Accepted Manuscript

On $x(ax+1)+y(by+1)+z(cz+1)$ and $x(ax+b)+y(ay+c)+z(az+d)$

Zhi-Wei Sun

PII: S0022-314X(16)30207-4
DOI: http://dx.doi.org/10.1016/j.jnt.2016.07.024
Reference: YJNTH 5545

To appear in: *Journal of Number Theory*

Received date: 28 April 2016
Revised date: 6 July 2016
Accepted date: 6 July 2016

Please cite this article in press as: Z.-W. Sun, On $x(ax+1)+y(by+1)+z(cz+1)$ and $x(ax+b)+y(ay+c)+z(az+d)$, *J. Number Theory* (2017), http://dx.doi.org/10.1016/j.jnt.2016.07.024

Final version of JNT-D-16-00273 for publication in J. Number Theory

ON $x(ax+1)+y(by+1)+z(cz+1)$ AND $x(ax+b)+y(ay+c)+z(az+d)$

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
`http://math.nju.edu.cn/~zwsun`

Abstract. In this paper we first investigate for what positive integers a,b,c every nonnegative integer n can be written as $x(ax+1)+y(by+1)+z(cz+1)$ with x,y,z integers. We show that (a,b,c) can be either of the following seven triples

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4),$$

and conjecture that any triple (a,b,c) among

$$(2,2,6),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10)$$

also has the desired property. For integers $0\leqslant b\leqslant c\leqslant d\leqslant a$ with $a>2$, we prove that any nonnegative integer can be written as $x(ax+b)+y(ay+c)+z(az+d)$ with x,y,z integers, if and only if the quadruple (a,b,c,d) is among

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3).$$

1. Introduction

Let $\mathbb{N}=\{0,1,2,\ldots\}$. Throughout this paper, for $f(x,y,z)\in\mathbb{Z}[x,y,z]$ we set

$$E(f(x,y,z))=\{n\in\mathbb{N}:\ n\neq f(x,y,z)\text{ for any }x,y,z\in\mathbb{Z}\}.$$

If $E(f(x,y,z))=\emptyset$, then we call $f(x,y,z)$ *universal over* $\mathbb{Z}$. The classical Gauss-Legendre theorem (cf. [N96, pp. 3-35]) states that

$$E(x^2+y^2+z^2)=\{4^k(8l+7):\ k,l\in\mathbb{N}\}.$$

Recall that those $T_x=x(x+1)/2$ with $x\in\mathbb{Z}$ are called *triangular numbers*. As $T_{-x-1}=T_x$, $T_{2x}=x(2x+1)$ and $T_{2x-1}=x(2x-1)$, we see that

$$\{T_x:\ x\in\mathbb{Z}\}=\{T_x:\ x\in\mathbb{N}\}=\{x(2x+1):\ x\in\mathbb{Z}\}.\tag{1.1}$$

2010 *Mathematics Subject Classification*. Primary 11E25; Secondary 11D85, 11E20.
Keywords. Representations of integers, universal sums, quadratic polynomials.
Supported by the National Natural Science Foundation (Grant No. 11571162) of China.

By the Gauss-Legendre theorem, any $n \in \mathbb{N}$ can be written as the sum of three triangular numbers (equivalently, $8n+3$ is the sum of three odd squares). In view of (1.1), this says that

$$\{x(2x+1)+y(2y+1)+z(2z+1):\ x,y,z \in \mathbb{Z}\} = \mathbb{N}. \tag{1.2}$$

Motivated by this, we are interested in finding all those $a,b,c \in \mathbb{Z}^+ = \{1,2,3,\ldots\}$ satisfying

$$\{x(ax+1)+y(by+1)+z(cz+1):\ x,y,z \in \mathbb{Z}\} = \mathbb{N}. \tag{1.3}$$

In the following theorem we determine all possible candidates $a,b,c \in \mathbb{Z}^+$ with (1.3) valid.

Theorem 1.1. *Let $a,b,c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$. If $x(ax+1)+y(by+1)+z(cz+1)$ is universal over $\mathbb{Z}$, then (a,b,c) is among the following 17 triples:*

$$\begin{gathered}(1,1,2),\ (1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,5),\\ (2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,2,6),\\ (2,3,3),\ (2,3,4),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).\end{gathered} \tag{1.4}$$

Remark 1.1. As proved by Liouville (cf. [D99, p. 23]),

$$\{2T_x+2T_y+T_z:\ x,y,z \in \mathbb{N}\} = \{2T_x+T_y+T_z:\ x,y,z \in \mathbb{N}\} = \mathbb{N}.$$

By [S15, Theorem 1.14], $T_x+T_y+2p_5(z)$ with $p_5(z)=z(3z-1)/2$ is also universal over $\mathbb{Z}$. These, together with (1.1) and (1.2), indicate that (1.3) holds for $(a,b,c) = (1,1,2),(1,2,2),(2,2,2),(2,2,3)$.

In Section 2 we will prove Theorem 1.1 as well as the following related result.

Theorem 1.2. (1.3) *holds if (a,b,c) is among the following 7 triples:*

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4).$$

In view of Theorems 1.1-1.2 and Remark 1.1, we have reduced the converse of Theorem 1.1 to our following conjecture.

Conjecture 1.1. (1.3) *holds if (a,b,c) is among the following six triples:*

$$(2,2,6),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).$$

Remark 1.2. It is easy to show that (1.3) holds for $(a,b,c)=(2,3,7)$ if and only if for any $n \in \mathbb{N}$ we can write $168n+41$ as $21x^2+14y^2+6z^2$ with $x,y,z \in \mathbb{Z}$.

Inspired by (1.2), we want to know for what $a,b,c,d \in \mathbb{N}$ with $b \leqslant c \leqslant d \leqslant a$ we have

$$\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z \in \mathbb{Z}\} = \mathbb{N}. \tag{1.5}$$

We achieve this in the following theorem which will be proved in Section 3.

Theorem 1.3. *Let $a>2$ be an integer and let $b,c,d\in\mathbb{N}$ with $b\leqslant c\leqslant d\leqslant a$. Then* (1.5) *holds if and only if (a,b,c,d) is among the following five quadruples:*

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3). \tag{1.6}$$

Remark 1.3. For $a\in\{1,2\}$ and $b,c,d\in\mathbb{N}$ with $b\leqslant c\leqslant d\leqslant a$, we can easily show that if (1.5) holds then (a,b,c,d) is among the following five quadruples:

$$(1,0,0,1),\ (1,0,1,1),\ (2,0,0,1),\ (2,0,1,1),\ (2,1,1,1).$$

The converse also holds since

$$x^2+y^2+2T_z,\ x^2+2T_y+2T_z,\ 2x^2+2y^2+T_z,\ 2x^2+T_y+T_z,\ T_x+T_y+T_z$$

are all universal over $\mathbb{Z}$ (cf. [S07]).

We also note some other universal sums. For example, we have

$$\{x^2+y(3y+1)+z(3z+2):x,y,z\in\mathbb{Z}\}=\{x^2+y(4y+1)+z(4z+3):x,y,z\in\mathbb{Z}\}=\mathbb{N}$$

which can be easily proved.

Based on our computation, we formulate the following conjecture for further research.

Conjecture 1.2. (i) *Any positive integer $n\neq 225$ can be written as $p(p-1)/2+q(q-1)/2+r(r-1)/2$ with p prime and $q,r\in\mathbb{Z}^+$.*

(ii) *Each $n\in\mathbb{N}$ can be written as $x^2+y(3y+1)/2+z(2z-1)$ with $x,y,z\in\mathbb{N}$. Also, any $n\in\mathbb{N}$ can be written as $x^2+y(3y+1)/2+z(5z+3)/2$ with $x,y,z\in\mathbb{N}$.*

(iii) *Every $n\in\mathbb{Z}^+$ can be written as $x^3+y^2+T_z$ with $x,y\in\mathbb{N}$ and $z\in\mathbb{Z}^+$. We also have $\{x^2+y(y+1)+z(z^2+1):\ x,y,z\in\mathbb{N}\}=\mathbb{N}$.*

(iv) *Any $n\in\mathbb{N}$ can be written as $x^4+y(3y+1)/2+z(7z+1)/2$ with $x,y,z\in\mathbb{Z}$.*

2. Proofs of Theorems 1.1-1.2

Proof of Theorem 1.1. For $x\in\mathbb{Z}\setminus\{0\}$, clearly $ax^2+x\geqslant|x|(a|x|-1)\geqslant a-1$. As $1=x(ax+1)+y(by+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $a\leqslant 2$.

Case 1. $a=b=1$.

As $1\notin\{x(x+1)+y(y+1):\ x,y\in\mathbb{Z}\}$, we must have $1\in\{z(cz+1):\ z\in\mathbb{Z}\}$ and hence $c=2$. (Note that if $c>2$ then $cz^2+z\geqslant c-1>1$ for all $z\in\mathbb{Z}\setminus\{0\}$.)

Case 2. $a=1<b$.

If $b>2$, then $y(by+1)\geqslant b-1>1$ and $z(cz+1)\geqslant c-1>1$ for all $y,z\in\mathbb{Z}\setminus\{0\}$. As $1=x(x+1)+y(by+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $b=2$. It is easy to see that $4\notin\{x(x+1)+y(2y+1):\ x,y\in\mathbb{Z}\}$. If $c>5$, then $z(cz+1)\geqslant c-1>4$ for all $z\in\mathbb{Z}\setminus\{0\}$. As $4=x(x+1)+y(2y+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $c\in\{2,3,4,5\}$.

Case 3. $a = b = 2$.

In view of (1.1),

$$5 \notin \{T_x + T_y : x, y \in \mathbb{N}\} = \{x(2x+1) + y(2y+1) : x, y \in \mathbb{Z}\}.$$

If $c > 6$, then $z(cz+1) \geqslant c - 1 > 5$ for all $z \in \mathbb{Z} \setminus \{0\}$. As $5 = x(2x+1) + y(2y+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $c \in \{2, 3, 4, 5, 6\}$.

Case 4. $a = 2 < b$.

Clearly, $2 \notin \{x(2x+1) : x \in \mathbb{Z}\}$. If $b > 3$, then $y(by+1) \geqslant b - 1 > 2$ and $z(cz+1) \geqslant c - 1 > 2$ for all $y, z \in \mathbb{Z} \setminus \{0\}$. As $2 = x(2x+1) + y(by+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $b = 3$. Note that $x(2x+1) + y(3y+1) \neq 9$ for all $x, y \in \mathbb{Z}$. If $c > 10$, then $z(cz+1) \geqslant c - 1 > 9$ for all $z \in \mathbb{Z} \setminus \{0\}$. Since $9 = x(2x+1) + y(3y+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $c \leqslant 10$. Note that $48 \neq x(2x+1) + y(3y+1) + z(6z+1)$ for all $x, y, z \in \mathbb{Z}$. So $c \in \{3, 4, 5, 7, 8, 9, 10\}$.

In view of the above, we have completed the proof of Theorem 1.1. $\Box$

Lemma 2.1. *Let u and v be integers with $u^2 + v^2$ a positive multiple of 5. Then $u^2 + v^2 = x^2 + y^2$ for some $x, y \in \mathbb{Z}$ with $5 \nmid xy$.*

Proof. Let a be the 5-adic order of $\gcd(u, v)$, and write $u = 5^a u_0$ and $v = 5^a v_0$ with $u_0, v_0 \in \mathbb{Z}$ not all divisible by 5. Choose $\delta, \varepsilon \in \{\pm 1\}$ such that $u_0' \not\equiv 2v_0' \pmod 5$, where $u_0' = \delta u_0$ and $v_0' = \varepsilon v_0$. Clearly, $5^2(u_0^2 + v_0^2) = u_1^2 + v_1^2$, where $u_1 = 3u_0' + 4v_0'$ and $v_1 = 4u_0' - 3v_0'$. Note that u_1 and v_1 are not all divisible by 5 since $u_1 \not\equiv v_1 \pmod 5$. Continue this process, we finally write $u^2 + v^2 = 5^{2a}(u_0^2 + v_0^2)$ in the form $x^2 + y^2$ with $x, y \in \mathbb{Z}$ not all divisible by 5. As $x^2 + y^2 = u^2 + v^2 \equiv 0 \pmod 5$, we must have $5 \nmid xy$. This concludes the proof. $\Box$

With the help of Lemma 2.1, we are able to deduce the following result.

Lemma 2.2. *For any $n \in \mathbb{N}$ and $r \in \{6, 14\}$, we can write $20n + r$ as $5x^2 + 5y^2 + z^2$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid z$.*

Proof. As $20n + r \equiv r \equiv 2 \pmod 4$, by the Gauss-Legendre theorem we can write $20n + r$ as $(2w)^2 + u^2 + v^2$ with $u, v, w \in \mathbb{Z}$ and $2 \nmid uv$. If $(2w)^2 \equiv -r \pmod 5$, then $u^2 + v^2 \equiv 2r \pmod 5$ and hence $u^2 \equiv v^2 \equiv r \pmod 5$. If $(2w)^2 \equiv r \pmod 5$, then $u^2 + v^2 \equiv 2 \pmod 4$ is a positive multiple of 5 and hence by Lemma 2.1 we can write it as $s^2 + t^2$, where s and t are odd integers with $s^2 \equiv -r \pmod 5$ and $t^2 \equiv r \pmod 5$. If $5 \mid w$, then one of u^2 and v^2 is divisible by 5 and the other is congruent to r modulo 5.

By the above, we can always write $20n + r = x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$, $2 \nmid z$ and $z^2 \equiv r \pmod 5$. Note that $x^2 \equiv -y^2 = (\pm 2y)^2 \pmod 5$. Without loss of generality, we assume that $x \equiv 2y \pmod 5$ and hence $2x \equiv -y \pmod 5$. Set $\bar{x} = (x - 2y)/5$ and $\bar{y} = (2x + y)/5$. Then

$$20n + r = x^2 + y^2 + z^2 = 5\bar{x}^2 + 5\bar{y}^2 + z^2.$$

This concludes the proof. $\square$

Remark 2.1. Let $n\in\mathbb{N}$ and $r\in\{6,14\}$. In contrast with Lemma 2.2, we conjecture that $20n+r$ can be written as $5x^2+5y^2+(2z)^2$ with $x,y,z\in\mathbb{Z}$ unless $r=6$ and $n\in\{0,11\}$, or $r=14$ and $n\in\{1,10\}$.

Lemma 2.3. (i) *For any positive integer $w=x^2+2y^2$ with $x,y\in\mathbb{Z}$, we can write w in the form u^2+2v^2 with $u,v\in\mathbb{Z}$ such that u or v is not divisible by* 3.

(ii) *$w\in\mathbb{N}$ can be written as $3x^2+6y^2$ with $x,y\in\mathbb{Z}$, if and only if $3\mid w$ and $w=u^2+2v^2$ for some $u,v\in\mathbb{Z}$.*

(iii) *Let $n\in\mathbb{N}$ with $6n+1$ not a square. Then, for any $\delta\in\{0,1\}$ we can write $6n+1$ as $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv\delta\pmod 2$.*

Remark 2.2. Part (i) first appeared in the middle of a proof given on page 173 of [JP] (see also [S15, Lemma 2.1] for other similar results). Parts (ii) and (iii) are Lemmas 3.1 and 3.3 of the author [S15].

Proof of Theorem 1.2. Let us fix a nonnegative integer n.

(i) As $24n+11\equiv 3\pmod 8$, by the Gauss-Legendre theorem there are odd integers u,v,w such that $24n+11=u^2+v^2+w^2=w^2+2\bar{u}^2+2\bar{v}^2$, where $\bar{u}=(u+v)/2$ and $\bar{v}=(u-v)/2$. As $2(\bar{u}^2+\bar{v}^2)\equiv 11-w^2\equiv 10\equiv 2\pmod 8$, we have $\bar{u}\not\equiv\bar{v}\pmod 2$. Without loss of generality, we assume that $2\mid\bar{u}$ and $2\nmid\bar{v}$. If $3\nmid\bar{v}$, then $\gcd(6,\bar{v})=1$. When $3\mid\bar{v}$, we have $3\nmid\bar{u}$ (since $w^2\not\equiv 11\pmod 3$), and $w^2+2\bar{v}^2$ is a positive multiple of 3, thus by Lemma 2.3(i) there are $s,t\in\mathbb{Z}$ with $3\nmid st$ such that $s^2+2t^2=w^2+2\bar{v}^2\equiv 3\pmod 8$ and hence $2\nmid st$. Anyway, $24n+11$ can be written as $r^2+2s^2+2t^2$ with $r,s,t\in\mathbb{Z}$ and $\gcd(6,t)=1$. Since $r^2+2s^2\equiv 11-2t^2\equiv 0\pmod 3$, by Lemma 2.3(ii) we may write $r^2+2s^2=3r_0^2+6s_0^2$ with $r_0,s_0\in\mathbb{Z}$. Since $3r_0^2+6s_0^2=r^2+2s^2\equiv 11-2t^2\equiv 9\pmod 8$, we have $r_0^2+2s_0^2\equiv 3\pmod 8$ and hence $2\nmid r_0s_0$. Write $s_0=2x+1$, r_0 or $-r_0$ as $4y+1$, and t or $-t$ as $6z+1$, where $x,y,z\in\mathbb{Z}$. Then

$$24n+11=6(2x+1)^2+3(4y+1)^2+2(6z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(3z+1)$. This proves (1.3) for $(a,b,c)=(1,2,3)$.

(ii) By the Gauss-Legendre theorem, there are $s,t,v\in\mathbb{Z}$ such that $32n+14=(2s+1)^2+(2t+1)^2+(2v)^2$ and hence $16n+7=(s+t+1)^2+(s-t)^2+2v^2$. As one of $s+t+1$ and $s-t$ is even, we have $16n+7=(2u)^2+w^2+2v^2$ for some $u,w\in\mathbb{Z}$. Clearly $2\nmid w$, $2v^2\equiv 7-w^2\equiv 2\pmod 4$, and $4u^2\equiv 7-2v^2-w^2\equiv 4\pmod 8$. So, u,v,w are all odd. Note that $w^2\equiv 7-4u^2-2v^2\equiv 7-4-2=1\pmod{16}$ and hence $w\equiv\pm1\pmod 8$. Now we can write u as $2x+1$, v or $-v$ as $4y+1$, w or $-w$ as $8z+1$, where x,y,z are integers. Thus

$$16n+7=4(2x+1)^2+2(4y+1)^2+(8z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(1,2,4)$.

(iii) By Dickson [D39, pp. 112-113] (or [JKS]),

$$E(10x^2+5y^2+2z^2)=\{8q+3:\ q\in\mathbb{N}\}\cup\bigcup_{k,l\in\mathbb{N}}\{25^k(5l+1),25^k(5l+4)\}.$$

So, there are $u,v,w\in\mathbb{Z}$ such that $40n+17=10u^2+5v^2+2w^2$. Clearly, $2\nmid v$, $2u^2+2w^2\equiv 17-5v^2\equiv 4 \pmod 8$ and hence $2\nmid uw$. Note that $2w^2\equiv 17\equiv 2 \pmod 5$ and hence $w\equiv \pm1 \pmod 5$. Thus, we can write $u=2x+1$, v or $-v$ as $4y+1$, and w or $-w$ as $10z+1$, where x,y,z are integers. Now we have

$$40n+17=10(2x+1)^2+5(4y+1)^2+2(10z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(5z+1)$. This proves (1.3) for $(a,b,c)=(1,2,5)$.

(iv) By the Gauss-Legendre theorem, there are $u,v,w\in\mathbb{Z}$ with $2\nmid w$ such that

$$16n+5=(2u)^2+(2v)^2+w^2=2(u+v)^2+2(u-v)^2+w^2.$$

As $w^2\equiv 1\not\equiv 5 \pmod 8$, both $u+v$ and $u-v$ are odd. Since $w^2\equiv 5-2-2=1 \pmod{16}$, we have $w\equiv\pm1 \pmod 8$. Now we can write $u+v$ or $-u-v$ as $4x+1$, $u-v$ or $v-u$ as $4y+1$, and w or $-w$ as $8z+1$, where $x,y,z\in\mathbb{Z}$. Thus

$$16n+5=2(4x+1)^2+2(4y+1)^2+(8z+1)^2$$

and hence $n=x(2x+1)+y(2y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(2,2,4)$.

(v) By Lemma 2.2, there are $u,v,w\in\mathbb{Z}$ with $2\nmid w$ such that $20n+6=5u^2+5v^2+w^2$. Clearly, $u\not\equiv v \pmod 2$, $w^2\equiv 1 \pmod 5$ and hence $w\equiv\pm1 \pmod 5$. Thus w or $-w$ has the form $10z+1$ with $z\in\mathbb{Z}$. Observe that

$$40n+12=10u^2+10v^2+2w^2=5(u+v)^2+5(u-v)^2+2(10z+1)^2.$$

As $u+v$ and $u-v$ are both odd, we may write $u+v$ or $-u-v$ as $4x+1$, and $u-v$ or $v-u$ as $4y+1$, where x and y are integers. Then

$$40n+12=5(4x+1)^2+5(4y+1)^2+2(10z+1)^2$$

and hence $n=x(2x+1)+y(2y+1)+z(5z+1)$. This proves (1.3) for $(a,b,c)=(2,2,5)$.

(vi) By Dickson [D39, pp. 112-113],

$$E(x^2+y^2+3z^2)=\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

So there are $u,v,w\in\mathbb{Z}$ such that $24n+7=u^2+v^2+3w^2$. As $u^2+v^2\not\equiv 7 \pmod 4$, we have $2\nmid w$ and hence $s=(u+v)/2\in\mathbb{Z}$ and $t=(u-v)/2\in\mathbb{Z}$.

Now $24n+7=2s^2+2t^2+3w^2$. As $2(s^2+t^2)\equiv 7-3w^2\equiv 4 \pmod 8$, we have $s^2+t^2\equiv 2 \pmod 4$ and hence $2\nmid st$. Note that $s^2+t^2\equiv (7-3)/2=2 \pmod 3$ and hence $3\nmid st$. Now we can write w or $-w$ as $4x+1$, s or $-s$ as $6y+1$, t or $-t$ as $6z+1$, where x,y,z are integers. Then

$$24n+7=3(4x+1)^2+2(6y+1)^2+2(6z+1)^2$$

and hence $n=x(2x+1)+y(3y+1)+z(3z+1)$. This proves (1.3) for $(a,b,c)=(2,3,3)$.

(vii) Note that $48n+13\equiv 1 \pmod 6$ but $48n+13\not\equiv 1 \pmod 8$. By Lemma 2.3(iii), there are $u,v,w\in\mathbb{Z}$ such that $48n+13=6u^2+(2v)^2+3w^2$. Clearly, $2\nmid w$ and $3\nmid v$. As $6u^2\equiv 13-3\equiv 6 \pmod 4$, we must have $2\nmid u$. Since $4v^2\equiv 13-6u^2-3w^2\equiv 4 \pmod 8$, we have $2\nmid v$. Observe that

$$3w^2\equiv 13-6u^2-4v^2\equiv 13-6-4=3 \pmod{16}$$

and hence $w\equiv\pm1 \pmod 8$. Now we can write u or $-u$ as $4x+1$, v or $-v$ as $6y+1$, and w or $-w$ as $8z+1$, where $x,y,z\in\mathbb{Z}$. Thus

$$48n+13=6(4x+1)^2+4(6y+1)^2+3(8z+1)^2$$

and hence $n=x(2x+1)+y(3y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(2,3,4)$.

So far we have completed the proof of Theorem 1.2. $\square$

3. Proof of Theorems 1.3

Lemma 3.1. *For any positive integer n, we can write $6n+1$ as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$ and $3\nmid xyz$.*

Remark 3.1. This is [S16, Lemma 4.3(ii)] proved by the author with the help of a result in [CL].

Proof of Theorem 1.3. (i) If $|x|\geqslant 2$, then

$$x(ax+b)\geqslant |x|(a|x|-b)\geqslant 2(2a-b)\geqslant 2a,$$

and similarly $x(ax+c)\geqslant 2a$ and $x(ax+d)\geqslant 2a$. So, if (1.5) holds then we must have

$$\{0,1,\ldots,2a-1\}\subseteq\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\{0,\pm1\}\}$$

and hence

$$2a\leqslant|\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\{0,\pm1\}\}|\leqslant 3^3=27.$$

Note that $a \in \{3, 4, \ldots, 13\}$ and $0 \leqslant b \leqslant c \leqslant d \leqslant a$. Via a computer we find that if (a, b, c, d) is not among the five quadruples in (1.6) then one of $1, 2, \ldots, 17$ cannot be written as $x(ax+b)+y(ay+c)+z(az+d)$ with $x, y, z \in \mathbb{Z}$. For example, $x(4x+2)+y(4y+2)+z(4z+3) \neq 17$ for any $x, y, z \in \mathbb{Z}$. This proves the "only if" part of Theorem 1.3.

(ii) Now we turn to prove the "if" part of Theorem 1.3. Let us fix a nonnegative integer n.

(a) By [S15, Theorem 1.7(iv)], there are $u, v, x \in \mathbb{Z}$ such that $12n + 5 = u^2 + v^2 + 36x^2$. Clearly $u \not\equiv v \pmod 2$ and $3 \nmid uv$. Without loss of generality, we assume that $u \equiv \pm 1 \pmod 6$ and $v \equiv \pm 2 \pmod 6$. We may write u or $-u$ as $6y + 1$, and v or $-v$ as $6z + 2$, where y and z are integers. Thus

$$12n + 5 = 36x^2 + (6y + 1)^2 + (6z + 2)^2$$

and hence $n = 3x^2 + y(3y + 1) + z(3z + 2)$. This proves (1.5) for $(a, b, c, d) = (3, 0, 1, 2)$.

(b) Let $\delta \in \{0, 1\}$. By the Gauss-Legendre theorem, $12n+6+3\delta$ can be written as the sum of three squares. In view of [S16, Lemma 2.2], there are $u, v, w \in \mathbb{Z}$ with $3 \nmid uvw$ such that $12n + 6 + 3\delta = u^2 + v^2 + w^2$. Clearly, u, v, w are neither all odd nor all even. Without loss of generality, we assume that $2 \nmid u$ and $2 \mid w$. Then $v \not\equiv \delta \pmod 2$. Obviously, $u \equiv \pm 1 \pmod 6$, $v \equiv \pm(1 + \delta) \pmod 6$ and $w \equiv \pm 2 \pmod 6$. Thus we may write u or $-u$ as $6x + 1$, v or $-v$ as $6y + 1 + \delta$, and w or $-w$ as $6z + 2$, where $x, y, z \in \mathbb{Z}$. Therefore,

$$12n + 6 + 3\delta = (6x + 1)^2 + (6y + 1 + \delta)^2 + (6z + 2)^2$$

and hence $n = x(3x + 1) + y(3y + 1 + \delta) + z(3z + 2)$. This proves (1.5) for $(a, b, c, d) = (3, 1, 1, 2), (3, 1, 2, 2)$.

(c) By Lemma 3.1, there are $u, v, w \in \mathbb{Z}$ with $3 \nmid uvw$ such that $6n + 7 = u^2 + v^2+2w^2$ and hence $12n+14 = (u+v)^2+(u-v)^2+(2w)^2$. As $(u+v)^2+(u-v)^2 \equiv 2 \pmod 4$, both $u + v$ and $u - v$ are odd. Since $(u + v)^2 + (u - v)^2 \equiv 14 - 1 \equiv 1 \pmod 3$, without loss of generality we may assume that $u + v \equiv \pm 1 \pmod 6$ and $u - v \equiv 3 \pmod 6$. Now we may write $u + v$ or $-u - v$ as $6x + 1$, w or $-w$ as $3y + 1$, and $u - v$ as $6z + 3$, where x, y, z are integers. Then

$$12n + 14 = (6x + 1)^2 + (6y + 2)^2 + (6z + 3)^2$$

and hence $n = x(3x+1)+y(3y+2)+z(3z+3)$. This proves (1.5) for $(a, b, c, d) = (3, 1, 2, 3)$.

(d) As $16n + 14 \equiv 2 \pmod 4$, by the Gauss-Legendre theorem $16n + 14 = u^2 + v^2 + w^2$ for some $u, v, w \in \mathbb{Z}$ with $2 \nmid uv$ and $2 \mid w$. Since $w^2 \equiv 14 - u^2 - v^2 \equiv 12 \equiv 4 \pmod 8$, $w/2$ or $-w/2$ has the form $4y + 1$ with $y \in \mathbb{Z}$. Thus $u^2 + v^2 \equiv 14 - 4(w/2)^2 \equiv 10 \pmod{16}$. Without loss of generality, we assume that $u \equiv \pm 1 \pmod 8$ and $v \equiv \pm 3 \pmod 8$. Write u or $-u$ as $8x+1$, and v or $-v$ as $8z+3$, where $x, z \in \mathbb{Z}$. Then

$$16n + 14 = (8x+1)^2 + (8y+2)^2 + (8z+3)^2$$

and hence $n = x(4x+1) + y(4y+2) + z(4z+3)$. This proves (1.5) for $(a,b,c,d) = (4,1,2,3)$.

In view of the above, we have completed the proof of Theorem 1.3. $\square$

Acknowledgments. The author would like to thank Dr. Hao Pan for his comments on the proof of Lemma 2.2, and the referee for his/her helpful suggestions.

References

[CL] S. Cooper and H. Y. Lam, *On the diophantine equation* $n^2 = x^2 + by^2 + cz^2$, J. Number Theory **133** (2013), 719–737.

[D39] L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939.

[D99] L. E. Dickson, *History of the Theory of Numbers,* Vol. II, AMS Chelsea Publ., 1999.

[JKS] W. C. Jagy, I. Kaplansky and A. Schiemann, *There are 913 regular ternary forms*, Mathematika **44** (1997), 332–341.

[JP] B. W. Jones and G. Pall, *Regular and semi-regular positive ternary quadratic forms*, Acta Math. **70** (1939), 165–191.

[N96] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Grad. Texts in Math., Vol. 164, Springer, New York, 1996.

[S07] Z.-W. Sun, *Mixed sums of squares and triangular numbers*, Acta Arith. **127** (2007), 103–113.

[S15] Z.-W. Sun, *On universal sums of polygonal numbers*, Sci. China Math. **58** (2015), 1367–1396.

[S16] Z.-W. Sun, *A result similar to Lagrange's theorem*, J. Number Theory **162** (2016), 190–211.