Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Number Theory 115 (2005) 381–393
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

Products of Hecke eigenforms ☆

Brad A. Emmons

Department of Mathematics, Utica College, 1600 Burrstone Road, Utica, NY 13502, USA

Received 10 May 2004; revised 14 December 2004

Available online 23 September 2005

Communicated by W. Duke

Abstract

Let $f(z)$ and $g(z)$ be Hecke eigenforms for $\Gamma_0(p)$, where p is a prime. If both $f(z)$ and $g(z)$ are non-cuspidal forms and $p \geqslant 7$, then the product is a Hecke eigenform only if it comes trivially from a level 1 solution. If $g(z)$ is a cuspform and $p \geqslant 5$, then in addition to the level 1 solutions, there are 8 new cases where the product of Hecke eigenforms is a Hecke eigenform.

Keywords: Modular form; Hecke eigenform; Eisenstein series; Fricke involution

1. Introduction

For $k \geqslant 4$ even, let

$$\begin{aligned} E_k(z) &= \frac{1}{2\zeta(k)} \sum_{m,n}{}' (mz+n)^{-k} \\ &= 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \end{aligned}$$

☆ The results in this paper were completed in partial fulfillment of the author's Ph.D. at the Indiana University.

E-mail address: bemmons@utica.edu.

0022-314X/$ - see front matter
doi:10.1016/j.jnt.2005.02.006

be the normalized Eisenstein series of weight k in $M_k(\Gamma)$. The identities

$$E_4E_4 = E_8, \quad E_4E_6 = E_{10}, \quad E_4E_{10} = E_{14}, \quad E_6E_8 = E_{14} \tag{1}$$

are well-known and follow from the fact that the vector space of modular forms of weight k for the full modular group $\Gamma = SL_2(\mathbb{Z})$ is one-dimensional when $k \in \{4, 6, 8, 10, 14\}$ [5]. In addition, if we let $\Delta_l(z)$ be the unique normalized cusp form in $S_l(\Gamma)$ for $l \in \{12, 16, 18, 20, 22, 26\}$, then we get the following product identities:

$$\begin{array}{llll} E_4\Delta_{12} = \Delta_{16}, & E_4\Delta_{16} = \Delta_{20}, & E_4\Delta_{18} = \Delta_{22}, & E_4\Delta_{22} = \Delta_{26}, \\ E_6\Delta_{12} = \Delta_{18}, & E_6\Delta_{16} = \Delta_{22}, & E_6\Delta_{20} = \Delta_{26}, & E_8\Delta_{12} = \Delta_{20}, \\ E_8\Delta_{18} = \Delta_{26}, & E_{10}\Delta_{12} = \Delta_{22}, & E_{10}\Delta_{16} = \Delta_{26}, & E_{14}\Delta_{12} = \Delta_{26}. \end{array} \tag{2}$$

Duke showed that these are the only cases where the product of two non-cuspidal eigenforms is another Hecke eigenform for the full modular group Γ [2]. In [4], Ghate considered the problem of looking at products of newforms in $M_k(\Gamma_1(N))$ where N is square-free, where it is shown that the product is not an eigenform except when forced to by dimension considerations. Both results use the Rankin–Selberg method.

In this paper, we consider modular forms $f(z)$ and $g(z)$ for the congruence subgroups $\Gamma_0(p)$ where p is a prime and drop the condition that f and g are newforms. The proofs of the main theorems consider the relations on the Fourier coefficients of Hecke eigenforms. For each solution $f(z)g(z) = h(z)$ in Eqs. (1) and (2), we get a trivial solution of the form $f(pz)g(pz) = h(pz)$. The question is whether we get any solutions which are not trivial oldform solutions.

In Section 2, we consider $f(z)g(z) = h(z)$ where f, g and h are non-cuspidal Hecke eigenforms in $\Gamma_0(p)$. We say a modular form $f(z) \in M_k(\Gamma_0(N))$ is a Hecke eigenform if $f(z)$ is an eigenform for all Hecke operators $T_k(p)$ where $(p, N) = 1$.

Theorem 1. *Let $p \geqslant 7$ be a prime, and let $f(z) \in M_k(\Gamma_0(p))$ and $g(z) \in M_l(\Gamma_0(p))$ be non-cuspidal Hecke eigenforms. If $h(z) = f(z)g(z)$ is a non-cuspidal Hecke eigenform, then*

$$(l, k) \in \{(4, 4), (4, 6), (4, 10), (6, 4), (6, 8), (8, 6), (10, 4)\}.$$

Moreover, if $(l, k) \neq (4, 4)$, then $f(z) = E_k(z)$ and $g(z) = E_l(z)$ or $f(z) = E_k(pz)$ and $g(z) = E_l(pz)$. If $(l, k) = (4, 4)$, then we get a 1-parameter family of solutions of the form

$$(E_4(z) + b\,E_4(pz))(E_4(z) - b\,E_4(pz)) = E_8(z) - b^2\,E_8(pz).$$

Theorem 2. *Let $p \geqslant 5$ be a prime, and let $f(z) \in M_k(\Gamma_0(p))$ and $g(z) \in S_l(\Gamma_0(p))$ be Hecke eigenforms. If the product $h(z) = f(z)g(z)$ is a Hecke eigenform, then $h(z)$*

is an oldform solution, *or*

$$(p, k, l) \in \{(11, 4, 2), (7, 2, 6), (7, 4, 4), (5, 2, 4), (5, 2, 8), (5, 4, 4), (5, 4, 6), (5, 6, 4)\}$$

in which case there is a unique solution.

The proofs of both theorems make use of the fact that if $f(z)$ is a non-cuspidal eigenform of weight k for $\Gamma_0(p)$ then $f(z) = a\, E_k(z) + b\, E_k(p\, z)$ for constants a and b, see [6].

When considering the dimension of the vector space $S_{k+l}(\Gamma_0(p))$ we would expect only the trivial oldform solutions. However, $S_k(\Gamma_0(p))$ decomposes into eigenspaces of the Fricke involution w_p,

$$S_k(\Gamma_0(p)) = S_k^+(\Gamma_0(p)) \oplus S_k^-(\Gamma_0(p))$$

and one of the $S_k^{\pm}$ may be one-dimensional. In that case, let $g_k^{\pm}(z)$ be the unique normalized cuspform in the respective space.

If we have a pair of spaces $S_l^{\pm}$ and $S_{k+l}^{\pm}$ which are one-dimensional, and if $k \geqslant 4$ we can construct

$$f(z) = E_k(z) \pm p^{k/2} E_k(p\, z) \tag{3}$$

which is in $M_k(\Gamma_0(p))$ and is an eigenvector of the Fricke involution. And so necessarily we have the identity

$$\left(E_k(z) \pm p^{k/2} E_k(p\, z)\right) g_l^{\pm}(z) = g_{k+l}^{\pm}(z). \tag{4}$$

If $k = 2$ we may only construct

$$f(z) = E_k(z) - p^{k/2} E_k(p\, z)$$

and so the signs in $S_l^{\pm}$ and $S_{2+l}^{\mp}$ must be of opposite parity.

For example, the spaces $S_2^-(\Gamma_0(11))$ and $S_6^+(\Gamma_0(11))$ are one-dimensional. We need to multiply the weight 2 cusp form $g_2^-(z)$ by a weight 4 modular form in $M_4^-(\Gamma_0(11))$ to get the weight 6 cusp form $g_6^+(z)$. The weight 4 modular form is

$$f(z) = E_4(z) - 121\, E_4(11\, z),$$

which accounts for the solution $(p, k, l) = (11, 4, 2)$. All of the non-trivial solutions in Theorem 2 are obtained this way, where the relevant spaces are

$$\begin{gathered} S_2^+(\Gamma_0(11)),\ S_6^+(\Gamma_0(11)),\ S_4^+(\Gamma_0(7)),\ S_8^-(\Gamma_0(7)),\ S_6^+(\Gamma_0(7)), \\ S_4^+(\Gamma_0(5)),\quad S_6^-(\Gamma_0(5)),\quad S_8^-(\Gamma_0(5)),\ S_{10}^+(\Gamma_0(5)). \end{gathered}$$

2. Proof of Theorem 1

We are looking for solutions to the equation

$$f(z)g(z) = h(z),$$

where f, g and h are non-cuspidal Hecke eigenforms. In what follows we would like to be able to normalize the factors $f(z)$ and $g(z)$ so that the coefficients of q are both 1. It is not immediately obvious that we would be able to do this. The following lemma shows us that it is possible for the cases we want.

Lemma 1. *Let $p > 2$ be a prime and let $f(z) \in M_k(\Gamma_0(p))$ and $g(z) \in M_l(\Gamma_0(p))$ be non-cuspidal eigenforms such that $f(z)g(z)$ is an eigenform. Then $f(z), g(z)$ and $f(z)g(z)$ are normalizable, or $w_p(f)(z), w_p(g)(z)$ and $w_p(fg)(z)$ are normalizable.*

Proof. Let $f(z), g(z)$ and $h(z)$ be as in the statement of the lemma. Then

$$f(z) = a\,E_k(z) + b\,E_k(p\,z),$$

$$g(z) = c\,E_l(z) + d\,E_l(p\,z)$$

and

$$h(z) = \alpha\,E_{k+l}(z) + \beta\,E_{k+l}(p\,z),$$

where $f(z)g(z) = h(z)$. Both $f(z)$ and $g(z)$ are normalizable when $a \neq 0$ and $c \neq 0$. If either $a = 0$ or $c = 0$, we consider the effect of the involution w_p on the product, where

$$w_p(f)(z) = f|_k \left[\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \right](z).$$

If $f(z)$ is not normalizable, then $w_p(f)(z)$ is normalizable.

If both $a = 0$ and $c = 0$, then applying the involution makes $w_p(f)$ and $w_p(g)$ both normalizable.

The problem occurs when $b = 0$ and $c = 0$. Here $g(z)$ is not normalizable, but when we apply the involution to the product, $w_p(f)$ is no longer normalizable. But in this case, $f(z) = E_k(z)$ and $g(z) = E_l(p\,z)$, and so

$$\begin{aligned} f(z)g(z) &= E_k(z)E_l(p\,z) \\ &= \left(D_k + q + \sigma_{k-1}(2)\,q^2 + \cdots\right)\left(D_l + q^p + \cdots\right) \\ &= D_k\,D_l + D_l\,q + D_l\,\sigma_{k-1}(2)\,q^2 + \cdots, \end{aligned}$$

where

$$D_k = \frac{(k-1)!\zeta(k)}{(2\pi i)^k}.$$

By comparing this product with

$$\begin{aligned} h(z) &= \alpha\, E_{k+l}(z) + \beta\, E_{k+l}(p\, z) \\ &= (\alpha+\beta)\ D_{k+l} + \alpha\, q + \alpha\, \sigma_{k+l-1}(2)\, q^2 + \cdots, \end{aligned}$$

we see that $\alpha = D_l$ and so $\sigma_{k-1}(2) = \sigma_{k+l-1}(2)$. This is only true if $l = 0$. Thus we have that either $f(z)$ and $g(z)$ are both normalizable, or both $w_p(f)(z)$ and $w_p(g)(z)$ are normalizable. In either case, when we consider the product

$$f(z)g(z) = D_k\, D_l + (D_k + D_l)\ q + \cdots,$$

this can only fail to be normalizable if $D_k = -D_l$ for some k and l. Since this is never true, the product $h(z)$ is also normalizable. $\square$

We may now assume that $f(z), g(z)$ and $h(z)$ are all normalizable in what follows. We multiply

$$\begin{aligned} f(z) &= E_k(z) + b\, E_k(p\, z) \\ &= x + \sum_{n=1}^{6} \sigma_{k-1}(n)\, q^n + O(q^7) \end{aligned}$$

and

$$\begin{aligned} g(z) &= E_l(z) + c\, E_l(p\, z) \\ &= y + \sum_{n=1}^{6} \sigma_{l-1}(n)\, q^n + O(q^7) \end{aligned}$$

and compare the coefficients of the product to that of

$$\begin{aligned} \alpha\, h(z) &= \alpha\ (E_{k+l}(z) + d\, E_{k+l}(p\, z)) \\ &= z + \alpha \sum_{n=1}^{6} \sigma_{k+l-1}(n)\, q^n + O(q^7). \end{aligned}$$

Since $p \geqslant 7$, the constants b, c and d will only make a contribution to the constant coefficients of f, g and h, and the coefficients of the terms of higher order.

We get a system of five equations in x, y, k, l and α. We see that $\alpha = (x + y)$, and so we can set up a linear system in x and y. This system of equations only has a solution when the pair (l, k) are as in the statement of the theorem. By substituting each of these pairs, except for $(4, 4)$, into the original equations we get values for the constant coefficients of $f(z)$ and $g(z)$. In each case, we note that these values are exactly the values of the constant coefficients of the Eisenstein series. Hence $f(z) = E_k(z)$ and $g(z) = E_l(z)$ for the cases when $k \neq l$.

For the case $k = l = 4$, consider the equation

$$(E_4(z) + b\,E_4(p\,z))(E_4(z) + c\,E_4(p\,z)) = \alpha(E_8(z) + d\,E_8(p\,z)).$$

The constant coefficients of the factors on the left-hand side are

$$x = \frac{1}{240}(1 + b), \quad y = \frac{1}{240}(1 + c).$$

By comparing the q coefficients we have $\alpha = x + y$, and by comparing the q^2 coefficients, we have

$$120\,x + 120\,y = 1.$$

Hence $c = -b$. This gives us a 1 parameter family of solutions of the form

$$(E_4(z) + b\,E_4(p\,z))(E_4(z) - b\,E_4(p\,z)) = E_8(z) - b^2\,E_8(p\,z)$$

as claimed. This completes the proof of Theorem 1. $\square$

3. Proof of Theorem 2

If

$$f(z) = E_k(z) + b\,E_k(p\,z)$$

is a normalized non-cuspidal eigenform, then the normalized image under the Fricke involution is

$$w_p(f)(z) = E_k(z) + b^{-1}\,p^k E_k(p\,z).$$

The constant coefficient of the non-cuspidal eigenform may therefore be taken as either $(1 + b)D_k$ or $(1 + b^{-1}p^k)D_k$ in what follows.

The following lemma gives a finite list of possible weight combinations for the non-cuspidal eigenform and the cuspidal eigenform.

Lemma 2. *Let $p \geqslant 5$ be a prime, and let $f(z) \in M_k(\Gamma_0(p))$ and $g(z) \in S_l(\Gamma_0(p))$ be Hecke eigenforms. If $f(z)g(z)$ is an eigenform, then one of the following holds: $k = 2$ and $l \leqslant 20$, $k = 4$ and $l \leqslant 24$, $k = 6$ and $l \leqslant 10$, $k = 8$ and $l \leqslant 18$, $k = 10$ and $l \leqslant 16$, or $k = 14$ and $l \leqslant 12$.*

Proof. If

$$f(z)\, g(z) = h(z), \tag{5}$$

where $g(z) = q + \sum_{n=2}^{\infty} a_n\, q^n$ and $h(z) = q + \sum_{n=2}^{\infty} b_n\, q^n$, then

$$\alpha\,(D_k\, \theta + q + \cdots)(q + a_2\, q^2 + \cdots) = (q + b_2\, q^2 + \cdots),$$

where $\theta = (1 + b)$ or $\theta = (1 + b^{-1} p^k)$. So $\alpha = \frac{1}{D_k\,\theta}$, and $\alpha(1 + a_2\, D_k\, \theta) = b_2$, so $\frac{1}{D_k\,\theta} + a_2 = b_2$. Since a_2 and b_2 are algebraic integers, $\frac{1}{D_k\,\theta}$ is an algebraic integer. When $k \notin \{2, 4, 6, 8, 10, 14\}$, there is a prime ℓ such that $\mathrm{ord}_\ell(D_k) > 0$. It can be shown that $\mathrm{ord}_\ell((D_k\theta)^{-1}) < 0$ for $\theta = (1+b)$ or $(1+b^{-1}p^k)$. This contradicts the fact that $(D_k\theta)^{-1}$ is an algebraic integer. Hence, the weight k of the non-cuspidal eigenform must be in $\{2, 4, 6, 8, 10, 14\}$, as claimed.

Since g is a normalized cusp form of weight l, we have

$$g(z) = q + a_2\, q^2 + a_3\, q^3 + a_4\, q^4 + a_5\, q^5 + \cdots.$$

Since g is an eigenform for $T_l(2)$, $a_2^2 = a_4 + 2^{l-1}$. And

$$f(z) = x + q + (1 + 2^{k-1})\, q^2 + (1 + 3^{k-1})\, q^3 + \cdots,$$

where $x = D_k\, \theta$. So

$$\begin{aligned} f(z)g(z) &= x\, q + (1 + a_2\, x\,)q^2 + ((1 + 2^{k-1}) + a_2 + a_3\, x)\, q^3 \\ &\quad + ((1 + 3^{k-1}) + a_2\,(1 + 2^{k-1}) + a_3 + a_4\, x)\, q^4 + \cdots \end{aligned}$$

and hence

$$\begin{aligned} T_{k+l}(2)(f(z)\, g(z)) &= (1 + a_2\, x)\, q \\ &\quad +((1 + 3^{k-1}) + a_2\,(1 + 2^{k-1}) + a_3 + a_4\, x + 2^{k+l-1}\, x)\, q^2 + \cdots . \end{aligned}$$

Since $f(z)\, g(z)$ is an eigenform for $T_{k+l}(2)$ with eigenvalue $\frac{1+a_2\,x}{x}$, we have

$$\frac{(1 + a_2\, x)^2}{x} = (1 + 3^{k-1}) + a_2\,(1 + 2^{k-1}) + a_3 + (a_2^2 - 2^{l-1})\, x + 2^{k+l-1}\, x.$$

Solving for 2^l we obtain

$$\begin{aligned} 2^l &= \frac{2}{x\,(1-2^k)}\left(\left(\frac{-1}{x}\right) + a_2\,(2^{k-1}-1) + (1+3^{k-1}) + a_3\right) \\ &\leqslant \left|\frac{2}{x\,(1-2^k)}\right| \left(\left|\frac{1}{x}\right| + \left|a_2\,(2^{k-1}-1)\right| + \left|1+3^{k-1}\right| + |a_3|\right) \\ &\leqslant \frac{8}{|x|\,(2^k-1)} \sup\left(\left|\frac{1}{x}\right|, |a_2|\,(2^{k-1}-1), (1+3^{k-1}), |a_3|\right). \end{aligned} \tag{6}$$

By the Weil–Petersson estimate, for $p' \neq p$, we have $\left|a_{p'}\right| \leqslant 2\,p'^{\frac{k-1}{2}}$ [1]. Hence, if we compare 2^l to each term in (6), we get the following inequalities:

$$\left(\frac{2}{\sqrt{3}}\right)^l \leqslant \frac{16}{\sqrt{3}\,|D_k|\,(2^k-1)}, \tag{7}$$

$$(\sqrt{2})^l \leqslant \frac{8\sqrt{2}}{|D_k|}, \tag{8}$$

$$2^l \leqslant \frac{8}{|D_k|^2\,(2^k-1)} \tag{9}$$

and

$$2^l \leqslant \frac{8}{|D_k|\,(2^k-1)}(3^k+1). \tag{10}$$

By substituting each $k \in \{2, 4, 6, 8, 10, 14\}$ into Eqs. (7)–(10) we get an upper bound for the weight of the cusp form for each k. And by substitution, we determine that the second line of (6) is not actually satisfied for $22 \leqslant l \leqslant 28$, $26 \leqslant l \leqslant 34$, $22 \leqslant l \leqslant 38$, $20 \leqslant l \leqslant 24$, $18 \leqslant l \leqslant 22$, or $14 \leqslant l \leqslant 16$, respectively. So we get the bounds as in the statement of the lemma and this completes the proof of Lemma 2. □

The following lemma tells us that as long as the level $p \geqslant 17$, we have no new non-trivial level p solutions.

Lemma 3. *Let $f(z)$ and $g(z)$ be as in Lemma 2. If $p \geqslant 17$ and $f(z)g(z)$ is an eigenform, then the solution comes from a level 1 solution.*

Proof. Since $p \geqslant 17$,

$$f(z) = D_k\,(1+b) + q + \sum_{i=1}^{16} \sigma_{k-1}(n)\,q^n + O(q^{17}) \tag{11}$$

and $g(z) = q + \sum_{n=2}^{\infty} a_n\, q^n$ where the coefficients satisfy the identities

$$\begin{aligned} a_{n\,m} &= a_n\, a_m \quad \text{ if } \gcd(n, m) = 1, \\ a_{\ell^r}\, a_\ell &= a_{\ell^{r+1}} + \ell^{k-1}\, a_{\ell^{r-1}} \quad \text{ for } \ell \text{ prime, } \ell \neq p. \end{aligned} \tag{12}$$

Since $p \geqslant 17$ we can write the first 16 coefficients a_n in terms of the a_{p_i}, where the p_i are the primes less than or equal to 13.

If we multiply $f(z)$ by $g(z)$, and normalize so that the q coefficient is 1, we get another expression

$$h(z) = \frac{f(z)\, g(z)}{D_k\,(1+b)} = q + \sum_{n=2}^{\infty} b_n\, q^n,$$

where each b_n is an expression involving b and a_{p_i} for all primes p_i such that $p_i \leqslant n$. Also, the coefficients b_n must satisfy the relations in (12) the same way the coefficients of $g(z)$ did. Each one of these relations gives us an equation, and each prime p_i gives us a new unknown a_{p_i}. By the q^{15} term, we get 8 equations, and 7 unknowns. We can include the coefficient of q^{16} and get another equation.

The solutions to this system have (k, l, b) consistent with the known level 1 solutions, except for

$$(k, l, b) \in \{(2, 12, 0), (4, 4, 1), (4, 6, 0)\}.$$

The others all have $b = 0$, hence the non-cuspidal eigenform is just the level 1 Eisenstein series $E_k(z)$. The cuspidal eigenforms $g(z)$ each agree with the unique level 1 cusp form Δ_l for $l \in \{12, 16, 18, 20, 22, 26\}$ up to the q^{13} (and hence the q^{16}) coefficient. To show that our results actually give Δ_l and not some other cusp form whose coefficients agree up the 16th coefficient, we have the following lemma:

Lemma 4. *Let* $l \in \{12, 16, 18, 20, 22, 26\}$, *and* $(N, 2) = 1$, *and let* $g(z) \in S_l(\Gamma_0(N))$. *If* $g(z)$ *and* $E_k(z)g(z)$ *are eigenforms*, *and* $\operatorname{ord}(g - \Delta_l) > 2$, *then* $g(z) = \Delta_l(z)$.

Proof. For the values of l given, and for $k \in \{4, 6, 8, 10, 14\}$, we have for $E_k(z) = D_k + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$ that $E_k(z)\,\Delta_l(z) = D_k\,\Delta_{k+l}(z)$. We let

$$g(z) = q + \sum_{n=2}^{\infty} a_n\, q^n,$$

$$\Delta_l(z) = q + \sum_{n=2}^{\infty} b_n\, q^n,$$

$$\Delta_{k+l}(z) = q + \sum_{n=2}^{\infty} c_n\, q^n$$

and

$$E_k(z)\, g_l(z) = D_k \left(q + \sum_{n=2}^{\infty} d_n\, q^n \right).$$

Suppose that $g \neq \Delta_l$, and let $m = \operatorname{ord}(g - \Delta_l) < \infty$. So for all $n < m$, we have $a_n = b_n$. Since $m > 2$ and $\gcd(N, 2) = 1$, m must be a power of an odd prime. By comparing coefficients in the equation $E_k(z)\, \Delta_l(z) = D_k\, \Delta_{k+l}(z)$ we have for all n,

$$D_k\, c_n = D_k\, b_n + \sum_{i=1}^{n-1} \sigma_{k-1}(i)\, b_{n-i}. \tag{13}$$

And from $E_k(z)\, g_l(z) = D_k \left(q + \sum_{n=2}^{\infty} d_n\, q^n \right)$, we get

$$D_k\, d_n = D_k\, a_n + \sum_{i=1}^{n-1} \sigma_{k-1}(i)\, a_{n-i}. \tag{14}$$

Since $a_n = b_n$ for all $n < m$, we see that $c_n = d_n$ for all $n < m$ as well. Since m must be a power of some odd prime, $m+1$ must be even, in which case we can write $m + 1 = 2^r \cdot s$, where s is odd and $r \geqslant 1$. If $s = 1$, then we have

$$b_{m+1} = b_{2^r} = b_{2^{r-1}}\, b_2 - 2^l\, b_{2^{r-1}} = a_{2^{r-1}}\, a_2 - 2^l\, a_{2^{r-1}} = a_{m+1},$$

since $g(z)$ and $\Delta_l(z)$ are eigenforms for $T_l(2)$ and $a_n = b_n$ for $n < m$. Similarly, we have $c_{m+1} = d_{m+1}$ since both Δ_{k+l} and $E_k(z)\, g(z)$ are eigenforms for $T_{k+l}(2)$. And if $s > 1$, then we have

$$b_{m+1} = b_{2^r s} = b_{2^r}\, b_s = a_{2^r}\, a_s = a_{m+1}$$

and also $c_{m+1} = d_{m+1}$. By substituting $n = m + 1$ in Eqs. (13) and (14), we have

$$D_k\, c_{m+1} = D_k\, b_{m+1} + \sum_{i=1}^{m} \sigma_{k-1}(i)\, b_{m+1-i}$$

and

$$D_k\, d_{m+1} = D_k\, a_{m+1} + \sum_{i=1}^{m} \sigma_{k-1}(i)\, a_{m+1-i},$$

and hence we have $b_m = a_m$, which contradicts our assumption that $m = \mathrm{ord}(g - \Delta_l)$. Thus we must have $g(z) = \Delta_l(z)$. $\square$

The solutions

$$(k, l, b) \in \{(2, 12, 0), (4, 4, 1), (4, 6, 0)\}$$

do not correspond to a level 1 solution, and I claim that they do not give us new solutions. The solution $(k, l, b) = (2, 12, 0)$ indicates that the non-cuspidal eigenform $f(z)$ is exactly the weight 2 Eisenstein series $f(z) = E_2(z)$, and the cusp form in each case is the weight 12 cusp form for the full modular group $g(z) = \Delta_{12}(z)$, by Lemma 4. But

$$q \left(\frac{d}{dq}\right) \Delta_{12}(z) = -24\, E_2(z) \Delta_{12}(z). \tag{15}$$

In general, if f is a modular form $q \frac{d}{dq} f$ is not a modular form [7]. Hence $k = 2, l = 12$, and $b = 0$ do not give solutions to the problem.

The solution $(k, l, b) = (4, 4, 1)$ gives us the equation

$$(E_4(z) + E_4(p\, z))\, (E_4(z) - E_4(p\, z)) = E_8(z) - E_8(p\, z),$$

where the first factor is indicated by the fact that $k = 4$ and $b = 1$, and the second factor has no constant coefficient in its q-expansion at ∞, hence it looked like a cusp form to our computations.

The solution $(k, l, b) = (4, 6, 0)$ can be realized by applying $q \left(\frac{d}{dq}\right)$ to both sides of $E_4\, E_4 = E_8$. This gives us

$$2\, E_4 \left(q \frac{d}{dq}\right) E_4 = \left(q \frac{d}{dq}\right) E_8,$$

which is the solution that we obtained, where $q \frac{d}{dq} E_4$ and $q \frac{d}{dq} E_8$ are multiplicative, but not modular forms. This completes the proof of Lemma 3. $\square$

For levels $p = 5, 7, 11$ and 13, the systems of equations will be almost identical to the system of equations we had for the level $p \geqslant 17$ case, except $E_k(p\, z)$ will contribute to the $q^{p\, m}$ terms. Each of these low primes have all of the solutions which are the known level 1 solutions. Similar to the levels $p \geqslant 17$ case, we need to rule out the possibility that a new solution agrees with these known solutions up to the q^{16} coefficient. This follows from Lemma 4. Most of the new solutions will not give new product identities for the same reason they did not give product identities for the $p \geqslant 17$ case.

There are two new solutions which do not give product identities which must be addressed. The case $(p, k, l, b) = (11, 2, 2, 11)$ does not give a new solution. Here

$$f(z) = E_2(z) + 11\, E_2(11\, z)$$

and

$$g(z) = \eta(z)^2 \eta(11\, z)^2,$$

which is a cusp form for $\Gamma_0(11)$ [3]. Here the product

$$f(z)\, g(z) = -\tfrac{1}{2}\, g'(z),$$

where $g'(z)$ is multiplicative, but not modular. Hence this is not a solution.

Similarly it can be shown that the solution where $(p, k, l, b) = (5, 2, 4, 5)$ is $f(z) = E_2(z) + 5\, E_2(5\, z)$, $g(z) = \eta(z)^4\, \eta(5\, z)^4$, and

$$f(z)g(z) = -\tfrac{1}{4}\, g'(z)$$

where $g'(z)$ is multiplicative, but not modular. Hence this is not a solution.

The remaining 8 solutions each give a new case where the product of two Hecke eigenforms is another eigenform. The argument for why this is so is given in the introduction. $\square$

4. New product identities

In the previous section we obtained 8 new product identities for prime levels $p \geqslant 5$. The discussion in the introduction can be used to find solutions to the problem for levels 2 and 3. The following is a list of vector spaces of cusp forms which are one-dimensional [8] and hence consist of Hecke eigenforms.

$$S_6^-(\Gamma_0(3)),\;\; S_8^+(\Gamma_0(3)),\;\; S_{10}^-(\Gamma_0(3)),\;\; S_{10}^+(\Gamma_0(3)),\;\; S_{14}^+(\Gamma_0(3)).$$

For pairs of vector spaces whose weights differ by at least 4, and for pairs whose weights differ by 2 with opposite parity, we can construct an eigenform in $M_k(\Gamma_0(p))$ as in (3). If we let $g_l^{\pm}(z)$ be the unique normalized cusp form in $S_l^{\pm}(\Gamma_0(p))$, then we get eight new product identities similar to that in equation (4).

Similarly in level 2 the following vector spaces of cusps forms are one-dimensional [8], and hence consist entirely of eigenforms:

$$\begin{gathered} S_8^+(\Gamma_0(2)),\;\; S_{10}^-(\Gamma_0(2)),\;\; S_{12}^+(\Gamma_0(2)),\\ S_{12}^-(\Gamma_0(2)),\;\; S_{14}^+(\Gamma_0(2)),\;\; S_{14}^-(\Gamma_0(2)). \end{gathered}$$

Similar to the level 3 discussion, we have 10 new level 2 identities.

It is interesting to note that we have 2-level 2 solutions where the non-cuspidal eigenform is of weight 4, and the cusp form is of weight 8. Each of the factors and the product can be identified in terms of known functions, and we get the interesting identities

$$(E_4(z) - 4\,E_4(2\,z)) \cdot \left(\eta^8(z)\eta^8(2\,z)\right) = -3\ (\Delta_{12}(z) - 64\,\Delta_{12}(2\,z))$$

and

$$(E_4(z) + 4\,E_4(2\,z)) \cdot \left(\eta^8(z)\eta^8(2\,z)\right) = 5\ (\Delta_{12}(z) + 64\,\Delta_{12}(2\,z))\,.$$

The eigenforms $E_4(z) - 4\,E_4(2\,z)$ and $E_4(z) + 4\,E_4(2\,z)$ span the set of all non-cuspidal eigenforms of weight 4 and level 2. Since they are equivalent eigenforms and the product of each one of these with $\eta^8(z)\eta^8(2\,z)$ is an eigenform in $S_{12}(\Gamma_0(2))$ which are themselves equivalent, any linear combination $a\,E_4(z) + b\,E_4(2\,z)$ times $\eta^8(z)\eta^8(2\,z)$ is an eigenform in $S_{12}(\Gamma_0(2))$.

Acknowledgments

B.A. Emmons is especially grateful to his advisor Michael Larsen for his help during the preparation of this work. He also thank the anonymous referee for the corrections and stylistic suggestions.

References

[1] P. Deligne, La conjecture de Weil: I, Inst. Hautes Études Sci. Publ. Math. (43) (1974) 273–307.
[2] W. Duke, When is the product of two Hecke eigenforms an eigenform? in: K. Gyory, H. Iwaniec, J. Urbanowicz (Eds.), Number Theory in Progress, vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 737–741.
[3] D. Dummit, H. Kisilevsky, J. McKay, Multiplicative products of η-functions, in: J. McKay (Ed.), Finite groups—Coming of Age (Montreal, Que., 1982), Contemporary Mathematics, vol. 45, American Mathematical Society, Providence, RI, 1985.
[4] E. Ghate, On products of eigenforms, Acta Arith. 102 (1) (2002) 27–44.
[5] A.W. Knapp, Elliptic Curves, Princeton University Press, Princeton, NJ, 1992.
[6] B. Mazur, Modular curves and the Eisenstein ideal, Inst. Hautes Études Sci. Publ. Math. (47) (1977) 33–186.
[7] S. Ramanujan, On certain arithmetical functions, Trans. Cambridge Philos. Soc. (1916) 159–184.
[8] W. Stein, The Modular Forms Database, `http://modular.fas.harvard.edu/Tables`, 2003.