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Deterministically generating Picard groups of hyperelliptic curves over finite fields[☆]



Michiel Kusters

UCI, United States

ARTICLE INFO

Article history:

Received 12 June 2014

Received in revised form 10

September 2015

Accepted 10 September 2015

Available online 2 November 2015

Communicated by David Goss

MSC:

11G20

14H25

14C22

11R58

Keywords:

Picard group

Hyperelliptic curve

Finite field

Shape parameter

Deterministic algorithm

ABSTRACT

Let $\epsilon > 0$. In this article we will present a deterministic algorithm which does the following. The input is a hyperelliptic curve C of genus g over a finite field k of cardinality q given by $y^2 + h(x)y = f(x)$ such that the x -coordinate map is ramified at ∞ . In time $O(g^{2+\epsilon}q^{1/2+\epsilon})$ the algorithm outputs a set of generators of the Picard group $\text{Pic}_k^0(C)$. This extends results which others have obtained when $g = 1$.

In this article we introduce a combinatorial tool, the *shape parameter*, which we use together with character sum estimates from class field theory to deduce the statement.

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Contents

1. Introduction	740
2. Preliminaries	741

[☆] This article covers some of the results of my PhD thesis written under the supervision of Hendrik Lenstra at the Universiteit Leiden. For more details, see [3, Chapter 6, Chapter 7].

E-mail address: kusters@gmail.com.

URL: <https://sites.google.com/site/kusters/>.

2.1.	Curves and function fields	741
2.2.	Class field theory	742
2.3.	Hyperelliptic curves	744
3.	Shape parameter	745
4.	Applications of the shape parameter to hyperelliptic curves	747
4.1.	Main statements	747
4.2.	Realizing Galois groups	750
4.3.	Character sum estimates	753
4.4.	Proof of theorem	755
5.	The algorithm	756
	References	758

1. Introduction

An algorithmic problem in arithmetic geometry is to explicitly find the group structure of the Picard group of a curve of genus g over a finite field of size $q = p^m$. A related problem is to find a generating set of this Picard group. Let $\epsilon > 0$. In this article we describe a deterministic way of finding a generating set, when the curve is hyperelliptic, in time $O(g^{2+\epsilon}q^{1/2+\epsilon})$.

Let C be a hyperelliptic curve of genus g over a finite field k of cardinality q and characteristic p given by an equation $y^2 + h(x)y = f(x)$. We require that (f, h) satisfies certain conditions (see Subsection 2.3) and we assume that the natural projection map to the projective line by taking the x -coordinate is ramified at ∞ . Our main theorem is the following.

Theorem 1.1. *For any $\epsilon > 0$ there is a deterministic algorithm which on input a hyperelliptic curve C of genus g over a finite field k of cardinality q outputs a set of generators of the Picard group $\text{Pic}_k^0(C)$ in time $O(g^{2+\epsilon}q^{1/2+\epsilon})$.*

Such a generating set can then be used in other algorithms to deterministically determine the group structure of $\text{Pic}_k^0(C)$.

Let us discuss one of the main ingredients of the proof of Theorem 1.1. Let ∞' be the point above ∞ . Let $\varphi_C : C(k) \rightarrow \text{Pic}_k^0(C)$ be the map given by $P \mapsto [P] - [\infty']$. For a subset S of k put $C_S = \{P \in C(k) : x(P) \in S\}$. An interval I of k is a subset of the form $B + \alpha[s, \dots, s+r]$ where B is an additive subgroup of k , $\alpha \in k$ and $s, r \in \mathbb{Z}_{\geq 0}$ (or more precisely, see Definition 3.4).

Kohel and Shparlinski [2, Corollary 2] have shown the following for $g = 1$. For S an interval of k of cardinality greater than $15(1 + \log(p))q^{1/2}$ they deduce that $\langle \varphi_C(C_S) \rangle = \text{Pic}_k^0(C)$. We generalize and improve their result in the following ways. This possible generalization was already suggested in [2].

Theorem 1.2. *Assume that $\#C(k) > (2g - 2)\sqrt{q}$. Assume that $p \neq 2$ or $p = 2$ and $\deg(h) < g$. Let $S \subseteq k$ be a coset of a subgroup or an interval. Put $s = 2$ if $p = 2$ and $s = 3$ if $p \neq 2$. Put $t = 1$ if S is a coset of a subgroup and $t = 2$ if S is an interval which*

is not a coset of a subgroup. Assume that

$$\#S \geq 2t(2g - 2 + s)\sqrt{q}.$$

Then we have $\langle \varphi_C(C_S) \rangle = \text{Pic}_k^0(C)$.

The above theorem improves the results of [2] in the following ways.

- We allow hyperelliptic curves of any genus.
- We obtain similar theorems for subsets of $S \subseteq k$ which are not intervals or subgroups. For this reason we introduce the notion of the shape parameter of such a subset S .
- Our constants, as can be seen above, are a bit better. This improvement is already partially suggested in [2]. Furthermore, we do not have a $(1 + \log p)$ factor. This improvement is also suggested in [2].
- We look at the case $p = 2$ in the above theorem, even when $\deg(h) = g$. This case requires more work and there are exceptional cases. In [2], this case is avoided by finding a similar result for the y -coordinate. In the end our estimates are better when $p = 2$, but there are exceptional sets coming from certain morphisms (see Remark 4.5).

In [2] the authors use the aforementioned corollary [2, Corollary 2] to give a deterministic algorithm to find the group structure of the set of rational points of an elliptic curve over a finite field of size q in $O(q^{1/2+\epsilon})$. We use Theorem 1.2 just to find a generating set of the Picard group. We deduce Theorem 1.1 from Theorem 1.2 by using intervals which are large enough.

The strategy of the proof of Theorem 1.2 is the following. First we translate our problem to the calculation of certain character sums on the finite abelian group $k^+ \times \text{Pic}_k^0(C)$. We then construct, using class field theory, a finite geometric abelian extension of function fields M of $k(C)$ with group $G = k^+ \times \text{Pic}_k^0(C)$, which for a point $P \in C(k) \setminus \{\infty'\}$ has Frobenius element $(P, M/k(C)) = (x(P), [P] - [\infty']) \in G$. Theorems from class field theory allow us to estimate the character sums after we have calculated conductors of certain subextensions of $M/k(C)$. In certain exceptional cases, our proof does not work. The extension $M/k(C)$ we obtain either has Galois group which is smaller than $k^+ \times \text{Pic}_k^0(C)$ or $M/k(C)$ is not geometric. With a bit more work, one can still work out these cases.

2. Preliminaries

2.1. Curves and function fields

We assume that the reader is familiar with the theory of curves and function fields (see for example [6,10]). In this subsection we introduce some notation and recall some facts.

Let k be a field. A function field over k is a finitely generated field extension of k of transcendence degree 1. There is an anti-equivalence of categories between the category of normal projective curves over k with finite morphisms and the category of function fields over k with finite morphisms (see [5, Proposition 3.13]). A curve C is mapped to its function field $k(C)$ and a map $C \rightarrow D$ of curves induces an inclusion $k(D) \subseteq k(C)$. We will mostly study normal projective curves by looking at their function fields. The set of non-generic points up to Galois equivalence of such a curve C correspond to places $\mathcal{P}_{k(C)/k}$ of $k(C)$, that is, the valuation rings of $k(C)$ which contain k but are not equal to $k(C)$. Note that $C(k)$ corresponds to the subset of valuation rings in $\mathcal{P}_{k(C)/k}$ with residue field k . We say that such a place has degree 1.

Let K be a function field k . The full constant field of K is the integral closure of k in K . We say that K is geometrically irreducible if the full constant field is k . The genus of K is denoted by $g(K)$. Let $\text{div}_k(K)$ be the free abelian group on $\mathcal{P}_{K/k}$. An element $D \in \text{div}_k(K)$ is called a divisor of K . If D is a divisor on K , we denote by $\deg_k(D)$ its k -degree. If $P \in \mathcal{P}_{K/k}$ we denote by $D_P \in \mathbb{Z}$ the coefficient of D corresponding to P . The divisors of degree 0 are denoted by $\text{div}_k^0(K)$. An element f of K^* gives rise to a divisor of degree 0, denoted by (f) . The Picard group, $\text{Pic}_k^0(K)$, is defined by the exactness of the sequence

$$K^* \rightarrow \text{div}_k^0(K) \rightarrow \text{Pic}_k^0(K) \rightarrow 0.$$

If k is finite, then $\text{Pic}_k^0(K)$ is finite. If L/K is a finite field extension, then by $\text{disc}(L/K)$ we denote its discriminant. Let $P \in \mathcal{P}_{L/k}$. Then by $P|_K$ we denote the restriction of P to K . We set $\mathfrak{f}(P/P|_K) = \frac{\deg_k(P)}{\deg_k(P|_K)}$. We say that L/K is geometric if the full constant fields of L and K are the same.

2.2. Class field theory

We assume that the reader is already familiar with class field theory (see [1,6,7]). We recall some notation and statements. Let k be a finite field. Let K be a function field over k .

The aim of class field theory is to describe abelian extensions of K . Let L/K be a finite abelian Galois extension of K with group G . Class field theory associates to this extension a divisor $\mathfrak{f}(L/K) \in \text{div}_k(K)$, called the conductor. This divisor gives information about the ramified places in L/K . If L is the compositum of L_1/K and L_2/K , then one has $\mathfrak{f}(L/K) = \text{lcm}(\mathfrak{f}(L_1/K), \mathfrak{f}(L_2/K))$. Let M/K be a finite Galois extension with group G and let $\chi \in \text{Hom}(G, \mathbb{C}^*)$. We set $\mathfrak{f}(\chi) = \mathfrak{f}(L^{\ker(\chi)}/K)$. The set of unramified primes in $\mathcal{P}_{K/k}$ is denoted by $\text{unr}(L/K)$. The places of degree 1 in $\text{unr}(L/K)$ are denoted by $\text{unr}^1(L/K)$. For a prime $P \in \text{unr}(L/K)$ we denote by $(P, L/K) \in G$ its Frobenius element. We have a group morphism $\text{Norm}_{L/K} : \text{div}_k^0(L) \rightarrow \text{div}_k^0(K)$ which maps a place P to $\mathfrak{f}(P/P|_K)P|_K$. This map induces a map $\text{Norm}_{L/K} : \text{Pic}_k^0(L) \rightarrow \text{Pic}_k^0(K)$.

Suppose M/K is a finite abelian extension in some algebraic closure of L . Then one has for $D' \in \operatorname{div}_k(L)$ the equality

$$(D', LM/L)|_M = (\operatorname{Norm}_{L/K}(D'), M/K).$$

Class field theory gives us the following.

Proposition 2.1. *Let K be a function field over k and let $D \in \operatorname{div}_k(K)$ be of degree 1. Then the maximal abelian unramified extension of K is the compositum of the following two disjoint extensions: $\bar{k} \cdot K$ and a unique finite subextension $K_{[D]}$ with Galois group isomorphic to $\operatorname{Pic}_k^0(K)$ such that $(D, K_{[D]}/K) = 0$. For $D' \in \operatorname{div}_k(K)$ we have $(D', K_{[D]}/K) = [D'] - \deg_k(D')[D] \in \operatorname{Pic}_k^0(K)$.*

Proof. See [3, Chapter 2, Proposition 3.10]. \square

Corollary 2.2. *Let k' be a finite extension of k . Let K be a function field over k . Then the map $\operatorname{Norm}_{Kk'/K} : \operatorname{Pic}_{k'}^0(Kk') \rightarrow \operatorname{Pic}_k^0(K)$ is surjective.*

Proof. Proposition 2.1 gives a surjective map $\operatorname{Pic}_{k'}^0(Kk') \rightarrow \operatorname{Pic}_k^0(K)$ and one easily checks that it agrees with the norm. \square

Theorem 2.3. *Let L/K be a geometric Galois extension of function fields over k with group G . Assume that we have an injective morphism $\chi \in \operatorname{Hom}(G, \mathbb{C}^*)$. Then we have*

$$\left| \sum_{P \in \operatorname{unr}^1(L/K)} \deg_k(P) \chi((P, L/K)) \right| \leq m q^{1/2},$$

where $m = 2g(K) - 2 + \deg_k(\mathfrak{f}(\chi))$. It is an equality if $m = 1$.

Proof. This follows from [6, Theorem 9.16B]. \square

Later we will need to compute some conductors. The following lemma is useful.

Lemma 2.4. *Let K be a function field over k . Let K_s be a separable closure of K . Let L, M be finite abelian Galois extensions of K inside K_s of prime degree p respectively prime degree l with $L \cap M = K$. Let $v \in \mathcal{P}_{K/k}$ and suppose that $r = \mathfrak{f}(L/K)_v \in \mathbb{Z}_{\geq 1}$ and $s = \mathfrak{f}(M/K)_v \in \mathbb{Z}_{\geq 1}$. Let w be the unique extension of v to L . Assume that LM/L is ramified at w if $p = l$ and $r = s$. Then the following hold:*

- i. LM/K is totally ramified at v ;
- ii. if $p \neq l$ or $r \neq s$, we have $\mathfrak{f}(LM/L)_w = (p-1) \max(0, s-r) + s$;
- iii. if $p = l$ and $r = s$, we have $r \geq \mathfrak{f}(LM/L)_w \geq t$ where $t = 2$ if p is the residue field characteristic of v and 1 otherwise.

Proof. The full proof can be found in [3, Chapter 2, Lemma 3.19]. We prove i and ii. The proof of iii is similar.

If $p \neq l$, the ramification indices are coprime, and hence LM/K is totally ramified. If $p = l$, $r \neq s$ then both extensions have different conductors. An easy calculation shows that LM/K is totally ramified at v (here we use that l and p are prime). Hence in all cases LM/K is totally ramified at v .

The Führerdiskriminantenproduktformel (see [7]) says that in a cyclic extension of prime degree p' the conductor is $1/(p' - 1)$ times the discriminant of this extension. Hence we will compute conductors using discriminants.

We use a well-known identity for towers of fields [7, Proposition 8 of Chapter III]:

$$\text{disc}(LM/K) = \text{Norm}_{L/K}(\text{disc}(LM/L)) + l \cdot \text{disc}(L/K).$$

We will calculate the discriminants $\text{disc}(L/K)$ and $\text{disc}(LM/K)$ at the prime v . This gives us $\text{disc}(L/K)_v = (p - 1)r$.

Assume first that $l \neq p$. Then LM/K has one cyclic subextension of degree 1 with conductor 0 (at infinity), 1 of degree p with conductor r , 1 of degree l with conductor s and one of degree $p \cdot l$ of conductor $\max(r, s)$. Assume next that $r \neq s$ but $p = l$. Then LM/K has one cyclic subextension of degree 1 with conductor 0, 1 of degree p with conductor s , one of degree p with conductor r , and $p - 1$ of degree p with conductor $\max(r, s)$. We find if $l \neq p$ or $r \neq s$

$$\text{disc}(LM/K)_v = (p - 1)r + (l - 1)s + (p - 1)(l - 1)\max(r, s).$$

The result follows from the Führerdiskriminantenproduktformel. \square

One has the following lemma.

Lemma 2.5. *Let K/k be a function field where k is a finite field. Let L/K be a finite abelian Galois extension with group G . Let $\chi, \chi' \in \text{Hom}(G, \mathbb{C}^*)$. Then we have $\mathfrak{f}(\chi \cdot \chi') \leq \text{lcm}(\mathfrak{f}(\chi), \mathfrak{f}(\chi'))$, with equality at $P \in \mathcal{P}_{K/k}$ if we have $\mathfrak{f}(\chi)_P \neq \mathfrak{f}(\chi')_P$ or if the orders of χ and χ' are coprime.*

Proof. The proof follows easily from the definition of \mathfrak{f} (see [3, Chapter 2, Lemma 3.20] for a proof). \square

2.3. Hyperelliptic curves

The results of this subsection can be partially found in [5, Subsection 7.4.3] and the details can be found in [3, Chapter 2, Section 4]. For a polynomial $f \in k[x]$ we define f_j by $f = \sum_i f_i x^i$.

Let k be a perfect field. A function field K/k is called *hyperelliptic* if it has full constant field k , if the genus satisfies $g(K) \geq 1$, and there exists $x \in K$ with $[K : k(x)] = 2$.

Let $g \in \mathbb{Z}_{\geq 1}$. Consider $(f, h) \in k[x]^2$ with the following properties:

- i. $\deg(f) \in \{2g + 1, 2g + 2\}$;
- ii. $y^2 + hy - f$ is separable and irreducible in $k(x)[y]$;
- iii. if $\text{char}(k) \neq 2$ the following hold:
 - (a) $h = 0$;
 - (b) f is separable in $k[x]$;
- iv. if $\text{char}(k) = 2$, then the following hold:
 - (a) $\deg(h) \leq g + 1$;
 - (b) $(h, h'^2 f + f'^2) = k[x]$;
 - (c) $(h_{g+1}, h_g^2 f_{2g+2} + f_{2g+1}^2) = k$.

Set $K_{f,h} = k(x)[y]/(y^2 + hy - f)$ with natural inclusion $k(x) \subseteq K_{f,h}$. Then $K_{f,h}$ is a hyperelliptic function field of genus g . Furthermore, set $U' = \text{Spec}(k[x, y]/(y^2 + h(x)y - f(x)))$, $V' = \text{Spec}(k[x', y']/(y'^2 + h_\infty(x')y' - f_\infty(x')))$ where $h_\infty(x') = h(1/x')x'^{g+1}$ and $f_\infty(x') = f(1/x')x'^{2g+2}$. Let $X = U' \cup V'$ glued together by $D(x) \cong D(x')$ with relations $x = 1/x'$ and $y = x^{g+1}y'$. Then X is a smooth model for the curve corresponding to $K_{f,h}$. For the discriminant, one has

$$\text{disc}(K_{f,h}/k(x)) = \begin{cases} \infty + (f) & \text{if } \text{char}(k) \neq 2, \deg(f) = 2g + 1 \\ (f) & \text{if } \text{char}(k) \neq 2, \deg(f) = 2g + 2 \\ (2g + 2)\infty + 2(h) & \text{if } \text{char}(k) = 2. \end{cases}$$

Conversely, any hyperelliptic function field K of genus g is isomorphic to $K_{f,h}$ for certain (f, h) as above.

3. Shape parameter

In this section, let G be a finite abelian group which we denote multiplicatively. Let $\mathbb{C}[G]$ be the group ring of G over \mathbb{C} . For $\chi \in G^\vee = \text{Hom}(G, \mathbb{C}^*)$ and $f = \sum_{g \in G} c_g g \in \mathbb{C}[G]$ where $c_g \in \mathbb{C}$ we set

$$f_\chi = \sum_{g \in G} c_g \chi(g^{-1}).$$

Proposition 3.1. *Let $f \in \mathbb{C}[G]$. Then one has*

$$f = \frac{1}{\#G} \sum_{g \in G} \sum_{\chi \in G^\vee} f_\chi \chi(g) g.$$

Proof. This is a well-known fact and can be seen as a Fourier transform (see [3, Chapter 6, Proposition 2.2]). \square

For a subset $S \subseteq G$ we set $\mathbb{C}[S] = \{\sum_{s \in S} c_s s : c_s \in \mathbb{C}\} \subseteq \mathbb{C}[G]$, which is a \mathbb{C} -vector space. Let χ_0 be the identity element of G^\vee . We define the shape parameter of S , which we denote by $\text{sh}_G(S)$, as follows:

$$\text{sh}_G(S) = \frac{\#S}{\#G} \cdot \inf_{f \in \mathbb{C}[S]: f_{\chi_0} \neq 0} \frac{\sum_{\chi \in G^\vee} |f_\chi|}{|f_{\chi_0}|}.$$

The following proposition gives some basic properties.

Proposition 3.2. *Let $S \subseteq G$ be non-empty. Then the following hold:*

- i. *For $\alpha \in \text{Aut}(G)$ and $b \in G$ we have $\text{sh}(b \cdot \alpha(S)) = \text{sh}(S)$.*
- ii. *We have $1 \leq \text{sh}(S) \leq \#S$. Furthermore we have $\text{sh}(S) = 1$ if and only if S is a coset of a subgroup of G . We have $\text{sh}(S) = \#S$ if and only if $\#S = 1$.*
- iii. *For $S \subseteq S'$ we have $\text{sh}(S') \leq \frac{\#S'}{\#S} \text{sh}(S)$.*

Let G' be a finite abelian group and let $S' \subseteq G'$ be non-empty. Then the following hold:

- iv. *Let $i : G \rightarrow G'$ be an injective group morphism. Then one has $\text{sh}_G(S) = \text{sh}_{G'}(i(S))$.*
- v. *Let $\pi : G \rightarrow G'$ be a surjective morphism of groups. Then the equality $\text{sh}_G(\pi^{-1}(S')) = \text{sh}_{G'}(S')$ holds.*
- vi. *We have $\text{sh}_{G \times G'}(S \times S') \leq \text{sh}_G(S) \times \text{sh}_{G'}(S')$.*

Proof. See [3, Chapter 6, Proposition 3.3] for the details. We will give a proof of the first two parts of ii.

Let $f = \sum_{s \in S} f_s s \in \mathbb{C}[S]$. We need to show that $|f_{\chi_0}| \leq \frac{\#S}{\#G} \sum_{\chi} |f_\chi|$. One obtains, using Proposition 3.1,

$$|f_{\chi_0}| = \left| \sum_{s \in S} f_s \right| = \left| \frac{1}{\#G} \sum_{s \in S} \sum_{\chi} f_\chi \chi(s) \right| \leq \frac{\#S}{\#G} \sum_{\chi} |f_\chi|$$

Assume that we have an equality. By the translation property, i, we may assume that $1 \in S$ and we may assume that $f_{\chi_0} = 1$. We have $f_{\chi_0} \chi_0(1) = 1$. Hence we see that we have an equality if and only if $f_\chi \chi(s) \in \mathbb{R}_{\geq 0}$ for all $\chi \in G^\vee$ and $s \in S$. Using the case $s = 0$, one obtains $f_\chi \geq 0$. We see that if $f_\chi > 0$, then for any $s \in S$ we have $\chi(s) = 1$. We obtain that for any $t, t' \in \langle S \rangle$ and χ with $f_\chi > 0$ we see that $\chi(t) = 1$ and hence $f_t = \frac{1}{\#G} \sum_{\chi} f_\chi \chi(t) = \frac{1}{\#G} \sum_{\chi} f_\chi \chi(t') = f_{t'}$. Hence our function is constant on $\langle S \rangle$, and as it is nonzero ($f_{\chi_0} = 1$), it is non-negative. As f has support on S , we see that $S = \langle S \rangle$. Actually, we obtain $f = \frac{1}{\#S} \cdot 1_S$. For the converse, assume that $S = \langle S \rangle$. With the help of the function $f = \frac{1}{\#S} \cdot 1_S$ it is easy to see that $\text{sh}(S) = 1$. \square

If $S \subseteq G$ is non-empty, we set

$$SS^{-1} = \{st^{-1} : s, t \in S\}.$$

Lemma 3.3. *We have*

$$\text{sh}(SS^{-1}) \leq \frac{\#(SS^{-1})}{\#S}.$$

Proof. The function $(\sum_{s \in S} s) \cdot (\sum_{s \in S} s^{-1})$ with support in SS^{-1} gives the upper bound. See [3, Chapter 6, Lemma 3.4] for the details. \square

Definition 3.4. An *interval* of \mathbb{Z} is a non-empty set $S \subseteq \mathbb{Z}$ such that there are $n, m \in \mathbb{R}$ with $S = [n, m] \cap \mathbb{Z}$.

Let $G = \mathbb{Z}/n\mathbb{Z}$. A *standard interval* of G is defined to be the image of an interval of \mathbb{Z} under the natural map $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$.

Let G be a finite abelian group. A subset $S \subseteq G$ is called a *full interval* if there exist $n \in \mathbb{Z}_{\geq 1}$, a surjective morphism $\pi : G \rightarrow \mathbb{Z}/n\mathbb{Z}$ and a standard interval T of $\mathbb{Z}/n\mathbb{Z}$ such that $\pi^{-1}(T) = S$. A full interval of a subgroup of G is called an *interval* of G .

Lemma 3.5. *For an interval $S \subseteq G$ we have $\text{sh}(S) \leq 2$.*

Proof. Using Proposition 3.2iv and v, we reduce to the case where $G = \mathbb{Z}/n\mathbb{Z}$, for which we use additive notation, and where S is a standard interval. First of all assume that the size of S is odd, then we may assume (after shifting) $S = \{-\bar{m}, -\bar{m} + 1, \dots, 0, \dots, \bar{m} - 1, \bar{m}\}$ for some $m \in \mathbb{Z}_{\geq 0}$ with $m \leq \frac{n-1}{2}$. Let $T = \{0, 1, \dots, \bar{m}\}$. Then $T - T = S$ and hence we find by Lemma 3.3

$$\text{sh}(S) \leq \frac{\#S}{\#T} = \frac{2m+1}{m+1} < 2.$$

The proof when $\#S$ is even is similar. \square

Lemma 3.6. *Let V be a vector space over \mathbb{F}_p of dimension n . Let $c \in \{1, \dots, p-1\}$ and $0 \leq i < n$ or $(c, i) = (1, n)$. Then there is an interval S in V with $\#S = cp^i$.*

Proof. If $(c, i) = (1, n)$, the statement is obviously true. Assume $i < n$ and let W be a subspace of dimension $i+1$ of V and consider a nonzero map $f \in W^\vee = \text{Hom}(W, \mathbb{F}_p)$. Pick an interval S_0 of \mathbb{F}_p of length c and set $S = f^{-1}(S_0)$. \square

4. Applications of the shape parameter to hyperelliptic curves

4.1. Main statements

This subsection contains the statements of the main results. Proofs following in later subsections.

Convention 1. In this article we assume that a hyperelliptic curve C of genus g is given by an equation $y^2 + h(x)y = f(x)$, where (f, h) are as in Subsection 2.3. Furthermore, we assume that ∞ is ramified in the extension $k(C)/k(x)$. This is equivalent to:

- $\text{char}(k) \neq 2$: $\deg(f) = 2g + 1$;
- $\text{char}(k) = 2$: $1 \leq \deg(h) \leq g$.

We let ∞' be the point above ∞ of $k(C)$.

Let k be a finite field of cardinality $q = p^m$ with p prime. Let C be a hyperelliptic curve over k given by a pair (f, h) following our conventions above. Then we have an injective map

$$\begin{aligned}\varphi_C : C(k) &\rightarrow \text{Pic}_k^0(C) \\ P &\mapsto [P] - [\infty'].\end{aligned}$$

Let $S \subseteq k$ be a subset. Set $C_S = \{P \in C(k) : x(P) \in S\}$. We will give conditions on $\#S$ and $\text{sh}_k(S)$ such that $\text{Pic}_k^0(C) = \langle \varphi_C(C_S) \rangle$.

Let $P \neq \infty$ be a prime of $k(x)$, the function field of the projective line over k , corresponding to the polynomial $\sum_{i=0}^n a_i x^i$ with $a_n = 1$. We put $T(P) = -a_{n-1} \in k$.

Theorem 4.1. *Let C over k be a hyperelliptic curve of genus g given according to our assumptions as above such that $\#C(k) > (2g - 2)\sqrt{q}$. Put $s = 2$ if $p = 2$ and $s = 3$ if $p \neq 2$. Let $S \subseteq k^+$ be such that*

$$q^{3/2} \cdot 2(2g - 2 + s) \cdot \text{sh}_k(S) < (\#C(k) + (2g - 2 + 2s)\sqrt{q}) \cdot \#S.$$

Then the following hold:

- Assume that $p \neq 2$ or $p = 2$ and $\deg(h) < g$. Then we have $\langle \varphi_C(C_S) \rangle = \text{Pic}_k^0(C)$.
- Assume that $p = 2$ and $\deg(h) = g$. Define the following:

$$\begin{aligned}d_i &= f_{2g+i} + \sqrt{f_{2g+2}h_{g-1+i}} \in k \text{ (for } i \in \{0, 1\}); \\ \epsilon_C &= (-1)^{\text{tr}_{k/\mathbb{F}_2}(d_0/h_g^2)} \in \mathbb{C}; \\ \lambda_2 &\in \text{Hom}(k^+, \mathbb{C}^*), \quad c \mapsto (-1)^{\text{tr}_{k/\mathbb{F}_2}(cd_1/h_g^2)}; \\ H_C &= \{x \in k : \lambda_2(x) = -\epsilon_C\} \subseteq k.\end{aligned}$$

Then we have:

- $\langle \varphi_C(C_S) \rangle \in \{\text{Pic}_k^0(C), \ker(\psi_C)\}$;
- if $S \cap H_C = \emptyset$, then $\langle \varphi_C(C_S) \rangle = \ker(\psi_C)$;

(c) if $S \cap H_C \neq \emptyset$, then $\langle \varphi_C(C_S) \rangle = \text{Pic}_k^0(C)$ if

$$q^{3/2}(2g - 2 + s) \text{sh}_k(S \cap H_C) < (\#C(k) + (2g - 2 + 2s)\sqrt{q}) \cdot \#(S \cap H_C).$$

Remark 4.2. Similar results can be obtained for $S \subseteq k^*$ when one takes the shape with respect to k^* .

Remark 4.3. Theorem 4.1 depends on C because of the dependence on $\#C(k)$. By Hasse–Weil we have $\#C(k) \geq q + 1 - 2g\sqrt{q}$ and hence, by using this bound, we can obtain a statement which does not depend on C . See for example Theorem 1.2.

From the above theorem we deduce one of the theorems of the introduction.

Proof of Theorem 1.2. This follows from Theorem 4.1, Hasse–Weil and bounds on the shape (Proposition 3.2ii and Lemma 3.5). \square

The exceptions in Theorem 4.1 come from the following morphism.

Proposition 4.4. Assume that $p = 2$ and that $\deg(h) = g$. Let d_i be as in Theorem 4.1. Then we have a surjective morphism of groups

$$\psi_C : \text{Pic}_k^0(C) \rightarrow \mathbb{F}_2$$

defined as follows: Let $P \neq \infty'$ be a prime of $k(C)$. Then we have:

$$\psi_C([P] - \deg_k(P)[\infty']) = \text{tr}_{k/\mathbb{F}_2} \left(\frac{f(P/P|_K) T(P|_K) d_1 + \deg_k(P) d_0}{h_g^2} \right) \in \mathbb{F}_2.$$

Remark 4.5. Assume that E is an elliptic curve over a finite field k of characteristic 2 given by $y^2 + a_1xy + a_3 = x^3 + a_2x^2 + a_4x + a_6$ with $a_1 \neq 0$. Then the map

$$\begin{aligned} \psi_E : E(k) &\rightarrow \mathbb{F}_2 \\ P &\mapsto \text{tr}_{k/\mathbb{F}_2}((x(P) + a_2)/a_1^2) \\ \infty &\mapsto 0 \end{aligned}$$

is a surjective morphism of groups with kernel $2E(k)$ (Proposition 4.4). Hence if we take $S = \{s \in k : \text{tr}_{k/\mathbb{F}_2}((s + a_2)/a_1^2) = 0\}$, a coset of a subgroup k of cardinality $q/2$, then $\langle P \in E(k) : x(P) \in S \rangle = 2E(k)$.

This morphism also plays an important role in showing that the so-called first fall degree of a certain Weil descent system, which is used for solving the elliptic curve discrete logarithm, is surprisingly small [4].

4.2. Realizing Galois groups

The goal in this subsection is to realize $k^+ \times \text{Pic}_k^0(C)$ as the Galois group of an extension M of $k(C)$ such that for $P \in \mathcal{P}_{k(C)/k}$ of degree 1 we have $(P, M/k(C)) = (x(P), [P] - [\infty'])$.

Let us realize k^+ first. Set $K = k(x)$.

Proposition 4.6. *Let $K_+ = K[Y]/(Y^q - Y - x)$ and let $y = \bar{Y} \in K_+$. Then K_+/K is a Galois extension of fields for which the following hold:*

- i. *the map $\varphi : k \rightarrow \text{Gal}(K_+/K)$, $c \mapsto (y \mapsto y + c)$ is an isomorphism of groups;*
- ii. *the extension is totally ramified at ∞ , and is unramified at all the other primes;*
- iii. *the extension is geometric;*
- iv. *for $P \in \mathcal{P}_{K/k} \setminus \{\infty\}$ we have $(P, K_+/K) = \varphi(\text{T}(P)) \in \text{Gal}(K_+/K)$;*
- v. *$\mathfrak{f}(K_+/K) = 2\infty$, $\text{disc}(K_+/K) = 2(q-1)\infty$; the conductor of any nontrivial subextension of K_+/K is 2∞ ;*
- vi. *$g(K_+) = 0$.*

Proof. This is a calculation which involves Riemann–Hurwitz (see [10]) and the Führerdiskriminantenproduktformel (see [7]). Full details can be found in [3, Chapter 7, Proposition 2.3]. \square

Proposition 4.7. *Let K_+ be as in the previous proposition (Proposition 4.6). For $c \in k^*$ put $z_c = (cy) + (cy)^p + (cy)^{p^2} + \dots + (cy)^{p^{m-1}}$. For $\bar{c} \in k^*/\mathbb{F}_p^*$ set $K_{\bar{c}} = K(z_c)$. Let $\tau_c : k \rightarrow \mathbb{F}_p^*$ be defined by $a \mapsto \text{tr}_{k/\mathbb{F}_p}(ca)$. Then the following hold:*

- i. *z_c is a zero of the irreducible polynomial $f_c = X^p - X - cx \in k(x)[X]$;*
- ii. *$K_{\bar{c}}/K$ is Galois, the map $\varphi_c : \mathbb{F}_p \rightarrow \text{Gal}(K_{\bar{c}}/K)$, $a \mapsto (z_c \mapsto z_c + a)$ is an isomorphism and the following diagram is commutative:*

$$\begin{array}{ccc} \text{Gal}(K_+/K) & \xrightarrow{\sim} & k \\ \downarrow & & \downarrow \tau_c \\ \text{Gal}(K_{\bar{c}}/K) & \xrightarrow{\sim} & \mathbb{F}_p; \end{array}$$

- iii. *for $P \in \mathcal{P}_{K/k} \setminus \{\infty\}$ we have*

$$(P, K_{\bar{c}}/K) = \varphi_c(\text{tr}_{k/\mathbb{F}_p}(c \text{T}(P))) \in \text{Gal}(K_{\bar{c}}/K);$$

- iv. *the map*

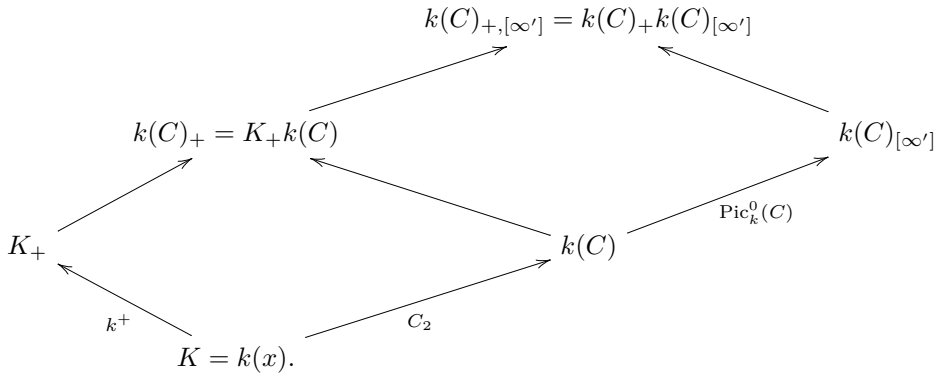
$$\begin{aligned} k^*/\mathbb{F}_p^* &\rightarrow \{L : K \subseteq L \subseteq K_+, [L : K] = p\} \\ \bar{c} &\mapsto K_{\bar{c}} \end{aligned}$$

is a bijection.

Proof. For $a \in k$ we have $\varphi(a)(z_c) = z_c + \text{tr}_{k/\mathbb{F}_p}(ca)$. As $\text{tr}_{k/\mathbb{F}_p}$ is surjective, it follows that $[K_{\bar{c}} : F] = p$. As z_c is a zero of $f_c = X^p - X - cx$, it follows that f_c the minimal polynomial of z_c over K . Hence statement i follows. Statement ii now follows easily. Statement iii follows directly from the definition of Frobenius elements, ii and Proposition 4.6 iv.

We will now prove statement iv. Both sets have the same size, hence it is enough to show that the map is injective. Using ii it is equivalent to show that for $c, c' \in k^*$ we have $\ker(\tau_c) = \ker(\tau_{c'})$ iff $c/c' \in \mathbb{F}_p^*$. But this follows easily from the fact that the trace form is non-degenerate. \square

We are now ready to realize certain Galois groups. Proposition 2.1 gives us an extension $k(C)_{[\infty']}/k(C)$ which is unramified with Galois group $\text{Pic}_k^0(C)$ and the Frobenius at a rational point P is $[P] - [\infty']$. Proposition 4.6 gives us an extension K_+/K with group k^+ . Consider the following diagram of function fields:



We will first study $\text{Gal}(k(C)_+/K)$.

First of all, the extension $k(C)/K$ is Galois with group C_2 and totally ramified at ∞ and at some more points. The extension K_+/K is geometric and Galois with group k^+ and totally ramified at ∞ . Consider the extension $k(C)_+/K$. As K_+ and $k(C)$ are linearly disjoint by genus considerations (Riemann–Hurwitz), $k(C)_+/K$ is Galois with group $k^+ \times C_2$. Also $k(C)_+/k(C)$ is Galois with group k^+ . We claim that $k(C)_+/K$ is geometric. If $\text{char}(k) \neq 2$, then as $(\#k, 2) = 1$, the extension $k(C)_+/K$ is totally ramified at ∞ . Assume that $\text{char}(k) = 2$ and that $\deg(h) < g$. The conductor at ∞ of $k(C)/K$ is $2(g + 1 - \deg(h))\infty$, which is more than the conductor of K_+/K at ∞ , which is 2∞ . Hence $k(C)_+/K$ is totally ramified at ∞ and $k(C)_+/k(C)$ is totally ramified at ∞' . Assume that $\text{char}(k) = 2$ and that $\deg(h) = g$. In this case, take a prime of K , not ∞ , dividing h . Then $k(C)/K$ is ramified at this prime, but K_+/K is not. Hence $k(C)_+/K_+$ is ramified at a prime above such a prime, and it cannot be a constant field extension. We conclude that $k(C)_+/K$ is always geometric.

The only possible ramification in $k(C)_+/k(C)$ is at ∞' . We have already shown that it is totally ramified at ∞' if $\text{char}(k) \neq 2$ or $\text{char}(k) = 2$ and $\deg(h) = g$. One knows that the maximal abelian extension of K_∞ , the completion of K at ∞ , which is totally

ramified of conductor 2 has degree g . Hence if $\text{char}(k) = 2$ and $\deg(h) = g$, we see that $k(C)_+/K$ cannot be totally ramified at ∞ . Hence in this case $k(C)_+/k(C)$ cannot be totally ramified. There is a unique field L with $k(C) \subseteq L \subseteq k(C)_+$ with $[L : k(C)] = 2$ which is unramified at ∞' , and hence unramified.

Lemma 4.8. *Let k be a finite field of characteristic p and let $a \in k$. Then $f_a = x^p - x - a \in k[x]$ is irreducible if and only if $\text{tr}_{k/\mathbb{F}_p}(a) \neq 0$.*

Proof. We leave the proof as an exercise for the reader (see [3, Theorem 7, Lemma 5.1]). \square

The following lemma explicitly describes L .

Lemma 4.9. *Assume that $p = 2$ and that $\deg(h) = g$. For $i = 0, 1$ put $d_i = f_{2g+i} + \sqrt{f_{2g+2}h_{g-1+i}}$. Then the unique unramified subextension L of $k(C)_+/k(C)$ comes from the subextension of K_+/K given by $z^2 - z - cx$ with $c = \frac{d_1}{h_g^2}$. This extension $L/k(C)$ is totally split at ∞' if and only if $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 0$.*

Proof. Let v be the normalized valuation at ∞' of $k(C)$. Then $v(x) = -2$ as $k(C)/K$ is ramified. We have $\deg(f) \in \{2g+1, 2g+2\}$. Put $y' = y + \sqrt{f_{2g+2}x^{g+1}} \in k(C)$. Then we have $y'^2 + hy' = f_{\text{new}}$ where $f_{\text{new}} = f + f_{2g+2}x^{2g+2} + \sqrt{f_{2g+2}h}x^{g+1}$. Note that $f_{\text{new}, 2g+1} = d_1$ is nonzero, as its square is nonzero by our assumptions on (f, h) . Hence f_{new} is of degree $2g+1$. From the equation which y' satisfies, one easily obtains $v(y') = -(2g+1)$.

Let z be an element of K_+ satisfying $z^2 - z - d_1x/h_g^2 = 0$ (Proposition 4.6 for the existence). Notice that $y'' = y'/(h_gx^g)$ satisfies

$$\begin{aligned} y''^2 + y'' &= \frac{f_{\text{new}}(x) + (h(x) - h_gx^g)y'}{h_g^2x^{2g}} \\ &= \frac{d_1x}{h_g^2} + \frac{d_0}{h_g^2} + \frac{(f_{\text{new}} - d_1x^{2g+1} - d_0x^{2g}) + (h(x) - h_gx^g)y'}{h_g^2x^{2g}}. \end{aligned}$$

Hence we have

$$(y' + z)^2 + (y' + z) = \frac{d_0}{h_g^2} + \frac{(f_{\text{new}}(x) - f_{2g+1}x^{2g+1} - f_{2g}x^{2g}) + (h(x) - h_gx^g)y'}{h_g^2x^{2g}}.$$

The valuation of the right hand side at infinity is non-negative and the part in the fraction has a positive valuation. The theorem of Kummer ([10, Chapter 2, Theorem 3.7]) gives the following. It shows that the extension $L/k(C)$ is unramified at infinity, and that the extension splits completely at infinity if and only if the polynomial $x^2 + x + \frac{d_0}{h_g^2}$ is not irreducible in $k[x]$. This happens if and only if $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 0$ by Lemma 4.8. \square

The following lemma gives us the conductor of subextensions of $k(C)_+/k(C)$.

Lemma 4.10. *Let L' be a subextension of degree p of $k(C)_+/k(C)$ which is totally ramified at ∞' . Then one has*

$$f(L'/k(C)) = \begin{cases} 2\infty' & p = 2 \\ 3\infty' & p \neq 2. \end{cases}$$

Proof. This follows from [Lemma 2.4](#) and [Proposition 4.6](#). \square

The next step is to study the extension $k(C)_{+,[\infty']}/k(C)$. If $p \neq 2$ or $p = 2$ and $\deg(h) < g$, then we have seen above that $k(C)_+/k(C)$ is totally ramified at ∞' . As $k(C)_{[\infty']}/k(C)$ is unramified, it shows that $k(C)_+$ and $k(C)_{[\infty']}$ are disjoint over $k(C)$. In this case we have $\text{Gal}(k(C)_{+,[\infty']}/k(C)) = k^+ \times \text{Pic}_k^0(C)$.

Assume that $p = 2$ and that $\deg(h) = g$. We want to understand the Galois extension $k(C)_{+,[\infty']}/k(C)$. Using [Lemma 4.9](#) and [Proposition 2.1](#), we see that two things can happen: If $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 0$, then one obtains $L \subseteq k(C)_{[\infty']}$ (there is a unique maximal extension where ∞' splits). This means that there is a surjective homomorphism $\text{Pic}_k^0(C) \rightarrow \text{Gal}(L/k(C))$. One has $\text{Gal}(k(C)_{+,[\infty']}/k(C)) = k^+ \times_{\text{Gal}(L/k(C))} \text{Pic}_k^0(C)$. If $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 1$, then $k(C)_+$ and $k(C)_{[\infty']}$ are disjoint, and $\text{Gal}(k(C)_{+,[\infty']}/k(C)) = k^+ \times \text{Pic}_k^0(C)$. Unfortunately, the extension is not geometric. There is a degree 2 extension of k inside $k(C)_{+,[\infty']}$ ([Proposition 2.1](#)). Also in this case one can produce a surjective homomorphism $\text{Pic}_k^0(C) \rightarrow \text{Gal}(L/k(C))$.

Proof of Proposition 4.4. Assume first $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 0$. Then $L \subseteq k(C)_{[\infty']}$ and this gives a surjective map ψ_C on the Galois groups. To see what it does, we look at the Frobenius elements. Let P be a prime of degree n in $k(C)$. One has $(P, k(C)_{[\infty']}/k(C)) = [P] - n[\infty'] \in \text{Pic}_k^0(C)$ ([Proposition 2.1](#)). This Frobenius maps to $(P, L/k(C)) = \text{tr}_{k/\mathbb{F}_2}(\frac{f(P/P|_K)T(P|_K)d_1}{h_g^2}) = \text{tr}_{k/\mathbb{F}_2}(\frac{f(P/P|_K)T(P|_K)d_1 + \deg_k(P)d_0}{h_g^2})$ ([Proposition 4.7](#) and [Lemma 4.9](#)).

Assume $\text{tr}_{k/\mathbb{F}_2}(\frac{d_0}{h_g^2}) = 1$. Let L' be the third degree 2 extension in the V_4 extension Lk' over $k(C)$ where k' is the unique degree 2 extension of k . Then we have a natural map $\text{Pic}_k^0(C) \rightarrow \text{Gal}(L'/k(C)) = \mathbb{F}_2$ ([Proposition 2.1](#)). Let P be a prime of $k(C)$ of degree n . Note that there is a unique maximal extension in $Lk'/k(C)$ where P is totally split. Assume that n is even. Then P splits in $L'/k(C)$ iff it splits in $L/k(C)$. If n is odd, then P splits in $L'/k(C)$ iff it does not split in $L/k(C)$. This gives the required map. \square

4.3. Character sum estimates

Put

$$C(k)^* = C(k) \setminus \{\infty'\} = \text{unr}^1(k(C)_{+,[\infty']}/k(C)).$$

Let $\lambda \in k^\vee$ and $\chi \in \text{Pic}_k^0(C)^\vee$. Since we have a natural map $\text{Gal}(k(C)_{+,[\infty']}/k(C)) \rightarrow k^+ \times \text{Pic}_k^0(C)$, we can view (λ, χ) as a character of $\text{Gal}(k(C)_{+,[\infty']}/k(C))$ by taking the product. We put

$$c_{(\lambda, \chi)} = \sum_{P \in C(k)^*} (\lambda, \chi)(P, k(C)_{+,[\infty']}/k(C)) = \sum_{P \in C(k)^*} \lambda(x(P))\chi(\varphi_C(P))$$

(we avoid the only ramification at ∞'). Our goal is to estimate these $c_{(\lambda, \chi)}$. Put $s = 2$ if $p = 2$ and $s = 3$ if $p \neq 2$.

4.3.1. Case 1

Assume that $p \neq 2$ or $p = 2$ and $\deg(h) < g$.

Lemma 4.11. *The following hold for $\lambda \in k^\vee$ and $\chi \in \text{Pic}_k^0(C)^\vee$.*

- i. if $\lambda \neq \chi_0$, then $|c_{(\lambda, \chi)}| \leq (2g - 2 + s)\sqrt{q}$;
- ii. $c_{\chi_0, \chi_0} = \#C(k) - 1$;
- iii. if $\chi \neq \chi_0$, then $|c_{(\chi_0, \chi)} + 1| \leq (2g - 2)\sqrt{q}$.

Proof. i: The degree of the conductor of the corresponding extension is s (see [Lemma 4.10](#) and [Lemma 2.5](#)). Hence the result follows from [Theorem 2.3](#).

ii: Obvious.

iii: The degree of the conductor of the corresponding extension is 0 ([Lemma 4.10](#), [Lemma 2.5](#)). Hence the result follows from [Theorem 2.3](#). \square

4.3.2. Case 2

Assume that $p = 2$ and $\deg(h) = g$. Let λ_2 be the special character of k^+ corresponding to the unramified subextension of $L/k(C)$ of degree 2. More explicitly, we define $\lambda_2 \in k^\vee$, $c \mapsto (-1)^{\text{tr}_{k/\mathbb{F}_2}(cd_1/h_g^2)} \in \mathbb{C}^*$ ([Lemma 4.9](#) and [Proposition 4.7](#)). Put $\epsilon_C = (-1)^{\text{tr}_{k/\mathbb{F}_2}(d_0/h_g^2)}$ (it is -1 if there is a constant field extension). Let $\chi_2 = (-1)^{\psi_C} \in \text{Pic}_k^0(C)^\vee$.

Lemma 4.12. *The following hold for $\lambda \in k^\vee$ and $\chi \in \text{Pic}_k^0(C)^\vee$.*

- i. $c_{(\lambda, \chi) \cdot (\lambda_2, \chi_2)} = \epsilon_C c_{(\lambda, \chi)}$;
- ii. if $\lambda \neq \chi_0, \lambda_2$, then $|c_{(\lambda, \chi)}| \leq (2g - 2 + s)\sqrt{q}$;
- iii. $c_{(\chi_0, \chi_0)} = \#C(k) - 1$;
- iv. $c_{(\lambda_2, \chi_2)} = \epsilon_C (\#C(k) - 1)$;
- v. if $\chi \neq \chi_0$, then $|c_{(\chi_0, \chi)} + 1| \leq (2g - 2)\sqrt{q}$;
- vi. if $\chi \neq \chi_2$, then $|c_{(\lambda_2, \chi)} + \epsilon_C| \leq (2g - 2)\sqrt{q}$.

Proof. i: Let $P \in C(k)^*$. We have $\lambda_2(x(P))\chi_2(\varphi_C(P)) = \epsilon_C$ by construction. Indeed, if $\epsilon_C = 1$, $\lambda_2(x(P))$ and $\chi_2(\varphi_C(P))$ are equal. If $\epsilon_C = -1$, then a rational point splits in

one extension iff it does not split in the other one, and hence they differ by a sign. The result follows.

ii: The degree of the conductor of the corresponding extension is s (Lemma 4.10, Lemma 2.5). Hence the result follows from Theorem 2.3.

iii: Obvious.

iv: Follows from ii and i.

v: The degree of the conductor of the corresponding extension is 0 (Lemma 4.10, Lemma 2.5). Hence the result follows from Theorem 2.3.

vi: Follows from v and i. \square

4.4. Proof of theorem

We can finally prove Theorem 4.1.

Proof of Theorem 4.1. Suppose $\langle \varphi_C(C_S) \rangle \subsetneq \text{Pic}_k^0(C)$. Then there exists a subgroup $H \subseteq \text{Pic}_k^0(C)$ of prime index l such that $\varphi_C(C_S) \subseteq H$. Let $\chi \in \text{Pic}_k^0(C)^\vee$ be a character with kernel H . Let $f = \sum_{a \in k} f_a a \in \mathbb{C}[S] \subseteq \mathbb{C}[k]$. Then for $a \in k$ we have $f_a = \frac{1}{q} \sum_{\lambda \in k^\vee} f_\lambda \lambda(a)$ (Proposition 3.1).

By construction we have

$$\begin{aligned} 0 &= \sum_{P \in C(k)^*} f(x(P))(\chi - 1)(\varphi_C(P)) = \frac{1}{q} \sum_{P \in C(k)^*} \sum_{\lambda \in k^\vee} f_\lambda \lambda(x(P))(\chi - 1)(P) \\ &= \frac{1}{q} \sum_{\lambda \in k^\vee} f_\lambda (c_{(\lambda, \chi)} - c_{(\lambda, 1)}). \end{aligned}$$

Assume that $\chi \neq \chi_2$ if $p = 2$ and $\deg(h) = g$. Choose f such that $\text{sh}_k(S) = \#S/q \cdot C_k(f)$. Rewrite our equation in the following way:

$$f_1(c_{(1,1)} - c_{(1,\chi)}) = \sum_{\lambda \in k^\vee, \lambda \neq 1} f_\lambda (c_{(\lambda, \chi)} - c_{(\lambda, 1)}).$$

We will now put in the estimates of Lemma 4.12. Notice first

$$\begin{aligned} |c_{(1,1)} - c_{(1,\chi)}| &= |(c_{(1,1)} + 1) - (c_{(1,\chi)} + 1)| = |\#C(k) - (c_{(1,\chi)} + 1)| \\ &\geq \#C(k) - (2g - 2)\sqrt{q} > 0. \end{aligned}$$

Taking absolute values gives

$$|f_1|(\#C(k) + (2g - 2 + 2s)\sqrt{q}) \leq 2(2g - 2 + s)\sqrt{q} \sum_{\lambda \in k^\vee} |f_\lambda|.$$

Pick f such that $C(f) = q/\#S \cdot \text{sh}_k(S)$. Then we obtain

$$\frac{q}{\#S} \cdot \text{sh}_k(S) = C(f) \geq \frac{\#C(k) + (2g - 2 + 2s)\sqrt{q}}{2(2g - 2 + s)\sqrt{q}}$$

and this gives us the required result.

Assume that $p = 2$, $\deg(h) = g$ and that $\chi = \chi_2$. Then one has

$$0 = \sum_{\lambda \in k^\vee} f_\lambda (c_{(\lambda, \chi_2)} - c_{(\lambda, 1)}) = \sum_{\lambda \pmod{\langle \lambda_2 \rangle}} (f_\lambda - \epsilon_C f_{\lambda \lambda_2}) (c_{(\lambda, \chi_2)} - c_{(\lambda, 1)}).$$

Hence we have

$$(f_1 - \epsilon_C f_{\lambda_2}) (c_{(1, 1)} - c_{(1, \chi_2)}) = 1/2 \sum_{\lambda \in k^\vee, \lambda \neq 1, \lambda_2} (f_\lambda - \epsilon_C f_{\lambda \lambda_2}) (c_{(\lambda, \chi_2)} - c_{(\lambda, 1)}).$$

The estimates of [Lemma 4.12](#) give

$$|f_1 - \epsilon_C f_{\lambda_2}|(\#C(k) + (2g - 2 + 2s)\sqrt{q}) \leq (2g - 2 + s)\sqrt{q} \sum_{\lambda \in k^\vee} |f_\lambda - \epsilon_C f_{\lambda \lambda_2}|.$$

Consider the expression $f_\lambda - \epsilon_C f_{\lambda \lambda_2}$. It is not hard to see that the image of the map $\mathbb{C}[S] \rightarrow \mathbb{C}[S]$, $f \mapsto f_\lambda - \epsilon_C f_{\lambda \lambda_2}$ is $\mathbb{C}[H_C \cap S]$ where $H_C = \{x \in k : \lambda_2(x) = -\epsilon_C\}$. If $H_C \cap S = \emptyset$, then we have $C_S \subseteq \ker(\psi_C) \stackrel{2}{\subseteq} \text{Pic}_k^0(C)$ ([Proposition 4.4](#)). We can interpret our equation as a shape of $H_C \cap S$ and by choosing the function which obtains the shape of $H_C \cap S$ we obtain:

$$\frac{\#C(k) + (2g - 2 + 2s)\sqrt{q}}{(2g - 2 + s)\sqrt{q}} \leq \frac{q}{\#(S \cap H_C)} \cdot \text{sh}_k(S \cap H_C). \quad \square$$

5. The algorithm

In this section we will describe how to find generators for $\text{Pic}_k^0(C)$, that is, we give the proof of [Theorem 1.1](#). We make a few assumptions:

- i. We can do operations in k , a finite field of cardinality q , as addition and multiplication in time polynomial in $\log(q)$.
- ii. Our hyperelliptic curve C is given as in Subsection 2.3 and $k(C)/k(x)$ is totally ramified at ∞ .
- iii. Divisors on $\text{Pic}_k^0(C)$ are represented as Galois-invariant divisors of $\text{div}_{\bar{k}}^0(C_{\bar{k}})$, where divisors on $\text{div}_{\bar{k}}^0(C_{\bar{k}})$ are represented in $\mathbb{Z}^{(C(\bar{k}))}$.

Proof of Theorem 1.1. Put $t = (2^4(2g + 1) + 2^2)^2$. Deterministically construct k' , a finite field extension of k of cardinality q^i where $tq > q^i \geq t$. This can be done in time $O(q^{1/2}i^4)$

(see [8]), which is in $O(q^{1/2}g^2)$. Addition and multiplication can then be done in k' in time polynomial in $\log(g)$ and $\log(q)$.

Construct an interval S of k' with the following properties:

- i. $\#S \geq \lceil 4(2g+1)q^{i/2} \rceil = r$;
- ii. $\#S = O(g^2q^{1/2})$;
- iii. if $p = 2$ and $\deg(h) = g$, then $S \subseteq H_C$ (see Theorem 4.1).

This can be done for the following reason. We claim that there are intervals of length between r and $2r$. Indeed, write r in basis $p = \text{char}(k)$, say with main term $a_s p^s$. We claim that there is an interval of cardinality $r' = 2a_s p^s$. Note that $r \leq r' \leq 2r$. We want to apply Lemma 3.6 (for H_C in the special case), and for this it is enough to show that $4r \leq q^i$. Indeed, we have

$$q^i \geq q^{i/2} t^{1/2} = q^{i/2} (2^4(2g+1) + 2^2) \geq 4(4(2g+1)q^{i/2} + 1) \geq 4r.$$

We claim that $\#S = O(g^2q^{1/2})$. Indeed, $gq^{i/2} \leq gt^{1/2}q^{1/2}$, which is of order $O(g^2q^{1/2})$ and the result follows.

We will apply Theorem 4.1 with our interval S . We have $\text{sh}_{k^+}(S) \leq 2$ (Lemma 3.5) and $q^i \geq (4g-2)^2$ and hence $\#S \geq 2\text{sh}_{k^+}(S)(2g-2+s)q^{i/2}$. Theorem 4.1 (see Remark 4.3) gives $\langle \varphi_{C_{k'},S} \rangle = \text{Pic}_{k'}^0(C_{k'})$.

We will construct $C_{k',S}$. For all $x \in S$ we look at the equation $y^2 + h(x)y = f(x)$ and we have to solve this in y (note that we have a smooth model of C).

Assume that $p = 2$. Note that $h \neq 0$. If x is fixed, we need to find y with

$$\left(\frac{y}{h(x)} \right)^2 - \frac{y}{h(x)} = \left(\frac{f(x)}{h(x)} \right)^2.$$

This is an Artin–Schreier equation and solutions can easily be obtained by linear algebra. Each step here can be done in polynomial time in $O(q^i)$, hence polynomial time in $\log(g)$ and $\log(q)$. Hence the total cost of this is $O(g^{2+\epsilon}q^{1/2+\epsilon})$.

Assume that $p \neq 2$. Then for $x \in S$ we need to solve $y^2 = f(x)$. First calculate a quadratic non-residue in time $O(q^{i/4+\delta})$, that is, in time $O(\log(g)^{1/2}q^{1/4+\delta})$ (see [9]). Then we apply Tonelli–Shanks to solve the equation for a fixed x in time polynomial in $\log(q)$ (see [11, Lemma 3.4]). Hence in total the cost of this step is again $O(g^{2+\epsilon}q^{1/2+\epsilon})$.

Hence we have calculated $C_{k',S}$. Let ∞'' be the point at infinity of $C_{k'}$. The image of $C_{k',S}$ under $\varphi_{C_{k'}} : C_{k'}(k') \rightarrow \text{Pic}_k^0(C_{k'})$ generates the group $\text{Pic}_k^0(C_{k'})$. It maps P to $[P] - [\infty'']$. Since the norm map $\text{Norm}_{k'(C)/k(C)} : \text{Pic}_k^0(C_{k'}) \rightarrow \text{Pic}_k^0(C)$ is surjective (Corollary 2.2), a generating set of $\text{Pic}_k^0(C)$ is given by

$$\text{Norm}_{k'(C)/k(C)} (\varphi_{C_{k'}}(C_{k',S})).$$

More explicitly, for $P \in C_{k',S}$ we have

$$\text{Norm}_{k'(C)/k(C)}(\varphi_{C_{k'}}(P)) = -[k' : k][\infty'] + \sum_{g \in \text{Gal}(k'/k)} [g(P)]. \quad \square$$

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