_{K,S_0} that sends each integer m to $\sum_{w\in S_0(K)} mw$ makes the square

$$\begin{array}{ccc} \mathbb{Z}^* & \xrightarrow{\epsilon^*} & Y^*_{K,S_0} \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z} & \longrightarrow & Y_{K,S_0} \end{array}$$

commute. Hence we in fact have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^* & \xrightarrow{\epsilon^*} & Y^*_{K,S_0} & \longrightarrow & X^*_{K,S_0} & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & Y_{K,S_0} & \longrightarrow & Y_{K,S_0}/\langle\sum_{w\in S_0(K)} w\rangle & \longrightarrow & 0 \end{array}$$

and in particular the required canonical isomorphism. □

Now (18) induces an exact sequence

$$Y_{K,S_0}\Big/\Big\langle \sum_{w\in S_0(K)} w\Big\rangle \to \mathrm{Cl}(\mathcal{O}_K)_S \to \mathrm{Cl}_S \to 0$$

which then by Lemma 5.1 gives an exact sequence

$$X^*_{K,S_0} \xrightarrow{\psi} \mathrm{Cl}(\mathcal{O}_K)_S \xrightarrow{\varphi} \mathrm{Cl}_S \to 0. \tag{19}$$

Theorem 5.2.

(i) $|G|\cdot \mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)') \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)_S)$.

(ii) *For any prime p not dividing $|G|$, we have that* $\mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)')_{(p)} \subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)_S)_{(p)}$.

Proof. In order to prove claim (i), we first obtain an idempotent by setting $e_0 := \sum_{\chi\in\Upsilon_0} e_\chi$ where $\Upsilon_0 := \{\chi \in \widehat{G}\colon\ e_\chi(\mathbb{C}\otimes X_{K,S_0}) = 0\}$. Υ_0 is closed under the natural action of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ on $\widehat{G}$, so $e_0 \in \mathbb{Q}[G]$ and hence $|G|e_0 \in \mathbb{Z}[G]$. Since $|G|e_0 \cdot X_{K,S_0} = 0$, we have that $|G|e_0 \in \mathrm{Ann}_{\mathbb{Z}[G]}(X^*_{K,S_0})$. We also have that

$$\begin{aligned} \mathrm{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'\big) &\subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'\big) \\ &\subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\big(\mathrm{Cl}^T_S\big) \\ &\subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_S), \end{aligned}$$

where the second inclusion follows from (17) and the final inclusion follows from the fact that Cl^T_S surjects onto Cl_S.

Note now that if $e_\chi(\mathbb{C}\otimes H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)') = 0$ then (17) implies that $\chi \in \Upsilon_0$, and hence by exactly the same argument as in the last paragraph of the proof of [8, Theorem 8.2.1], we deduce that we have $e_0 \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)') = \mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$. Fix $x \in \mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$. Then $x = e_0 y = e_0^2 y = e_0(e_0 y) = e_0 x$ for some $y \in \mathrm{Fitt}_{\mathbb{Z}[G]}(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)')$.

Fix $c \in \mathrm{Cl}(\mathcal{O}_K)_S$. Then $\varphi(xc) = x\varphi(c) = 0$ so $xc \in \ker(\varphi) = \mathrm{im}(\psi)$ by (19), and hence $xc = \psi(z)$ for some $z \in X^*_{K,S_0}$. But then $(|G|x)c = (|G|e_0 x)c = (|G|e_0)(xc) = (|G|e_0)\psi(z) = \psi(|G|e_0 z) = \psi(0) = 0$, and this shows that

$$|G| \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S'_0}, \mathbb{G}_m)'\big) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\big(\mathrm{Cl}(\mathcal{O}_K)_S\big).$$

In order to prove claim (ii), we note first that if $p \nmid |G|$, then for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of $\mathbb{Z}_{(p)}[G]$-modules, we have

$$\mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}(M_2) = \mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}(M_1)\,\mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}(M_3).$$

Using the exact sequences (17) and (19), the fact that Cl_S^T surjects onto Cl_S and the fact that $\mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}((X_{K,S_0})_{(p)}) = \mathrm{Fitt}_{\mathbb{Z}_{(p)}[G]}((X^*_{K,S_0})_{(p)})$ (this follows from the proof of Lemma 5.1, in particular from the first two displayed short exact sequences and the fact that Y_{K,S_0} is self-dual), we finally get that

$$\begin{aligned}
\mathrm{Fitt}_{\mathbb{Z}[G]}\big(H^1_{W,T}(\mathcal{O}_{K,S}, \mathbb{G}_m)'\big)_{(p)} &= \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathrm{Cl}_S^T\big)_{(p)}\,\mathrm{Fitt}_{\mathbb{Z}[G]}(X_{K,S_0})_{(p)} \\
&\subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}_S)_{(p)}\,\mathrm{Fitt}_{\mathbb{Z}[G]}\big(X^*_{K,S_0}\big)_{(p)} \\
&\subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}_S)_{(p)}\,\mathrm{Fitt}_{\mathbb{Z}[G]}\big(\ker(\varphi)\big)_{(p)} \\
&= \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathrm{Cl}(\mathcal{O}_K)_S\big)_{(p)},
\end{aligned}$$

where the last equality is implied by the tautological short exact sequence

$$0 \to \ker(\varphi) \to \mathrm{Cl}(\mathcal{O}_K)_S \xrightarrow{\varphi} \mathrm{Cl}_S \to 0. \qquad \square$$

Claims (ii) and (iii) of Corollary 2.3 are now immediate consequences of Theorem 4.1 and Theorem 5.2.

6. Connections to recent work

6.1. The conjecture of Emmons and Popescu

In [20], Emmons and Popescu formulate a refinement of Stark's conjecture for finite abelian extensions of global fields K/k with Galois group G in terms of the value at 0 of the r-th derivative of a truncated $\mathbb{C}[G]$-valued L-function of K/k for which they replace the standard hypothesis on r, namely that r places of the set of places of k that plays the role of our S should split completely in the extension, by a weaker one. In this subsection, we explore the connection between this refinement and ours under this weaker hypothesis.

In order to state this conjecture, we say that a pair (Σ, T) is appropriate for K/k if Σ and T are finite, non-empty, disjoint sets of places of k such that Σ contains all the archimedean and all the K/k-ramified places of k and $\mathcal{O}^\times_{K,\Sigma,T}$ has no $\mathbb{Z}$-torsion. Fix such a pair. As in [20, Definition 2.1], if Σ' is a subset of Σ, Π is a subset of $\widehat{G}$ and r is a non-negative integer, we say Σ' is an r-cover for Π if the following two conditions are satisfied:

1. For all $\chi \in \Pi$, there exist (at least) r distinct places $v \in \Sigma'$ such that $\chi(G_v) = \{1\}$ (where G_v denotes the decomposition group in K/k of any place of K above v).
2. If the trivial character 1_G belongs to Π, then $|\Sigma'| \geqslant r + 1$.

We now fix a non-negative integer r such that Σ is an r-cover for $\widehat{G}$ and note that, by (7), r is less than or equal to $r_\Sigma(\chi) := \mathrm{ord}_{s=0} L_{K/k,\Sigma}(s,\chi)$ for every character $\chi \in \widehat{G}$. Let $\widehat{G}_{r,\Sigma} := \{\chi \in \widehat{G}\colon r_\Sigma(\chi) = r\}$. For any $\mathbb{Z}[G]$-module M with no $\mathbb{Z}$-torsion, let $M_{r,\Sigma} := \{m \in M\colon e_\chi m = 0$ in $\mathbb{C}M$ for all $\chi \in \widehat{G} \setminus \widehat{G}_{r,\Sigma}\}$. Note that in particular $\mathbb{C}[G]_{r,\Sigma} = \theta^{(r)}_{K/k,\Sigma,T}(0) \cdot \mathbb{C}[G]$. As in [20, Definition 2.4], we define $\Sigma_{\min}$ as the intersection of all the subsets of Σ which are r-covers for $\widehat{G}_{r,\Sigma} \setminus \{1_G\}$ (by [20, Lemma 2.3], $\Sigma_{\min}$ happens to be the unique minimal r-cover for $\widehat{G}_{r,\Sigma} \setminus \{1_G\}$). For any place v in Σ, we fix a place $w(v)$ of K above v. We introduce and fix an order on Σ, which in particular induces an order on each of its subsets. If $\widehat{G}_{r,\Sigma} = \{1_G\}$ (as explained in [20, Example 1], this happens if and only if Σ consists precisely of $r+1$ completely split places), we let $I(\Sigma) := \{v_1,\dots,v_r\}$, assuming that $v_1 < \dots < v_r < v_{r+1}$ are the elements of Σ. For any $I \subseteq \Sigma$ of cardinality r, we define a $\mathbb{C}[G]$-linear regulator map $R_I : \mathbb{C} \wedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,\Sigma,T} \to \mathbb{C}[G]$ by setting

$$R_I(u_1 \wedge \dots \wedge u_r) := \det\left(\frac{1}{|G_v|}\sum_{\sigma\in G} \log\bigl|u_j^{\sigma^{-1}}\bigr|_{w(v)} \cdot \sigma\right)_{v\in I,\, 1\leqslant j\leqslant r}$$

for all $u_1,\dots,u_r \in \mathcal{O}^\times_{K,\Sigma,T}$ and then extending by $\mathbb{C}$-linearity. Finally, we define the regulator map

$$\mathcal{R} = \mathcal{R}_{r,\Sigma} := \begin{cases} \sum_{I\in\wp_r(\Sigma_{\min})} R_I, & \text{if } \widehat{G}_{r,\Sigma} \neq \{1_G\}, \\ R_{I(\Sigma)}, & \text{if } \widehat{G}_{r,\Sigma} = \{1_G\} \end{cases}$$

where the summation over all the subsets of cardinality r of $\Sigma_{\min}$ is by definition equal to 0 if $\Sigma_{\min} = \emptyset$. By [20, Proposition 3.2], $\mathcal{R}$ restricts to give a $\mathbb{C}[G]$-isomorphism $(\mathbb{C} \wedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,\Sigma,T})_{r,\Sigma} \xrightarrow{\cong} (\mathbb{C}[G])_{r,\Sigma}$.

For any $\mathbb{Z}[G]$-module M, we define a $\mathbb{C}[G]$-linear pairing

$$\mathbb{C} \wedge^r_{\mathbb{Z}[G]} \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(M, \mathbb{Z}[G]\bigr) \times \mathbb{C} \wedge^r_{\mathbb{Z}[G]} M \to \mathbb{C}[G]$$

by setting $(\phi_1 \wedge \dots \wedge \phi_r)(u_1 \wedge \dots \wedge u_r) = \det_{1\leqslant i,j\leqslant r}(\phi_i(u_j))_{1\leqslant i,j\leqslant r}$ for all $\phi_1,\dots,\phi_r \in \mathrm{Hom}_{\mathbb{Z}[G]}(M,\mathbb{Z}[G])$ and all $u_1,\dots,u_r \in M$ and then extending by $\mathbb{C}$-linearity.

In the rest of this section we abbreviate $\theta^{(r)}_{K/k,\Sigma,T}(0)$ by denoting it $\theta^{(r)}_T(0)$.

Definition 6.1. We define the following two subsets of $\mathbb{R}[G]$:

$$\mathcal{I} := \bigl\{\theta^{(r)}_T(0)R(\phi)\colon \phi \in \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(\mathcal{O}^\times_{K,\Sigma,T}, X_{K,\Sigma}\bigr)\bigr\},$$

$$\mathcal{J} := \bigl\{\bigl(\wedge_{i=1}^{i=r}\varphi_i\bigr)\bigl(\mathcal{R}^{-1}\bigl(\theta^{(r)}_T(0)\bigr)\bigr)\colon \varphi_1,\dots,\varphi_r \in \mathrm{Hom}_{\mathbb{Z}[G]}\bigl(\mathcal{O}^\times_{K,\Sigma,T}, \mathbb{Z}[G]\bigr)\bigr\}.$$

Then [20, Conjecture 3.8] can be restated as follows.

Conjecture 6.2 *(Emmons–Popescu). If (Σ,T) is appropriate for K/k and Σ is an r-cover for $\widehat{G}$, then $\mathcal{J} \subseteq \mathbb{Z}[G]$.*

Remark 6.3. As already noted in [20, Remark 3.9], it is a straightforward consequence of the above construction that, if (at least) r places of Σ split completely in the extension K/k, then Conjecture 6.2 is equivalent to Rubin's Conjecture B′ (and therefore also to Rubin's Conjecture B) for the set of data $(K/k,\Sigma,T,r)$ in [35]. Theorem 6.4 (i) below therefore justifies our assertion that the predicted containment (2) is stronger than Rubin's conjecture. Theorem 6.4 (ii) below then combines with Corollary 4.2 to suggest that, under the more general hypotheses of Emmons and Popescu, and modulo certain possible denominators bounded by $|G|^r$, it would be reasonable to expect the set $\mathcal{J}$ to be contained not only in $\mathbb{Z}[G]$ but also in the Fitting ideal of the G-module $H^1_{W,T}(\mathcal{O}_{K,S},\mathbb{G}_m)'$.

For any element x and subset A of $\mathbb{C}[G]$, xA simply denotes the subset $\{xa\colon a \in A\}$ of $\mathbb{C}[G]$.

Theorem 6.4.

(i) *If Σ contains (at least) r places which split completely in K/k (and in particular whenever $|\Sigma| = r+1$, by [20, Lemma 2.2]), we have that $\mathcal{I} = \mathcal{J}$.*
(ii) *For every $\chi \in \widehat{G}$, we have that $e_\chi \mathcal{I} \subseteq \frac{1}{|\ker(\chi)|^r} e_\chi \mathcal{J}$ and that $e_\chi \mathcal{J} \subseteq \frac{1}{|\ker(\chi)|^r} e_\chi \mathcal{I}$.*

Proof. The case $r = 0$ is clear because, with $e_0 = e_r$ defined as in the proof of Corollary 4.2, one finds that (7) implies that $\theta_T^{(0)}(0)R(\phi) = \theta_T^{(0)}(0)e_0R(\phi) = \theta_T^{(0)}(0)$ for any ϕ. We assume henceforth that $r > 0$. In order to prove claim (i), let $v_1, \ldots, v_r \in \Sigma$ split completely in K/k. Put $I = \{v_1, \ldots, v_r\}$, let $v_0 \in \Sigma \setminus I$ and write $w_i = w(v_i)$ for all i. Then by [20, Remark 3.1], $\mathcal{R}$ coincides with the regulator map R_η defined by Rubin in [35, §2.1] for $\eta = w_1^* \wedge \cdots \wedge w_r^*$ where the w_i^* are the elements of $\mathrm{Hom}_{\mathbb{Z}[G]}(Y_{K,\Sigma}, \mathbb{Z}[G])$ given by $w_i^*(w) = \sum_{gw_i = w} g$ for each $w \in \Sigma(K)$. Hence

$$\begin{aligned}
\theta_T^{(r)}(0) \wedge^r R_{K,\Sigma}^{-1}\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big) &= \mathcal{R}^{-1}\big(\theta_T^{(r)}(0)\mathcal{R}\big(\wedge^r R_{K,\Sigma}^{-1}\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big)\big)\\
&= \mathcal{R}^{-1}\big(\theta_T^{(r)}(0)\big(w_1^* \wedge \cdots \wedge w_r^*\big)\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big)\\
&= \mathcal{R}^{-1}\big(\theta_T^{(r)}(0)\big),
\end{aligned} \tag{20}$$

where the second equality follows from the definition of R_η and the third equality holds because $(w_1^* \wedge \cdots \wedge w_r^*)(\wedge_{i=1}^{i=r}(w_i - w_0))$ is just the determinant of the identity $r \times r$ matrix with entries in $\mathbb{Z}[G]$.

In order to prove inclusion one way, let first $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,\Sigma,T}^\times, X_{K,\Sigma})$. For $1 \leqslant i \leqslant r$, let $f_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(X_{K,\Sigma}, \mathbb{Z}[G])$ be defined by letting the direct sum of the f_i be the composite homomorphism

$$X_{K,\Sigma} \xrightarrow{\alpha} Y_{K,I} \xrightarrow{\cong} \mathbb{Z}[G]^r$$

where the second map sends each w_i to the element of $\mathbb{Z}[G]^r$ with 1 in the i-th component and zeros in the other components (and $\mathbb{Z}[G]^r$ denotes the direct sum of r copies of $\mathbb{Z}[G]$). Then clearly $f_i(w_j - w_0) = \delta_{ij}$ (the Kronecker Delta). For $1 \leqslant i \leqslant r$, let $\varphi_i := f_i \circ \phi$. Then (20) implies that

$$\begin{aligned}
\theta_T^{(r)}(0)R(\phi) &= \theta_T^{(r)}(0)R(\phi)\det\big(f_i(w_j - w_0)\big)_{1 \leqslant i,j \leqslant r}\\
&= \theta_T^{(r)}(0)\det_{\mathbb{R}[G]}\big(\wedge^r \phi_{\mathbb{R}} \circ \wedge^r R_{K,\Sigma}^{-1}\big)\big(\wedge_{i=1}^{i=r} f_i\big)\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\\
&= \theta_T^{(r)}(0)\big(\wedge_{i=1}^{i=r} f_i\big)\big(\big(\wedge^r \phi_{\mathbb{R}} \circ \wedge^r R_{K,\Sigma}^{-1}\big)\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big)\\
&= \theta_T^{(r)}(0)\big(\wedge_{i=1}^{i=r}(f_i \circ \phi)\big)\big(\wedge^r R_{K,\Sigma}^{-1}\big(\wedge_{i=1}^{i=r}(w_i - w_0)\big)\big)\\
&= \big(\wedge_{i=1}^{i=r}\varphi_i\big)\big(\mathcal{R}^{-1}\big(\theta_T^{(r)}(0)\big)\big) \in \mathcal{J},
\end{aligned}$$

and we have proved inclusion one way.

In order to prove the second required inclusion, we now let $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,\Sigma,T}^\times, \mathbb{Z}[G])$. Let F be the composite homomorphism

$$\mathbb{Z}[G]^r \xrightarrow{\cong} Y_{K,I} \xrightarrow{\beta} X_{K,\Sigma}$$

where the first map is the inverse of the one described above and $\alpha \circ \beta$ is the identity on $Y_{K,I}$ (such a β exists because $Y_{K,I}$ is a free $\mathbb{Z}[G]$-module and α is surjective). Define $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,\Sigma,T}, X_{K,\Sigma})$ by $\phi := F \circ \bigoplus_{1\leqslant i\leqslant r}\varphi_i$. Then clearly we have $\varphi_i = f_i \circ \phi$ for each i. But then, by the same argument as above, we get that

$$\begin{aligned}\left(\wedge_{i=1}^{i=r}\varphi_i\right)\left(\mathcal{R}^{-1}\left(\theta_T^{(r)}(0)\right)\right) &= \left(\wedge_{i=1}^{i=r}(f_i\circ\phi)\right)\left(\mathcal{R}^{-1}\left(\theta_T^{(r)}(0)\right)\right)\\ &= \theta_T^{(r)}(0)R(\phi)\in\mathcal{I},\end{aligned}$$

and we have proved inclusion the other way.

In order to prove claim (ii), we note first that if $|\Sigma| = r+1$, then by [20, Lemma 2.2] Σ contains (at least) r places which split completely in K/k and the result follows from claim (i), so we may and will assume henceforth that $|\Sigma| > r+1$. If $r_\Sigma(\chi) > r$, then $e_\chi\mathcal{I} = 0 = e_\chi\mathcal{J}$ by (7), so the result is trivially true for each such χ. We henceforth fix $\chi \in \widehat{G}_{r,\Sigma}$. By (7), we know that χ cannot be the trivial character and furthermore that (exactly) r places $v_{1,\chi},\ldots,v_{r,\chi}$ of Σ split completely in the fixed field of $\ker(\chi)$, which we will denote by K_χ. Put $I_\chi := \{v_{1,\chi},\ldots,v_{r,\chi}\}$, let $v_{0,\chi}\in\Sigma\setminus I_\chi$ and write $w_{i,\chi} = w(v_{i,\chi})$ for all i. We also write N_χ for $N_{\ker(\chi)}$, the algebraic norm attached to $\ker(\chi)$, set $\Gamma_\chi := \mathrm{Gal}(K_\chi/k) \cong G/\ker(\chi)$, and let $\alpha_\chi : X_{K_\chi,\Sigma} \to Y_{K_\chi,I_\chi}$ be the map α defined in Section 3.3 corresponding to the triple $(K_\chi/k, \Sigma, I_\chi)$.

For any $z \in (\mathbb{C}\wedge^r_{\mathbb{Z}[G]}\mathcal{O}^{\times}_{K,\Sigma,T})_{r,\Sigma}$, we have by [20, Lemma 3.10(2)] that $e_\chi\mathcal{R}(z) = e_\chi R_{I_\chi}(z)$, and by [35, Lemma 2.2] that $R_{I_\chi}(z) = \prod_{i=1}^{i=r}\frac{1}{|G_{v_{i,\chi}}|}R_\eta(z)$ for $\eta = w^*_{1,\chi}\wedge\cdots\wedge w^*_{r,\chi}$, where again R_η is the regulator map defined by Rubin in [35, §2.1] and $G_{v_{i,\chi}}$ denotes the decomposition group in G of any place of K above $v_{i,\chi}$. Hence

$$\begin{aligned}&e_\chi\theta_T^{(r)}(0)\wedge^r R^{-1}_{K,\Sigma}\left(\wedge_{i=1}^{i=r}(w_{i,\chi}-w_{0,\chi})\right)\\ &\quad= \mathcal{R}^{-1}\left(\theta_T^{(r)}(0)e_\chi\mathcal{R}\left(\wedge^r R^{-1}_{K,\Sigma}\left(\wedge_{i=1}^{i=r}(w_{i,\chi}-w_{0,\chi})\right)\right)\right)\\ &\quad= \mathcal{R}^{-1}\left(\theta_T^{(r)}(0)e_\chi\left(\prod_{i=1}^{i=r}\frac{1}{|G_{v_{i,\chi}}|}\right)\left(w^*_{1,\chi}\wedge\cdots\wedge w^*_{r,\chi}\right)\left(\wedge_{i=1}^{i=r}(w_{i,\chi}-w_{0,\chi})\right)\right)\\ &\quad= e_\chi\left(\prod_{i=1}^{i=r}\frac{N_{G_{v_{i,\chi}}}}{|G_{v_{i,\chi}}|}\right)\mathcal{R}^{-1}\left(\theta_T^{(r)}(0)\right)\\ &\quad= e_\chi\mathcal{R}^{-1}\left(\theta_T^{(r)}(0)\right),\end{aligned}\tag{21}$$

where the second equality follows from the definition of R_η, the third equality holds because $(w^*_{1,\chi}\wedge\cdots\wedge w^*_{r,\chi})(\wedge_{i=1}^{i=r}(w_{i,\chi}-w_{0,\chi}))$ is just the determinant of the $r\times r$ diagonal matrix with $N_{G_{v_{i,\chi}}}$ in the (i,i) entry for each i and zeros elsewhere and the last equality holds because $he_\chi = e_\chi$ for any $h\in\ker(\chi)$ and $G_{v_{i,\chi}}\subseteq\ker(\chi)$ for $1\leqslant i\leqslant r$.

In order to prove inclusion one way, let first $\phi\in\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{K,\Sigma,T}, X_{K,\Sigma})$. For $1\leqslant i\leqslant r$, let $f_i\in\mathrm{Hom}_{\mathbb{Z}[G]}(X_{K,\Sigma},\mathbb{Z}[G])$ be defined by letting the direct sum of the f_i be the composite homomorphism

$$X_{K,\Sigma}\to N_\chi X_{K,\Sigma}\xrightarrow{\cong}X_{K_\chi,\Sigma}\xrightarrow{\alpha_\chi}Y_{K_\chi,I_\chi}\xrightarrow{\cong}\mathbb{Z}[\Gamma_\chi]^r\to\mathbb{Z}[G]^r$$

where the first map is given by the action of N_χ on $X_{K,\Sigma}$, the second map is the one defined in [39, §I.6.5], the fourth map sends, for each i, the place of K_χ below w_i to the element of $\mathbb{Z}[\Gamma_\chi]^r$ with 1 in the i-th component and zeros in the other components, and the fifth map sends the coset of any $g\in G$ in $G/\ker(\chi)$ to $N_\chi g$ (after identifying Γ_χ with $G/\ker(\chi)$). It is then straightforward to check that $f_i(w_{j,\chi} - w_{0,\chi}) = \delta_{ij} N_\chi$. For $1 \leqslant i \leqslant r$, let $\varphi_i := f_i \circ \phi$. Then, using the fact that $N_\chi e_\chi = |\ker(\chi)| e_\chi$, the equality (21) implies that

$$\begin{aligned}
e_\chi \theta_T^{(r)}(0) R(\phi) &= \left(\frac{N_\chi}{|\ker(\chi)|}\right)^r e_\chi \theta_T^{(r)}(0) R(\phi) \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \theta_T^{(r)}(0) R(\phi) \det\bigl(f_i(w_{j,\chi} - w_{0,\chi})\bigr)_{1 \leqslant i,j \leqslant r} \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \theta_T^{(r)}(0) \det_{\mathbb{R}[G]}\bigl(\wedge^r \phi_{\mathbb{R}} \circ \wedge^r R_{K,\Sigma}^{-1}\bigr)\bigl(\wedge_{i=1}^{i=r} f_i\bigr)\bigl(\wedge_{i=1}^{i=r}(w_{i,\chi} - w_{0,\chi})\bigr) \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \theta_T^{(r)}(0) \bigl(\wedge_{i=1}^{i=r} f_i\bigr)\bigl(\bigl(\wedge^r \phi_{\mathbb{R}} \circ \wedge^r R_{K,\Sigma}^{-1}\bigr)\bigl(\wedge_{i=1}^{i=r}(w_{i,\chi} - w_{0,\chi})\bigr)\bigr) \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \theta_T^{(r)}(0) \bigl(\wedge_{i=1}^{i=r}(f_i \circ \phi)\bigr)\bigl(\wedge^r R_{K,\Sigma}^{-1}\bigl(\wedge_{i=1}^{i=r}(w_{i,\chi} - w_{0,\chi})\bigr)\bigr) \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \bigl(\wedge_{i=1}^{i=r} \varphi_i\bigr)\bigl(\mathcal{R}^{-1}\bigl(\theta_T^{(r)}(0)\bigr)\bigr) \in \frac{1}{|\ker(\chi)|^r} e_\chi \mathcal{J},
\end{aligned}$$

and we have proved the first inclusion.

In order to prove the second required inclusion, we now let $\varphi_1, \dots, \varphi_r \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,\Sigma,T}^\times, \mathbb{Z}[G])$. Let F be the composite homomorphism

$$\mathbb{Z}[G]^r \to \mathbb{Z}[\Gamma_\chi]^r \xrightarrow{\cong} Y_{K_\chi, I_\chi} \xrightarrow{\beta_\chi} X_{K_\chi,\Sigma} \xrightarrow{\cong} N_\chi X_{K,\Sigma} \to X_{K,\Sigma}$$

where the second and fourth maps are the inverses of the respective maps described above, the first map is induced by the natural surjection from G to Γ_χ, the fifth map is the obvious inclusion and $\alpha_\chi \circ \beta_\chi$ is the identity on Y_{K_χ, I_χ} (such a β_χ exists because Y_{K_χ, I_χ} is a free $\mathbb{Z}[\Gamma_\chi]$-module and α_χ is surjective). Define $\phi \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,\Sigma,T}^\times, X_{K,\Sigma})$ by $\phi := F \circ \bigoplus_{1 \leqslant i \leqslant r} \varphi_i$. It is then straightforward to check that for any $u \in \mathcal{O}_{K,\Sigma,T}^\times$, we have $(f_i \circ \phi)(u) = |\ker(\chi)| N_\chi \varphi_i(u)$. But then, by the same argument as above, and again using that $N_\chi e_\chi = |\ker(\chi)| e_\chi$, we get that

$$\begin{aligned}
e_\chi \bigl(\wedge_{i=1}^{i=r} \varphi_i\bigr)\bigl(\mathcal{R}^{-1}\bigl(\theta_T^{(r)}(0)\bigr)\bigr) &= e_\chi \left(\frac{|\ker(\chi)| N_\chi}{|\ker(\chi)|^2}\right)^r \bigl(\wedge_{i=1}^{i=r} \varphi_i\bigr)\bigl(\mathcal{R}^{-1}\bigl(\theta_T^{(r)}(0)\bigr)\bigr) \\
&= e_\chi \frac{1}{|\ker(\chi)|^{2r}} \bigl(\wedge_{i=1}^{i=r}(f_i \circ \phi)\bigr)\bigl(\mathcal{R}^{-1}\bigl(\theta_T^{(r)}(0)\bigr)\bigr) \\
&= \frac{1}{|\ker(\chi)|^r} e_\chi \theta_T^{(r)}(0) R(\phi) \in \frac{1}{|\ker(\chi)|^r} e_\chi \mathcal{I},
\end{aligned}$$

and we have proved the second inclusion. □

6.2. *The fractional Galois ideal of Buckingham*

In this section we prove the validity of the containment (3) relating elements of the form $\theta_{K/k,S,T}^{(r)}(0) R(\phi)$ to Buckingham's fractional Galois ideal $\mathcal{J}(K/k, S, T)$. As explained in Section 2, this comparison is interesting of its own accord and furthermore is required in order to establish the validity of Proposition 2.7.

We recall that e_r is the idempotent defined in the proof of Corollary 4.2 and, for the reader's convenience, recall the definition of the $\mathbb{Z}[G]$-submodule of $\mathbb{R}[G]$ $\mathcal{J}(K/k, S, T)$ from [3]. In order to do so, we once again fix a finite set T of places of k disjoint from S such that $\mathcal{O}_{K,S,T}^\times$ has no $\mathbb{Z}$-torsion.

We are still assuming that r places of S split completely in K/k. Let $r_S(\chi)$ (for every $\chi \in \widehat{G}$) and the $\mathbb{C}[G]$-linear pairing $\mathbb{C} \wedge^t \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \times \mathbb{C} \wedge^t \mathcal{O}_{K,S,T}^\times \to \mathbb{C}[G]$ be defined as in Section 6.1. For any $t \geqslant 0$, set

$$\begin{aligned}\Omega_{S,T,t} := \big\{ & u \in \wedge^t \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}\colon\ (\phi_1 \wedge \cdots \wedge \phi_t)(u) \in \mathbb{Z}[G] \text{ for any } \phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]\big), \\ & e_\chi u = 0 \text{ for all } \chi \in \widehat{G} \text{ with } r_S(\chi) \neq t\big\}.\end{aligned}$$

$\Omega_{S,T,t}$ is clearly a $\mathbb{Z}[G]$-submodule of $e_t \wedge^t \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}$, where $e_t := \sum_{r_S(\chi)=t} e_\chi$. We define finally

$$\begin{aligned}\mathcal{J}(K/k, S, T) := \theta_{K/k,S,T}^*(0)\Bigg\{ & \det_{\mathbb{R}[G]}\Bigg((\beta \otimes_{\mathbb{Q}} \mathbb{R}) \circ \bigoplus_{t=0}^{\infty} e_t \wedge^t R_{K,S}^{-1}\Bigg)\colon \\ & \beta \in \mathrm{Hom}_{\mathbb{Q}[G]}\Bigg(\bigoplus_{t=0}^{\infty} e_t \wedge^t \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}, \bigoplus_{t=0}^{\infty} e_t \wedge^t X_{K,S} \otimes \mathbb{Q}\Bigg), \\ & \beta(\Omega_{S,T,t}) \subseteq e_t \wedge_{\mathbb{Z}[G],\mathrm{tf}}^t X_{K,S} \text{ for all } t \geqslant 0\Bigg\}\end{aligned}$$

where $\wedge_{\mathbb{Z}[G],\mathrm{tf}}^t X_{K,S}$ denotes the image of $\wedge_{\mathbb{Z}[G]}^t X_{K,S}$ in $(\wedge_{\mathbb{Z}[G]}^t X_{K,S}) \otimes \mathbb{Q}$.

Theorem 6.5. *For any $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, X_{K,S})$, we have*

$$\theta_{K/k,S,T}^{(r)}(0) R(\phi) \in e_r \mathcal{J}(K/k, S, T).$$

Proof. Fix $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, X_{K,S})$, and let $\widetilde{e_r \wedge^r \phi_{\mathbb{Q}}}$ denote the element of $\mathrm{Hom}_{\mathbb{Q}[G]}(\bigoplus_{t=0}^{\infty} e_t \wedge^t \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}, \bigoplus_{t=0}^{\infty} e_t \wedge^t X_{K,S} \otimes \mathbb{Q})$ given by $e_r \wedge^r (\phi \otimes \mathbb{Q})$ on $e_r \wedge^r \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}$ and by 0 on $e_t \wedge^t \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}$ for any $t \neq r$. Then, using the proof of Corollary 4.2, we get that

$$\begin{aligned}\theta_{K/k,S,T}^{(r)}(0) R(\phi) &= \theta_{K/k,S,T}^*(0) e_r \det_{\mathbb{R}[G]}\big(\wedge^r \phi_{\mathbb{R}} \circ \wedge^r R_{K,S}^{-1}\big) \\ &= \theta_{K/k,S,T}^*(0) e_r \det_{\mathbb{R}[G]}\big(e_r \wedge^r \phi_{\mathbb{R}} \circ e_r \wedge^r R_{K,S}^{-1}\big) \\ &= \theta_{K/k,S,T}^*(0) e_r \det_{\mathbb{R}[G]}\Bigg(\big(\widetilde{e_r \wedge^r \phi_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{R}\big) \circ \bigoplus_{t=0}^{\infty} e_t \wedge^t R_{K,S}^{-1} \,\Big|\, e_r \wedge^r X_{K,S} \otimes \mathbb{Q}\Bigg) \\ &= \theta_{K/k,S,T}^*(0) e_r \det_{\mathbb{R}[G]}\Bigg(\big(\widetilde{e_r \wedge^r \phi_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{R}\big) \circ \bigoplus_{t=0}^{\infty} e_t \wedge^t R_{K,S}^{-1}\Bigg),\end{aligned}$$

where the fourth equality follows from the general fact that for any finitely generated $\mathbb{R}[G]$-module M, any idempotent $e \in \mathbb{R}[G]$ and any $\mathbb{R}[G]$-endomorphism α of M, one has $e \det_{\mathbb{R}[G]}(\alpha) = e \det_{\mathbb{R}[G]}(\alpha|_{eM})$.

Hence we only need to show that $e_r \wedge^r (\phi \otimes \mathbb{Q})(\Omega_{S,T,r}) \subseteq e_r \wedge_{\mathbb{Z}[G],\mathrm{tf}}^r X_{K,S}$ in order to complete the proof of the theorem. If $r = 0$, this statement is trivial, so we assume henceforth that $r > 0$. For $i \in \{1, \ldots, r\}$, define $\phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$ by $\phi_i(u) := x_i$ if $\phi(u) = \sum_{i=1}^{i=r} x_i(w_i - w_0) + \sum_{w \in S_0(K)} y_w w$ with all the x_i and y_w in $\mathbb{Z}[G]$. Fix $u = (u_1 \wedge \cdots \wedge u_r) \otimes q \in \Omega_{S,T,r}$ with $u_1, \ldots, u_r \in \mathcal{O}_{K,S,T}^\times$, $q \in \mathbb{Q}$ (note that every element of $e_r \wedge^r \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Q}$ can be written as $e_r u$ for a u of this form). Then we have that

$$
\begin{aligned}
\wedge^r(\phi\otimes\mathbb{Q})(u_1\wedge\cdots\wedge u_r) &= \wedge_{j=1}^{j=r}\left(\sum_{i=1}^{i=r}\phi_i(u_j)(w_i-w_0)+\sum_{w\in S_0(K)}y_w^j w\right)\\
&= \left(\sum_{\tau\in S_r}\operatorname{sgn}(\tau)\prod_{k=1}^{k=r}\phi_{\tau(k)}(u_k)\right)\wedge_{i=1}^{i=r}(w_i-w_0)+\sum_{\omega}z_\omega\omega\\
&= \det\bigl(\phi_i(u_j)\bigr)_{1\leqslant i,j\leqslant r}\wedge_{i=1}^{i=r}(w_i-w_0)+\sum_{\omega}z_\omega\omega\\
&= \bigl((\phi_1\wedge\cdots\wedge\phi_r)(u_1\wedge\cdots\wedge u_r)\bigr)\wedge_{i=1}^{i=r}(w_i-w_0)+\sum_{\omega}z_\omega\omega
\end{aligned}
$$

where ω runs over monomials $w_1'\wedge\cdots\wedge w_r'$ where at least one of the w_i' belongs to $S_0(K)$, all the y_w^j and z_ω belong to $\mathbb{Z}[G]$ and S_r denotes the group of permutations of $\{1,\dots,r\}$. Now, by the proof of [35, Lemma 2.6], we deduce that

$$
\begin{aligned}
&e_r\left(\bigl((\phi_1\wedge\cdots\wedge\phi_r)(u_1\wedge\cdots\wedge u_r)\bigr)\wedge_{i=1}^{i=r}(w_i-w_0)+\sum_{\omega}z_\omega\omega\right)\\
&\quad= e_r\bigl((\phi_1\wedge\cdots\wedge\phi_r)(u_1\wedge\cdots\wedge u_r)\bigr)\wedge_{i=1}^{i=r}(w_i-w_0),
\end{aligned}
$$

and hence, by the integrality condition imposed by definition on elements of $\Omega_{S,T,r}$, we finally have that

$$
\begin{aligned}
e_r\wedge^r(\phi\otimes\mathbb{Q})(u) &= e_r\bigl((\phi_1\wedge\cdots\wedge\phi_r)(u_1\wedge\cdots\wedge u_r)\bigr)\wedge_{i=1}^{i=r}(w_i-w_0)\otimes q\\
&= e_r\bigl((\phi_1\wedge\cdots\wedge\phi_r)(u)\bigr)\wedge_{i=1}^{i=r}(w_i-w_0)\\
&\in e_r\wedge^r_{\mathbb{Z}[G],\mathrm{tf}}X_{K,S},
\end{aligned}
$$

as required. $\square$

Acknowledgments

I would like to thank David Burns for providing much encouragement and many helpful suggestions, and Paul Buckingham for some stimulating discussions concerning this project.

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