

Accepted Manuscript

Addition formula and γ -factors for local fields

Yongchang Zhu

PII: S0022-314X(18)30302-0
DOI: <https://doi.org/10.1016/j.jnt.2018.10.003>
Reference: YJNTH 6144

To appear in: *Journal of Number Theory*

Received date: 23 April 2018
Revised date: 15 October 2018
Accepted date: 15 October 2018

Please cite this article in press as: Y. Zhu, Addition formula and γ -factors for local fields, *J. Number Theory* (2019), <https://doi.org/10.1016/j.jnt.2018.10.003>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



ADDITION FORMULA AND γ -FACTORS FOR LOCAL FIELDS

YONGCHANG ZHU

ABSTRACT. We prove a formula that shows explicitly the additive structure of a local field is determined by the multiplicative structure and γ factors. And we prove a formula that relates the γ -factors of different local fields. The unramified coefficients of our addition formula appear as string amplitudes.

1. INTRODUCTION

Let F be a local field and let $\psi : F \rightarrow S^1$ be a non-trivial additive character of F , dx be the self-dual Haar measure determined by ψ . For a quasi-character c of F , Tate introduced the L -function and ρ -factor $\rho(c)$ (also called γ -factor) [Ta] (see also [Iw]). The ρ -factors and L -functions are generalized by R.Godement and H.Jacquet [GJ] for irreducible admissible representations of $GL_n(F)$. On the other hand side, Langlands and Deligne have defined local epsilon factors (therefore γ -factors) [L] [D] for complex representations of the Weil group (more generally, of the Weil-Deligne group); these local invariants are compatible with the L -functions of complex representations of the global Weil group. The local Langlands correspondence for irreducible representations of GL_n and n -dimensional representations of Weil-Deligne groups preserves the gamma factors.

This work studies the gamma factors for GL_1 . Its very definition [Ta] already indicates these invariants encode the information of the additive structure of F . The purpose of this note is to make this more explicit by proving the following addition formula (1.1) and use it to prove some new relations of gamma factors.

Theorem 1.1. *For quasi-characters c_1 and c_2 of F^* with $1 > \text{re } c_1, \text{re } c_2 > 0$ for $F = \mathbb{R}$, $\frac{1}{2} > \text{re } c_1, \text{re } c_2 > 0$ for $F = \mathbb{C}$, or $\text{re } c_1, \text{re } c_2 > 0$ for F non-Archimedean, and for all $x, y \in F^*$,*

$$(c_1 c_2)(x + y) - (c_1 c_2)(x) - (c_1 c_2)(y) = \int_{\widehat{F^*}} \frac{\rho(|_F c_1 c_2)}{\rho(|_F c_1 \chi) \rho(|_F c_2 \chi^{-1})} (c_1 \chi)(x) (c_2 \chi^{-1})(y) d\chi. \quad (1.1)$$

where $\widehat{F^*}$ is the character group of F^* and $d\chi$ is the Haar measure on $\widehat{F^*}$ dual to the Haar measure $d^*x = |x|_F^{-1} dx$ on F^* .

The condition $\text{re } c_i > 0$ is imposed for the convergence of the integral in the right hand side of (1.1). We will prove the convergence is uniform when c_1, c_2 varies on certain compact subsets containing no poles of the integrand (Proposition 3.4). The upper bound condition for $\text{re } c_i$ for the case $F = \mathbb{R}$ or \mathbb{C} is imposed to avoid more complicated pole structure of the integrand. It can be relaxed but there will be more terms in the left hand side of (1.1).

In the notation used in [GGPS], the coefficients in (1.1) are beta functions for quasi-characters of F^* ,

$$\beta(c_1^{-1}, c_2^{-1}) \stackrel{\text{def}}{=} \frac{\rho(|_F c_1 c_2)}{\rho(|_F c_1) \rho(|_F c_2)}. \quad (1.2)$$

The ρ -factors $\rho(c)$ and the measure $d\chi$ on \widehat{F}^* are uniquely determined by the choice of ψ , we can show that the beta measure

$$\beta(c_1^{-1}\chi^{-1}, c_2^{-1}\chi)d\chi = \frac{\rho(|_F c_1 c_2)}{\rho(|_F c_1 \chi)\rho(|_F c_2 \chi^{-1})}d\chi \quad (1.3)$$

is independent of ψ . We view (1.1) as an analytic analog of binomial expansion $(x+y)^n = \sum_i \binom{n}{i} x^i y^{n-i}$ for algebraic character x^n and the beta functions (1.2) as analogs of binomial coefficients. A familiar analytic addition formula is

$$(x+y)^{s_1+s_2} - x^{s_1+s_2} - y^{s_1+s_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(-s_1-it)\Gamma(-s_2+it)}{\Gamma(-s_1-s_2)} x^{s_1+it} y^{s_2-it} dt \quad (1.4)$$

for $x, y \in \mathbb{R}_{\geq 0}$ and $1 > \operatorname{re} s_1, \operatorname{re} s_2 > 0$, which is an addition formula for $(\mathbb{R}_{\geq 0}, +, \cdot)$, the usual beta functions appear as coefficients. Our formula (1.1) is an addition formula for general local fields, the usual beta functions are now replaced by beta functions (1.2).

It is known that if F is a finite extension of \mathbb{Q}_p , the topological group F^* is isomorphic to $K^* = \mathbb{Z} \oplus \mu_{q-1} \oplus \mu_{p^a} \oplus \mathbb{Z}_p^d$, where q is the number of elements in the residue field, μ_m denotes the cyclic group of order m , and $d = [K : \mathbb{Q}_p]$ (see e.g. [Neukirch], page 140). It may happen that two non-isomorphic local fields have the isomorphic multiplicative groups. The formula (1.1) shows that the additive structure is determined by the beta measures, more precisely we have

Corollary 1.2. *Let F and F' be local fields, $\alpha : F'^* \rightarrow F^*$ be an isomorphism of topological groups that preserves the beta measures (1.3), then α is an local field isomorphism.*

The formula (1.1) can be used to derive new relations of beta factors for different local fields. If E is a finite extension of F , $x, y \in F$, c_1, c_2 are quasi-characters of E as in Theorem 1.1. We can expand $(c_1 c_2)(x+y)$ in two different ways, using (1.1) for E or F . The resulting identity can be used to prove the following formula that relates the ρ -factors ρ_E of E and ρ_F of F . We have

Theorem 1.3. *Let E be a finite extension of F , c_1 and c_2 be quasi-characters of E with $\operatorname{re} c_1, \operatorname{re} c_2$ satisfying conditions in Theorem 1.1. We use c_1, c_2 to denote their restrictions on F^* . Let $\pi : \widehat{E}^* \rightarrow \widehat{F}^*$ be the restriction homomorphism for the embedding $F^* \subset E^*$, and $d\alpha$ be the Haar measure on $Ker = \operatorname{Ker}(\pi)$ determined by the equation*

$$\int_{\widehat{E}^*} f(\beta) d\beta = \int_{Ker} \int_{\widehat{F}^*} f(\alpha\chi) d\alpha d\chi. \quad (1.5)$$

Then

$$\int_{Ker} \frac{\rho_E(|_E c_1 c_2)}{\rho_E(|_E c_1 \chi)\rho_E(|_E c_2 \chi^{-1})} d\chi = \frac{\rho_F(|_F c_1 c_2)}{\rho_F(|_F c_1)\rho_F(|_F c_2)}. \quad (1.6)$$

In the case $F = \mathbb{R}$, $E = \mathbb{C}$, the kernel $Ker = \{\delta_{2n} \mid n \in \mathbb{Z}\}$, where $\delta_n(re^{i\theta}) = e^{in\theta}$. The measure on Ker determined by the compatibility relation (1.5) is $\frac{1}{2\pi}$ times the counting measure. We take $c_1(x) = \delta_u(x)|x|_{\mathbb{C}}^{s_1}$ and $c_2(x) = \delta_v(x)|x|_{\mathbb{C}}^{s_2}$, (1.6) for this case is

$$\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{\rho_{\mathbb{C}}(\delta_{u+v} |_{\mathbb{C}}^{s_1+s_2+1})}{\rho_{\mathbb{C}}(\delta_{u+2n} |_{\mathbb{C}}^{s_1+1})\rho_{\mathbb{C}}(\delta_{v-2n} |_{\mathbb{C}}^{s_2+1})} = \frac{\rho_{\mathbb{R}}(\operatorname{sign}^{u+v} |_{\mathbb{R}}^{1+2s_1+2s_2})}{\rho_{\mathbb{R}}(\operatorname{sign}^u |_{\mathbb{R}}^{1+2s_1})\rho_{\mathbb{R}}(\operatorname{sign}^v |_{\mathbb{R}}^{1+2s_2})}. \quad (1.7)$$

where sign is the character of \mathbb{R}^* given by $\operatorname{sign}(x) = \frac{x}{|x|_{\mathbb{R}}}$.

Notice that the restriction of δ_n on \mathbb{R}^* is sign^n . There are four cases for u, v according to their parities. The case $u = v = 0$ of the above formula implies that

$$1 + 2 \sum_{n=1}^{\infty} \frac{(-s_1)_n (-s_2)_n}{(1+s_1)_n (1+s_2)_n} = \sqrt{\pi} \frac{\Gamma(1+s_1) \Gamma(1+s_2) \Gamma(\frac{1}{2} + s_1 + s_2)}{\Gamma(\frac{1}{2} + s_1) \Gamma(\frac{1}{2} + s_2) \Gamma(1+s_1+s_2)}. \quad (1.8)$$

This formula can be also proved directly using the following Dougall-Ramanujan formula (see [BE], page 191):

$$1 + 2 \sum_{n=1}^{\infty} \frac{(-s_1)_n (-s_2)_n (-s_3)_n}{(1+s_1)_n (1+s_2)_n (1+s_3)_n} = \frac{\Gamma(1+s_1) \Gamma(1+s_2) \Gamma(1+s_3) \Gamma(1+s_1+s_2+s_3)}{\Gamma(1+s_1+s_3) \Gamma(1+s_2+s_3) \Gamma(1+s_1+s_2)} \quad (1.9)$$

which is valid for $\operatorname{re}(s_1 + s_2 + s_3 + 1) > 0$. We set $s_3 = -\frac{1}{2}$, (1.9) reduces to (1.8). For the other three cases of u, v , (1.7) can be also proved using (1.9).

If F is non-Archimedean with q elements in the residue field, and suppose E has q' elements in the residue field. The kernel Ker of the restriction map $\widehat{\mathbb{E}^*} \rightarrow \widehat{\mathbb{F}^*}$ is countable and the measure on Ker satisfying (1.5) is $e^{-1}(1-q^{-1})(1-q'^{-1})^{-1}$ times the counting measure, here e is the ramification index of E over F . The formula (1.6) is

$$e^{-1}(1-q^{-1})(1-q'^{-1})^{-1} \sum_{\chi \in \operatorname{Ker}} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \chi|) \rho_E(|_E c_2 \chi^{-1}|)} = \frac{\rho_F(|_F c_1 c_2|)}{\rho_F(|_F c_1|) \rho_F(|_F c_2|)}. \quad (1.10)$$

Notice that $\rho(c)$ for ramified c is a Gauss sum (see (2.7)), (1.10) is a new identity of Gauss sums, which is a non-Archimedean analog of (1.7) and (1.8). Since (1.8) is a specialization of Dougall-Ramanujan formula (1.9), a natural question is if (1.9) has a non-Archimedean analog. It seems that one needs to extend the concept of hypergeometric functions to non-Archimedean fields in order to formulate the problem accurately; such functions should be \mathbb{C} -valued, defined on F with parameters as quasi-characters of F^* , similar to hypergeometric functions over finite fields studied in [FLRST] recently.

As an application of (1.1) and its variant Theorem 3.6, we will give a new proof of Barnes' identity for local fields. The first Barnes lemma states

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s) \Gamma(b-s)}{\Gamma(a+b)} \frac{\Gamma(c+s) \Gamma(d-s)}{\Gamma(c+d)} ds = \frac{\Gamma(a+c) \Gamma(b+d)}{\Gamma(a+b+c+d)}, \quad (1.11)$$

where the path of integration from $-i\infty$ to $i\infty$ is curved so that the poles of $\Gamma(a+s) \Gamma(c+s)$ lie on the left and the poles of $\Gamma(b-s) \Gamma(d-s)$ lie on the right. In [Li], W.-C. Li formulated and proved a generalization of (1.11) for non-Archimedean field F in terms of representations of $GL_2(F)$: for each irreducible admissible infinite dimensional representation of $GL_2(F)$, there is an identity of Barnes' type. The generalization of (1.11) associated to representations of $GL_2(\mathbb{R})$ is given in [Pr] later. Our view of (1.11) is that it is an analytic generalization of

$$\sum_{i+j=k} \binom{m}{i} \binom{n}{j} = \binom{m+n}{k}. \quad (1.12)$$

which can be proved using the addition formula for $(x+y)^{m+n}$, while (1.11) can be proved using (1.4). In Theorem 5.1, we will show that the same method, using (1.1), proves the Barnes' identity associated to principal series representations in [Li] [Pr].

The unramified coefficient of the expansion (1.1) also appeared in string theory. We have the beta-integral

$$\frac{\rho(|_F^{s_1+s_2+1}|)}{\rho(|_F^{s_1+1}|) \rho(|_F^{s_2+1}|)} = \int_F |x|_F^{-s_1-1} |1-x|_F^{-s_2-1} dx \quad (1.13)$$

where the convergence region is $\operatorname{re} s_i > 0$ and $\operatorname{re}(s_1 + s_2) < 1$, which is a 4-point Veneziano string amplitude of tree level when $F = \mathbb{R}$ and is a 4-point Virasoro-Shapiro string amplitude when $F = \mathbb{C}$. The p -adic string amplitudes were introduced in [FO], the 4-point amplitudes of tree level are exactly (1.13). See [Za] for a path integral approach to p -adic string amplitudes and [BFMO] for a discussion of N -point amplitude. We refer to [GKPSW] and references there for recent works in this topic. The fact that the same term (1.13) appear in two different contexts suggests some relation between the two.

This paper is organized as follows. In Section 2, we recall basics about the local part of Iwasawa-Tate theory [Iw][Ta] and identify the Haar measure on \widehat{F}^* . In Section 3, we prove Theorem 1.1, Theorem 1.3 and a variant of Theorem 1.1 (Theorem 3.6). In Section 4, we give the explicit formulas of (1.1) for unramified quasi-characters. In Section 5, we give a new proof of Barnes' identity for principal series of GL_2 .

2. LOCAL ZETA INTEGRALS AND ρ -FACTORS

Our exposition follows mostly [Lang]. For a fixed non-trivial character ψ of F , the self-dual Haar measure dx is the unique Haar measure on F such that the Fourier transform defined using dx ,

$$\mathcal{F}f(y) = \int_F f(x)\psi(xy)dx$$

satisfies $\mathcal{F}^2 f(y) = f(-y)$. Recall a zeta integral on F is

$$\int_F f(x)c(x)|x|^{-1}dx$$

where f is a Schwartz function on F , c is a quasi-character $c : F^* \rightarrow \mathbb{C}^*$. The integral converges when $\operatorname{re} c > 0$ and can be meromorphically continued to all quasi-characters c . The ρ -factor $\rho(c)$ for c is defined by the functional equation

$$\int f(x)c(x)|x|^{-1}dx = \rho(c) \int \mathcal{F}f(x)c^{-1}(x)dx. \quad (2.1)$$

To indicate the dependence of ρ on ψ , we write $d_\psi x$, $\rho_\psi(c)$ for dx , $\rho(c)$. If $\psi_a(x) = \psi(ax)$, it is easy to prove that $d_{\psi_a} = |a|^{\frac{1}{2}}d_\psi x$ and $\rho_{\psi_a}(c) = |a|^{\frac{1}{2}}c^{-1}(a)\rho_\psi(c)$, therefore the measure in (1.3) is independent of ψ .

We will list the explicit formulas for $\rho(c)$ under the following choice of ψ . For $F = \mathbb{R}$, $\psi(x) = e^{2\pi i x}$. For $F = \mathbb{C}$, $\psi(z) = e^{4\pi i \operatorname{re} z} = e^{4\pi i x}$. For F non-Archimedean, let \mathcal{O} be the ring of integers, $\pi \in \mathcal{O}$ be a prime, q the cardinality of residue field. We take $\psi : F \rightarrow S^1$ with $\psi|_{\mathcal{O}} = 1$ and $\psi|_{\pi^{-1}\mathcal{O}} \neq 1$. Then the self-dual Haar measure dx on \mathbb{R} is the Lebesgue measure; dx on \mathbb{C} is 2 times usual Lebesgue measure; dx on a non-Archimedean field has the property that $\operatorname{vol}(\mathcal{O}) = 1$.

We denote the group of quasi-characters of F^* by \widehat{F}^* and its subgroup of characters by \widehat{F} . We denote by $|\cdot|_F$ the quasi-character defined by $\operatorname{vol}(aU) = |a|_F \operatorname{vol}(U)$ where vol denote the volume of the additive Haar measure. This is the same as $|\cdot|_F$ used in the introduction. We emphasize that for $F = \mathbb{C}$, $|z| = z\bar{z}$, the square of the usual norm.

Recall that the groups of quasi-characters and characters in different cases. For $F = \mathbb{R}$,

$$\widehat{\mathbb{R}}^* = \{0, 1\} \times \mathbb{C}, \quad \widehat{\mathbb{R}} = \{0, 1\} \times i\mathbb{R},$$

where (ϵ, s) represents the quasi-character $x \mapsto \text{sign}(x)^\epsilon |x|^s$. For $F = \mathbb{C}$,

$$\widehat{\mathbb{C}}^* = \mathbb{Z} \times \mathbb{C}, \quad \widehat{\mathbb{C}}^* = \mathbb{Z} \times i\mathbb{R},$$

where (n, s) represents the quasi-character $z \mapsto \delta_n(z)|z|^s = (z/\sqrt{z\bar{z}})^n (z\bar{z})^s$. For F non-Archimedean, F^* is isomorphic to $\pi^{\mathbb{Z}} \times \mathcal{O}^*$. Every character χ of \mathcal{O}^* gives a character of F^* which we still denote by χ : $\chi(\pi^n u) = \chi(u)$. Every quasi-character of F^* is $\pi^n u \mapsto \chi(u)|\pi^n|^s$ for $s \in \mathbb{C}/(\frac{2\pi}{\log q}\mathbb{Z}i)$. So we have

$$\widetilde{F}^* = \widehat{\mathcal{O}}^* \times \mathbb{C}/(\frac{2\pi}{\log q}\mathbb{Z}i), \quad \widehat{F}^* = \widehat{\mathcal{O}}^* \times \mathbb{R}i/(\frac{2\pi}{\log q}\mathbb{Z}i).$$

The above identifications of \widetilde{F}^* introduce an analytic structure on \widetilde{F}^* for all the cases $F = \mathbb{R}, \mathbb{C}$ or non-Archimedean.

We take the Haar measure of F^* as $d^*x = \frac{1}{|x|}dx$. It determines the dual measure $d\chi$ on the Pontryagin dual group \widehat{F}^* by the condition that the Fourier-Mellin transform $L^2(F^*) \rightarrow L^2(\widehat{F}^*)$, $h \mapsto \mathcal{M}h(\chi) = \int h(g)\chi(g)d^*g$ is an isometry.

For $F = \mathbb{R}$, the dual measure on $\widehat{\mathbb{R}}^* = \{0, 1\} \times i\mathbb{R}$ is

$$\int_{\widehat{\mathbb{R}}^*} f d\chi = \frac{1}{4\pi} \int_{-\infty}^{\infty} f(0, it) dt + \frac{1}{4\pi} \int_{-\infty}^{\infty} f(1, it) dt. \quad (2.2)$$

To see this formula, we let h be a function on \mathbb{R}^* supported on $\mathbb{R}_{>0}$. Then $\mathcal{M}h(\epsilon, it) = \int_0^\infty h(x)x^{it-1}dx$. The Mellin inversion formula shows that our measure given in (2.2) is the dual measure.

For $F = \mathbb{C}$, the measure $|z|^{-1}dz$ we take on \mathbb{C}^* , under the identification $\widehat{\mathbb{C}}^* = S^1 \times \mathbb{R}_{>0}$, is the product of the probability measure on unit circle S^1 and $\frac{4\pi}{r}dr$ on $r \in \mathbb{R}_{>0}$. The dual measure on $\widehat{\mathbb{C}}^* = \mathbb{Z} \times i\mathbb{R}$ is the product of the counting measure on \mathbb{Z} and the measure $\frac{1}{4\pi^2}dt$ on $it \in i\mathbb{R}$. That is, the integral of a function $f(n, it)$ on $\widehat{\mathbb{C}}^* = \mathbb{Z} \times i\mathbb{R}$ is

$$\int_{\widehat{\mathbb{C}}^*} f(n, it) d\chi = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(n, it) dt. \quad (2.3)$$

For F non-Archimedean field. The measure $\frac{1}{|x|}dx$ we choose on F^* , under the identification $F^* = \pi^{\mathbb{Z}}\mathcal{O}^* = \mathbb{Z} \times \mathcal{O}^*$, is the product measure of the counting measure on \mathbb{Z} with $(1 - q^{-1})$ times the probability measure on \mathcal{O}^* . So the dual measure on $\widehat{F}^* = \widehat{\mathcal{O}}^* \times \mathbb{R}i/(\frac{2\pi}{\log q}\mathbb{Z}i)$ is the product of the probability measure on the second factor with $(1 - q^{-1})^{-1}$ times the counting measure on $\widehat{\mathcal{O}}^*$.

We list the formulas for ρ -factors. For $F = \mathbb{R}$,

$$\rho(|x|^s) = \frac{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}, \quad \rho(|x|^s \text{sign}(x)) = -i \frac{\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})}{\pi^{-\frac{(1-s)+1}{2}} \Gamma(\frac{(1-s)+1}{2})}. \quad (2.4)$$

For $F = \mathbb{C}$,

$$\rho(\delta_n |z|_{\mathbb{C}}^s) = \rho(\delta_n |z\bar{z}|^s) = (-i)^{|n|} \frac{(2\pi)^{1-s} \Gamma(s + \frac{|n|}{2})}{(2\pi)^s \Gamma((1-s) + \frac{|n|}{2})}. \quad (2.5)$$

For F non-Archimedean,

$$\rho(|x|^s) = \frac{1 - q^{-(1-s)}}{1 - q^{-s}}. \quad (2.6)$$

For χ a character of \mathcal{O}^* of conductor $m \geq 1$, i.e., $\chi(1 + \pi^m \mathcal{O}) = 1$ and $\chi(1 + \pi^{m-1} \mathcal{O}) \neq 1$, we view χ as a character of F^* as before. Then

$$\rho(\chi(x)|x|^s) = q^{ms-m}\tau(\chi) \quad (2.7)$$

where $\tau(\chi)$ is the Gauss sum defined as

$$\tau(\chi) = \sum_{u \in (\mathcal{O}/\pi^m \mathcal{O})^*} \chi(u) \psi(-\frac{u}{\pi^m}). \quad (2.8)$$

We will also need the Stirling's asymptotic formula of $\Gamma(z)$, for $\sigma_1 \leq a \leq \sigma_2$ and $b \rightarrow \infty$,

$$|\Gamma(a + bi)| = \sqrt{2\pi}|b|^{a-1/2} e^{-\pi|b|/2} \left(1 + O\left(\frac{1}{|b|}\right)\right). \quad (2.9)$$

where the implied constant O depends only on σ_1 and σ_2 (see e.g. [AAR], page 21).

3. PROOFS OF THEOREM 1.1 AND THEOREM 1.3

We first prove some lemmas.

Lemma 3.1. *Let $f_1, f_2 \in \mathcal{S}(F)$, $f_1 * f_2$ be the convolution product defined as*

$$f_1 * f_2(x) = \int f_1(x-y)f_2(y)dy.$$

Then for quasi-characters c_1, c_2 with $\operatorname{re} c_1, \operatorname{re} c_2 < 0$, we have

$$\int_{\widehat{F^*}} \frac{\zeta(f_1, |c_1 \chi)}{\rho(|c_1 \chi)} \frac{\zeta(f_2, |c_2 \chi^{-1})}{\rho(|c_2 \chi^{-1})} d\chi = \frac{\zeta(f_1 * f_2, |c_1 c_2)}{\rho(|c_1 c_2)} \quad (3.1)$$

Proof. By the local functional equation (2.1), we have

$$\frac{\zeta(f_1, |c_1 \chi)}{\rho(|c_1 \chi)} = \zeta(\mathcal{F}f_1, c_1^{-1} \chi^{-1}), \quad \frac{\zeta(f_2, |c_2 \chi^{-1})}{\rho(|c_2 \chi^{-1})} = \zeta(\mathcal{F}f_2, c_2^{-1} \chi)$$

Since $\operatorname{re} c_i < 0$, $\operatorname{re} c_i^{-1} > 0$, $\mathcal{F}f_i(x)c_i(x)^{-1} \in L^1(F^*) \cap L^2(F^*)$. From

$$\zeta(\mathcal{F}f_2, c_2^{-1} \chi) = \int_{F^*} \mathcal{F}f_2(x)c_2^{-1}(x)\chi(x)d^*x,$$

we see that $\zeta(\mathcal{F}f_2, c_2^{-1} \chi)$, as a function of χ , is the Fourier-Mellin transform of $\mathcal{F}f_2(x)c_2^{-1}(x)$. Similarly,

$$\overline{\zeta(\mathcal{F}f_1, c_1^{-1} \chi^{-1})} = \int_{F^*} \overline{\mathcal{F}f_1(x)c_1^{-1}(x)\chi(x)}d^*x$$

is the Fourier-Mellin transform of $\overline{\mathcal{F}f_1(x)c_1^{-1}(x)}$. So as a function of χ , both $\overline{\zeta(\mathcal{F}f_1, c_1^{-1} \chi^{-1})}$ and $\zeta(\mathcal{F}f_2, c_2^{-1} \chi)$ are in $L^2(\widehat{F^*})$, their product is integrable. By Plancherel formula,

$$\left(\overline{\zeta(\mathcal{F}f_1, c_1^{-1} \chi^{-1})}, \zeta(\mathcal{F}f_2, c_2^{-1} \chi) \right)_{L^2(\widehat{F^*})} = \left(\mathcal{F}f_1(x)c_1^{-1}(x), \mathcal{F}f_2(x)c_2^{-1}(x) \right)_{L^2(F^*)},$$

which is the same as

$$\int_{\widehat{F^*}} \zeta(\mathcal{F}f_1, c_1^{-1} \chi^{-1}) \zeta(\mathcal{F}f_2, c_2^{-1} \chi) d\chi = \zeta(\mathcal{F}f_1 \mathcal{F}f_2, c_1^{-1} c_2^{-1}). \quad (3.2)$$

Then using $\mathcal{F}f_1 \mathcal{F}f_2 = \mathcal{F}(f_1 * f_2)$, we have

$$\zeta(\mathcal{F}f_1 \mathcal{F}f_2, c_1^{-1}c_2^{-1}) = \zeta(\mathcal{F}(f_1 * f_2), c_1^{-1}c_2^{-1}) = \frac{\zeta(f_1 * f_2, ||c_1c_2)}{\rho(||c_1c_2)}.$$

This proves (3.1). \square

We give a heuristic argument how (3.1) can be used to derive a formula that expresses $(c_1c_2)(x+y)$ in terms of $(c_1\chi)(x)(c_2\chi^{-1})(y)$ and ρ -factors. Assume

$$B(x, y) \stackrel{\text{def}}{=} \int_{\widehat{F^*}} \frac{\rho(||c_1c_2)}{\rho(||c_1\chi)\rho(||c_2\chi^{-1})} (c_1\chi)(x)(c_2\chi^{-1})(y) d\chi \quad (3.3)$$

converges and consider it as a function of $(x, y) \in F \times F$, then

$$\int_{F \times F} B(x, y) f(x) g(y) dx dy = \rho(||c_1c_2) \int_{\widehat{F^*}} \frac{\zeta(f_1, ||c_1\chi)}{\rho(||c_1\chi)} \frac{\zeta(f_2, ||c_2\chi^{-1})}{\rho(||c_2\chi^{-1})} d\chi \quad (3.4)$$

On the other hand side, we have

$$\int_{F \times F} (c_1c_2)(x+y) f(x) g(y) dx dy = \int_F (c_1c_2)(t) f * g(t) dt = \zeta(f * g, ||c_1c_2) \quad (3.5)$$

By (3.1), the right hand sides of (3.4) and (3.5) would be equal, we would have $B(x, y) = c_1c_2(x+y)$. However a direct computation shows that the integral in (3.3) is not convergent on the region $\text{re } c_1 < 0, \text{re } c_2 < 0$. We will prove that it is convergent on the certain region (Proposition 3.4). The idea for the proof of Theorem 1.1 below is to make the above heuristic argument rigorous in the region $\text{re } c_1 > 0, \text{re } c_2 > 0$. We now extend (3.1) to this case.

To extend (3.1), we interpret the integral in the left hand side of (3.1) as a contour integral. The connected component $\widetilde{F^*}_0$ of $\widetilde{F^*}$ containing the trivial character is a Riemann surface: for $F = \mathbb{R}, \mathbb{C}$, $\widetilde{F^*}_0 = \mathbb{C}$, where $s \in \mathbb{C}$ represents the quasi-character $x \mapsto |x|^s$; for F non-Archimedean, $\widetilde{F^*}_0 = \mathbb{C}/\frac{2\pi}{\log q} \mathbb{Z}i$. In all the three cases, $\widetilde{F^*}_0$, the connect component of the character group $\widetilde{F^*}$ containing 1 is a curve in $\widetilde{F^*}_0$, we view it as a contour with direction going upwards. And we will denote this contour by C_0 . The connected components of $\widetilde{F^*}$ are parametrized by the characters of maximal compact subgroup K of F^* . For $F = \mathbb{R}$, $K = \{1, -1\}$. For $F = \mathbb{C}$, $K = S^1$. For F is non-Archimedean, $K = \mathcal{O}^*$. For a character $\alpha : K \rightarrow S^1$, we extend it to a character $\alpha : F^* \rightarrow S^1$ as follows: if $F = \mathbb{R}$, we put $\alpha(x) = \alpha(x/|x|)$; if $F = \mathbb{C}$, we put $\alpha(re^{i\theta}) = \alpha(e^{i\theta})$; for F non-Archimedean, $\alpha(\pi^n u) = \alpha(u)$, where $u \in \mathcal{O}^*$, as we already defined in Section 2. Then $\widetilde{F^*} = \sqcup_{\alpha \in \widehat{K}} \alpha \widetilde{F^*}_0$ is the decomposition of $\widetilde{F^*}$ into connected components. A quasi-characters c is in $\alpha \widetilde{F^*}_0$ iff

$$c(kg) = \alpha(k)c(g)$$

for $k \in K$. If $f \in \mathcal{S}(F)$ satisfies

$$f(kx) = \alpha^{-1}(k)f(x) \quad (3.6)$$

for $k \in K$, then the zeta integral $\zeta(f, c) = 0$ unless c is in the connected components of α . If f is a compactly supported smooth function on F^* , $\zeta(f, c)$ converges on all $c \in \widetilde{F^*}$ and is a holomorphic function on c .

Suppose $\Phi(c)$ is a meromorphic function on $c \in \widetilde{F^*}_0$ with no poles in the line $\widetilde{F^*}_0$, then the integral $\int_{\widehat{F^*}_0} \Phi(\chi) d\chi$ can be written as the contour integral $\lambda_F \int_{C_0} \Phi(||^s) ds$, where

$$\lambda_{\mathbb{R}} = \frac{1}{4\pi i}, \quad \lambda_{\mathbb{C}} = \frac{1}{4\pi^2 i}, \quad \lambda_F = \frac{\log q}{(1 - q^{-1})2\pi i} \quad \text{for } F \text{ non-Archimedean.} \quad (3.7)$$

These formulas follow from our descriptions of the Haar measure $d\chi$ on $\widehat{F^*}$. For c_1 (resp., c_2) in the connected component of α_1 (resp., α_2), and f_1, f_2 satisfies

$$f_1(kx) = \alpha_1^{-1}(k)f_1(x); \quad f_2(kx) = \alpha_2^{-1}(k)f_2(x). \quad (3.8)$$

for $k \in K$, then (3.1) can be reformulated as

$$\lambda_F \int_{C_0} \frac{\zeta(f_1, c_1 | |^{1+s})}{\rho(c_1 | |^{1+s})} \frac{\zeta(f_2, c_2 | |^{1-s})}{\rho(c_2 | |^{1-s})} ds = \frac{\zeta(f_1 * f_2, | |_{c_1 c_2})}{\rho(| |_{c_1 c_2})}. \quad (3.9)$$

Our assumption that f_1 and f_2 are compactly supported on F^* implies that $\zeta(f_i, c_i | |^{1+s})$ is an entire function on s . Since $\text{re } c_1 < 0$, we can write $c_1 = | |^r \chi$, $r < 0, \chi \in \widehat{F^*}$, we have

$$\frac{1}{\rho(c_1 | |^{1+s})} = c_1(-1)\rho(c_1^{-1} | |^{-s}) = c_1(-1)\rho(\chi^{-1} | |^{-(s+r)}).$$

By (2.4), (2.5), (2.6), (2.7), we see that, the meromorphic function (of variable s) $\frac{1}{\rho(c_1 | |^{1+s})}$ is regular on $\text{re } s < -r$, i.e., all poles of $\frac{1}{\rho(c_1 | |^{1+s})}$ (if any) is on the half space $\text{re } s \geq -r > 0$, which are on the right to C_0 . Similarly, all the poles of $\frac{1}{\rho(c_2 | |^{1-s})}$ is one the left to C_0 by the condition $\text{re } c_2 < 0$.

Lemma 3.2. *Let α_1, α_2 be the characters of K . Assume f_1 and f_2 are compactly supported smooth functions on F^* that satisfy (3.8) and assume c_1 and c_2 are in the connected components of α_1 and α_2 respectively. Let C be a contour in the complex plane from $-\infty i$ to ∞i for $F = \mathbb{R}, \mathbb{C}$, or from the bottom to the top of the strip $0 \leq \text{Im } s \leq 2\pi/\log q$ with the starting point and the ending point representing an equal quasi-character of F^* when F is non-Archimedean. Assume that all the poles of meromorphic function $\frac{1}{\rho(c_1 | |^{1+s})}$ of s are on the right to C and all the poles of meromorphic function $\frac{1}{\rho(c_2 | |^{1-s})}$ of s are on the left to C (See Figure 1 below), then*

$$\lambda_F \int_C \frac{\zeta(f_1, c_1 | |^{1+s})}{\rho(c_1 | |^{1+s})} \frac{\zeta(f_2, c_2 | |^{1-s})}{\rho(c_2 | |^{1-s})} ds = \frac{\zeta(f_1 * f_2, | |_{c_1 c_2})}{\rho(| |_{c_1 c_2})}, \quad (3.10)$$

where λ_F is as in (3.7).

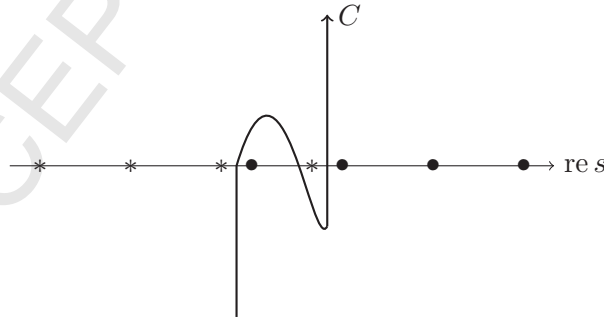


FIGURE 1. • are poles of $\frac{1}{\rho(c_1 | |^{1+s})}$, * are poles of $\frac{1}{\rho(c_2 | |^{1-s})}$.

Proof. We view $\frac{\zeta(f_1, c_1 | |^{1+s})}{\rho(c_1 | |^{1+s})} \frac{\zeta(f_2, c_2 | |^{1-s})}{\rho(c_2 | |^{1-s})}$ as a meromorphic functions of three variables c_1, c_2 and $s \in \mathbb{C}$. We let D be the domain of pairs $(c_1, c_2) \in \alpha_1 \widehat{F^*}_0 \times \alpha_2 \widehat{F^*}_0$ such that the poles of $\frac{1}{\rho(c_1 | |^{1+s})}$ and $\frac{1}{\rho(c_2 | |^{1-s})}$ as meromorphic functions of s don't overlap. It is clear that D is connected. Since f_1 and f_2 are compactly supported on F^* , $\zeta(f_1, | |_{c_1 \chi})$ and $\zeta(f_2, | |_{c_2 \chi^{-1}})$ are rapidly decay as

$\text{Im} \chi \rightarrow \infty$. So the integral converges and the result is a meromorphic functions of $(c_1, c_2) \in D$. By Lemma 3.1, the formula (3.10) holds when $\text{re } c_1, \text{re } c_2 < 0$ and C is the contour $C_0 = \widehat{F}^*$. Notice that C_0 separate the poles of $\frac{1}{\rho(c_1|^{1+s})}$ and $\frac{1}{\rho(c_2|^{1-s})}$ as C , so (3.10) holds under the condition of the Lemma by analytic continuation. \square

Lemma 3.3. *Assume f_1 and f_2 as in Lemma 3.2, and assume c_1 and c_2 are in the connected components of α_1 and α_2 respectively. Suppose $1 > \text{re } c_1, \text{re } c_2 > 0$ for $F = \mathbb{R}$, $\frac{1}{2} > \text{re } c_1, \text{re } c_2 > 0$ for $F = \mathbb{C}$, or $\text{re } c_1, \text{re } c_2 > 0$ for F non-Archmedean. Then*

$$\begin{aligned} & \int_{\widehat{F}^*} \frac{\zeta(f_1, |c_1 \chi|)}{\rho(|c_1 \chi|)} \frac{\zeta(f_2, |c_2 \chi^{-1}|)}{\rho(|c_2 \chi^{-1}|)} d\chi \\ &= \frac{\zeta(f_1 * f_2, |c_1 c_2|)}{\rho(|c_1 c_2|)} - \frac{\zeta(f_1, |c_1|) \zeta(f_2, |c_2|)}{\rho(|c_1 c_2|)} - \frac{\zeta(f_2, |c_2|) \zeta(f_1, |c_1|)}{\rho(|c_1 c_2|)}. \end{aligned} \quad (3.11)$$

Proof. The left hand side is equal to the contour integral

$$\lambda_F \int_{C_0} \frac{\zeta(f_1, c_1 |^{1+s})}{\rho(c_1 |^{1+s})} \frac{\zeta(f_2, c_2 |^{1-s})}{\rho(c_2 |^{1-s})} ds.$$

Let C be a contour as in Lemma 3.2, we compare the difference of integrals

$$\Delta \stackrel{\text{def}}{=} \left(\int_{C_0} - \int_C \right) \frac{\zeta(f_1, c_1 |^{1+s})}{\rho(c_1 |^{1+s})} \frac{\zeta(f_2, c_2 |^{1-s})}{\rho(c_2 |^{1-s})} ds.$$

Because our assumption of the upper bounds for $\text{re } s_1, \text{re } s_2$ for $F = \mathbb{R}, \mathbb{C}$, we see that there are two poles of

$$\frac{1}{\rho(c_1 |^{1+s}) \rho(c_2 |^{1-s})} = c_1(-1) c_2(-1) \rho(c_1^{-1} |^{-s}) \rho(c_2^{-1} |^s)$$

between C_0 and C , they are $|^s = -c_1$ and $|^s = c_2$. When we deform C_0 to C , we obtains two possible residues at $|^s = -c_1$ and $|^s = c_2$, see Figure 2 below.

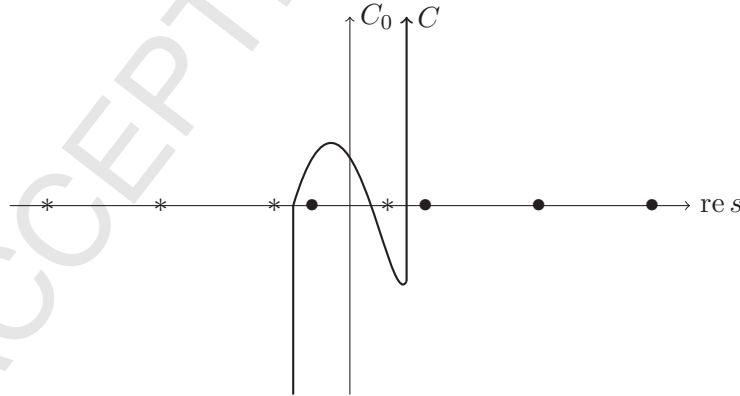


FIGURE 2. \bullet are poles of $\frac{1}{\rho(c_1|^{1+s})}$, $*$ are poles of $\frac{1}{\rho(c_2|^{1-s})}$.

So Δ is given as the sum of two residues:

$$\Delta = 2\pi i \lambda_F (\text{Res}_{|s=-c_1} - \text{Res}_{|s=c_2}) \frac{\zeta(f_1, c_1 |^{1+s})}{\rho(c_1 |^{1+s})} \frac{\zeta(f_2, c_2 |^{1-s})}{\rho(c_2 |^{1-s})}$$

One obtain the result by a direct computation of the residues. \square

Proposition 3.4. *Let $S \subset \widetilde{F}^* \times \widetilde{F}^*$ be the region of quasi-characters given by*

$$S = \{(c_1, c_2) \mid \operatorname{re} c_1 + \operatorname{re} c_2 > 0\}$$

for $F \neq \mathbb{C}$ and for $F = \mathbb{C}$,

$$S = \{(c_1, c_2) \mid \operatorname{re} c_1 > 0, \operatorname{re} c_2 > 0\}.$$

then

$$\int_{\widetilde{F}^*} \frac{1}{\rho(|c_1\chi|)\rho(|c_2\chi^{-1}|)} d\chi \quad (3.12)$$

is convergent for every $(c_1, c_2) \in S$ that is not a pole of the integrand. And the convergence is uniform when (c_1, c_2) varies on any compact subset not containing the pole of the integrand.

Proof. We prove the lemma in three cases: F is a non-archimedean field; $F = \mathbb{R}$; and $F = \mathbb{C}$. In the first case, we write $F^* = \langle \pi \rangle \mathcal{O}^*$, $c_1 = \psi_1 |\cdot|^{s_1}$, $c_2 = \psi_2 |\cdot|^{s_2}$, where $\psi_1, \psi_2 \in \widehat{\mathcal{O}^*}$. The integral (3.12) is a sum of integrals over countably many connected components of \widehat{F}^* :

$$(3.12) = \sum_i \int_{\widehat{F}_0^*} \frac{1}{\rho(|^{s_1+1}\psi_1\chi_i\chi|)\rho(|^{s_2+1}\psi_2\chi_i^{-1}\chi^{-1}|)} d\chi$$

where χ_i runs through $\widehat{\mathcal{O}^*}$. Since the integrand is continuous and \widehat{F}_0^* is compact, each summand is integrable. We will denote the conductor of a character α of \mathcal{O}^* by $N(\alpha)$. If $N(\chi_i) > m \stackrel{\text{def}}{=} \max(N(\psi_1), N(\psi_2))$, then $N(\chi_i\psi_1) = N(\chi_i\psi_2) = N(\chi_i)$. For this case, using (2.7), we have

$$\begin{aligned} & \left| \frac{1}{\rho(|^{s_1+1}\psi_1\chi_i\chi|)\rho(|^{s_2+1}\psi_2\chi_i^{-1}\chi^{-1}|)} \right| \\ &= \frac{1}{|q^{N(\psi_1\chi_i)s_1}\tau(\psi_1\chi_i)q^{N(\psi_2\chi_i^{-1})s_2}\tau(\psi_2\chi_i^{-1})|} \\ &= \frac{1}{q^{N(\chi_i)(\operatorname{re} s_1 + \operatorname{re} s_2 + 1)}} \end{aligned}$$

where we used $|\tau(\psi_1\chi_i)| = q^{\frac{1}{2}N(\psi_1\chi_i)} = q^{\frac{1}{2}N(\chi_i)}$ and $|\tau(\psi_2\chi_i^{-1})| = q^{\frac{1}{2}N(\chi_i)}$. The sum of terms with $N(\chi_i) > m \stackrel{\text{def}}{=} \max(N(\psi_1), N(\psi_2))$ is bounded by a scalar times

$$\sum_{N=m+1}^{\infty} C(N)q^{-N(\operatorname{re} s_1 + \operatorname{re} s_2 + 1)} \quad (3.13)$$

where $C(N)$ denotes the number of characters of \mathcal{O}^* with conductor N . We have

$$C(1) = q - 2, C(2) = q^2 - q + 1, \dots, C(N) = q^N - q^{N-1} \text{ for } N \geq 2.$$

It is clear that (3.13) converges. There are only finitely many χ_i 's with $N(\chi_i) \leq m$, they do not cause harm to our result.

Case $F = \mathbb{R}$. Let $c_1 = |\cdot|^{s_1} \operatorname{sign}^{\epsilon_1}$, $c_2 = |\cdot|^{s_2} \operatorname{sign}^{\epsilon_2}$, ϵ_i ($i = 1, 2$) is 0 or 1. The integral (3.12) is a sum of two integrals on the two components of $\widehat{\mathbb{R}^*}$, each is of the form

$$\int_{\mathbb{R}} \frac{\Gamma((1 - (1 + s_1 + it) + \epsilon_1)/2)}{\Gamma((1 + s_1 + it + \epsilon_1)/2)} \frac{\Gamma((1 - (1 + s_2 - it) + \epsilon_2)/2)}{\Gamma((1 + s_2 + it + \epsilon_2)/2)} dt.$$

We write the above integral as $\int_{|t| \leq 1} + \int_{|t| > 1}$. The first terms cause no harm. For the second term, we apply Stirling's asymptotic formula (2.9), we see that the integrand is bounded by a scalar times

$$|t|^{-\operatorname{re} s_1 - \operatorname{re} s_2 - 1}.$$

The result follows from the assumption $\operatorname{re} s_1 + \operatorname{re} s_2 > 0$.

Case $F = \mathbb{C}$. Let $c_1(re^{i\theta}) = e^{im_1\theta}r^{2s_1}$, $c_2(re^{i\theta}) = e^{im_2\theta}r^{2s_2}$, i.e., $c_i = \delta_{m_i}|\cdot|_{\mathbb{C}}^{s_i}$ ($i = 1, 2$). Using (2.5), we see that the integral (3.12) is a scalar times

$$\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\Gamma(-s_1 - it + |m_1 + n|/2) \Gamma(-s_2 + it + |n - m_2|/2)}{\Gamma(1 + s_1 + it + |m_1 + n|/2) \Gamma(1 + s_2 - it + |n - m_2|/2)} dt$$

We will prove the case that both m_1 and m_2 are even. The other three cases about the parities of m_1 and m_2 have similar proof. The sum is

$$\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\Gamma(-s_1 - it + |2\delta + n|/2) \Gamma(-s_2 + it + |n|/2)}{\Gamma(1 + s_1 + it + |2\delta + n|/2) \Gamma(1 + s_2 - it + |n|/2)} dt = \sum_{n \in 2\mathbb{Z}} \int_{\mathbb{R}} + \sum_{n \in 2\mathbb{Z}+1} \int_{\mathbb{R}}$$

where $2\delta = m_1 + m_2$ is even. We will prove the first sum is convergent and the convergence is uniform when (s_1, s_2) on a compact set, the proof for the second sum is similar. The absolute value of the first sum is bounded by

$$\begin{aligned} & \int_{\mathbb{R}} \left| \frac{\Gamma(-s_1 - it + |\delta|) \Gamma(-s_2 + it)}{\Gamma(1 + s_1 + it + |\delta|) \Gamma(1 + s_2 - it)} \right| dt \\ & + \int_{\mathbb{R}} \sum_{k=1}^{\infty} \left| \frac{\Gamma(-s_1 - it + |\delta + k|) \Gamma(-s_2 + it + k)}{\Gamma(1 + s_1 + it + |\delta + k|) \Gamma(1 + s_2 - it + k)} \right| dt \\ & + \int_{\mathbb{R}} \sum_{k=1}^{\infty} \left| \frac{\Gamma(-s_1 - it + |\delta - k|) \Gamma(-s_2 + it + k)}{\Gamma(1 + s_1 + it + |\delta - k|) \Gamma(1 + s_2 - it + k)} \right| dt \end{aligned}$$

The first term causes no harm, thanks to (2.9) and the assumption $\operatorname{re} s_i > 0$. We only prove

$$\int_{\mathbb{R}} \sum_{k=1}^{\infty} \left| \frac{\Gamma(-s_1 - it + |\delta + k|) \Gamma(-s_2 + it + k)}{\Gamma(1 + s_1 + it + |\delta + k|) \Gamma(1 + s_2 - it + k)} \right| dt \quad (3.14)$$

converges and the sum is bounded on $\sigma_1 \leq \operatorname{re} s_1, \operatorname{re} s_2 \leq \sigma_2$. The proof of the similar statement for the third term is similar. Using $\Gamma(z+1) = z\Gamma(z)$, we have, for a positive integer k ,

$$\frac{\Gamma(-s - it + k)}{\Gamma(1 + s + it + k)} = \frac{\Gamma(-s - it)}{\Gamma(s + it)} \frac{1}{s + it + k} \prod_{j=0}^{k-1} \frac{-s - it + j}{s + it + j}$$

For $\operatorname{re} s > 0$,

$$\left| \frac{-s - it + j}{s + it + j} \right| < 1.$$

Our integrand in (3.14) is bounded by

$$\int_{\mathbb{R}} \frac{|\Gamma(-s_1 - it + |\delta|) \Gamma(-s_2 + it)|}{|\Gamma(1 + s_1 + it + |\delta|) \Gamma(1 + s_2 - it)|} \sum_{n=1}^{\infty} \frac{1}{|(s_1 + it + |k + \delta|)(s_2 - it + k)|} dt$$

By the lemma below, it is bounded above by

$$C \int_{\mathbb{R}} \frac{|\Gamma(-s_1 - it + |\delta|) \Gamma(-s_2 - it)|}{|\Gamma(1 + s_1 + it + |\delta|)|} (|t| + 1)^{-2} dt$$

for some scalar C . Applying the Stirling's asymptotic formula (2.9) to the Gamma functions, we prove the convergence result. \square

Lemma 3.5. Suppose $0 < \sigma_1 < \operatorname{re} s_1, \operatorname{re} s_2 < \sigma_2$, let

$$h(s_1, s_2, t) = \sum_{k=1}^{\infty} \frac{1}{|(s_1 + it + |k + \delta|)(s_2 - it + k)|}$$

Then $|h(s_1, s_2, t)| = O(|t|^{-1})$ as $|t| \rightarrow \infty$, where the implied constant in O depends only on σ_1, σ_2 and the bounds of $\operatorname{im} s_1, \operatorname{im} s_2$.

Proof. Let $s_1 = r_1 + iy_1$ and $s_2 = r_2 + iy_2$, so $r_1, r_2 > 0$. It is enough to prove $|t + y_1|^{\frac{1}{2}}|t - y_2|^{\frac{1}{2}}|h(s_1, s_2, t)| = O(1)$ as $t \rightarrow \infty$.

$$|t - y_2|^{\frac{1}{2}} \frac{1}{|s_2 - it + k|} = \frac{1}{\sqrt{(r_2 + k)^2 |t - y_2|^{-1} + |t - y_2|}} \leq \frac{1}{|r_2 + k|}$$

Similarly we have

$$|t + y_1|^{\frac{1}{2}} \frac{1}{|(s_1 + it + |k + \delta|)|} \leq \frac{1}{|(r_1 + |k + \delta|)|}.$$

So

$$|t + y_1|^{\frac{1}{2}}|t - y_2|^{\frac{1}{2}}|h(s_1, s_2, t)| \leq \sum_{n=1}^{\infty} \frac{1}{|r_1 + |k + \delta|| |r_2 + k|} = O(1).$$

\square

Proof of Theorem 1.1. Both sides of (1.1) are continuous functions on $F^* \times F^*$. Denote the right hand side of (1.1) by $B(x, y)$. It is enough to prove

$$\int_{F \times F} (c_1 c_2(x + y) - (c_1 c_2)(x) - (c_1 c_2)(y)) f(x) g(y) dx dy = \int_{F \times F} B(x, y) f(x) g(y) dx dy \quad (3.15)$$

for all compactly supported smooth functions f, g on F^* as in Lemma 3.2 and Lemma 3.3. The left hand side is

$$\begin{aligned} & \int_{F \times F} c_1 c_2(x + y) f(x) g(y) dx dy - \zeta(g, | \cdot |) \zeta(f, | \cdot | c_1 c_2) - \zeta(f, | \cdot |) \zeta(g, | \cdot | c_1 c_2) \\ &= \int_{F \times F} c_1 c_2(u) f(u - y) g(y) du dy - \zeta(g, | \cdot |) \zeta(f, | \cdot | c_1 c_2) - \zeta(f, | \cdot |) \zeta(g, | \cdot | c_1 c_2) \\ &= \int_F c_1 c_2(u) (f * g)(u) du - \zeta(g, | \cdot |) \zeta(f, | \cdot | c_1 c_2) - \zeta(f, | \cdot |) \zeta(g, | \cdot | c_1 c_2) \\ &= \zeta(f * g, | \cdot | c_1 c_2) - \zeta(g, | \cdot |) \zeta(f, | \cdot | c_1 c_2) - \zeta(f, | \cdot |) \zeta(g, | \cdot | c_1 c_2). \end{aligned}$$

The right hand side is

$$\begin{aligned} & \int_{F \times F} B(x, y) f(x) g(y) dx dy \\ &= \int_{\widehat{F^*}} \frac{\rho(| \cdot | c_1 c_2)}{\rho(| \cdot | c_1 \chi) \rho(| \cdot | c_2 \chi^{-1})} \zeta(f, | \cdot | c_1 \chi) \zeta(g, | \cdot | c_2 \chi^{-1}) d\chi. \end{aligned}$$

The identity (3.15) holds by Lemma 3.3. \square

We now prove a variant of Theorem 1.1.

Theorem 3.6. *For quasi-characters (c_1, c_2) in the domain S given in Proposition 3.4, for all $x, y \in F^*$,*

$$(c_1 c_2)(x + y) = \lambda_F \sum_i \int_C \frac{\rho(|_F c_1 c_2|)}{\rho(|_F c_1 \chi_i|^s) \rho(|_F c_2 \chi_i^{-1}|^{-s})} (c_1 \chi_i|^s)(x) (c_2 \chi_i^{-1}|^{-s})(y) ds. \quad (3.16)$$

where χ_i runs through all characters of K , C is a contour as in Lemma 3.2, i.e., it separate the poles of $\frac{1}{\rho(|_F c_1 \chi_i|^s)}$ and $\frac{1}{\rho(|_F c_2 \chi_i^{-1}|^{-s})}$ and the constant λ_F is given in (3.7).

Proof. The proof is similar to that of Theorem 1.1. We multiply the both sides of (3.16) by $f(x)$ and $g(y)$, then take integrals on $(x, y) \in F \times F$. The results follows from Lemma 3.2. \square

Proof of Theorem 1.3. Since two local fields E and F appear, we use notations $||_E$ and $||_F$. For $x, y \in F^*$, we expand $c_1 c_2(x + y) - c_1 c_2(x) - c_1 c_2(y)$ in two different ways, using (1.1) for local fields F or E , we get

$$\begin{aligned} & \int_{\widehat{F}^*} \frac{\rho_F(|_F c_1 c_2|)}{\rho_F(|_F c_1 \chi|) \rho_F(|_F c_2 \chi^{-1}|)} (c_1 \chi)(x) (c_2 \chi^{-1})(y) d\chi \\ &= \int_{\widehat{E}^*} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \beta|) \rho_E(|_E c_2 \beta^{-1}|)} (c_1 \beta)(x) (c_2 \beta^{-1})(y) d\beta. \end{aligned}$$

By (1.5),

$$\begin{aligned} & \int_{\widehat{E}^*} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \beta|) \rho_E(|_E c_2 \beta^{-1}|)} (c_1 \beta)(x) (c_2 \beta^{-1})(y) d\beta \\ &= \int_{\alpha \in Ker} \int_{\chi \in F^*} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \alpha \chi|) \rho_E(|_E c_2 \alpha^{-1} \chi^{-1}|)} (c_1 \alpha \chi)(x) (c_2 \alpha^{-1} \chi^{-1})(y) d\alpha d\chi \quad (\because \alpha(x) = \alpha(y) = 1) \\ &= \int_{\chi \in F^*} \left(\int_{\alpha \in Ker} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \alpha \chi|) \rho_E(|_E c_2 \alpha^{-1} \chi^{-1}|)} d\alpha \right) (c_1 \chi)(x) (c_2 \chi^{-1})(y) d\chi \end{aligned}$$

So we have

$$\begin{aligned} & \int_{\widehat{F}^*} \frac{\rho_F(|_F c_1 c_2|)}{\rho_F(|_F c_1 \chi|) \rho_F(|_F c_2 \chi^{-1}|)} (c_1 \chi)(x) (c_2 \chi^{-1})(y) d\chi \\ &= \int_{\chi \in \widehat{F}^*} \left(\int_{\alpha \in Ker} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \alpha \chi|) \rho_E(|_E c_2 \alpha^{-1} \chi^{-1}|)} d\alpha \right) (c_1 \chi)(x) (c_2 \chi^{-1})(y) d\chi \end{aligned}$$

Therefore

$$\int_{\widehat{F}^*} \frac{\rho_F(|_F c_1 c_2|)}{\rho_F(|_F c_1 \chi|) \rho_F(|_F c_2 \chi^{-1}|)} \chi\left(\frac{x}{y}\right) d\chi = \int_{\chi \in \widehat{F}^*} \left(\int_{\alpha \in Ker} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \alpha \chi|) \rho_E(|_E c_2 \alpha^{-1} \chi^{-1}|)} d\alpha \right) \chi\left(\frac{x}{y}\right) d\chi \quad (3.17)$$

This implies

$$\frac{\rho_F(|_F c_1 c_2|)}{\rho_F(|_F c_1 \chi|) \rho_F(|_F c_2 \chi^{-1}|)} = \int_{\alpha \in Ker} \frac{\rho_E(|_E c_1 c_2|)}{\rho_E(|_E c_1 \alpha \chi|) \rho_E(|_E c_2 \alpha^{-1} \chi^{-1}|)} d\alpha.$$

as two sides have the equal Fourier-Mellin transform by (3.17) \square

4. EXAMPLES

In this section, we give the explicit examples of (1.1) for unramified c_1 and c_2 .

For $F = \mathbb{R}$, $c_i(x) = |x|^{s_i}$ ($i = 1, 2$) with $1 > \operatorname{re} s_1, \operatorname{re} s_2 > 0$. Using

$$\rho(|x|^s)^{-1} = 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2}, \quad \rho(|x|^s \operatorname{sign})^{-1} = i 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{(s+1)\pi}{2}, \quad (4.1)$$

we obtain

$$\frac{\rho(|x|^{1+s_1+s_2})}{\rho(|x|^{1+s_1+it})\rho(|x|^{1+s_2-it})} = \frac{2\Gamma(-s_1-it)\Gamma(-s_2+it) \cos \frac{\pi(s_1+it)}{2} \cos \frac{\pi(s_2-it)}{2}}{\Gamma(-s_1-s_2) \cos \frac{\pi(s_1+s_2)}{2}} \quad (4.2)$$

and

$$\frac{\rho(|x|^{1+s_1+s_2})}{\rho(|x|^{1+s_1+it} \operatorname{sign})\rho(|x|^{1+s_2-it} \operatorname{sign})} = \frac{-2\Gamma(-s_1-it)\Gamma(-s_2+it) \sin \frac{\pi(s_1+it)}{2} \sin \frac{\pi(s_2-it)}{2}}{\Gamma(-s_1-s_2) \cos \frac{\pi(s_1+s_2)}{2}}. \quad (4.3)$$

Using the description of the Haar measure on $\widehat{\mathbb{R}^*}$ in (2.1), we see that (1.1) gives

$$\begin{aligned} & |x+y|^{s_1+s_2} - |x|^{s_1+s_2} - |y|^{s_1+s_2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(-s_1-it)\Gamma(-s_2+it) \cos \frac{\pi(s_1+it)}{2} \cos \frac{\pi(s_2-it)}{2}}{\Gamma(-s_1-s_2) \cos \frac{\pi(s_1+s_2)}{2}} |x|^{s_1+it} |y|^{s_2-it} dt + \\ & - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(-s_1-it)\Gamma(-s_2+it) \sin \frac{\pi(s_1+it)}{2} \sin \frac{\pi(s_2-it)}{2}}{\Gamma(-s_1-s_2) \cos \frac{\pi(s_1+s_2)}{2}} |x|^{s_1+it} |y|^{s_2-it} \operatorname{sign}(x/y) dt. \end{aligned} \quad (4.4)$$

For the case $x, y > 0$, this formula can be simplified as (1.2).

For $F = \mathbb{C}$, $c_i(x) = |x|_{\mathbb{C}}^{s_i} = |x\bar{x}|^{s_i}$ ($i = 1, 2$) with $\frac{1}{2} > \operatorname{re} s_1, \operatorname{re} s_2 > 0$, using (2.5) and the measure on $\widehat{\mathbb{C}^*}$ in (2.2), (1.1) is

$$\begin{aligned} & ((x+y)(\bar{z}+\bar{w}))^{s_1+s_2} - (x\bar{x})^{s_1+s_2} - (y\bar{y})^{s_1+s_2} \\ &= \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(s_1+s_2+1)\Gamma(-s_1-it+\frac{|n|}{2})\Gamma(-s_2+it+\frac{|n|}{2})}{\Gamma(-s_1-s_2)\Gamma(1+s_1+it+\frac{|n|}{2})\Gamma(1+s_2-it+\frac{|n|}{2})} (x\bar{x})^{s_1+it} (y\bar{y})^{s_2+it} \delta_n\left(\frac{x}{y}\right) dt. \end{aligned} \quad (4.5)$$

For F non-Archimedean, $c_i(x) = |x|^{s_i}$ ($i = 1, 2$) with $\operatorname{re} s_1, \operatorname{re} s_2 > 0$, using (2.6) and (2.7), (1.1) is

$$\begin{aligned} & |x+y|^{s_1+s_2} - |x|^{s_1+s_2} - |y|^{s_1+s_2} \\ &= \frac{\log q}{2\pi(1-q^{-1})} \int_0^{\frac{2\pi}{\log q}} \frac{(1-q^{s_1+s_2})(1-q^{-s_1-it-1})(1-q^{-s_2+it-1})}{(1-q^{-s_1-s_2-1})(1-q^{s_1+it})(1-q^{s_2-it})} |x|^{s_1+it} |y|^{s_2-it} dt \\ & + \frac{\log q}{2\pi(1-q^{-1})} \frac{1-q^{s_1+s_2}}{1-q^{-s_1-s_2-1}} \int_0^{\frac{2\pi}{\log q}} |x|^{s_1+it} |y|^{s_2-it} dt \sum_{\chi} q^{-N(\chi)(s_1+s_2+1)} \chi(-x/y). \end{aligned} \quad (4.6)$$

where χ runs through all non-trivial characters of \mathcal{O}^* , and $N(\chi)$ is the conductor of χ .

5. AN NEW PROOF OF BARNES' IDENTITY FOR PRINCIPAL SERIES OF GL_2 .

In this section, our local field $F \neq \mathbb{C}$, let S be the set of pairs of quasi-characters as in Proposition 3.4. We have

Theorem 5.1. *For $(c_1, c_2), (d_1, d_2) \in S$,*

$$\lambda_F \sum_{\chi_i} \int_C \frac{\rho(|_F c_1 c_2|)}{\rho(|_F c_1 \chi_i^{-1}|^{-s}) \rho(|_F c_2 \chi_i|^{-s})} \frac{\rho(|_F d_1 d_2|)}{\rho(|_F d_1 \chi_i|^{-s}) \rho(|_F d_2 \chi_i^{-1}|^{-s})} ds = \frac{\rho(|_F c_1 d_1 c_2 d_2|)}{\rho(|_F c_1 d_1|) \rho(|_F c_2 d_2|)} \quad (5.1)$$

where λ_F is as in (3.7), C is a path of s as follows: for $F = \mathbb{R}$ (non-Archimedean field), it is from $-\infty$ to $i\infty$ (from 0 to $\frac{2\pi i}{\log q}$) that separate the poles of $\frac{1}{\rho(|_F c_1 \chi_i^{-1}|^{-s}) \rho(|_F d_2 \chi_i^{-1}|^{-s})}$ and the poles of $\frac{1}{\rho(|_F c_2 \chi_i|^{-s}) \rho(|_F d_1 \chi_i|^{-s})}$

Our proof is similar to the proof of (1.12) by expanding $(x+y)^{m+n}$ in two different ways.

Proof. The convergence of (5.1) follows from Proposition 3.4. We may assume $\text{re } c_i$ and $\text{re } d_i$ satisfy the conditions of Theorem 1.1 and $\text{re } d_1 > \text{re } c_2$, $\text{re } d_2 > \text{re } c_1$, the general case follows from analytic continuation. So (1.1) holds and it holds for d_1, d_2 in place of c_1, c_2 :

$$(d_1 d_2)(x+y) - (d_1 d_2)(x) - (d_1 d_2)(y) = \int_{\widehat{F}^*} \frac{\rho(|_F d_1 d_2|)}{\rho(|_F d_1 \mu|) \rho(|_F d_2 \mu^{-1}|)} (d_1 \mu)(x) (d_2 \mu^{-1})(y) d\mu. \quad (5.2)$$

Multiply (1.1) and (5.2), the left hand side is

$$\begin{aligned} & c_1 c_2 d_1 d_2 (x+y) - c_1 c_2 d_1 d_2 (x) - c_1 c_2 d_1 d_2 (y) \\ & - c_1 c_2 (x) (d_1 d_2 (x+y) - d_1 d_2 (x) - d_1 d_2 (y)) \\ & - c_1 c_2 (y) (d_1 d_2 (x+y) - d_1 d_2 (x) - d_1 d_2 (y)) \\ & - (c_1 c_2 (x+y) - c_1 c_2 (x)) d_1 d_2 (x) \\ & - (c_1 c_2 (x+y) - c_1 c_2 (y)) d_1 d_2 (y) \\ & = c_1 c_2 d_1 d_2 (x+y) - c_1 c_2 d_1 d_2 (x) - c_1 c_2 d_1 d_2 (y) + (\Delta) \end{aligned} \quad (5.3)$$

where (Δ) denotes the sum of 2nd, 3rd, 4th and 5th line of the left hand side of (5.3). The right hand side is

$$\begin{aligned} & \int_{\widehat{F}^* \times \widehat{F}^*} \frac{\rho(|_F c_1 c_2|)}{\rho(|_F c_1 \chi|) \rho(|_F c_2 \chi^{-1}|)} \frac{\rho(|_F d_1 d_2|)}{\rho(|_F d_1 \mu|) \rho(|_F d_2 \mu^{-1}|)} (c_1 d_1 \chi \mu)(x) (c_2 d_2 \chi^{-1} \mu^{-1})(y) d\chi d\mu \\ & = \int_{\widehat{F}^*} B(\tau) (c_1 d_1 \tau)(x) (c_2 d_2 \tau^{-1})(y) d\tau \end{aligned} \quad (5.4)$$

where

$$B(\tau) = \int_{\widehat{F}^*} \frac{\rho(|_F c_1 c_2|)}{\rho(|_F c_1 \tau \mu^{-1}|) \rho(|_F c_2 \tau^{-1} \mu|)} \frac{\rho(|_F d_1 d_2|)}{\rho(|_F d_1 \mu|) \rho(|_F d_2 \mu^{-1}|)} d\mu.$$

We consider $B(\tau)$ as a contour integral. Let $C(\tau)$ be the contour integral with the same integrand as $B(\tau)$, but the contour is $\sqcup \chi_i C$ as in (5.1), i.e.,

$$C(\tau) = \lambda_F \sum_{\chi_i} \int_C \frac{\rho(|_F c_1 c_2|)}{\rho(|_F c_1 \tau \chi_i^{-1}|^{-s}) \rho(|_F c_2 \tau^{-1} \chi_i|^{-s})} \frac{\rho(|_F d_1 d_2|)}{\rho(|_F d_1 \tau^{-1} \chi_i|^{-s}) \rho(|_F d_2 \tau \chi_i^{-1}|^{-s})} ds.$$

There are four poles between the contours of $B(\tau)$ and $C(\tau)$: $\mu = c_1\tau, c_2^{-1}\tau, d_1^{-1}, d_2$. By the residue theorem, we see that

$$B(\tau) = C(\tau) \quad (5.5)$$

$$-\frac{\rho(|_F d_1 d_2)}{\rho(|_F c_1 d_1 \tau) \rho(|_F c_1^{-1} d_2 \tau^{-1})} - \frac{\rho(|_F d_1 d_2)}{\rho(|_F c_2^{-1} d_1 \tau) \rho(|_F c_2 d_2 \tau^{-1})} \quad (5.6)$$

$$-\frac{\rho(|_F c_1 c_2)}{\rho(|_F c_1 d_1 \tau) \rho(|_F c_2 d_1^{-1} \tau^{-1})} - \frac{\rho(|_F c_1 c_2)}{\rho(|_F c_1 d_2^{-1} \tau) \rho(|_F c_2 d_2 \tau^{-1})} \quad (5.7)$$

Substitute this into the right hand side of (5.4), and use (1.1) for the four integrals corresponding the terms in (5.6) and (5.7) (we need to use a variant of (1.1) for two of the integrals), we find (5.4) is

$$\int_{\widehat{F^*}} C(\tau)(c_1 d_1 \tau)(x)(c_2 d_2 \tau^{-1})(y) d\tau + (\Delta).$$

So (5.3)=(5.4) implies that

$$c_1 c_2 d_1 d_2(x+y) - c_1 c_2 d_1 d_2(x) - c_1 c_2 d_1 d_2(y) = \int_{\widehat{F^*}} C(\tau)(c_1 d_1 \tau)(x)(c_2 d_2 \tau^{-1})(y) d\tau. \quad (5.8)$$

Compare this with the following identity which is (1.1) for $c_1 d_1$ and $c_2 d_2$

$$c_1 d_1 c_2 d_2(x+y) - c_1 d_1 c_2 d_2(x) - c_1 d_1 c_2 d_2(y) = \int_{\widehat{F^*}} \frac{\rho(|_F c_1 d_1 c_2 d_2)}{\rho(|_F c_1 d_1 \chi) \rho(|_F c_2 d_2 \chi^{-1})} (c_1 d_1 \chi)(x)(c_2 d_2 \chi^{-1})(y) d\chi, \quad (5.9)$$

we obtain

$$C(\tau) = \frac{\rho(|_F c_1 d_1 c_2 d_2)}{\rho(|_F c_1 d_1 \tau) \rho(|_F c_2 d_2 \tau^{-1})},$$

the case $\tau = 1$ of this identity is (5.1). □

REFERENCES

- [AAR] G.E.Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge Univ. Press. 1999
- [BE] H. Bateman, A. Erdélyi, *Higher Transcendental Functions*, bateman manuscript project ed. vol. 1, McGraw-Hill, New York (1955).
- [BFMO] L. Brekke, P. G.O. Freund, E. Melzer, M. Olson, *Adelic string N-point amplitudes*, Phys. Lett. B, 216, 1-2 (1989) 53-58.
- [D] P. Deligne: Les constantes des équations fonctionnelles des fonctions, in P. Deligne and W. Kuyk (eds.), *Modular functions of one variable, II*, Lecture Notes in Math. 349, 501-597 (1973).
- [FO] P. G. O. Freund and M. Olson, *Nonarchimedean Strings*, Phys. Lett. B199 (1987) 186-190.
- [FW] P. G. O. Freund and E. Witten, *Adelic String Amplitudes*, Phys. Lett. B199 (1987) 191.
- [FLRST] J. Fuselier, L. Long, R. Ramakrishna, H. Swisher, F-T. Tu, *Hypergeometric Functions over Finite Fields*, 2017, arXiv:1510.02575
- [GGPS] I.M.Gelfand, M.I.Graev, and I.I.Piatetskii-Shapiro, *Representation Theory and Automorphic Functions* (Saunders, Philadelphia, PA, 1996)
- [GJ] R.Godement and H.Jacquet, *Zeta Functions of Simple Algebras*, Lecture Notes in Mathematics, Vol. 260, Springer-Verlag, Berlin-New York, 1972.
- [GKPSW] S. Gubser1, J. Knaute, S. Parikh1, A. Samberg, P. Witaszczyk, *A p-adic version of AdS/CFT*. Commun. Math. Phys. 352, 1019–1059 (2017)
- [H] G. Henniart, Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires. Invent. Math. 113, no. 2 (1993) 339-350.
- [Iw] K. Iwasawa, *A Note on Functions*, Proc. ICM 1950.
- [Lang] S. Lang, *Algebraic Number Theory*, 2nd edition, Springer-Verlag, 1994.

- [L] R.P. Langlands: *On the functional equation of Artin's L-functions*, unpublished manuscript.
- [Li] W.-C. Li, *Barnes' identities and representations of GL_2 , II: Non archimedean local field case*, J. Reine Angew. Math. 345 (1983) 69–92.
- [Neukirch] J. Neukirch, *Algebraic Number Theory*, Springer, 1992.
- [Pr] H.E. Prado, *Reduced Weil Models for Representationis of $GL(2, \mathbb{R})$ and barnes' Lemma*. J. London Math. Soc. (2) 55 (1997), 159-169.
- [Ta] *Fourier analysis in number fields, and Hecke's zeta-functions*, Princeton, May 1950, thesis; reproduced in Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965) pp. 305-347, Acad, Press 1967
- [Za] A. V. Zabrodin, *Nonarchimedean Strings and Bruhat-Tits Trees*, Commun. Math. Phys. 123 (1989) 463-483.

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG

E-mail address: mazhu@ust.hk