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# Integral isosceles triangle–parallelogram and Heron triangle–rhombus pairs with a common area and common perimeter

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In this paper we show that there are infinitely many pairs of integer isosceles triangles and integer parallelograms with a common (integral) area and common perimeter. We also show that there are infinitely many Heron triangles and integer rhombuses with common area and common perimeter. As a corollary, we show there does not exist any Heron triangle and integer square which have a common area and common perimeter.

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## 1. Introduction

The study of geometrical objects is a very ancient problem. There are many questions in number theory which are related to triangles, rectangles, squares, polygons, and so forth. For example, there is the well-known congruent number problem which asks: given a positive integer $n$, does there exist a right triangle with rational side lengths whose area is $n$? As a second example, several researchers have related various types of triangles and quadrilaterals to the theory of elliptic curves. Both Goins and Maddox [5] and Dujella and Peral [4] constructed elliptic curves over $\mathbb{Q}$ coming from Heron triangles. Izadi, Khoshnam, and Moody later generalized their notions to Heron quadrilaterals [7]. In [9] Naskręcki constructed elliptic curves associated to Pythagorian triplets, and Izadi et al. similarly studied curves arising from Brahmagupta quadrilaterals [8].

Another problem connecting geometrical objects with number theory is devoted to the construction of triangles with area, perimeter or side lengths with certain arithmetic properties. Bill Sands asked his colleague R.K. Guy if there were triangles with integer sides associated with rectangles having the same perimeter and area. In 1995, Guy [6] showed that the answer was affirmative, but that there is no non-degenerate right triangle and rectangle pair with the same property. In that same paper, Guy also showed that there are infinitely many such isosceles triangle and rectangle pairs. Several other works in this direction have been solved, all involving pairs of geometric shapes having a common area and common perimeter: two distinct Heron triangles by A. Bremner [1], Heron triangle and rectangle pairs by R.K. Guy and Bremner [2], integer right triangle and parallelogram pairs by Y. Zhang [13], and integer right triangle and rhombus pairs by S. Chern [3].

In this paper we continue this line of study. The first problem we examine regards integer isosceles triangles and integer parallelograms which share a common area and common perimeter. We then consider Heron triangle and integer rhombus pairs. Using the theory of elliptic curves we are able to prove that there are infinitely many examples of each type.

## 2. Integral isosceles triangle and parallelogram pairs

We first address the case of integral isosceles triangles and parallelograms which have a common (integral) area and common perimeter. As we are requiring the area of the isosceles triangle to be integral, then necessarily the altitude to the non-isosceles side of the triangle must be rational. By the general solution to the Pythagorean equation, we may take the equal legs of the isosceles triangle to have length $m^2 + n^2$, with the base being $2(m^2 - n^2)$ and the altitude $2mn$, for some rational $m, n$. The perimeter of the triangle is $4m^2$, with an area of $2mn(m^2 - n^2)$. See Fig. 1.

For the parallelogram, we let $p, q$ be the consecutive side lengths, with their intersection angle $\theta$. The perimeter of the parallelogram is $2(p + q)$, while the area is $pq \sin\theta$. In order for the two areas to be equal, then $\sin\theta$ must necessarily be rational. We as-

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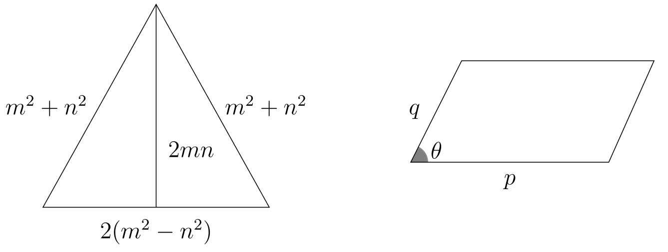


**Fig. 1.** An isosceles triangle and parallelogram.

sume a stronger condition, namely that $\sin\theta = 2t/(1+t^2)$, from which it follows that $\cos\theta = (1-t^2)/(1+t^2)$ is also rational.[1] So we will actually produce isosceles triangles with the same area and perimeter as parallelograms whose angles have rational values for sine and cosine. We call a polygon *$\theta$-integral* if the polygon has integer side lengths, integral area, and its interior angles have rational values for sine and cosine.

**Theorem 2.1.** *Let $P = (X, Y)$ be a rational point on the elliptic curve*

$$Y^2 = X^3 - 16t^2(1+t^2)^2X + 64t^4(1+t^2)^2 \tag{1}$$

*such that*

$$0 < |X| < 4t(1+t^2), \quad Y < 8t^2(1+t^2) \tag{2}$$

*for some rational $0 < t \leq 1$. Then there exist infinitely many integer isosceles triangle and $\theta$-integral parallelogram pairs with a common area and a common perimeter for this value of $t$, i.e. with $\sin\theta = 2t/(1+t^2)$.*

**Proof.** Equating the perimeters and areas, we have

$$\begin{aligned} 2mn(m^2-n^2) &= pq\sin\theta, \\ 2m^2 &= p+q, \end{aligned} \tag{3}$$

where $\theta$ is chosen such that $\sin\theta = f(t) = 2t/(1+t^2)$, with $t$ a positive rational number. Since $f(t) = f(1/t)$ we may assume that $0 < t \leq 1$.

Equation (3) can be transformed into a family of elliptic curves as follows. Starting with

$$(p-q)^2 = (p+q)^2 - 4pq,$$

[1] We note that if we do not require $\sin\theta$ and $\cos\theta$ to both be rational, it is trivial to produce a parallelogram with the same perimeter and same area as any given triangle. If a given triangle has perimeter $P$ and area $A$, we could simply set $p = q = P/4$. Then for the areas to be equal, we would need $(P^2/16)\sin\theta = A$, or in other words $\theta = \sin^{-1}(16A/P^2)$. In order for $\cos\theta$ to be rational, an easy calculation shows that $P^4 - 16A^2$ would need to be square, a condition which will not hold in general.

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then from equation (3), we obtain

$$(p-q)^2 = 4m^4 - 8mn(m^2-n^2)(t^2+1)/(2t),$$

or equivalently

$$((p-q)/(m^2))^2 = 4 - 4((n/m)-(n/m)^3)((1+t^2)/t).$$

Setting $\frac{p-q}{m^2} = \frac{Y}{4t^2(1+t^2)}$ and $\frac{n}{m} = \frac{X}{4t(1+t^2)}$, the resulting equation becomes

$$E_t : Y^2 = X^3 - 16t^2(1+t^2)^2X + 64t^4(1+t^2)^2. \tag{4}$$

The discriminant of $E_t$ is

$$\Delta(t) = -2^{12}(1+t^2)^4(4t^4-19t^2+4).$$

For rational $0 < t \leq 1, \Delta(t) \neq 0$, hence $E_t$ is nonsingular and defines an elliptic curve.

It is easy to check the following three rational points are on $E_t$:

$$P_1(t) = (0, 8t^2(1+t^2)),$$
$$P_2(t) = (-4t(t^2-1), 8t^2(t^2+2t-1)),$$
$$P_3(t) = (4(t^2+1), 8(t^2+1)^2).$$

In fact, we can show these three points are independent, and that the rank of $E_t$ over $\mathbb{Q}(t)$ is $\geq 3$ for all but finitely many values of $t$. We do so using the Silverman Specialization Theorem [11, Theorem 11.4]. If we find a value $t = t_0$ such that $P_1(t_0), P_2(t_0)$, and $P_3(t_0)$ are linearly independent on $E_{t_0}$, then the three points $P_1(t), P_2(t)$, and $P_3(t)$ are necessarily independent over $\mathbb{Q}(t)$. Setting $t_0 = 3$, the curve $E_3$ is then $y^2 = x^3 - 14400x + 518400$, with the three points $(0, 720), (-96, 1008), (40, 80)$. We check they are independent by computing the determinant of their height pairing matrix, which is non-zero $\approx 0.3266$, as computed by SAGE [10]. Thus the rank of $E_t$ is at least 3 (over $\mathbb{Q}(t)$), and we have that there are infinitely many points on $E_t$, for all but finitely many values of $t$.

Given a point $(X, Y)$ on the curve $E_t$, we can reverse the correspondence to construct the isosceles triangle and parallelogram pairs. From the transformations used above, we have

$$n = \frac{mX}{4t(t^2+1)}, \quad p = \frac{m^2(8t^4+8t^2+Y)}{8t^2(t^2+1)},$$

and

$$q = \frac{m^2(8t^4+8t^2-Y)}{8t^2(t^2+1)}.$$

The sides of the isosceles triangle then become (up to scaling),

$$\left\{X^2 + 16t^2(t^2+1)^2, X^2 + 16t^2(t^2+1)^2, 2(16t^2(t^2+1)^2 - X^2)\right\}.$$

The sides $\{p, q\}$ of the parallelogram are given by

$$\left\{2(t^2+1)(8t^2(t^2+1)+Y), 2(t^2+1)(8t^2(t^2+1)-Y)\right\}.$$

In order for all the side lengths to be positive rational numbers, we need the conditions

$$0 < |X| < 4t(t^2+1), \quad Y < 8t^2(t^2+1)$$

to be satisfied.

To show there are infinitely many rational points on $E_t$ meeting these constraints, we use a theorem of Poincaré and Hurwitz (see [12, p. 78]) about the density of rational points. Their theorem states that if an elliptic curve $E(\mathbb{Q})$ has positive rank and at most one torsion point of order two, then the set $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$. The same result holds if $E$ has three torsion points of order two, under the assumption that we have a rational point of infinite order on the bounded branch of the set $E(\mathbb{R})$. The curve $E_t$ always has one real 2-torsion point. We note that if the curve $E_t$ has three real points of order two (for some $0 < t \leq 1$), then the point $(0, 8t^2(1+t^2))$ is easily seen to be a point of infinite order on the bounded branch. Thus, if there is a rational point $P$ satisfying equation (2), then $E_t$ has infinitely many rational points satisfying equation (2) by the density theorem. □

**Corollary 2.2.** *For every rational number $0 < t \leq 1$, there are infinitely many pairs of integer isosceles triangles and $\theta$-integral parallelograms which have a common area and a common perimeter, with $\sin\theta = 2t/(1+t^2)$.*

**Proof.** Given any rational $0 < t \leq 1$, note the point $P(t) = (4t^2, 8t^3)$ is on the curve $E_t$. Considering its $x$ and $y$-coordinates, we see that $X = 4t^2 > 0$ for all $t \neq 0$, and also that $|X| = 4t^2 < 4t(1+t^2)$ which is true for all $0 < t \leq 1$. In addition, we have $Y = 8t^3 < 8t^2(1+t^2)$ which is similarly always true for $0 < t \leq 1$.

Thus the conditions of Theorem 2.1 are satisfied by the point $P$ for this value of $t$, and the conclusion immediately follows. □

As a special case of the corollary, we have a new proof of the following result, first established by Guy [6].

**Corollary 2.3.** *There are infinitely many integer isosceles triangles and integer rectangles which have a common (integral) area and common perimeter.*

**Proof.** By setting $t = 1$, then $\theta = \pi/2$, and the parallelogram is a rectangle. The result is immediate from Corollary 2.2. □

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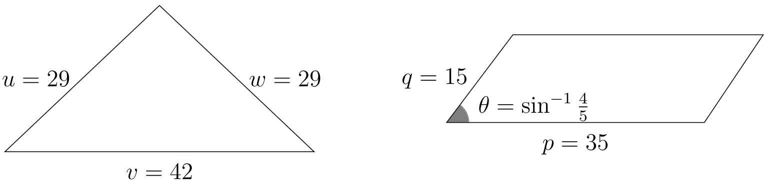


**Fig. 2.** An example of an integral isosceles triangle and parallelogram, both of which have a perimeter of 100 and an area of 420.

We can use the proof of Corollary 2.2 to produce explicit examples of the desired isosceles triangles and parallelograms. Using $X = 4t^2$ and $Y = 8t^3$, we find that (after re-scaling) the sides of the isosceles triangle may be taken to be

$$\left\{1 + 3t^2 + t^4, 1 + 3t^2 + t^4, 2(1 + t^2 + t^4)\right\},$$

and the corresponding sides of the parallelogram are

$$\left\{(1 + t + t^2)(1 + t^2), (1 - t + t^2)(1 + t^2)\right\}.$$

The perimeter of each is $4(1 + t^2)^2$, while the area is $2t(1 + t^2)(1 - t + t^2)(1 + t + t^2)$.

As a concrete example, take $t = 1/2$, or equivalently $t = 2$. The resulting integer isosceles triangle has sides $\{29, 29, 42\}$ and the $\theta$-integral parallelogram has sides $\{15, 35\}$ with angle $\theta = \sin^{-1}(4/5)$. The triangle and parallelogram each have a perimeter of 100, and an area of 420 (Fig. 2).

We similarly can set $t = 1$ to obtain an isosceles triangle and rectangle pair. In this case, the triangle has side lengths $\{5, 5, 6\}$ with a perimeter of 16 and an area of 12. The rectangle with the same perimeter and area has sides of length 2 and 6.

## 3. Integral Heron triangle and rhombus pairs

In this section, we prove the result that there are infinitely many Heron triangles and $\theta$-integral rhombus pairs which have both a common area and common perimeter. Recall that a Heron triangle is a triangle whose side lengths and area are all integers. Every Heron triangle, as proved by Brahmagupta, has the sides of the form

$$\left\{(v + w)(u^2 - vw), v(u^2 + w^2), w(u^2 + v^2)\right\},$$

where $u, v, w \in \mathbb{Z}$.

Let $p$ be the length of the side of the rhombus, and $\theta$ its smallest interior angle. As before, we require the rhombus to be $\theta$-integral, or in other words, that both $\sin\theta$ and $\cos\theta$ are rational. Hence, we may write $\sin\theta = 2t/(t^2 + 1)$, for some $0 < t \leq 1$.

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**Theorem 3.1.** *Let $Q = (X, Y)$ be a rational point on the elliptic curve*

$$E'_t : Y^2 + 4(1 + t^2)XY + 16t^2(1 + t^2)Y = X^3 \tag{5}$$

*such that*

$$4(1 + t^2)Y > X^2, \tag{6}$$

*for some rational $0 < t \leq 1$. If the point $Q$ is not of finite order, then there exist infinitely many Heron triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter for this value of $t$, i.e. with $\sin\theta = 2t/(1 + t^2)$.*

**Proof.** If we equate the perimeters and areas of the Heron triangle and $\theta$-integral rhombus, then

$$uvw(v+w)(u^2 - vw) = p^2 \sin\theta,$$

$$u^2(v + w) = 2p.$$

Combining these equations, and writing $\sin\theta = 2t/(1 + t^2)$, we get

$$uvw(v + w)(u^2 - vw) = \frac{2tu^4(v + w)^2}{4(1 + t^2)}.$$

Setting $X_1 = \frac{w}{u}$, $Y_1 = \frac{2tw}{v}$, and noting that $\frac{2tX_1}{Y_1} = \frac{v}{u}$, the above equation transforms into

$$Y_1^2 - 4(1 + t^2)X_1Y_1 + 2tY_1 = -8t(1 + t^2)X_1^3. \tag{7}$$

Multiplying by $(8t(1+t^2))^2$, and substituting $X_2 = -8t(1+t^2)X_1$ and $Y_2 = 8t(1+t^2)Y_1$, we end up with

$$E'_t : Y_2^2 + 4(1 + t^2)X_2Y_2 + 16t^2(1 + t^2)Y_2 = X_2^3. \tag{8}$$

The discriminant of $E'_t$ is

$$\Delta(t) = 2^{16}t^6(1 + t^2)^4(4t^4 - 19t^2 + 4).$$

For rational $0 < t \leq 1$, $\Delta(t) \neq 0$, hence $E'_t$ is non-singular and defines an elliptic curve. We note that the curve $E'_t$ has no obvious rational points. In comparison to $E_t$ of Theorem 2.1, the curve $E'_t$ frequently has rank 0, meaning there are not infinitely many rational points.

Working the correspondence backwards, we find that after scaling we may take $(u, v, w) = (Y_1, 2tX_1, X_1Y_1)$. Re-scaling again, some straightforward calculation leads to the sides of the Heron triangle being

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$$\left\{(Y_1+2t)(Y_1-2tX_1^2), 2tY_1(1+X_1^2), Y_1^2+4t^2X_1^2\right\},$$

and the side of the rhombus as $Y_1(Y_1+2t)/2$.

In order for the rhombus to have positive side length, we need $Y_1 > 0$ or $Y_1 < -2t$. If we require $Y_1 < -2t$, then the Heron triangle side length $2tY_1(1+X_1^2)$ is always negative. If instead we take $Y_1 > 0$, then two of the three triangle sides are necessarily positive, with the third side being $(Y_1+2t)(Y_1-2tX_1^2)$. We thus take as our condition the inequality $Y_1 > 2tX_1^2$. In terms of $X_2, Y_2$, this is

$$4(1+t^2)Y_2 > X_2^2.$$

By the same density argument as used in Theorem 2.1, we can conclude that if we have a point $(X_2, Y_2)$ of infinite order on $E_t'$ satisfying the inequality, then we will have infinitely many pairs of integer Heron triangles and $\theta$-integral rhombuses which have a common area and common perimeter. □

In contrast to Corollary 2.2, we are not able to show that for every value of $t$, $0 < t < 1$, we can produce infinitely many pairs of integer Heron triangles and rhombuses with our desired properties. The difficulty comes when the rank of $E_t'$ is 0. However, we can show there are infinitely many values of $t$ for which the hypotheses of the previous theorem are satisfied.

**Corollary 3.2.** *For infinitely many values of $t$, with $0 < t < 0.4292535$, there exist infinitely many integer Heron triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter.*

**Proof.** We begin by showing that $E_t'$ has positive rank for infinitely many rational values of $t$. Suppose we require $X' = -16t^2(t^2+1)$ to be the $x$-coordinate of a rational point on $E_t'$. This constraint is equivalent to requiring $-(48t^4+40t^2-9)$ to be square. The equation $z^2 = -48t^4 - 40t^2 + 9$ is birationally equivalent to the elliptic curve

$$E' : y^2 = x^3 - x^2 + 5x - 14,$$

via the maps

$$x = -\frac{4t^2+3z-9}{8t^2}, \quad y = \frac{3(20t^2+3z-9)}{16t^3},$$

and

$$t = \frac{3y}{2(x^2+x+7)}, \quad z = \frac{3(x^2-4x-9)}{x^2+x+7}. \tag{9}$$

The curve $E'$ has rank 1, with generator $(6, 14)$ and therefore has an infinite number of rational points $(x, y)$. From which, we obtain an infinite number of values of $t$ for which

$X'$ will be a valid $x$-coordinate of a rational point on $E'_t$. It is easy to check this point $(X', Y')$ has infinite order, showing an infinite number of values of $t$ for which $E'_t$ has positive rank.

The upper bound we use for $t$ comes from solving $-48t^4 - 40t^2 + 9 = z^2 > 0$, which results in $|t| < \sqrt{-15 + 6\sqrt{13}}/6 \approx 0.4292535$. In order for $0 < t < 0.4292535$ as desired, we simply note that $(t, z)$ and $(-t, z)$ are additive inverses on the curve $z^2 = -48t^4 - 40t^2 + 9$, as seen by the birational maps (9). Thus $(t, z)$ is of infinite order if and only if $(-t, z)$ has infinite order. We thus obtain infinitely many values for $t$, with $0 < t < 0.4292535$, such that $E'_t$ has positive rank.

The point with $x$-coordinate $X' = -16t^2(1 + t^2)$ corresponds to $X_1 = 2t$ on the curve defined by (7). We claim that the resulting point $(X_1, Y_1)$ will always satisfy the inequality $Y_1 > 2tX_1^2$, which (as we saw in the proof of the Theorem 3.1) is equivalent to (6).

The point on $E'_t$ with $x$-coordinate $X' = -16t^2(t^2 + 1)$ is $(X', Y')$, with $Y' = 8t^2(t^2 + 1)(3 + 4t^2 \pm z)$. The inequality we need to check is $X'^2 < 4(1 + t^2)Y'$, or equivalently

$$256t^4(t^2 + 1)^2 - 32t^2(t^2 + 1)^2(3 + 4t^2 \pm z) < 0,$$

which simplifies to just $4t^2 - 3 < \pm z$. We note that

$$\begin{aligned} 0 < z^2 &= -48t^4 - 40t^2 + 9 \\ &= -3(4t^2 - 3)^2 - 28(4t^2 - 3) - 48, \end{aligned}$$

and so necessarily $4t^2 - 3 < 0$. Thus, for any point $(t, z)$ satisfying $z^2 = -48t^4 - 40t^2 - 9$, the pullback to $E'_t$ (8) of either $(t, z)$ or $(t, -z)$ will satisfy $X'^2 < 4(1 + t^2)Y'$.

Recall the inequality (6) was to ensure the sides of the triangle and rhombus had positive length. We see the resulting Heron triangle corresponding to $(X', Y')$ has side lengths

$$\left\{(3 - 4t^2 + z)(5 + 4t^2 + z), 2(1 + 4t^2)(3 + 4t^2 + z), 2(3 + 4t^2)(3 - 4t^2 + z)\right\},$$

while the rhombus has side length $4(t^2 + 1)(3 - 4t^2 + z)$. The common perimeter is $16(t^2 + 1)(-4t^2 + 3 + z)$, and the common area is $32t(t^2 + 1)(-4t^2 + 3 + z)^2$.

We have shown that we have infinitely many values of $t$, with $0 < t < .4292535$ such that the curve $E'_t$ has positive rank. For each of these values of $t$, (6) will be satisfied by $(X', Y')$. We observe that $(X', Y')$ does not have finite order by specialization. For $t = 3/7$, then we can take $z = 9/49$ and so $(X', Y') = (-8352/2401, 801792/117649)$. The point $(X', Y')$ is easily checked to have infinite order on $E'_{3/7}$. The conclusion then immediately follows from Theorem 3.1. □

We remark that the choice of $X' = -16t^2(t^2 + 1)$ was not the only possible choice for a point on $E'_t$. Other choices of $X'$ which lead to a positive rank curve $E'$ would similarly

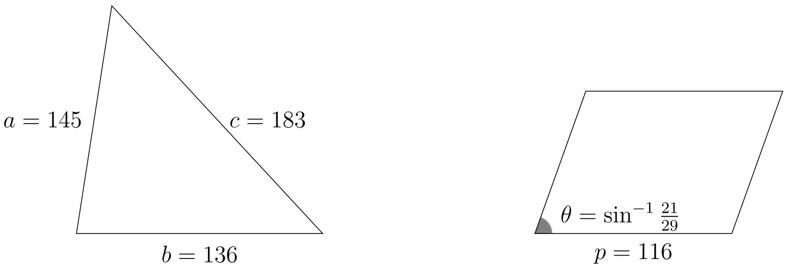


**Fig. 3.** An example of an integral Heron triangle and Rhombus, both of which have a perimeter of 464 and an area of 9744.

work as well. We chose $X'$ so that $X_1 = X'/(-8t(1+t^2))$ would have a simple expression. With another choice, it could be possible to extend the results of the corollary slightly past 0.4292535. However, for any $t$ in the interval $(3\sqrt{3} - \sqrt{11})/4 \approx 0.46988 < t \leq 1$, the discriminant of $E'_t$ satisfies $\Delta(t) < 0$ and the curve only has one connected component. A careful analysis shows the infinite (and only) component of (7) lies beneath $y = 2tx^2$. We therefore cannot find any rational points which will satisfy (6) for any $t$ in this interval.

We now give a concrete example of a Heron triangle and an integer rhombus.

**Example 3.3.** Corresponding to the generator $(6, 14)$, we get $t = 3/7$, and thus $(X_1, Y_1) = (6/7, 576/343)$. This leads to a Heron triangle with sides $(a, b, c) = (145, 136, 183)$ and the rhombus with side $p = 116$, and $\theta = \sin^{-1} = 21/29$. Both shapes have perimeter 464 and area 9744 (Fig. 3).

**Corollary 3.4.** *There is no pair of integer Heron triangle and integer square with the same area and same perimeter.*

**Proof.** We utilize the proof of Theorem 3.1. As a square has right angles, then $\sin\theta = 1$, and thus $t = 1$. We therefore put $t = 1$, in (8), and we have

$$E'_1 : y^2 + 8xy + 32y = x^3. \tag{10}$$

As computed by SAGE [10], the rank of $E'_1$ is zero and its torsion subgroup is isomorphic to $\mathbb{Z}_3$, with points of order three $(0, 0)$ and $(0, -32)$. If we map back these points via the transformations (7) and (8), we are led to $X_1 = 0$ and so $w = 0$, which is a degenerate case. Thus there is no solution. □

## 4. Conclusion

We have proved that there are infinitely many isosceles triangle and $\theta$-integral parallelogram pairs with common area and common perimeter. We have also shown that there are infinitely many Heron triangle and $\theta$-integral rhombus pairs with the same property, while there does not exist any such Heron triangle and square pair.

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In this direction, a more general question remains to be answered which asks whether there exist pairs of Heron triangles and (non-rhombus) parallelograms with a common area and common perimeter. This is also discussed in [13] and a partial solution in the affirmative is also given there.

We conclude by noting that while proving Theorem 2.1, we showed that the rank of $E_t$ is greater than or equal to 3 for infinitely many values of $t$. Therefore, it might be interesting to find whether there exist curves in this family with high rank. We performed some experiments, and the curves with the highest rank we found had rank 5.

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