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A near-optimal solution to the Gauss–Kuzmin–Lévy problem for θ-expansions

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A B S T R A C T

Chakraborty and Rao [4] considered the θ-expansions of numbers in $[0, \theta)$, where $0 < \theta < 1$. A Wirsing-type approach to the Perron–Frobenius operator of the generalized Gauss map under its invariant measure allows us to study the optimality of the convergence rate. Actually, we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss–Kuzmin–Lévy problem.

1. Introduction

Motivated by problems in random number generation, the study initiated by Bhattacharya and Goswami [1] leads to the interesting concept of θ-expansions of numbers in $[0, \theta)$, where $0 < \theta < 1$. We mention that the case $\theta = 1$ refers to regular continued fraction (RCF) expansions. Note that in the literature there exist several generalizations of the RCF expansions (see [6,17,11] for some background information).

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Our aim here is to give a solution to the Gauss–Kuzmin–Lévy problem for θ-expansions.

The outline of this paper is as follows. We start with a quick review of the theory of RCFs in Section 2. Among other things, we describe the dynamical system given by the Gauss map on the unit interval and mention some classical results of Kuzmin [12] and Lévy [16]. In Section 3 we describe the θ-expansions extensively studied by Chakraborty and Rao [4] and Chakraborty and Dasgupta [3]. We present the problem concerning the symbolic dynamics of the generalized Gauss map and the existence of an absolutely continuous invariant probability. It is only recently that Sebe and Lascu [24] proved the first Gauss–Kuzmin theorem for θ-expansions. The solution presented is based on the ergodic behavior of a certain random system with complete connections. Following the treatment in the case of the RCF, the Gauss–Kuzmin–Lévy problem for the new transformation can be approached in terms of the associated Perron–Frobenius operator [8]. In Section 4 we focus our study on the Perron–Frobenius operator under the invariant measure induced by the limit distribution function. In Section 5 we use as in [22,23] a Wirsing-type approach [25] to get close to the optimal convergence rate. The strategy is to restrict the domain of the Perron–Frobenius operator to the Banach space of functions which have a continuous derivative on $[0, \theta]$. Actually, in Theorem 5.3 of Section 5 we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss–Kuzmin–Lévy problem. The last section collects some concluding remarks.

2. Generalities on RCF expansions

Classically, the idea of continued fractions evolved as a method of representing positive real numbers by means of a terminating or non-terminating sequence of positive integers. The RCF representation of numbers in the unit interval is closely connected with the following dynamical system. Let $I = [0, 1]$ and consider the *RCF transformation* (or *Gauss map*) $T : I \to I$ defined as

$$T(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{2.1}$$

where $\lfloor \cdot \rfloor$ stands for integer part. Any irrational $0 < x < 1$ can be written as the infinite RCF

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} := [a_1, a_2, a_3, \ldots], \tag{2.2}$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$. Writing T^n for the n-th iterate of T, where $n \in \mathbb{N}$, with T^0 being the identity map, the positive integers $a_n(x) = a_1(T^{n-1}(x))$, $n \in \mathbb{N}_+$, with $a_1(x) = \left\lfloor \frac{1}{x} \right\rfloor$, are the RCF *digits* (also known as *incomplete quotients*) of x.

One of the major discoveries in the context of the above dynamical system was a result of Gauss' investigation (around 1800) of the measure $\lambda\left((T^n)^{-1}\right)$ on I, where λ denotes the Lebesgue measure on I. Gauss wrote that (in modern notation)

$$\lim_{n\to\infty} \lambda\left(T^n \leq x\right) = \frac{\log(1+x)}{\log 2}, \quad x \in I. \tag{2.3}$$

In 1812 Gauss asked Laplace [2] to estimate the n-th error term $e_n(x)$ defined by

$$e_n(x) = \lambda(T^{-n}[0,x]) - \frac{\log(1+x)}{\log 2}, \quad n \geq 1,\ x \in I. \tag{2.4}$$

This has been called *Gauss' problem*. The first known proof of this is due to Kuzmin [12], who showed in 1928 that $e_n(x) = \mathcal{O}(q^{\sqrt{n}})$ as $n \to \infty$, uniformly in x with some (unspecified) $0 < q < 1$. This has been called the Gauss–Kuzmin theorem. Lévy [16] improved Kuzmin's result by showing that $|e_n(x)| \leq q^n$ for $n \in \mathbb{N}_+$, $x \in I$, with $q = 0.67157\ldots$. For such historical reasons, the Gauss–Kuzmin–Lévy theorem is regarded as the first basic result in the rich metrical theory of RCFs. An advantage of the Gauss–Kuzmin–Lévy theorem relative to the Gauss–Kuzmin theorem is the determination of the value of q.

Recall that the Gauss measure γ on I defined by

$$\mathrm{d}\gamma(x) = \frac{1}{\log 2}\frac{\mathrm{d}x}{1+x} \tag{2.5}$$

is invariant for the transformation T, i.e., $\gamma = \gamma T^{-1}$, and the dynamical system (I, T) is ergodic under γ.

In the results of Kuzmin and Lévy the constants are far from optimal. In 1974 a decisive step in the final solution of Gauss' problem was taken by Wirsing [25]. Full proofs of these results and all the details in the classical theory are discussed by Iosifescu and Kraaikamp [8].

Apart from the RCF expansion there are many other continued fraction expansions. Such a development has led to the appearance of various studies on the Gauss' problem for non-RCFs. We mention just a few recent as [10,13,14,19–21].

3. θ-expansions and the generalized Gauss map

For a fixed $\theta \in (0,1)$, Chakraborty and Rao [4] have considered a generalization of the Gauss map (2.1), $T_\theta : [0,\theta] \to [0,\theta]$ defined as

$$T_\theta(x) := \begin{cases} \dfrac{1}{x} - \theta\left\lfloor \dfrac{1}{x\theta}\right\rfloor & \text{if } x \in (0,\theta], \\ 0 & \text{if } x = 0. \end{cases} \tag{3.1}$$

Note that the classical Gauss map is just the special case $\theta = 1$. The transformation T_θ is connected with the θ-expansion for a number in $(0, \theta)$ as follows. The numbers $\theta \left\lfloor \frac{1}{y\theta} \right\rfloor$ obtained by taking y successively equal to x, $T_\theta(x)$, $T_\theta^2(x)$, ..., lead to the θ-expansion of x as

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \ddots}}} = [a_1\theta, a_2\theta, a_3\theta \ldots], \tag{3.2}$$

where $a_n \in \mathbb{N}_+$. The positive integers $a_n = a_n(x) = a_1(T_\theta^{n-1}(x))$, $n \in \mathbb{N}_+$, with $T_\theta^0(x) = x$ and $a_1 = a_1(x) = \left\lfloor \frac{1}{x\theta} \right\rfloor$ are also called *digits* or *incomplete quotients* of x with respect to the θ-expansion in (3.2).

It was shown in [4] that this expansion has many of the usual properties of RCFs. A natural question is whether the dynamical system given by the transformation T_θ admits an absolutely continuous invariant probability like the Gauss measure (2.5) in the case $\theta = 1$. Chakraborty and Rao [4] have identified that for certain values of θ (for example, if $\theta^2 = 1/m$, $m \in \mathbb{N}_+$) the invariant measure for the generalized Gauss transformation T_θ as

$$\mathrm{d}\gamma_\theta = \frac{1}{\log(1+\theta^2)} \frac{\theta \mathrm{d}x}{1 + x\theta}. \tag{3.3}$$

Moreover, if $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, $[a_1\theta, a_2\theta, a_3\theta \ldots]$ is the θ-expansion of any $x \in (0, \theta)$ if and only if the following conditions hold:

(i) $a_n \geq m$ for any $m \in \mathbb{N}_+$;
(ii) in case when x has a finite expansion, i.e., $x = [a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta]$, then $a_n \geq m + 1$.

It was proved in [4] that the dynamical system $([0, \theta], T_\theta)$ is ergodic and the measure γ_θ is invariant under T_θ, that is, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0,\theta]}$, where $\mathcal{B}_{[0,\theta]}$ denotes the σ-algebra of all Borel subsets of $[0, \theta]$.

Similar to classical results on RCFs, using the ergodicity of T_θ and Birkhoff's ergodic theorem [5], a number of results were obtained in [4]. It should be stressed that the ergodic theorem does not yield any information on the convergence rate in the Gauss problem that amounts to the asymptotic behavior of $\mu((T_\theta)^{-n})$ as $n \to \infty$, where μ is an arbitrary probability measure on $\mathcal{B}_{[0,\theta]}$. So, that a Gauss–Kuzmin theorem is needed.

Until now, the estimate of the convergence rate remains an open question. The first attempt of a version of a Gauss–Kuzmin theorem was made by Sebe and Lascu [24]. Using the natural extension for θ-expansions, we obtained an infinite-order-chain representation of the sequence of the incomplete quotients of these expansions. Together with the ergodic behavior of a certain homogeneous random system with complete connections (see [7] for the general theory), this allowed us to solve a variant of the Gauss–Kuzmin problem. Then Lascu and Nicolae [15] obtained another solution to this problem applying the method of Rockett and Szüsz [18]. We emphasize that an important tool in these approaches is the Perron–Frobenius operator under the invariant measure γ_θ.

4. The associated Perron–Frobenius operator

Let μ be a probability measure on $([0,\theta], \mathcal{B}_{[0,\theta]})$ such that $\mu((T_\theta)^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for any $A \in \mathcal{B}_{[0,\theta]}$. In particular, this condition is satisfied if T_θ is μ-preserving, that is, $\mu(T_\theta)^{-1} = \mu$. Let

$$L^1_\mu := \{f : [0,\theta] \to \mathbb{C} : \int_0^\theta |f| \mathrm{d}\mu < \infty\}.$$

The *Perron–Frobenius operator* of T_θ under μ is defined as the bounded linear operator U_μ which takes the Banach space L^1_μ into itself and satisfies the equation

$$\int_A U_\mu f \,\mathrm{d}\mu = \int_{(T_\theta)^{-1}(A)} f \,\mathrm{d}\mu \quad \text{for all } A \in \mathcal{B}_{[0,\theta]},\ f \in L^1_\mu. \tag{4.1}$$

Throughout the paper we will assume that $\theta^2 = 1/m$, $m \in \mathbb{N}_+$. Recall a result obtained in [24].

Proposition 4.1.

(i) The Perron–Frobenius operator $U := U_{\gamma_\theta}$ of T_θ under the invariant probability measure γ_θ is given a.e. in $[0,\theta]$ by the equation

$$Uf(x) = \sum_{j \geq m} P_j(x) f(u_j(x)), \quad m \in \mathbb{N}_+,\ f \in L^1_{\gamma_\theta}, \tag{4.2}$$

where

$$P_j(x) := \frac{x\theta + 1}{(x + j\theta)(x + (j+1)\theta)} \quad \text{and} \quad u_j(x) := \frac{1}{x + j\theta},$$

with $j \geq m$ and $x \in [0,\theta]$.

(ii) Let μ be a probability measure on $([0,\theta], \mathcal{B}_{[0,\theta]})$ such that μ is absolutely continuous with respect to the Lebesgue measure λ_θ on $([0,\theta], \mathcal{B}_{[0,\theta]})$ and let $h := d\mu/d\lambda_\theta$ a.e. in $[0,\theta]$. For any $n \in \mathbb{N}$ and $A \in \mathcal{B}_{[0,\theta]}$, one has

$$\mu\left((T_\theta)^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}\gamma_\theta(x), \tag{4.3}$$

where $f(x) := (\log(1+\theta^2)) \frac{x\theta+1}{\theta} h(x)$, $x \in [0,\theta]$.

Remark 4.2. In hypothesis of Proposition 4.1(ii) it follows that

$$\mu((T_\theta)^{-n}(A)) - \gamma_\theta(A) = \int_A (U^n f(x) - 1)\mathrm{d}\gamma_\theta(x), \tag{4.4}$$

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_{[0,\theta]}$, where $f(x) := (\log(1+\theta^2))\frac{x\theta+1}{\theta}h(x)$, $x \in [0,\theta]$. The last equation shows that the asymptotic behavior of $\mu((T_\theta)^{-n}(A)) - \gamma_\theta(A)$ as $n \to \infty$ is given by the asymptotic behavior of the n-th power of the Perron–Frobenius operator U on $L^1_{\gamma_\theta}$.

5. A Wirsing-type approach

Let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}$ put $F^n_\theta(x) = \mu(T^n_\theta < x)$, $x \in [0,\theta]$, where T^0_θ is the identity map. As $(T^n_\theta < x) = T^{-n}_\theta((0,x))$, by Proposition 4.1 (ii) we have

$$F^n_\theta(x) = \int_0^x \frac{U^n f^0_\theta(u)}{1+u\theta}\theta \mathrm{d}u, \quad n \in \mathbb{N}, \tag{5.1}$$

with $f^0_\theta(x) = \frac{x\theta+1}{\theta}(F^0_\theta)'(x)$, $x \in [0,\theta]$, where $(F^0_\theta)' = d\mu/d\lambda_\theta$.

We will assume that $(F^0_\theta)' \in C^1([0,\theta])$. So, we study the behavior of U^n as $n \to \infty$, assuming that the domain of U is $C^1([0,\theta])$, the collection of all functions $f : [0,\theta] \to \mathbb{C}$ which have a continuous derivative.

Let $f \in C^1([0,\theta])$. Then the series (4.2) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. We get

$$\begin{aligned}
(Uf)'(x) &= \sum_{j\geq m}\left[(P_j(x))' f\left(\frac{1}{x+j\theta}\right) - P_j(x) f'\left(\frac{1}{x+j\theta}\right)\frac{1}{(x+j\theta)^2}\right] \\
&= \sum_{j\geq m}\left[\left(\frac{j\theta - \frac{1}{\theta}}{(x+j\theta)^2} - \frac{(j+1)\theta - \frac{1}{\theta}}{(x+(j+1)\theta)^2}\right) f\left(\frac{1}{x+j\theta}\right)\right. \\
&\quad \left. - P_j(x)\frac{1}{(x+j\theta)^2} f'\left(\frac{1}{x+j\theta}\right)\right] \\
&= -\sum_{j\geq m}\left[\frac{(j+1)\theta - \frac{1}{\theta}}{(x+(j+1)\theta)^2}\left[f\left(\frac{1}{x+j\theta}\right) - f\left(\frac{1}{x+(j+1)\theta}\right)\right]\right. \\
&\quad \left. + P_j(x)\frac{1}{(x+j\theta)^2} f'\left(\frac{1}{x+j\theta}\right)\right], \quad x \in [0,\theta].
\end{aligned} \tag{5.2}$$

Thus, we can write

$$(Uf)' = -Vf', \quad f \in C^1([0,\theta]), \tag{5.3}$$

where $V : C([0,\theta]) \to C([0,\theta])$ is defined by

$$Vg(x) = \sum_{j\geq m}\left(\frac{(j+1)\theta - \frac{1}{\theta}}{(x+(j+1)\theta)^2}\int\limits_{\frac{1}{x+(j+1)\theta}}^{\frac{1}{x+j\theta}} g(u)\mathrm{d}u + P_j(x)\frac{1}{(x+j\theta)^2}g\left(\frac{1}{x+j\theta}\right)\right) \tag{5.4}$$

with $g \in C([0,\theta))$ and $x \in [0,\theta]$. Clearly, $(U^n f)' = (-1)^n V^n f'$, $n \in \mathbb{N}_+$, $f \in C^1([0,\theta])$.

We are going to show that V^n takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

Proposition 5.1. *There are positive constants $v_\theta < w_\theta < 1$ and a real-valued function $\varphi_\theta \in C([0,\theta])$ such that*

$$v_\theta\varphi_\theta \leq V\varphi_\theta \leq w_\theta\varphi_\theta, \quad \theta^2 = 1/m,\, m \in \mathbb{N}_+. \tag{5.5}$$

Proof. Let $h_\theta : \mathbb{R}_+ \to \mathbb{R}$, with $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, be a continuous bounded function such that $\lim_{x\to\infty} h_\theta(x) < \infty$. We look for a function $g_\theta : (0,\theta] \to \mathbb{R}$ such that $Ug_\theta = h_\theta$, assuming that the equation

$$Ug_\theta(x) = \sum_{j\geq m} P_j(x) g_\theta\left(u_j(x)\right) = h_\theta(x) \tag{5.6}$$

holds for $x \in \mathbb{R}_+$. By reducing the terms of the series involved (5.6) yields

$$\frac{h_\theta(x)}{x\theta+1} - \frac{h_\theta(x+\theta)}{\theta(x+\theta)+1} = \frac{1}{(x+m\theta)(x+(m+1)\theta)} g_\theta\left(\frac{1}{x+m\theta}\right), \quad x \in \mathbb{R}_+. \tag{5.7}$$

Hence

$$g_\theta(u) = \left(\frac{1}{u\theta}+1\right)h_\theta\left(\frac{1}{u} - m\theta\right) - \frac{1}{u\theta}h_\theta\left(\frac{1}{u} - (m-1)\theta\right), \quad u \in (0,\theta], \tag{5.8}$$

and we indeed have $Ug_\theta = h_\theta$ since

$$\begin{aligned} Ug_\theta(x) &= \sum_{j\geq m}\frac{x\theta+1}{(x+j\theta)(x+(j+1)\theta)}g_\theta\left(\frac{1}{x+j\theta}\right) \\ &= \sum_{j\geq m}\frac{x\theta+1}{\theta}\left[\frac{h_\theta(x+j\theta-m\theta)}{x+j\theta} - \frac{h_\theta(x+(j+1)\theta-m\theta)}{x+(j+1)\theta}\right] \\ &= \frac{x\theta+1}{\theta}\left(\frac{h_\theta(x)}{x+m\theta} - \lim_{j\to\infty}\frac{h_\theta(x+(j+1)\theta-m\theta)}{x+(j+1)\theta}\right) = h_\theta(x), \quad x \in \mathbb{R}_+. \end{aligned} \tag{5.9}$$

In particular, for any fixed $a_\theta \in [0,\theta]$ we consider the function $h_{a_\theta} : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$h_{a_\theta}(x) = \frac{1}{e_\theta x + a_\theta + 1}, \quad x \in \mathbb{R}_+ \tag{5.10}$$

where the coefficient e_θ will be specified later. By the above, the function $g_{a_\theta} : (0, \theta] \to \mathbb{R}$ defined as

$$\begin{aligned} g_{a_\theta}(x) &= \left(\frac{1}{x\theta} + 1\right) h_{a_\theta}\left(\frac{1}{x} - m\theta\right) - \frac{1}{\theta x} h_{a_\theta}\left(\frac{1}{x} - (m-1)\theta\right) \\ &= \left(\frac{1}{x\theta} + 1\right) \frac{1}{e_\theta\left(\frac{1}{x} - m\theta\right) + a_\theta + 1} - \frac{1}{x\theta} \frac{1}{e_\theta\left(\frac{1}{x} - (m-1)\theta\right) + a_\theta + 1} \\ &= \frac{1}{\theta}\left[\frac{x\theta + 1}{e_\theta + x(-e_\theta m\theta + a_\theta + 1)} - \frac{1}{e_\theta + x(-e_\theta(m-1)\theta + a_\theta + 1)}\right] \end{aligned} \tag{5.11}$$

for any $x \in (0, \theta]$ satisfies

$$U g_{a_\theta}(x) = h_{a_\theta}(x), \quad x \in [0, \theta]. \tag{5.12}$$

Setting

$$\varphi_{a_\theta}(x) = g'_{a_\theta}(x) = \frac{1}{\theta}\left[\frac{e_\theta(m+1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta m\theta + a_\theta + 1))^2} - \frac{e_\theta(m-1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta(m-1)\theta + a_\theta + 1))^2}\right] \tag{5.13}$$

we have

$$V\varphi_{a_\theta}(x) = -(U g_{a_\theta})'(x) = -(h_{a_\theta})'(x) = \frac{e_\theta}{(e_\theta x + a_\theta + 1)^2}, \quad x \in [0, \theta]. \tag{5.14}$$

We choose a_θ by asking that $(\varphi_{a_\theta}/V\varphi_{a_\theta})(0) = (\varphi_{a_\theta}/V\varphi_{a_\theta})(\theta)$. Since

$$(\varphi_{a_\theta}/V\varphi_{a_\theta})(0) = \frac{2(a_\theta + 1)^2}{e_\theta^2} \tag{5.15}$$

and

$$(\varphi_{a_\theta}/V\varphi_{a_\theta})(\theta) = \frac{(a_\theta + 1 + e_\theta\theta)^2}{e_\theta\theta^3}\left[\frac{e_\theta(m+1)\theta - a_\theta - 1}{(a_\theta + 1)^2} - \frac{e_\theta(m-1)\theta - a_\theta - 1}{(a_\theta + 1 + e_\theta\theta)^2}\right], \tag{5.16}$$

this amounts to the equation

$$H_\theta(a_\theta) = 2\theta(a_\theta + 1)^4 - e_\theta^3\left[(2m+1)(a_\theta + 1) + e_\theta(m+1)\theta\right] = 0. \tag{5.17}$$

We choose the coefficient e_θ such that the equation $H_\theta(x) = 0$, $x \in [0, \theta]$, yields a unique solution $a_\theta \in [0, \theta]$. Asking that

$$H_\theta(0) < 0, \quad H_\theta(\theta) > 0, \quad \text{and} \quad \frac{dH_\theta}{da_\theta} > 0, \tag{5.18}$$

we may determine e_θ (see Appendix A.1). For this unique acceptable solution $a_\theta \in [0, \theta]$ the function $\varphi_{a_\theta}/V\varphi_{a_\theta}$ attains its maximum equal to $2(a_\theta + 1)^2/e_\theta^2$ at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a_\theta})\left(x_{min}^\theta\right) > 1$ (see Appendix A.2). It follows that for $\varphi_\theta = \varphi_{a_\theta}$ we have

$$\frac{e_\theta^2 \varphi_\theta}{2(a_\theta + 1)^2} \le V\varphi_\theta \le \frac{\varphi_\theta}{m(a_\theta)}, \tag{5.19}$$

that is, $v_\theta \varphi_\theta \le V\varphi_\theta \le w_\theta \varphi_\theta$, where

$$v_\theta = \frac{e_\theta^2}{2(a_\theta + 1)^2} \quad \text{and} \quad w_\theta = \frac{1}{m(a_\theta)}. \qquad \square \tag{5.20}$$

Corollary 5.2. *Let $f_\theta^0 \in C^1([0, \theta])$ such that $(f_\theta^0)' > 0$. Put $\alpha_\theta = \min_{x\in[0,\theta]} \varphi_\theta(x)/(f_\theta^0)'(x)$ and $\beta_\theta = \max_{x\in[0,\theta]} \varphi_\theta(x)/(f_\theta^0)'(x)$. Then*

$$\frac{\alpha_\theta}{\beta_\theta} v_\theta^n (f_\theta^0)' \le V^n (f_\theta^0)' \le \frac{\beta_\theta}{\alpha_\theta} w_\theta^n (f_\theta^0)', \quad n \in \mathbb{N}_+. \tag{5.21}$$

Proof. Since V is a positive operator, we have

$$v_\theta^n \varphi_\theta \le V^n \varphi_\theta \le w_\theta^n \varphi_\theta, \quad n \in \mathbb{N}_+. \tag{5.22}$$

Noting that $\alpha_\theta (f_\theta^0)' \le \varphi_\theta \le \beta_\theta (f_\theta^0)'$, we can write

$$\frac{\alpha_\theta}{\beta_\theta} v_\theta^n (f_\theta^0)' \le \frac{v_\theta^n}{\beta_\theta}\varphi_\theta \le \frac{1}{\beta_\theta} V^n \varphi_\theta \le V^n (f_\theta^0)' \le \frac{1}{\alpha_\theta} w_\theta^n \varphi_\theta \le \frac{\beta_\theta}{\alpha_\theta} w_\theta^n (f_\theta^0)', \quad n \in \mathbb{N}_+, \tag{5.23}$$

which shows that (5.21) holds. $\square$

Theorem 5.3 *(Near-optimal solution to Gauss–Kuzmin–Lévy problem). Let $f_\theta^0 \in C^1([0, \theta])$ such that $(f_\theta^0)' > 0$ and let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}_+$ and $x \in [0, \theta]$ we have*

$$\begin{aligned} &(\log(1+\theta^2))^2 \frac{\alpha_\theta}{2\theta\beta_\theta} \min_{x\in[0,\theta]} (f_\theta^0)'(x) v_\theta^n G_\theta(x)(\theta - G_\theta(x)) \le |\mu(T_\theta^n < x) - G_\theta(x)| \\ &\le (\log(1+\theta^2))^2 \frac{(1+\theta^2)\beta_\theta}{2\theta\alpha_\theta} \max_{x\in[0,\theta]} (f_\theta^0)'(x) w_\theta^n G_\theta(x)(\theta - G_\theta(x)) \end{aligned} \tag{5.24}$$

where α_θ, β_θ, v_θ and w_θ are defined in Proposition 5.1 and Corollary 5.2, and

$$G_\theta(x) = \frac{\log(1 + x\theta)}{\log(1 + \theta^2)}. \tag{5.25}$$

Proof. For any $n \in \mathbb{N}$ and $x \in [0, \theta]$ set $d_n(G_\theta(x)) = \mu(T_\theta^n < x) - G_\theta(x)$. Then by (5.1) we have

$$d_n(G_\theta(x)) = \int_0^x \frac{U^n f_\theta^0(u)}{(\theta u + 1)} \theta \mathrm{d}u - G_\theta(x). \tag{5.26}$$

Differentiating twice with respect to x yields

$$d_n'(G_\theta(x)) \frac{1}{\log(1+\theta^2)} \frac{\theta}{\theta x + 1} = \frac{U^n f_\theta^0(x)}{\theta x + 1} \theta - \frac{1}{\log(1+\theta^2)} \frac{\theta}{\theta x + 1}, \tag{5.27}$$

$$(U^n f_\theta^0(x))' = \frac{\theta}{(\log(1+\theta^2))^2} \frac{d_n''(G_\theta(x))}{\theta x + 1}, \quad n \in \mathbb{N}, x \in [0, \theta]. \tag{5.28}$$

Hence by (5.3) we have

$$d_n''(G_\theta(x)) = (-1)^n \left(\log(1+\theta^2)\right)^2 \frac{\theta x + 1}{\theta} V^n (f_\theta^0)'(x), \tag{5.29}$$

for any $n \in \mathbb{N}$, $x \in [0, \theta]$. Since $d_n(0) = d_n(\theta) = 0$, a well-known interpolation formula yields

$$d_n(x) = -\frac{x(\theta - x)}{2} d_n''(\xi), \quad n \in \mathbb{N}, x \in [0, \theta], \tag{5.30}$$

for a suitable $\xi = \xi(n, x) \in [0, \theta]$. Therefore

$$\begin{aligned} &\mu(T_\theta^n < x) - G_\theta(x) \\ &\quad = (-1)^{n+1} \left(\log(1+\theta^2)\right)^2 \frac{\theta \xi_\theta + 1}{\theta} V^n (f_\theta^0)'(\xi_\theta) \frac{G_\theta(x)(\theta - G_\theta(x))}{2} \end{aligned} \tag{5.31}$$

for any $n \in \mathbb{N}$ and $x \in [0, \theta]$, and another suitable $\xi_\theta = \xi_\theta(n, x) \in [0, \theta]$. The result stated follows now from Corollary 5.2. $\square$

6. Final remarks

To conclude, we use the values obtained in Appendix A.

Let us consider the case $m = 2$. The equation $H_\theta(x) = 0$, with $e_\theta = 1$, has as unique acceptable solution $a_\theta = 0.6445398$. For this value of a_θ the function $\varphi_{a_\theta}/V\varphi_{a_\theta}$ attains its maximum equal to 5.409022308 at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a_\theta})(0.297421) = 5.28441$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_\theta^n)$ and $O(v_\theta^n)$ as $n \to \infty$, with $v_\theta > 0.184876294$ and $w_\theta < 0.189235884$.

For $m = 3$, the equation $H_\theta(x) = 0$, with $e_\theta = 0.67$, has as unique acceptable solution $a_\theta = 0.287897$. For this value of a_θ the function $\varphi_{a_\theta}/V\varphi_{a_\theta}$ attains its maximum equal to

7.389969626 at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a_\theta})(0.256122) = 7.29924$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_\theta^n)$ and $O(v_\theta^n)$ as $n \to \infty$, with $v_\theta > 0.135318553$ and $w_\theta < 0.137000564$.

Finally, let us consider the case $m = 4$. The equation $H_\theta(x) = 0$, with $e_\theta = 0.5772$, has as unique acceptable solution $a_\theta = 0.249911$. For this value of a_θ the function $\varphi_{a_\theta}/V\varphi_{a_\theta}$ attains its maximum equal to 9.378546393 at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a\theta})(0.228161) = 9.3072$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_\theta^n)$ and $O(v_\theta^n)$ as $n \to \infty$, with $v_\theta > 0.106626331$ and $w_\theta < 0.107443699$.

Obviously, the determination of the exact convergence rate remains an open question. We may derive it using the same strategy as in [9].

Appendix A

A.1. Imposing conditions (5.18) and using a mathematical software we obtain

m	e_θ	a_θ	m	e_θ	a_θ
2	1	0.6445398	17	0.266721	0.121272
3	0.6704	0.287897	18	0.258692	0.117853
4	0.5772	0.249911	19	0.251324	0.114708
5	0.513167	0.223606	20	0.244533	0.111805
6	0.465794	0.204125	25	0.217106	0.100000
7	0.429017	0.188983	30	0.197052	0.0912886
8	0.399444	0.176778	35	0.181587	0.0845132
9	0.375022	0.166667	40	0.169204	0.0790571
10	0.354429	0.158113	45	0.159005	0.0745384
11	0.336772	0.150756	50	0.150419	0.0707134
12	0.321422	0.144338	60	0.136674	0.0645535
13	0.307923	0.138677	70	0.126065	0.0597579
14	0.295935	0.133633	80	0.117564	0.0559056
15	0.285198	0.129098	90	0.110555	0.0527013
16	0.275514	0.125003	100	0.104651	0.0499983

A.2. Since

$$(\varphi_{a_\theta}/V\varphi_{a_\theta})'(x) = \frac{2}{\theta e_\theta}(e_\theta x + a_\theta + 1)\left[\frac{(A_\theta+\theta e_\theta)(e_\theta^2 + A_\theta(a_\theta+1))}{(e_\theta - xA_\theta)^3} - \frac{(A_\theta-\theta e_\theta)(e_\theta^2+(A_\theta-\theta e_\theta)(a_\theta+1))}{(e_\theta - x(A_\theta-\theta e_\theta))^3}\right]$$

where $A_\theta = \theta e_\theta m - a_\theta - 1$, the equation $(\varphi_{a_\theta}/V\varphi_{a_\theta})'(x) = 0$ has a unique positive solution in $(0, \theta)$

$$x_{min}^\theta = \frac{e_\theta(B_\theta - C_\theta)}{(A_\theta - \theta e_\theta)B_\theta - A_\theta C_\theta}$$

with

$$B_\theta = \sqrt[3]{(A_\theta + \theta e_\theta)(e_\theta^2 + (a_\theta + 1)A_\theta)}, \quad C_\theta = \sqrt[3]{(A_\theta - \theta e_\theta)(e_\theta^2 + (a_\theta + 1)(A_\theta - \theta e_\theta))}.$$

Since in the particular cases studied for $m = 2$, $m = 3$ and $m = 4$, the minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a_\theta})(x_{min}^\theta)$ has the following values

$m = 2$	$m(a_\theta) = 5.28441\ldots$
$m = 3$	$m(a_\theta) = 7.29924\ldots$
$m = 4$	$m(a_\theta) = 9.3072\ldots$

we may assume that $m(a_\theta) > 1$ for every $m \in \mathbb{N}_+$.

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