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Some new approximations of Glaisher–Kinkelin's and Bendersky–Adamchik's constants by continued fraction

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In this paper, some new continued fraction approximations, inequalities and rates of convergence of Glaisher–Kinkelin's and Bendersky–Adamchik's constants are provided. Finally, for demonstrating the superiority of our new convergent sequences over the classical sequences and Mortici's sequences, some numerical computations are also given.

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1. Introduction

In the theory of mathematical constants, an important concern is the definition of new sequences which converge to these fundamental constants with increasingly higher speed. These convergent sequences and constants play a key role in many areas of mathematics and science in general, as theory of probability, applied statistics, physics, special functions, number theory, or analysis.

One of the most useful convergent sequence in mathematics is

$$w_n = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln(n) + \frac{n^2}{4}, \tag{1.1}$$

which converges towards the well-known Glaisher–Kinkelin's constant $\ln(A)$ and $A \approx 1.282427130\ldots$.

This constant appeared in Barnes [2]. Up to now, it has been computed to great depth, which means that there must be some way to approximate it to high accuracy at modest computational cost. One such approach would be by way of Euler–Maclaurin summation. A good reference is Brendan McKay's article [1], available in the "Talk" page associated with the (defective in its discussion of error terms) Wikipedia page on Euler Maclaurin summation.

It is not hard to see that

$$\ln(A) - w_n = \sum_{k=n+1}^{\infty}\left[\frac{2k-1}{4} + \left(\frac{k^2-k}{2} + \frac{1}{12}\right)\ln(1-1/k)\right].$$

Replacing $\ln(1-1/k)$ with its series expansion and rearranging gives

$$\ln(A) - w_n = \sum_{j=3}^{\infty}\sum_{k=n+1}^{\infty} k^{-j}\left(-\frac{1}{12j} + \frac{1}{2(j+1)} - \frac{1}{2(j+2)}\right). \tag{1.2}$$

This last expression can be estimated term by term via Euler–Maclaurin techniques. The problem of estimating $\sum_{k=n}^{\infty} k^{-j}$ is known to history as the 'Basel problem' and it was for the express purpose of tackling it that Euler invented the method. (Maclaurin hit upon it separately, and with a different purpose in mind.) The upshot of this approach is that

$$\ln(A) - w_n = \frac{-1}{720}n^{-2} + \frac{1}{5040}n^{-4} - \frac{1}{10\,080}n^{-6} + \frac{1}{9504}n^{-8} + O[n^{-10}].$$

The expansion can be taken to arbitrary depth.

Related to Glaisher–Kinkelin's constant, the following sequences are defined,

$$s_n = \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln(n) + \frac{n^3}{9} - \frac{n}{12} \tag{1.3}$$

and

$$t_n = \sum_{k=1}^{n} k^3 \ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln(n) + \frac{n^4}{16} - \frac{n^2}{12}, \tag{1.4}$$

which converge towards the well-known Bendersky–Adamchik's constant $\ln(B)$ and $\ln(C)$, where $B \approx 1.03091675\ldots$ and $C \approx 0.97955746\ldots$. These two constants were considered by Choi and Srivastava in [3–5] in the theory of multiple gamma functions.

Up until now, many researchers made great efforts in the area of concerning the rate of convergence of the sequences $(w_n)_{n\geq 1}$, $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, and establishing sequences which converge faster to Glaisher–Kinkelin's and Bendersky–Adamchik's constants and had a lot of inspiring results. For example, in [13], Mortici provided some new inequalities for these constants as follows:

$$w_n - \frac{1}{720n^2} + \frac{1}{5040n^4} - \frac{1}{10\,080n^6} < \ln(A) < w_n - \frac{1}{720n^2} + \frac{1}{5040n^4}, \tag{1.5}$$

$$s_n + \frac{1}{360n} - \frac{1}{7560n^3} < \ln(B) < s_n + \frac{1}{360n}, \tag{1.6}$$

$$t_n + \frac{1}{5040n^2} - \frac{1}{33\,600n^4} < \ln(C) < t_n + \frac{1}{5040n^2}. \tag{1.7}$$

In addition, the method using continued fraction to approximate the well known constants often appeared in many literatures. For example, Mortici [12] and [14] used continued fraction to approximate gamma function. Lu [7] provided approximation of Euler's constant by using continued fraction. From above works, we can see that the approximations which used continued fraction have quicker convergence rates than the others which used polynomials. In view of this fact, in this paper, using continued fraction approximation, we provide some quicker convergent sequences for Glaisher–Kinkelin's and Bendersky–Adamchik's constants as follows:

Theorem 1.1. *For Glaisher–Kinkelin's constant, we have*

$$\ln(A) \approx w_n + \frac{1}{n} \frac{a_1}{n + \frac{a_2}{n + \frac{a_3}{n + \cdots}}}, \tag{1.8}$$

where

$$a_1 = -\frac{1}{720}, \; a_2 = \frac{1}{7}, \; a_3 = \frac{5}{14}, \ldots.$$

For Bendersky–Adamchik's constants, we have

$$\ln(B) \approx s_n + \frac{b_1}{n + \frac{b_2}{n + \frac{b_3}{n + \cdots}}} \tag{1.9}$$

and

$$\ln(C) \approx t_n + \frac{1}{n}\frac{c_1}{n + \frac{c_2}{n+\frac{c_3}{n+\cdots}}}, \tag{1.10}$$

where

$$b_1 = \frac{1}{360},\ b_2 = \frac{1}{21},\ b_3 = \frac{53}{210}, \ldots$$

and

$$c_1 = \frac{1}{5040},\ c_2 = \frac{3}{20},\ c_3 = \frac{703}{1980}, \ldots.$$

Remark 1.1. Let $c[r] = -1/(12r) + 1/(2(r+1)(r+2))$. A proof that the desired asymptotic continued fraction-type expansion exists for $\ln(A) - w_n$ would be at hand if only the following identity could be established: for integer q with $q \geq 2$,

$$\frac{1}{2q+1}c[2q+2] - \frac{1}{2}c[2q+1] + \frac{1}{2q+1}\sum_{s=1}^{q-1} B_{2s}\binom{2q+1}{2s}c[2q+2-2s] = 0. \tag{1.11}$$

However we do not provide a direct proof and more work needs to be done in this direction. It also seems a challenging problem to give a proof of the equality (1.11).

Next, using Theorem 1.1, we provide some inequalities for Glaisher–Kinkelin's and Bendersky–Adamchik's constants.

Theorem 1.2. *For all natural numbers $n \geq 1$, we have*

$$w_n + \frac{1}{n}\frac{-\frac{1}{720}}{n + \frac{\frac{1}{7}}{n+\frac{\frac{5}{14}}{n}}} < \ln(A) < w_n + \frac{1}{n}\frac{-\frac{1}{720}}{n + \frac{\frac{1}{7}}{n}}; \tag{1.12}$$

$$s_n + \frac{\frac{1}{360}}{n + \frac{\frac{1}{21}}{n}} < \ln(B) < s_n + \frac{\frac{1}{360}}{n + \frac{\frac{1}{21}}{n+\frac{\frac{53}{210}}{n}}}; \tag{1.13}$$

$$t_n + \frac{1}{n}\frac{\frac{1}{5040}}{n + \frac{\frac{3}{20}}{n}} < \ln(C) < t_n + \frac{1}{n}\frac{\frac{1}{5040}}{n + \frac{\frac{3}{20}}{n+\frac{\frac{703}{1980}}{n}}}. \tag{1.14}$$

Finally, to show that the three continued fraction approximations convergence faster, combining Theorems 1.1 and 1.2, we provide the rates of convergence of these three sequences as follows:

Theorem 1.3. *For all natural numbers $n \geq 2$, we have*

$$\frac{1}{14\,112(n+1)^6} < w_n + \frac{1}{n}\frac{-\frac{1}{720}}{n+\frac{\frac{1}{7}}{n}} - \ln(A) < \frac{1}{14\,112(n-1)^6}; \tag{1.15}$$

$$\frac{53}{1\,587\,600(n+1)^5} < \ln(B) - s_n - \frac{\frac{1}{360}}{n+\frac{\frac{1}{21}}{n}} < \frac{53}{1\,587\,600(n-1)^5}; \tag{1.16}$$

$$\frac{703}{66\,528\,000(n+1)^6} < \ln(C) - t_n - \frac{1}{n}\frac{\frac{1}{5040}}{n+\frac{\frac{3}{20}}{n}} < \frac{703}{66\,528\,000(n-1)^6}. \tag{1.17}$$

To obtain Theorem 1.1, we need the following lemma which was used in [8–12] and very useful for constructing asymptotic expansions.

Lemma 1.1. *If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{1.18}$$

with $s > 1$, then

$$\lim_{n\to\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{1.19}$$

Lemma 1.1 was first proved by Mortici in [11]. From Lemma 1.1, we can see that the speed of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value s satisfying (1.18).

The rest of this paper is arranged as follows: In Section 2, we provide the proof of Theorem 1.1. In Section 3, the proof of Theorem 1.2 is given. In Section 4, we complete the proof of Theorem 1.3. In Section 5, we give some numerical computations which demonstrate the superiority of our new convergent sequences over the classical sequences and Mortici's sequence.

2. Proof of Theorem 1.1

First, we deal with (1.8). Based on the argument of Theorem 2.1 in [12] or Theorem 5 in [14], we need to find the value $a_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$w_n^{(1)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln(n) + \frac{n^2}{4} + \frac{a_1}{n^2}. \tag{2.1}$$

To measure the accuracy of this approximation, a method is to say that an approximation (2.1) is better if $w_n^{(1)} - \ln(A)$ converges to zero faster. Using (2.1) and developing the power series in $1/n$, we have

$$w_n^{(1)} - w_{n+1}^{(1)} = \frac{720a_1+1}{360n^3} + \frac{-720a_1-1}{240n^4} + \frac{840a_1+1}{210n^5} + O\left(\frac{1}{n^6}\right). \tag{2.2}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(w_n^{(1)} - \ln(A))_{n\geq 1}$ is even higher as the value s satisfying (1.18). Thus, using Lemma 1.1, we have:

(i) If $a_1 \neq -1/720$, then the rate of convergence of the sequence $(w_n^{(1)} - \ln(A))_{n\geq 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2(w_n^{(1)} - \ln(A)) = \frac{720a_1+1}{720} \neq 0.$$

(ii) If $a_1 = -1/720$, then from (2.2), we have

$$w_n^{(1)} - w_{n+1}^{(1)} = -\frac{1}{1260}\frac{1}{n^5} + O\left(\frac{1}{n^6}\right)$$

and the rate of convergence of the sequence $(w_n^{(1)} - \ln(A))_{n\geq 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4(w_n^{(1)} - \ln(A)) = -\frac{1}{5040n^4}.$$

We know that the fastest possible sequence $(w_n^{(1)})_{n\geq 1}$ is obtained only for $a_1 = -1/720$.

Next, we define the sequence $(w_n^{(2)})_{n\geq 1}$ by the relation

$$w_n^{(2)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln(n) + \frac{n^2}{4} + \frac{1}{n}\frac{-\frac{1}{720}}{n+\frac{a_2}{n}}. \tag{2.3}$$

Using the same method from (2.1) to (2.2), we have

$$w_n^{(2)} - w_{n+1}^{(2)} = \frac{7a_2-1}{1260n^5} + \frac{1-7a_2}{504n^6} + \frac{140a_2-17-42a_2^2}{5040n^7} + O\left(\frac{1}{n^8}\right). \tag{2.4}$$

The fastest possible sequence $(w_n^{(2)})_{n\geq 1}$ is obtained only for $a_2 = 1/7$. Then, from (2.4), we have

$$w_n^{(2)} - w_{n+1}^{(2)} = \frac{1}{2352n^7} + O\left(\frac{1}{n^8}\right)$$

and the rate of convergence of the sequence $(w_n^{(2)} - \ln(A))_{n\geq 1}$ is n^{-6}, since

$$\lim_{n\to\infty} n^6(w_n^{(2)} - \ln(A)) = \frac{1}{14\,112}.$$

Similarly, we have $a_3 = 5/14, \cdots$, the new sequence (1.8) is obtained.

Next, we deal with (1.9). We need to find the value $b_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$s_n^{(1)} = \sum_{k=1}^{n} k^2 \ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \ln(n) + \frac{n^3}{9} - \frac{n}{12} + \frac{b_1}{n}. \tag{2.5}$$

Using (2.5) and developing the power series in $1/n$, we have

$$s_n^{(1)} - s_{n+1}^{(1)} = \frac{360b_1 - 1}{360n^2} + \frac{1 - 360b_1}{360n^3} + \frac{420b_1 - 1}{420n^4} + O\left(\frac{1}{n^5}\right). \tag{2.6}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(s_n^{(1)} - \ln(B))_{n\geq 1}$ is even higher as the value s satisfying (1.18). Thus, using Lemma 1.1, we have:

(i) If $b_1 \neq 1/360$, then the rate of convergence of the sequence $(s_n^{(1)} - \ln(B))_{n\geq 1}$ is n^{-1}, since

$$\lim_{n\to\infty} n(s_n^{(1)} - \ln(B)) = \frac{360b_1 - 1}{360} \neq 0.$$

(ii) If $b_1 = 1/360$, then from (2.6), we have

$$s_n^{(1)} - s_{n+1}^{(1)} = \frac{1}{2520n^4} + O\left(\frac{1}{n^5}\right)$$

and the rate of convergence of the sequence $(s_n^{(1)} - \ln(B))_{n\geq 1}$ is n^{-3}, since

$$\lim_{n\to\infty} n^3(s_n^{(1)} - \ln(B)) = \frac{1}{7560}.$$

We know that the fastest possible sequence $(s_n^{(1)})_{n\geq 1}$ is obtained only for $b_1 = 1/360$.

Similarly, we have $b_2 = 1/21$, $b_3 = 53/210, \cdots$, and the new sequence (1.9) is obtained.

Finally, we deal with (1.10). We need to find the value $c_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$t_n^{(1)} = \sum_{k=1}^{n} k^3 \ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln(n) + \frac{n^4}{16} - \frac{n^2}{12} + \frac{c_1}{n^2}. \tag{2.7}$$

Using the same method from (2.5) to (2.6), we have

$$t_n^{(1)} - t_{n+1}^{(1)} = \frac{5040c_1 - 1}{2520n^3} + \frac{1 - 5040c_1}{1680n^4} + \frac{100\,800c_1 - 17}{25\,200n^5} + O\left(\frac{1}{n^6}\right). \tag{2.8}$$

The fastest possible sequence $(t_n^{(1)})_{n\geq 1}$ is obtained only for $c_1 = 1/5040$. Then, from (2.8), we have

$$t_n^{(1)} - t_{n+1}^{(1)} = \frac{1}{8400n^5} + O\left(\frac{1}{n^6}\right)$$

and the rate of convergence of the sequence $(t_n^{(1)} - \ln(C))_{n\geq 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4(t_n^{(1)} - \ln(C)) = \frac{1}{33\,600}.$$

Similarly, we have $c_2 = 3/20$, $c_3 = 703/1980, \cdots$, and the new sequence (1.10) is obtained.

3. Proof of Theorem 1.2

First, we deal with (1.12). Let

$$w_n^{(3)} = w_n + \frac{1}{n}\frac{-\frac{1}{720}}{n + \frac{\frac{1}{7}}{n+\frac{\frac{5}{14}}{n}}}, \quad w_n^{(2)} = w_n + \frac{1}{n}\frac{-\frac{1}{720}}{n + \frac{\frac{1}{7}}{n}}.$$

Since $w_n^{(3)}$, $w_n^{(2)}$ converge to $\ln(A)$, we only need to show that $(w_n^{(3)})_{n\geq 1}$ is strictly increasing and $(w_n^{(2)})_{n\geq 1}$ is strictly decreasing.

Let $f_A(x) = w_x^{(3)} - w_{x+1}^{(3)}$, $g_A(x) = w_x^{(2)} - w_{x+1}^{(2)}$. By some calculations, we have

$$f_A'''(x) = \frac{F_A(x)}{210x^5(x+1)(2x^2+1)^4(2x^3+6x^2+7x+3)^4} > 0,$$

$$g_A'''(x) = -\frac{G_A(x)}{30x^3(x+1)^3(7x^2+1)^4(7x^2+14x+8)^4} < 0,$$

where

$$\begin{aligned} F_A(x) &= 23\,680x^{14} + 165\,760x^{13} + 556\,384x^{12} + 1\,183\,424x^{11} + 1\,792\,624x^{10} \\ &\quad + 2\,065\,680x^9 + 1\,891\,520x^8 + 1\,417\,184x^7 + 885\,344x^6 + 465\,184x^5 \\ &\quad + 204\,710x^4 + 73\,500x^3 + 20\,655x^2 + 4185x + 405, \\ G_A(x) &= 37\,059\,435x^{12} + 222\,356\,610x^{11} + 583\,202\,900x^{10} + 877\,745\,575x^9 \\ &\quad + 839\,828\,983x^8 + 538\,765\,192x^7 + 241\,633\,896x^6 + 79\,832\,221x^5 \\ &\quad + 20\,927\,802x^4 + 4\,831\,498x^3 + 1\,021\,440x^2 + 143\,360x + 20\,480. \end{aligned}$$

Combining $f_A''(\infty) = 0$, $g_A''(\infty) = 0$ and $f_A'''(x) > 0$, $g_A'''(x) < 0$, we have $f_A''(x) < 0$, $g_A''(x) > 0$ for $x \geq 1$. Thus, $f_A(x)$ is strictly concave, and $g_A(x)$ is strictly convex. Combining $f_A(\infty) = 0$ and $g_A(\infty) = 0$, we obtain $f_A(x) < 0$ and $g_A(x) > 0$ for $x \geq 1$. The proof of (1.12) is completed.

Next, we deal with (1.13). Let

$$s_n^{(2)} = s_n + \frac{\frac{1}{360}}{n + \frac{\frac{1}{21}}{n}}, \quad s_n^{(3)} = s_n + \frac{\frac{1}{360}}{n + \frac{\frac{1}{21}}{n + \frac{\frac{53}{210}}{n}}}.$$

We only need to show that $(s_n^{(2)})_{n\geq 1}$ is strictly increasing and $(s_n^{(3)})_{n\geq 1}$ is strictly decreasing. Let $f_B(x) = s_x^{(2)} - s_{x+1}^{(2)}$, $g_B(x) = s_x^{(3)} - s_{x+1}^{(3)}$. By similar calculations, we have $f_B'''(x) > 0$ and $g_B'''(x) < 0$. Combining $f_B''(\infty) = 0$, $g_B''(\infty) = 0$ and $f_B'''(x) > 0$, $g_B'''(x) < 0$, we have $f_B''(x) < 0$, $g_B''(x) > 0$ for $x \geq 1$. Thus, $f_B(x)$ is strictly concave, and $g_B(x)$ is strictly convex. Combining $f_B(\infty) = 0$ and $g_B(\infty) = 0$, we obtain $f_B(x) < 0$ and $g_B(x) > 0$ for $x \geq 1$. The proof of (1.13) is completed.

Finally, we deal with (1.14). Let

$$t_n^{(2)} = t_n + \frac{1}{n}\frac{\frac{1}{5040}}{n + \frac{\frac{3}{20}}{n}}, \quad t_n^{(3)} = t_n + \frac{\frac{1}{5040}}{n + \frac{\frac{3}{20}}{n + \frac{\frac{703}{1980}}{n}}}.$$

We only need to show that $(t_n^{(2)})_{n\geq 1}$ is strictly increasing and $(t_n^{(3)})_{n\geq 1}$ is strictly decreasing. Let $f_C(x) = t_x^{(2)} - t_{x+1}^{(2)}$, $g_C(x) = t_x^{(3)} - t_{x+1}^{(3)}$. By similar calculations, we have $f_C'''(x) > 0$ and $g_C'''(x) < 0$. Combining $f_C''(\infty) = 0$, $g_C''(\infty) = 0$ and $f_C'''(x) > 0$, $g_C'''(x) < 0$, we have $f_C''(x) < 0$, $g_C''(x) > 0$ for $x \geq 1$. Thus, $f_C(x)$ is strictly concave, and $g_C(x)$ is strictly convex. Combining $f_C(\infty) = 0$ and $g_C(\infty) = 0$, we obtain $f_C(x) < 0$ and $g_C(x) > 0$ for $x \geq 1$. The proof of (1.14) is completed.

4. Proof of Theorem 1.3

First, we prove (1.15). Based on the argument of Theorem 1 in [6], it is easy to have

$$w_n^{(2)} - \ln(A) = \sum_{k=n}^{\infty}(w_n^{(2)} - w_{n+1}^{(2)}) = \sum_{k=n}^{\infty} f_w(k). \tag{4.1}$$

By some calculations, we have

$$f_w'(x) = -F_w(x)/G_w(x), \tag{4.2}$$

where

$$G_w(x) = 360x(x+1)(7x^2+1)^2(7x^2+14x+8)^2$$

and

$$
\begin{aligned}
F_w(x) = {} & 1920 + 600\,663x^3 + 29\,809x - 74\,880\ln(x+1)x^2 + 864\,360\ln(x)x^{11} \\
& + 136\,762x^2 - 15\,188\,040\ln(x+1)x^8 - 7\,717\,500\ln(x+1)x^6 \\
& + 3\,582\,180\ln(n)x^5 + 864\,360x^{10} + 4\,321\,800x^9 + 9\,209\,550x^8 \\
& + 10\,907\,400x^7 + 1\,330\,560\ln(x)x^4 + 11\,520\ln(x)x - 4\,753\,980\ln(x+1)x^{10} \\
& - 1\,330\,560\ln(x+1)x^4 + 15\,188\,040\ln(x)x^8 + 1\,833\,174x^4 + 4\,288\,431x^5 \\
& + 8\,104\,257x^6 + 12\,947\,760\ln(x)x^7 - 12\,947\,760\ln(x+1)x^7 + 74\,880\ln(x)x^2 \\
& - 360\,720\ln(x+1)x^3 - 11\,520\ln(x+1)x - 864\,360\ln(x+1)x^{11} \\
& + 7\,717\,500\ln(x)x^6 - 11\,298\,420\ln(x+1)x^9 - 3\,582\,180\ln(x+1)x^5 \\
& + 360\,720\ln(x)x^3 + 4\,753\,980\ln(x)x^{10} + 11\,298\,420\ln(x)x^9.
\end{aligned}
$$

By some calculations, we have

$$G_w(x) - 336x^8F_w(x) > 0 \tag{4.3}$$

as $x \ge 1$. For the upper bound in (1.15), we have

$$-f_w'(x) = \frac{F_w(x)}{G_w(x)} < \frac{1}{336x^8}. \tag{4.4}$$

Since $f_w(\infty) = 0$, we have

$$f_w(k) = -\int_k^\infty f_w'(x)dx < \frac{1}{336}\int_k^\infty x^{-8}dx = \frac{1}{2352}k^{-7} < \frac{1}{2352}\int_{k-1}^{k} x^{-7}dx. \tag{4.5}$$

Combining (4.1) and (4.5), for all natural numbers $n \ge 2$, we have

$$w_n^{(2)} - \ln(A) < \sum_{k=n}^{\infty}\frac{1}{2352}\int_{k-1}^{k} x^{-7}dx = \frac{1}{2352}\int_{n-1}^{\infty} x^{-7}dx = \frac{1}{14\,112(n-1)^6}. \tag{4.6}$$

For the lower bound, combining (4.2), we have

$$-f_w'(x) = \frac{F_w(x)}{G_w(x)} > \frac{1}{336(x+1)^8}, \tag{4.7}$$

where we use the following fact, for $x \ge 1$,

$$G_w(x) - 336(x+1)^8F_w(x) < 0. \tag{4.8}$$

Combining (4.7), we have

$$f_w(k) = -\int_k^\infty f_w'(x)dx > \frac{1}{336}\int_k^\infty (x+1)^{-8}dx = \frac{1}{2352}(k+1)^{-7} > \frac{1}{2352}\int_{k+1}^{k+2} x^{-7}dx. \tag{4.9}$$

Combining (4.1) and (4.9), we have

$$w_n^{(2)} - \ln(A) > \sum_{k=n}^\infty \frac{1}{2352}\int_{k+1}^{k+2} x^{-7}dx = \frac{1}{2352}\int_{n+1}^\infty x^{-7}dx = \frac{1}{14\,112(n+1)^6}. \tag{4.10}$$

Combining (4.6) and (4.10), we complete the proof of (1.15).

Next, we prove (1.16). It is easy to have

$$\ln(B) - s_n^{(2)} = \sum_{k=n}^\infty (s_{k+1}^{(2)} - s_k^{(2)}) = \sum_{k=n}^\infty f_s(k). \tag{4.11}$$

For the upper bound in (1.16), by similar calculation, we have

$$-f_s'(x) < \frac{53}{52\,920x^7}, \tag{4.12}$$

for $x \geq 1$. Since $f_s(\infty) = 0$, we have

$$f_s(k) = -\int_k^\infty f_s'(x)dx < \frac{53}{52\,920}\int_k^\infty x^{-7}dx = \frac{53}{317\,520}k^{-6} < \frac{53}{317\,520}\int_{k-1}^k x^{-6}dx. \tag{4.13}$$

Combining (4.11) and (4.13), for all natural numbers $n \geq 2$, we have

$$\ln(B) - s_n^{(2)} < \sum_{k=n}^\infty \frac{53}{317\,520}\int_{k-1}^k x^{-6}dx = \frac{53}{317\,520}\int_{n-1}^\infty x^{-6}dx = \frac{53}{1\,587\,600(n-1)^5}. \tag{4.14}$$

For the lower bound, by similar calculation we have

$$-f_s'(x) > \frac{53}{52\,920(x+1)^7}, \tag{4.15}$$

for $x \geq 1$. Combining (4.15), we have

$$f_s(k) = -\int_k^\infty f_s'(x)dx > \frac{53}{52\,920}\int_k^\infty (x+1)^{-7}dx = \frac{53}{317\,520}(k+1)^{-6} > \frac{53}{317\,520}\int_{k+1}^{k+2} x^{-6}dx. \tag{4.16}$$

Combining (4.11) and (4.16), we have

$$\ln(B) - s_n^{(2)} > \sum_{k=n}^{\infty} \frac{53}{317\,520} \int_{k+1}^{k+2} x^{-6}dx = \frac{53}{317\,520} \int_{n+1}^{\infty} x^{-6}dx = \frac{53}{1\,587\,600(n+1)^5}. \tag{4.17}$$

Combining (4.14) and (4.17), we complete the proof of (1.16).

Finally, we prove (1.17).

$$\ln(C) - t_n^{(2)} = \sum_{k=n}^{\infty}(t_{k+1}^{(2)} - t_k^{(2)}) = \sum_{k=n}^{\infty} f_t(k). \tag{4.18}$$

For the upper bound in (1.17), by some calculation, we have

$$-f_t'(x) < \frac{703}{1\,584\,000x^8}, \tag{4.19}$$

for $x \geq 1$. Since $f_t(\infty) = 0$, we have

$$f_t(k) = -\int_k^{\infty} f_t'(x)dx < \frac{703}{1\,584\,000} \int_k^{\infty} x^{-8}dx = \frac{703}{11\,088\,000}k^{-7} < \frac{703}{11\,088\,000} \int_{k-1}^{k} x^{-7}dx. \tag{4.20}$$

Combining (4.18) and (4.20), for all natural numbers $n \geq 2$, we have

$$\ln(C) - t_n^{(2)} < \sum_{k=n}^{\infty} \frac{703}{11\,088\,000} \int_{k-1}^{k} x^{-7}dx = \frac{703}{11\,088\,000} \int_{n-1}^{\infty} x^{-7}dx = \frac{703}{66\,528\,000(n-1)^6}. \tag{4.21}$$

For the lower bound, by similar calculation, we have

$$-f_t'(x) > \frac{703}{1\,584\,000(x+1)^8}, \tag{4.22}$$

for $x \geq 1$. Combining (4.22), we have

$$\begin{aligned} f_t(k) &= -\int_k^{\infty} f_t'(x)dx > \frac{703}{1\,584\,000} \int_k^{\infty} (x+1)^{-8}dx \\ &= \frac{703}{11\,088\,000}(k+1)^{-7} > \frac{703}{11\,088\,000} \int_{k+1}^{k+2} x^{-7}dx. \end{aligned} \tag{4.23}$$

Table 1
Simulations for w_n, W_n and $w_n^{(2)}$.

n	$\frac{w_n - \ln(A)}{\ln(A)}$	$\frac{W_n - \ln(A)}{\ln(A)}$	$\frac{w_n^{(2)} - \ln(A)}{\ln(A)}$
10	5.5754×10^{-5}	3.9466×10^{-10}	2.8087×10^{-10}
25	8.9314×10^{-6}	1.6308×10^{-12}	1.1642×10^{-12}
50	2.2332×10^{-6}	2.5513×10^{-14}	1.8221×10^{-14}
100	5.5833×10^{-7}	3.9877×10^{-16}	2.8483×10^{-16}
250	8.9334×10^{-8}	1.6335×10^{-18}	1.1668×10^{-18}
1000	5.5834×10^{-9}	3.9881×10^{-22}	2.8487×10^{-22}

Combining (4.18) and (4.23), we have

$$\ln(C) - t_n^{(2)} > \sum_{k=n}^{\infty} \frac{703}{11\,088\,000} \int_{k+1}^{k+2} x^{-7} dx = \frac{703}{11\,088\,000} \int_{n+1}^{\infty} x^{-7} dx = \frac{703}{66\,528\,000(n+1)^6}. \tag{4.24}$$

Combining (4.21) and (4.24), we complete the proof of (1.17).

5. Numerical computation

In this section, we give three tables to demonstrate the superiority of our new convergent sequences

$$w_n^{(2)} = w_n + \frac{1}{n} \frac{-\frac{1}{720}}{n + \frac{\frac{1}{7}}{n}}, \quad s_n^{(2)} = s_n + \frac{\frac{1}{360}}{n + \frac{\frac{1}{21}}{n}}, \quad t_n^{(2)} = t_n + \frac{1}{n} \frac{\frac{1}{5040}}{n + \frac{\frac{3}{20}}{n}}$$

over the classical sequences w_n, s_n, t_n, and Mortici's sequences

$$W_n = w_n - \frac{1}{720n^2} + \frac{1}{5040n^4}, \quad S_n = s_n + \frac{1}{360n} - \frac{1}{7560n^3},$$
$$T_n = t_n + \frac{1}{5040n^2} - \frac{1}{33\,600n^4},$$

respectively.

Combining Theorem 1.1, Theorem 1.2 and Theorem 1.3, we have Table 1, Table 2 and Table 3.

In conclusion, we assert that the use of continued fractions in the problem of approximating the constants of Glaisher–Kinkelin type is more adequate than the use of classical asymptotic series as in Mortici [13], since more accurate approximations are obtained.

Moreover, sequences in (1.8)–(1.10) were obtained by using a step-by-step procedure. We propose as an open problem the finding of an systematical method to present the general term of the sequences which define the continued fractions (1.8)–(1.10).

Table 2
Simulations for s_n, S_n and $s_n^{(2)}$.

n	$\frac{ln(B)-s_n}{ln(B)}$	$\frac{ln(B)-S_n}{ln(B)}$	$\frac{ln(B)-s_n^{(2)}}{ln(B)}$
10	9.1186×10^{-3}	1.2935×10^{-8}	1.0868×10^{-8}
25	3.6489×10^{-3}	1.3329×10^{-10}	1.1211×10^{-10}
50	1.8245×10^{-3}	4.1692×10^{-12}	3.5072×10^{-12}
100	9.1228×10^{-4}	1.3032×10^{-13}	1.0963×10^{-13}
250	3.6492×10^{-4}	1.3345×10^{-15}	1.1227×10^{-15}
1000	9.1229×10^{-5}	1.3033×10^{-18}	1.0964×10^{-18}

Table 3
Simulations for t_n, T_n and $t_n^{(2)}$.

n	$\frac{ln(C)-t_n}{-ln(C)}$	$\frac{ln(C)-T_n}{-ln(C)}$	$\frac{ln(C)-t_n^{(2)}}{-ln(C)}$
10	9.5911×10^{-5}	7.2008×10^{-10}	5.0429×10^{-10}
25	1.5365×10^{-5}	2.9755×10^{-12}	2.0905×10^{-12}
50	3.8419×10^{-6}	4.6552×10^{-14}	3.2721×10^{-14}
100	9.6053×10^{-7}	7.2761×10^{-16}	5.1149×10^{-16}
250	1.5369×10^{-7}	2.9805×10^{-18}	2.0953×10^{-18}
1000	9.6054×10^{-9}	7.2768×10^{-22}	5.1156×10^{-22}

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References

[1] http://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula.
[2] E.W. Barnes, The theory of the G-function, Quart. J. Math. 31 (1899) 264–314.
[3] J. Choi, A set of mathematical constants arising naturally in the theory of the multiple Gamma functions, Abstr. Appl. Anal. 2012 (2012) 121795, http://dx.doi.org/10.1155/2012/121795.
[4] J. Choi, H.M. Srivastava, Certain classes of series involving the zeta function, J. Math. Anal. Appl. 231 (1999) 91–117.
[5] J. Choi, H.M. Srivastava, Certain classes of series associated with the zeta function and multiple Gamma functions, J. Comput. Appl. Math. 118 (2000) 87–109.
[6] D.W. DeTemple, A quicker convergences to Euler's constant, Amer. Math. Monthly 100 (5) (1993) 468–470.
[7] D. Lu, A new quicker sequence convergent to Euler's constant, J. Number Theory 136 (2014) 320–329.
[8] C. Mortici, Very accurate estimates of the polygamma functions, Asymptot. Anal. 68 (3) (2010) 125–134.
[9] C. Mortici, A quicker convergence toward the gamma constant with the logarithm term involving the constant e, Carpathian J. Math. 26 (1) (2010) 86–91.
[10] C. Mortici, On new sequences converging towards the Euler–Mascheroni constant, Comput. Math. Appl. 59 (8) (2010) 2610–2614.

[11] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly 117 (5) (2010) 434–441.
[12] C. Mortici, A new Stirling series as continued fraction, Numer. Algorithms 56 (1) (2011) 17–26.
[13] C. Mortici, Approximating the constants of Glaisher–Kinkelin type, J. Number Theory 133 (2013) 2465–2469.
[14] C. Mortici, A continued fraction approximation of the gamma function, J. Math. Anal. Appl. 402 (2013) 405–410.