ARTICLE IN PRESS YJNTH:5779

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The values of cubic forms at prime arguments

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ARTICLE INFO

Article history:
Received 6 November 2016
Received in revised form 1 March 2017
Accepted 23 May 2017
Available online xxxx
Communicated by the Principal Editors

MSC:
11P32
11P05
11P55

Keywords:
Diophantine inequalities
Primes
Exceptional set
Cubic forms

ABSTRACT

Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Let $\mathcal{V}$ be a well-spaced sequence, $\delta > 0$. If λ_1/λ_2 is irrational and algebraic, then we prove that $E(\mathcal{V}, X, \delta) \ll X^{17/18+2\delta+\varepsilon}$, where $E(\mathcal{V}, X, \delta)$ denotes the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality $|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < v^{-\delta}$ has no solution in primes p_1, p_2, p_3, p_4, p_5. Further, we assume that except for one, all other the ratios λ_k/λ_l $(1 \leq k < l \leq 5)$ are irrational and algebraic, then $17/18$ can be replaced by $11/12$. These improve the earlier results.

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1. Introduction

A formal application of the Hardy–Littlewood method suggests that whenever s and k are natural numbers with $s \geq k + 1$, then all large integers n satisfying appropriate local conditions should be represented as the sum of s kth powers of prime numbers. We write

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http://dx.doi.org/10.1016/j.jnt.2017.05.014
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Please cite this article in press as: W. Ge, F. Zhao, The values of cubic forms at prime arguments, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.05.014

$$\mathcal{N}_5 = \{n \in \mathbb{N} : n \equiv 1(\bmod 2),\ n \not\equiv 0, \pm 2(\bmod 9),\ n \not\equiv 0(\bmod 7)\},$$

and

$$E_5(N) = |\{1 \le n \le N : n \in \mathcal{N}_5 \text{ and } n \notin \mathcal{A}_5\}|,$$

where

$$\mathcal{A}_5 = \{p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 : p_1, p_2, p_3, p_4, p_5 \text{ are prime numbers}\}.$$

Hua [6] established that almost all numbers in $\mathcal{N}_5$ can be represented as sums of five cubes of prime numbers. Precisely, Hua proved that $E_5(N) \ll N \log^{-A} N$ for some positive number A. There have been also a series of recent advances (see [9,8,11,16,17]).

Davenport and Heilbronn first considered the Diophantine inequalities. Given $k \ge 1$ and s nonzero real numbers $\lambda_1, \cdots, \lambda_s$ (not all in rational ratio, not all negative), we write

$$F(\mathbf{p}) = \sum_{j=1}^{s} \lambda_j p_j^k,$$

where $\mathbf{p} = (p_1, \cdots, p_s)$ with each p_j a prime. Various authors have considered the distribution of values of such forms, for example, see [14]. Here we continue in the direction started by Brüdern, Cook and Perelli [1] and followed by Cook and Fox [3], Cook [2], Harman [5] and Cook and Harman [4]. We call a set of positive reals $\mathcal{V}$ a well-spaced set if there is a $c > 0$ such that

$$u, v \in \mathcal{V}, \quad u \ne v \quad \Rightarrow |u - v| > c.$$

We further assume that

$$|\{v \in \mathcal{V} : 0 \le v \le X\}| \gg X^{1-\varepsilon}.$$

In this paper, let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative, let $\mathcal{V}$ be a well-spaced sequence, and let $E(\mathcal{V}, X, \delta)$ denote the number of $v \in \mathcal{V}$ with $v \le X$ such that the inequality

$$|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4, p_5.

In [4], Cook and Harman show that if λ_1/λ_2 is irrational and algebraic, then one has

$$E(\mathcal{V}, X, \delta) \ll X^{1-\frac{2}{3}\rho(3)+2\delta+\varepsilon} \tag{1.1}$$

for any $\varepsilon > 0$, where $\rho(3) = \frac{1}{14}$, since they use bounds for the exponential sums which arise [10].

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First, using the latest bounds for the exponential sums in [17], we obtain stronger results as follows.

Theorem 1.1. *Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then*

$$E(\mathcal{V}, X, \delta) \ll X^{\frac{17}{18}+2\delta+\varepsilon}$$

for any $\varepsilon > 0$.

Theorem 1.2. *Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then there is a sequence $X_j \to \infty$ such that*

$$E(\mathcal{V}, X_j, \delta) \ll X_j^{\frac{17}{18}+2\delta+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy

$$q_{j+1}^{1-\omega} \ll q_j \quad \text{for some } \omega \in [0, 1), \tag{1.3}$$

then, for all $X \geq 1$,

$$E(\mathcal{V}, X, \delta) \ll X^{\frac{1+2\chi}{3}+2\delta+\varepsilon} \tag{1.4}$$

for any $\varepsilon > 0$ with

$$\chi = \max\left(\frac{3-\omega}{6-4\omega}, \frac{11}{12}\right). \tag{1.5}$$

Further, if we assume some stronger conditions, we remove 2/3 in (1.1). Our results are as follows.

Theorem 1.3. *Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Suppose that except for one, all other the ratios λ_k/λ_l $(1 \leq k < l \leq 5)$ are irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then*

$$E(\mathcal{V}, X, \delta) \ll X^{\frac{11}{12}+2\delta+\varepsilon} \tag{1.6}$$

for any $\varepsilon > 0$.

Theorem 1.4. *Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Suppose that except for one, all other the ratios λ_k/λ_l $(1 \leq k < l \leq 5)$ are irrational. Let $\mathcal{V}$ be a well-spaced sequence. If there exist some $\omega \in [0, 1)$ for all the convergent denominators $q_{k,l,j}$ of irrational λ_k/λ_l $(1 \leq k < l \leq 5)$ satisfying*

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$$q_{k,l,j+1}^{1-\omega} \ll q_{k,l,j}, \tag{1.7}$$

then, for all $X \geq 1$,

$$E(\mathcal{V}, X, \delta) \ll X^{\chi^*+2\delta+\varepsilon} \tag{1.8}$$

for any $\varepsilon > 0$ *with*

$$\chi^* = \max\left(\frac{4-\omega}{7-4\omega}, \frac{11}{12}\right). \tag{1.9}$$

Theorems 1.1 and 1.3 follow immediately from Theorems 1.2 and 1.4, respectively. Since, in the case of λ_k/λ_l algebraic, we can take $\omega = \varepsilon$. The reader should have no difficulties in deducing the following Corollary, which improves Corollary 1 of [4].

Corollary 1.5. *Let* λ_1, λ_2, λ_3, λ_4, λ_5 *be non-zero real numbers, not all of the same sign, with* λ_1/λ_2 *irrational,* ϖ *real and* $\varepsilon > 0$. *Then there are infinitely many solutions in primes* p_j *to the inequality*

$$|\sum_{j=1}^{9} \lambda_j p_j^3 + \varpi| < (\max p_j)^{-\frac{1}{12}+\varepsilon}.$$

Notation. Throughout the paper, the letter η denotes a sufficiently small, fixed positive number. The letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each fixed $\varepsilon > 0$. The letter p, with or without subscript, denotes a prime number. Constants, both explicit and implicit, in Vinogradov symbols may depend on λ_1, λ_2, λ_3, λ_4, λ_5. We write $e(x) = \exp(2\pi i x)$.

2. Outline of the method and preliminary lemmas

We follow the modification of the Hardy–Littlewood method first stated by Davenport and Heilbronn. Now let $0 < \tau < 1$, P be some (large) positive quantity to be chosen later (see equation (5.2) below in section 5) and $X = P^3$. We define

$$S(\alpha) = \sum_{\eta P \leq p < P} (\log p) e(\alpha p^3), \quad T(\alpha) = \sum_{\eta P \leq n < P} e(\alpha n^3), \tag{2.1}$$

$$K(\alpha) = \left(\frac{\sin \pi\tau\alpha}{\pi\alpha}\right)^2, \quad A(x) = \int_{-\infty}^{+\infty} K(\alpha) e(\alpha x) d\alpha. \tag{2.2}$$

Then, by [13], it is easy to show that

$$K(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad A(x) = \max(0, \tau - |x|). \tag{2.3}$$

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If we write

$$N_v = \frac{1}{\tau} \sum_{\eta P \le p_j < P} \Big(\prod_{j=1}^{5} \log p_j\Big) A(\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v),$$

then $0 \le N_v \le \psi(v)$, where $\psi(v)$ counts the number of the solutions to

$$|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < \tau,$$

weighted by a term $\prod_{j=1}^{5} \log p_j$. We shall restrict our attention to those v satisfying $X/2 \le v \le X$. In general, one can consider $X2^{-j} \le v \le X2^{1-j}$, $j = 1, 2, \dots$, and obtain a satisfactory bound for the exceptional set. Then, by (2.2), we have

$$N_v = \frac{1}{\tau} \int_{-\infty}^{+\infty} S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)S(\lambda_4\alpha)S(\lambda_5\alpha)K(\alpha)e(-\alpha v)d\alpha. \tag{2.4}$$

To estimate the integral in (2.4), we divide the real line into three parts: the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$ which are defined by

$$\mathfrak{M} = \{\alpha : |\alpha| \le \phi\}, \quad \mathfrak{m} = \{\alpha : \phi < |\alpha| \le \xi\}, \quad \mathfrak{t} = \{\alpha : |\alpha| > \xi\},$$

where $\phi = P^{-3+\frac{5}{12}-\varepsilon}$, $\xi = \tau^{-2}P^{1+2\varepsilon}$.

Lemma 2.1. *Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \quad 1 \le q \le P^{3/2} \quad \text{and} \quad |q\alpha - a| < P^{-3/2}.$$

Then one has

$$\sum_{P < p \le 2P} e(p^3\alpha) \ll P^{\frac{11}{12}+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{1/2}(1 + P^3|\alpha - a/q|)^{1/2}}.$$

Proof. This follows from Lemma 8.5 in [17] and Theorem 1.1 in [12] (also see Lemma 2.3 in [18]). □

Corollary 2.2. *Suppose that $P \ge Z \ge P^{\frac{11}{12}+\varepsilon}$ and $|S(\lambda_j\alpha)| \ge Z$. Then there are two coprime integers a, q satisfying*

$$1 \le q \ll (P/Z)^2 P^{\varepsilon}, \quad |q\lambda_j\alpha - a| \ll (P/Z)^2 P^{\varepsilon-3}.$$

Proof. Let $Q = P^{3/2}$, there exist two coprime integers a, q with $1 \le q \le Q$ and $|q\lambda_j\alpha - a| \le Q^{-1}$. By Lemma 2.1 and the hypothesis $Z \ge P^{\frac{11}{12}+\varepsilon}$, we have

$$P^{\frac{11}{12}+\varepsilon} \le Z \le |S(\lambda_j\alpha)| \ll P^{\frac{11}{12}+\frac{\varepsilon}{2}} + \frac{P^{1+\frac{\varepsilon}{2}}}{q^{1/2}(1+P^3|\lambda_j\alpha - a/q|)^{1/2}}.$$

Thus we have $1 \le q \ll (P/Z)^2P^{\varepsilon}$, $|q\lambda_j\alpha - a| \ll (P/Z)^2P^{\varepsilon-3}$. □

Lemma 2.3. *With the previous notation, we have*

$$\int_{-1}^{1} |S_j(\alpha)|^8 d\alpha \ll P^{5+\varepsilon}, \quad \int_{-1}^{1} |S_j(\alpha)|^4 d\alpha \ll P^{2+\varepsilon},$$

$$\int_{-\infty}^{+\infty} |S_j(\alpha)|^8 K(\alpha) d\alpha \ll \tau P^{5+\varepsilon}.$$

Proof. These immediately follow from Hua's inequality. □

We define the multiplicative function $w_3(q)$ by taking

$$w_3(p^{3u+v}) = \begin{cases} 3p^{-u-1/2}, & \text{when } u \ge 0 \text{ and } v = 1; \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le 3. \end{cases}$$

Weyl's inequality [15, lemma 2.4] and Lemmas 6.1–6.2 of Vaughan [15] together lead to the following conclusion (also one can see [17, lemma 2.3] or [7, Lemma 2.1]).

Lemma 2.4. *If α is a real number satisfying that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \le q \le P^{\frac{3}{4}}$ and $|q\alpha - a| \le P^{-\frac{9}{4}}$, then one has*

$$\sum_{P \le x < 2P} e(x^3\alpha) \ll \frac{w_3(q)P}{1 + P^3|\alpha - a/q|},$$

otherwise, one has $\sum_{P \le x < 2P} e(x^3\alpha) \ll P^{\frac{3}{4}+\varepsilon}$.

Lemma 2.5. *([17, Lemma 2.1]) Let c be a constant. For $Q \ge 2$, one has*

$$\sum_{1 \le q \le Q} d(q)^c w_3(q)^2 \ll (\log Q)^A,$$

where A is a positive constant, $d(q)$ is the divisor function.

3. The major arc

Lemma 3.1. *([4, Lemma 3]) We have*

$$\int_{\mathfrak{M}} S(\lambda_1\alpha) \cdots S(\lambda_5\alpha) e(-v\alpha) K(\alpha) d\alpha \gg \tau^2 P^2. \tag{3.1}$$

4. The trivial arc

By Lemma 2.3, we have

$$\begin{aligned}
&\int_{\mathfrak{t}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|K(\alpha)d\alpha \\
&\ll P\prod_{j=1}^{4}\Big(\int_{\xi}^{+\infty} |S(\lambda_j\alpha)|^4K(\alpha)d\alpha\Big)^{\frac14} \\
&\ll P\prod_{j=1}^{4}\Big(\sum_{n=[\xi]}^{+\infty}\int_{n}^{n+1} |S(\lambda_j\alpha)|^4\frac{1}{\alpha^2}d\alpha\Big)^{\frac14} \\
&\ll P\prod_{j=1}^{4}\Big(\sum_{n=[\xi]}^{+\infty}\frac{1}{n^2}\int_{0}^{1} |S(\lambda_j\alpha)|^4d\alpha\Big)^{\frac14} \\
&\ll \xi^{-1}P^{3+\varepsilon} \ll \tau^2P^{2-\varepsilon}.
\end{aligned} \tag{4.1}$$

5. The minor arc

We take $\sigma = 11/12$. Let $\mathfrak{m}' = \mathfrak{m}_1 \cup \mathfrak{m}_2$, $\hat{\mathfrak{m}} = \mathfrak{m} \setminus \mathfrak{m}'$, where

$$\mathfrak{m}_1 = \{\alpha \in \mathfrak{m} : |S(\lambda_1\alpha)| < P^{\sigma+\varepsilon}\}, \quad \mathfrak{m}_2 = \{\alpha \in \mathfrak{m} : |S(\lambda_2\alpha)| < P^{\sigma+\varepsilon}\}.$$

Lemma 5.1. *We have*

$$\int_{\mathfrak{m}'} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2K(\alpha)d\alpha \ll \tau P^{5+2\sigma+2\varepsilon}. \tag{5.1}$$

Proof. This follows from Lemma 2.3. □

Lemma 5.2. *Let q be a convergent to the continued fraction of λ_1/λ_2 and let*

$$P = q^{\frac38}. \tag{5.2}$$

Then we have

$$\int_{\hat{\mathfrak{m}}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2K(\alpha)d\alpha \ll \tau P^{\frac{13}{3}+6\varepsilon}. \tag{5.3}$$

Proof. We divide $\hat{\mathfrak{m}}$ into disjoint sets such that for $\alpha \in S(Z_1, Z_2, y)$, we have

$$Z_1 \le |S(\lambda_1\alpha)| < 2Z_1, \quad Z_2 \le |S(\lambda_2\alpha)| < 2Z_2, \quad y \le |\alpha| < 2y,$$

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where $Z_1 = P^{\frac{11}{12}+\varepsilon}2^{t_1}$, $Z_2 = P^{\frac{11}{12}+\varepsilon}2^{t_2}$, $y = \phi 2^s$ for some positive integers t_1, t_2, s. Thus, by Corollary, there exist two pairs of coprime integers (a_1, q_1), (a_2, q_2) with $a_1 a_2 \neq 0$ and

$$1 \leq q_j \ll (P/Z_j)^2 P^{\varepsilon}, \quad |q_j \lambda_j \alpha - a_j| \ll (P/Z_j)^2 P^{\varepsilon - 3}.$$

Then for any $\alpha \in S(Z_1, Z_2, y)$, we have

$$\left| \frac{a_j}{\alpha} \right| \ll q_j + (P/Z_j)^2 P^{\varepsilon-3} y^{-1} \ll q_j + P^{-\frac{1}{4}+\varepsilon} \ll q_j.$$

We further subdivide $\hat{\mathfrak{m}}$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j \leq q_j < 2Q_j$ on each set. Then, by a familiar argument (see p. 147 of [15] for example),

$$\begin{aligned} \left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| &= \left| \frac{a_2(q_1\lambda_1\alpha - a_1) + a_1(a_2 - q_2\lambda_2\alpha)}{\lambda_2 \alpha} \right| \\ &\ll Q_2 (P/Z_1)^2 P^{\varepsilon-3} + Q_1 (P/Z_2)^2 P^{\varepsilon-3} \\ &\ll (P/Z_1)^2 (P/Z_2)^2 P^{\varepsilon-3} \ll P^{-\frac{8}{3}-3\varepsilon}. \end{aligned}$$

Also

$$|a_2 q_1| \ll P^{2\varepsilon} y Q_1 Q_2.$$

If $|a_2 q_1|$ took on R distinct values, we could apply the pigeon-hole principle to deduce the existence of n with

$$\left\| n \frac{\lambda_1}{\lambda_2} \right\| \ll P^{-\frac{8}{3}-3\varepsilon}, \quad n \ll \frac{P^{2\varepsilon} y Q_1 Q_2}{R}.$$

Note that $P = q^{\frac{3}{8}}$ by (5.2), where a/q is a convergent to λ_1/λ_2. This forces that

$$R \ll \frac{P^{2\varepsilon} y Q_1 Q_2}{q}.$$

By the well-known bound on the divisor function, each value of $|a_2 q_1|$ corresponds to $\ll P^{\varepsilon}$ values of a_2, q_1. Hence we conclude that $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $\ll RP^{\varepsilon}$ intervals of length

$$\ll \min\left(\frac{1}{Q_1}\left(\frac{P}{Z_1}\right)^2 P^{\varepsilon-3}, \frac{1}{Q_2}\left(\frac{P}{Z_2}\right)^2 P^{\varepsilon-3} \right) \ll \frac{P^{\varepsilon-1}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}}.$$

Thus integrating over such a set gives

$$\begin{aligned}
&\int |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \\
&\ll \min(\tau^2, y^{-2}) Z_1^2 Z_2^2 P^6 \frac{P^{\varepsilon-1}}{Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}} \times \frac{P^{3\varepsilon} y Q_1 Q_2}{q} \\
&\ll \tau P^{7+5\varepsilon} q^{-1} \ll \tau P^{\frac{13}{3}+5\varepsilon}.
\end{aligned} \tag{5.4}$$

Summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$\int_{\hat{\mathfrak{m}}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \ll \tau P^{\frac{13}{3}+6\varepsilon}. \qquad \square \tag{5.5}$$

6. The proof of Theorem 1.2

We take $\tau = X^{-\delta}$. Let $\mathcal{E} = \mathcal{E}(\mathcal{V}, X, \delta)$ denote the set of $v \in \mathcal{V}$, $1 \le v \le X$ such that the inequality

$$|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < \tau$$

has no solution in primes p_1, p_2, p_3, p_4, p_5 and $E = E(\mathcal{V}, X, \delta) = |\mathcal{E}(\mathcal{V}, X, \delta)|$. Then by (3.1) and (4.1), we have

$$\left| \sum_{v\in\mathcal{E}} \int_{\mathfrak{m}} S(\lambda_1\alpha)\cdots S(\lambda_5\alpha) e(-v\alpha) K(\alpha)d\alpha \right| \gg \tau^2 P^2 E. \tag{6.1}$$

Now, we come to prove the first part of Theorem 1.2. Note that $P = q^{\frac{3}{8}}$ by (5.2), where a/q is a convergent to λ_1/λ_2. By Cauchy's inequality, (5.1) and (5.3), we get

$$\begin{aligned}
&\left| \sum_{v\in\mathcal{E}} \int_{\mathfrak{m}} S(\lambda_1\alpha)\cdots S(\lambda_5\alpha) e(-v\alpha) K(\alpha)d\alpha \right| \\
&\ll \left(\int_{-\infty}^{+\infty} \Big| \sum_{v\in\mathcal{E}} e(-v\alpha) \Big|^2 K(\alpha)d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \right)^{1/2} \\
&\ll \left(\tau P^{\frac{41}{6}+6\varepsilon}\right)^{1/2} \left(\sum_{v_1,v_2\in\mathcal{E}} \max(0, \tau - |v_1 - v_2|) \right)^{1/2} \ll \tau E^{1/2} P^{\frac{41}{12}+3\varepsilon}.
\end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2), we have

$$E(\mathcal{V}, X, \delta) \ll X^{\frac{17}{18}+2\delta+\varepsilon}. \tag{6.3}$$

Of course, there are infinitely many q we could have taken in since λ_1/λ_2 is irrational, and this gives the sequence $X_j \to \infty$. This completes the proof of the first part of Theorem 1.2.

Next, we come to prove the second part of Theorem 1.2. Now, if the convergent denominators for λ_1/λ_2 satisfy (1.3), then we can modify our work in Lemmas 5.1 and 5.2. We now let P be a sufficiently large number, and assume that

$$\min(Z_1, Z_2) > P^{\chi+\varepsilon},$$

with χ given by (1.5). We then obtain

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_2 q_1 \right| \ll P^{1-4\chi-3\varepsilon}.$$

However, we know from (1.3) that there is a convergent a/q to λ_1/λ_2 with

$$P^{(1-\omega)(4\chi-1)} \ll q \ll P^{4\chi-1}.$$

Then, the expression corresponding to (5.4) is now

$$\begin{aligned} &\int |S_1(\alpha)S_2(\alpha)S_3(\alpha)|^2 K(\alpha) d\alpha \\ &\ll \tau P^{7+5\varepsilon} q^{-1} \ll \tau P^{7-(1-\omega)(4\chi-1)+5\varepsilon} \ll \tau P^{5+2\chi+5\varepsilon} \end{aligned}$$

by our choice of χ. Thus we have

$$\int\limits_{\mathfrak{m}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha) d\alpha \ll \tau P^{5+2\chi+6\varepsilon}. \tag{6.4}$$

We work as (6.1), (6.2) and (6.3). Then the proof of Theorem 1.2 is now easily completed.

7. The proof of Theorem 1.4

We need only to re-estimate the integral in (2.4) on the minor arc $\mathfrak{m}$. First, we divide $\mathfrak{m}$ into two sets $\mathfrak{m}''$ and $\tilde{\mathfrak{m}} = \mathfrak{m} \setminus \mathfrak{m}''$ such that, for $\alpha \in \mathfrak{m}''$, we have that at least three of the sums $|S(\lambda_1\alpha)|, |S(\lambda_2\alpha)|, \dots, |S(\lambda_5\alpha)|$ do not exceed $P^{\chi^*+\varepsilon}$, where χ^* is defined by (1.9). Then for $\alpha \in \tilde{\mathfrak{m}}$, there are at least three of $S(\lambda_j\alpha)$ exceed $P^{\chi^*+\varepsilon}$. Since at most one of the ratios λ_i/λ_j $(1 \le i < j \le 5)$ is rational, there exist $1 \le k < l \le 5$ satisfying λ_k/λ_l is irrational, and

$$|S(\lambda_k\alpha)| \ge P^{\chi^*+\varepsilon}, \quad |S(\lambda_l\alpha)| \ge P^{\chi^*+\varepsilon}.$$

Now, if the convergent denominators for λ_k/λ_l satisfying (1.7), then we can modify our work in Lemma 5.2. We now let P be a sufficiently large number, then we know from

(1.7) that there is a convergent a/q to λ_k/λ_l with

$$P^{(1-\omega)(4\chi^*-1)} \ll q \ll P^{4\chi^*-1}.$$

Then, by the proof of Lemma 5.2, we can easily get

$$\int\limits_{\tilde{\mathfrak{m}}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \ll \tau P^{7-(1-\omega)(4\chi^*-1)+6\varepsilon} \ll \tau P^{4+3\chi^*+6\varepsilon}. \tag{7.1}$$

Now, we consider the integral on $\mathfrak{m}''$. Without loss of generality we need only consider the set

$$\mathfrak{m}^* = \{\alpha \in \mathfrak{m} : |S(\lambda_1\alpha)| \geq |S(\lambda_2\alpha)|, |S(\lambda_j\alpha)| \leq P^{\chi^*+\varepsilon}, j = 3, 4, 5\}.$$

Lemma 7.1. *We have*

$$\int\limits_{\mathfrak{m}^*} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \ll \tau P^{4+3\chi^*+6\varepsilon}. \tag{7.2}$$

Proof. For convenience, let $G(\alpha) = |S(\lambda_2\alpha)\cdots S(\lambda_5\alpha)|^2$. Therefore, we have

$$\begin{aligned} J(\mathfrak{m}^*) &:= \int\limits_{\mathfrak{m}^*} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \\ &= \sum_{\eta P \leq p < P} (\log p) \int\limits_{\mathfrak{m}^*} e(\alpha\lambda_1 p^3)G(\alpha)S(-\lambda_1\alpha)K(\alpha)d\alpha \\ &\leq \sum_{\eta P \leq p < P} (\log p) \Bigg| \int\limits_{\mathfrak{m}^*} e(\alpha\lambda_1 p^3)G(\alpha)S(-\lambda_1\alpha)K(\alpha)d\alpha \Bigg| \\ &\leq (\log P) \sum_{\eta P \leq n < P} \Bigg| \int\limits_{\mathfrak{m}^*} e(\alpha\lambda_1 n^3)G(\alpha)S(-\lambda_1\alpha)K(\alpha)d\alpha \Bigg|. \end{aligned}$$

Then, by Cauchy's inequality, we get

$$J(\mathfrak{m}^*) \ll P^{\frac{1}{2}} \log P \Bigg(\sum_{\eta P \leq n < P} \Bigg| \int\limits_{\mathfrak{m}^*} e(\alpha\lambda_1 n^3)G(\alpha)S(-\lambda_1\alpha)K(\alpha)d\alpha \Bigg|^2 \Bigg)^{1/2}. \tag{7.3}$$

We estimate the right sum in (7.3) and obtain

$$\begin{aligned} &\sum_{\eta P \leq n < P} \Bigg| \int\limits_{\mathfrak{m}^*} e(\alpha\lambda_1 n^3)G(\alpha)S(-\lambda_1\alpha)K(\alpha)d\alpha \Bigg|^2 \\ &= \sum_{\eta P \leq n < P} \int\limits_{\mathfrak{m}^*}\int\limits_{\mathfrak{m}^*} G(\alpha)S(-\lambda_1\alpha)K(\alpha)G(-\beta)S(\lambda_1\beta)K(\beta)e(\lambda_1 n^3(\alpha-\beta))d\alpha d\beta \end{aligned}$$

$$= \int_{\mathfrak{m}^*} G(-\beta)S(\lambda_1\beta)K(\beta)\Big(\int_{\mathfrak{m}^*} G(\alpha)S(-\lambda_1\alpha)T(\lambda_1(\alpha-\beta))K(\alpha)d\alpha\Big)d\beta$$

$$\le \int_{\mathfrak{m}^*} |G(-\beta)S(\lambda_1\beta)|F(\beta)K(\beta)d\beta, \tag{7.4}$$

where

$$T(x) = \sum_{\eta P \le n < P} e(xn^3),$$

and

$$F(\beta) = \int_{\mathfrak{m}^*} |G(\alpha)S(-\lambda_1\alpha)T(\lambda_1(\alpha-\beta))|K(\alpha)d\alpha. \tag{7.5}$$

Let $\mathcal{M}_\beta(r,b) = \{\alpha \in \mathfrak{m}^* : |r\lambda_1(\alpha-\beta)-b| \le P^{-\frac{9}{4}}\}$. Then, the set $\mathcal{M}_\beta(r,b) \neq \emptyset$ forces that

$$|b + r\lambda_1\beta| \le |r\lambda_1(\alpha-\beta)-b| + |r\lambda_1\alpha| \le P^{-\frac{9}{4}} + r|\lambda_1|\tau^{-2}P^{2\varepsilon}.$$

Let $\mathcal{A} = \{b \in \mathbb{Z} : |b + r\lambda_1\beta| \le P^{-\frac{9}{4}} + r|\lambda_1|\tau^{-2}P^{2\varepsilon}\}$. We divide $\mathcal{A}$ into two sets $\mathcal{A}_1 = \{b \in \mathbb{Z} : -r|\lambda_1|\tau^{-1} + P^{-9/4} \le b + r\lambda_1\beta \le r|\lambda_1|\tau^{-1} - P^{-9/4}\}$ and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$. Let

$$\mathcal{M}_\beta = \bigcup_{1 \le r \le P^{\frac{3}{4}}} \bigcup_{\substack{b\in\mathcal{A}\\(b,r)=1}} \mathcal{M}_\beta(r,b).$$

Then, by Lemma 2.4, we have

$$F(\beta) \ll P \int_{\mathcal{M}_\beta \cap \mathfrak{m}^*} |G(\alpha)S(-\lambda_1\alpha)| \frac{w_3(r)K(\alpha)}{1 + P^3|\lambda_1(\alpha-\beta) - b/r|} d\alpha$$

$$+ P^{\frac{3}{4}+\varepsilon} \int_{\mathfrak{m}^*} |G(\alpha)S(-\lambda_1\alpha)|K(\alpha)d\alpha. \tag{7.6}$$

For the first integral in (7.6), by Cauchy's inequality, we get

$$\int_{\mathcal{M}_\beta \cap \mathfrak{m}^*} |G(\alpha)S(-\lambda_1\alpha)| \frac{w_3(r)K(\alpha)}{1 + P^3|\lambda_1(\alpha-\beta) - b/r|} d\alpha$$

$$\ll \Big(\int_{\mathfrak{m}^*} |G(\alpha)|^2 K(\alpha)d\alpha\Big)^{1/2} \Big(\int_{\mathcal{M}_\beta} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 K(\alpha)}{(1 + P^3|\lambda_1(\alpha-\beta) - b/r|)^2} d\alpha\Big)^{1/2}. \tag{7.7}$$

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Next, we come to estimate the last integral in (7.7). First, we divide it into two parts.

$$\begin{aligned}
&\int_{\mathcal{M}_\beta} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 K(\alpha)}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2} d\alpha \\
&= \sum_{j=1,2} \sum_{1\le r\le P^{3/4}} \sum_{\substack{b\in\mathcal{A}_j\\(b,r)=1}} \int_{\mathcal{M}_\beta(r,b)} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 K(\alpha)}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2} d\alpha \\
&=: L_1(\beta)+L_2(\beta). \qquad (7.8)
\end{aligned}$$

For the first part, we get

$$\begin{aligned}
L_1(\beta) &\ll \tau^2 \sum_{1\le r\le P^{3/4}} \sum_{\substack{b\in\mathcal{A}_1\\(b,r)=1}} \int_{\mathcal{M}_\beta(r,b)} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2} d\alpha \\
&\ll \tau^2 \sum_{1\le r\le P^{3/4}} \sum_{\substack{b\in\mathcal{A}_1\\(b,r)=1}} \int_{|\lambda_1\gamma|\le \frac{1}{rP^{9/4}}} \frac{|S(\lambda_1(\beta+\gamma)+b/r)|^2 w_3(r)^2}{(1+P^3|\lambda_1\gamma|)^2} d\gamma \\
&\ll \tau^2 \sum_{1\le r\le P^{3/4}} w_3(r)^2 \int_{|\lambda_1\gamma|\le \frac{1}{rP^{9/4}}} \frac{U(\mathcal{A}_1^*)}{(1+P^3|\lambda_1\gamma|)^2} d\gamma,
\end{aligned}$$

where

$$U(\mathcal{A}_1^*) = \sum_{b\in\mathcal{A}_1^*} |S(\lambda_1(\beta+\gamma)+b/r)|^2,$$

and

$$\mathcal{A}_1^* = \{b\in\mathbb{Z} : -r([|\lambda_1|\tau^{-1}]+1) < b+r\lambda_1\beta \le r([|\lambda_1|\tau^{-1}]+2)\}.$$

Then, we have

$$\begin{aligned}
U(\mathcal{A}_1^*) &= \sum_{\eta P\le p_1,p_2<P} \sum_{b\in\mathcal{A}_1^*} e((\lambda_1(\beta+\gamma)+b/r)(p_1^3-p_2^3)) \\
&= \sum_{\eta P\le p_1,p_2<P} e(\lambda_1(\beta+\gamma)(p_1^3-p_2^3)) \sum_{b\in\mathcal{A}_1^*} e\left(\frac{b(p_1^3-p_2^3)}{r}\right) \\
&\le r(2|\lambda_1|\tau^{-1}+3) \sum_{\substack{\eta P\le p_1,p_2<P\\ p_1^3\equiv p_2^3 (\mathrm{mod} r)}} e(\lambda_1(\beta+\gamma)(p_1^3-p_2^3)) \\
&\ll r\tau^{-1} \sum_{\substack{\eta P\le p_1,p_2<P\\ p_1^3\equiv p_2^3 (\mathrm{mod} r)}} 1
\end{aligned}$$

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$$\ll r\tau^{-1}P^2r^{-2} \sum_{\substack{1\le b_1,b_2<r,(b_1b_2,r)=1\\ b_1^3\equiv b_2^3(\mathrm{mod} r)}} 1$$

$$\ll \tau^{-1}P^2 \sum_{\substack{1\le b<r\\ b^3\equiv 1(\mathrm{mod} r)}} 1 \ll \tau^{-1}P^2 d(r)^c.$$

Hence, by Lemma 2.5, we have

$$L_1(\beta) \ll \tau P^2 \sum_{1\le r\le P^{3/4}} w_3^2(r)d(r)^c \int\limits_{|\lambda_1\gamma|\le r^{-1}P^{-9/4}} \frac{1}{(1+P^3|\lambda_1\gamma|)^2}d\gamma$$

$$\ll \tau P^{-1} \sum_{1\le r\le P^{3/4}} w_3^2(r)d(r)^c \ll \tau P^{-1+\varepsilon}. \tag{7.9}$$

Now, we begin to estimate $L_2(\beta)$. First, without loss of generality we need only consider the set

$$\mathcal{A}'_2 = \{b\in\mathbb{Z} : r|\lambda_1|\tau^{-1} - P^{-9/4} < b + r\lambda_1\beta \le r|\lambda_1|\tau^{-2}P^{2\varepsilon} + P^{-9/4}\}$$

which falls in the set

$$\mathcal{A}_2^* = \{b\in\mathbb{Z} : r([|\lambda_1|\tau^{-1}]-1) < b + r\lambda_1\beta \le r([|\lambda_1|\tau^{-2}P^{2\varepsilon}]+2)\}.$$

Then, we obtain

$$L_2(\beta) \ll \sum_{1\le r\le P^{3/4}} \sum_{\substack{b\in\mathcal{A}_2^*\\ (b,r)=1}} \int\limits_{\mathcal{M}_\beta(r,b)} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 K(\alpha)}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2}d\alpha$$

$$\ll \sum_{1\le r\le P^{3/4}} \sum_{k=[|\lambda_1|\tau^{-1}]-1}^{[|\lambda_1|\tau^{-2}P^{2\varepsilon}]+1} \sum_{\substack{rk<b+r\lambda_1\beta\le r(k+1)\\ (b,r)=1}} \int\limits_{\mathcal{M}_\beta(r,b)} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 K(\alpha)}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2}d\alpha$$

$$\ll \sum_{1\le r\le P^{3/4}} \sum_{k=[|\lambda_1|\tau^{-1}]-1}^{[|\lambda_1|\tau^{-2}P^{2\varepsilon}]+1} \sum_{\substack{rk<b+r\lambda_1\beta\le r(k+1)\\ (b,r)=1}} \int\limits_{\mathcal{M}_\beta(r,b)} \frac{|S(-\lambda_1\alpha)|^2 w_3(r)^2 |\alpha|^{-2}}{(1+P^3|\lambda_1(\alpha-\beta)-b/r|)^2}d\alpha$$

$$\ll \sum_{1\le r\le P^{3/4}} \sum_{k=[|\lambda_1|\tau^{-1}]-1}^{[|\lambda_1|\tau^{-2}P^{2\varepsilon}]+1} \frac{1}{(k-1)^2} \int\limits_{|\lambda_1\gamma|\le r^{-1}P^{-9/4}} \frac{U(\mathcal{B}_k)w_3(r)^2}{(1+P^3|\lambda_1\gamma|)^2}d\gamma,$$

where $\mathcal{B}_k = \{b\in\mathbb{Z} : rk < b + r\lambda_1\beta \le r(k+1)\}$. Similar to the estimate of $U(\mathcal{A}_1^*)$, we have $U(\mathcal{B}_k) \ll P^2 d(r)^c$. Hence we get

$$\begin{aligned} L_2(\beta) &\ll P^2 \sum_{1\le r\le P^{3/4}} \frac{w_3(r)^2 d(r)^c}{([|\lambda_1|\tau^{-1}]-2)} \int\limits_{|\lambda_1\gamma|\le r^{-1}P^{-9/4}} \frac{1}{(1+P^3|\lambda_1\gamma|)^2} d\gamma \\ &\ll \tau P^{-1} \sum_{1\le r\le P^{3/4}} w_3(r)^2 d(r)^c \ll \tau P^{-1+\varepsilon}. \end{aligned} \tag{7.10}$$

Combining (7.6)–(7.10), we have

$$\begin{aligned} F(\beta) &\ll \tau^{1/2} P^{1/2+\varepsilon} \Big(\int\limits_{\mathfrak{m}^*} |G(\alpha)|^2 K(\alpha) d\alpha \Big)^{1/2} \\ &\quad + P^{3/4+\varepsilon} \int\limits_{\mathfrak{m}^*} |G(\alpha) S(\lambda_1\alpha)| K(\alpha) d\alpha. \end{aligned} \tag{7.11}$$

Thus, by (7.3), (7.4) and (7.11), we conclude that

$$\begin{aligned} J(\mathfrak{m}^*) &\ll \tau^{\frac14} P^{\frac34+\varepsilon} \Big(\int\limits_{\mathfrak{m}^*} |G(\alpha)|^2 K(\alpha) d\alpha \Big)^{1/4} \Big(\int\limits_{\mathfrak{m}^*} |G(\alpha) S(\lambda_1\alpha)| K(\alpha) d\alpha \Big)^{1/2} \\ &\quad + P^{7/8+\varepsilon} \int\limits_{\mathfrak{m}^*} |G(\alpha) S(\lambda_1\alpha)| K(\alpha) d\alpha. \end{aligned} \tag{7.12}$$

By Cauchy's inequality and Lemma 2.3, we have

$$\begin{aligned} \int\limits_{\mathfrak{m}^*} |G(\alpha) S(\lambda_1\alpha)| K(\alpha) d\alpha &\ll J(\mathfrak{m}^*)^{1/2} \Big(\int\limits_{\mathfrak{m}^*} |G(\alpha)| K(\alpha) d\alpha \Big)^{1/2} \\ &\ll J(\mathfrak{m}^*)^{1/2} \prod_{j=2}^{5} \Big(\int_{-\infty}^{+\infty} |S(\lambda_j\alpha)|^8 K(\alpha) d\alpha \Big)^{1/8} \\ &\ll \tau^{1/2} P^{5/2+\varepsilon} J(\mathfrak{m}^*)^{1/2}. \end{aligned} \tag{7.13}$$

By the definition of $\mathfrak{m}^*$, we have

$$\int\limits_{\mathfrak{m}^*} |G(\alpha)|^2 K(\alpha) d\alpha \ll P^{6\chi^*+6\varepsilon} J(\mathfrak{m}^*). \tag{7.14}$$

Then we conclude from (7.12), (7.13) and (7.14) that

$$J(\mathfrak{m}^*) \ll \tau^{1/2} P^{2+3\varepsilon} J(\mathfrak{m}^*)^{1/2} (P^{\frac32\chi^*} + P^{\frac{11}{8}}) \ll \tau^{1/2} P^{2+\frac32\chi^*+3\varepsilon} J(\mathfrak{m}^*)^{1/2}. \tag{7.15}$$

Therefore, (7.2) follows from (7.15). □

At last, combining (7.1) and (7.2), we have

$$\int_{\mathfrak{m}} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)|^2 K(\alpha)d\alpha \ll \tau P^{4+3\chi^*+6\varepsilon}.$$

Then we work as in the section 6, the proof of Theorem 1.4 is now easily completed.

Acknowledgments

We thank the referees for their time and comments. The research has been supported by the National Natural Science Foundation of China (Grant Nos. 11371122, 11471112).

References

[1] J. Brüdern, R.J. Cook, A. Perelli, The values of binary linear forms at prime arguments, in: Sieve Methods, Exponential Sums and Their Applications in Number Theory, Cambridge University Press, Cambridge, 1996, pp. 1–54.

[2] R.J. Cook, The value of additive forms at prime arguments, J. Théor. Nombres Bordeaux 14 (2001) 77–91.

[3] R.J. Cook, A. Fox, The values of ternary quadratic forms at prime arguments, Mathematika 48 (2001) 137–149.

[4] R.J. Cook, G. Harman, The values of additive forms at prime arguments, Rocky Mountain J. Math. 36 (4) (2006) 1153–1164.

[5] G. Harman, The values of ternary quadratic forms at prime arguments, Mathematika 51 (12) (2005) 83–96.

[6] L.K. Hua, On the representation of numbers as the sums of the powers of primes, Math. Z. 44 (1938) 335–346.

[7] K. Kawada, T.D. Wooley, On the Waring–Goldbach problem for fourth and fifth powers, Proc. Lond. Math. Soc. 83 (3) (2001) 1–50.

[8] K. Kawada, T.D. Wooley, Relations between exceptional sets for additive problems, J. Lond. Math. Soc. 82 (2010) 437–458.

[9] A.V. Kumchev, On the Waring–Goldbach problem: exceptional sets for sums of cubes and higher powers, Canad. J. Math. 57 (2005) 298–327.

[10] A.V. Kumchev, On Weyl sums over primes and almost primes, Michigan Math. J. 54 (2) (2006) 243–268.

[11] X.M. Ren, The Waring–Goldbach problem for cubes, Acta Arith. 94 (2000) 287–301.

[12] X.M. Ren, On exponential sums over primes and application in Waring–Goldbach problem, Sci. China Ser. A 48 (6) (2005) 785–797.

[13] R.C. Vaughan, Diophantine approximation by prime numbers (I), Proc. Lond. Math. Soc. 28 (3) (1974) 373–384.

[14] R.C. Vaughan, Diophantine approximation by prime numbers (II), Proc. Lond. Math. Soc. 28 (3) (1974) 385–401.

[15] R.C. Vaughan, The Hardy–Littlewood Method, 2nd edition, Cambridge University Press, Cambridge, 1997.

[16] T.D. Wooley, Slim exceptional sets for sums of cubes, Canad. J. Math. 54 (2002) 417–448.

[17] L.L. Zhao, On the Waring–Goldbach problem for fourth and sixth powers, Proc. Lond. Math. Soc. 108 (3) (2014) 1593–1622.

[18] L.L. Zhao, The additive problem with one cube and three cubes of primes, Michigan Math. J. 63 (2014) 763–779.

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