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General Section

On a class of Lebesgue-Ljunggren-Nagell type equations

Andrej Dąbrowski[a,*], Nursena Günhan[b], Gökhan Soydan[b]

[a] *Institute of Mathematics, University of Szczecin, 70-451 Szczecin, Poland*
[b] *Department of Mathematics, Bursa Uludağ University, 16059 Bursa, Turkey*

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ABSTRACT

Text. Given odd, coprime integers a, b ($a > 0$), we consider the Diophantine equation $ax^2 + b^{2l} = 4y^n$, $x, y \in \mathbb{Z}$, $l \in \mathbb{N}$, n odd prime, $\gcd(x, y) = 1$. We completely solve the above Diophantine equation for $a \in \{7, 11, 19, 43, 67, 163\}$, and b a power of an odd prime, under the conditions $2^{n-1}b^l \not\equiv \pm 1 (\text{mod } a)$ and $\gcd(n, b) = 1$. For other square-free integers $a > 3$ and b a power of an odd prime, we prove that the above Diophantine equation has no solutions for all integers x, y with ($\gcd(x, y) = 1$), $l \in \mathbb{N}$ and all odd primes $n > 3$, satisfying $2^{n-1}b^l \not\equiv \pm 1 (\text{mod } a)$, $\gcd(n, b) = 1$, and $\gcd(n, h(-a)) = 1$, where $h(-a)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-a})$.

Video. For a video summary of this paper, please visit https://youtu.be/Q0peJ2GmqeM.

* Corresponding author.
E-mail addresses: dabrowskiandrzej7@gmail.com, andrzej.dabrowski@usz.edu.pl (A. Dąbrowski), nursenagunhan@uludag.edu.tr (N. Günhan), gsoydan@uludag.edu.tr (G. Soydan).

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1. Introduction

The Diophantine equation $x^2 + C = y^n$ ($x \geq 1$, $y \geq 1$, $n \geq 3$) has a rich history. Lebesgue proved that this equation has no solution when $C = 1$, and Cohn solved the equation for several values of $1 \leq C \leq 100$. The remaining values of C in the above range were covered by Mignotte and de Weger, and finally by Bugeaud, Mignotte and Siksek. Barros in his PhD thesis considered the range $-100 \leq C \leq -1$. Also, several authors (Abu Muriefah, Arif, Dąbrowski, Le, Luca, Pink, Soydan, Togbé, Ulas,...) became interested in the case where only the prime factors of C are specified. Surveys of these and many other topics can be found in [1] and [5]. Some people studied the more general equation $ax^2 + C = 2^i y^n$, $a > 0$ and $i \leq 2$.

Given odd, coprime integers a, b ($a > 0$), we consider the Diophantine equation

$$ax^2 + b^{2l} = 4y^n, \quad x, y \in \mathbb{Z},\ l, n \in \mathbb{N},\ n\,odd\ prime, \gcd(x, y) = 1. \tag{1}$$

If $a \equiv 1 \pmod 4$, then reducing modulo 4 we trivially obtain that the equation (1) has no solution.

It is known (due to Ljunggren [18]) that the Diophantine equation $ax^2 + 1 = 4y^n$, $n \geq 3$, has no positive solution with $y > 1$ such that $a \equiv 3 (\mathrm{mod}\ 4)$ and the class number of the quadratic field $\mathbb{Q}(\sqrt{-a})$ is not divisible by n. When $a = 3$, then $3x^2 + 1 = 4y^n$ has the only positive solution $(x, y) = (1, 1)$.

As our first result, we completely solve the equation (1) for $a \in \{7, 11, 19, 43, 67, 163\}$, under the conditions $2^{n-1}b^l \not\equiv \pm 1 (\mathrm{mod}\ a)$ and $\gcd(n, b) = 1$.

Theorem 1. *Fix $p \in \{7, 11, 19, 43, 67, 163\}$ and $b = \pm q^r$, with q an odd prime different from p and $r \geq 1$.*

(i) The Diophantine equation

$$px^2 + b^{2l} = 4y^n,\ l \in \mathbb{N},\ \gcd(x, y) = 1 \tag{2}$$

has no solutions (p, x, y, b, l, n) with integers x, y and primes $n > 3$, satisfying the conditions $2^{n-1}b^l \not\equiv \pm 1 (mod\ p)$ and $\gcd(n, b) = 1$.

(ii) If $n = 3$ and $p \neq 7$, then the equation (2) has no solutions $(p, x, y, b, l, 3)$ satisfying the conditions $4b^l \not\equiv \pm 1 (mod\ p)$ and $\gcd(3, b) = 1$.

(iii) If $n = 3$ and $p = 7$, then the equation (2) leads to 6 infinite families of solutions, corresponding to solutions of Pell-type equations (4), (5), (6), (7), and satisfying the conditions $4b^l \not\equiv \pm 1 (mod\ 7)$ and $\gcd(3, b) = 1$.

Remarks. (i) The Diophantine equation (2) has many solutions (infinitely many?) satisfying the conditions $2^{n-1}b^l \equiv \pm 1 (\mathrm{mod}\ p)$ and $\gcd(n, b) = 1$. Examples include $(p, x, y, b, l, n) \in \{(7, \pm 1, 2, \pm 11, 1, 5), (11, \pm 1, 3, \pm 31, 1, 5), (7, \pm 7, 2, \pm 13, 1, 7),$

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$(11, \pm 253, 3, \pm 67, 1, 11)$, $(7, \pm 1, 2, \pm 181, 1, 13)$, $(11, \pm 1801, 3, \pm 21929, 1, 17)$, $(7, \pm 457, 2, \pm 797, 1, 19)$, $(7, \pm 967, 2, \pm 5197, 1, 23)\}$.

(ii) If b is divisible by at least two different odd primes, then the Diophantine equation (2) may have solutions satisfying the conditions $2^{n-1}b^l \not\equiv \pm 1 (\mathrm{mod}\ p)$. Examples include $(p, x, y, b, l, n) \in \{(7, 103820535541, 4, 10341108537, 1, 37)$, $(7, 4865, 46, 1320267, 1, 7)$, $(19, 315003, 49, 909715, 1, 7)$, $(19, 581072253, 49, 3037108805, 1, 11)\}$.

(iii) Write the equation (2) as $px^2 + b^{2l} = 4y(y^{(n-1)/2})^2$ (compare [18, p.116]). Now using $4y = u^2 + pv^2$, taking $u = \pm 1$, and multiplying the equation by p, we arrive at the equation

$$X^2 - p(1 + pv^2)Y^2 = -pb^{2l}. \tag{3}$$

If $b = \pm 1$, we obtain the equation (7') in [18]. Ljunggren used an old result by Mahler to deduce that, if $p > 3$, then (3) has no solution with $Y > 1$ such that any prime divisor of Y divides $p(1 + pv^2)$ as well.

(iv) Question: may we extend Ljunggren's idea to prove non-existence of solutions of our equation for some b^l?

For a family of positive square-free integers a with $h(-a) > 1$ we can prove the following result (a variant of the results by Bugeaud [9] and Arif and Al-Ali [3] in a case of the equation $ax^2 + b^{2l+1} = 4y^n$). Let $h(-a)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-a})$.

Theorem 2. *Fix a positive square-free integer a, different from 3, 7, 11, 19, 43, 67, 163, and $b = \pm q^r$, with q an odd prime not dividing a and $r \geq 1$. Then the Diophantine equation (1) has no solutions (a, x, y, b, l, n), with integers x, y and primes $n > 3$ satisfying the conditions $\gcd(n, h(-a)) = 1$, $2^{n-1}b^l \not\equiv \pm 1 (mod\ a)$, and $\gcd(n, b) = 1$.*

Remarks. (i) There are a lot of positive square-free integers a with $rad(h(-a))|6$ (hypothetically, infinitely many): 18 values of a with $h(-a) = 2$, 54 values of a with $h(-a) = 4$, 31 values of a with $h(-a) = 6$, etc. Here $rad(m)$ denotes the radical of a positive integer m, i.e. the product of all prime divisors of m.

(ii) For fixed a and b we can (in some cases) use MAGMA [8] to solve the Diophantine equation $ax^2 + b^{2l} = 4y^3$ (applying `SIntegralPoints` subroutine of MAGMA to associated families of elliptic curves). In a general case, one can try to prove a variant of Dahmen's result [14] saying that the above equation has no solution for a positive proportion of l's, not divisible by 3.

(iii) The following variant of a result by Laradji, Mignotte and Tzanakis (see [16, Theorem 2.3]) follows immediately from our Theorem 2 (note that always $h(-p) < p$). Let p, q be odd primes with $p \equiv 3 (\mathrm{mod}\ 8)$ and $p > 3$. Then the Diophantine equation $px^2 + q^{2l} = 4y^p$ has no solution (x, y, l) with positive integers x, y, l satisfying $\gcd(x, y) = 1$.

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(iv) Dieulefait and Urroz [15] used the method of Galois representations attached to $\mathbb{Q}$-curves to solve the Diophantine equation $3x^2 + y^4 = z^n$. The authors suggest that their method can be applied to solve this type of equations with 3 replaced by other values of a. We expect that their method can be extended to the case $ax^2 + y^4 = 4z^n$ with small a as well.

(v) We can solve the Diophantine $ax^2+b^{2l} = 4y^n$ for relatively small values of $a > 0$ (at least) in positive integers x, y, l, n, $\gcd(x, y) = 1$, $n \geq 7$ a prime dividing l, by using the Bennett-Skinner strategy [4]. We treat some examples in Section 3. Let us also mention that the smallest positive integer a with $h(-a) = 7$ is 71, and one needs to consider newforms of weight 2 and level 10082.

(vi) Pink [21] used estimates for linear forms in two logarithms in the complex and the p-adic case, to give an explicit bound for the number of solutions of the Diophantine equation $x^2 + (p_1^{\alpha_1} \cdots p_s^{\alpha_s})^2 = 2y^n$ in terms of s and $\max\{p_1, \cdots, p_s\}$. We can prove analogous result concerning the equations $px^2 + (p_1^{\alpha_1} \cdots p_s^{\alpha_s})^2 = 4y^n$, with $p \in \{7, 11, 19, 43, 67, 163\}$.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Below in the proof, b is a power of an odd prime $q \neq p$.

As the class number of $\mathbb{Q}(\sqrt{-p})$ with $p \in \{7, 11, 19, 43, 67, 163\}$ is 1, we have the following factorization

$$\frac{b^l + x\sqrt{-p}}{2} \cdot \frac{b^l - x\sqrt{-p}}{2} = y^n.$$

Now we have

$$\frac{b^l + x\sqrt{-p}}{2} = \left(\frac{u + v\sqrt{-p}}{2}\right)^n,$$

where u, v are odd rational integers. Note that necessarily $\gcd(u, v) = 1$. Equating real parts we get

$$2^{n-1}b^l = u \sum_{r=0}^{(n-1)/2} \binom{n}{2r} u^{n-2r-1}(-p)^r v^{2r}.$$

As u is odd, its possible values are among divisors of b^l. Here, we assume that $2^{n-1}b^l \not\equiv \pm 1 (\mathrm{mod}\ p)$.

(i) If $u = \pm 1$, then $2^{n-1}b^l = \sum_{r=0}^{(n-1)/2} \binom{n}{2r}(-p)^r v^{2r}$, and in particular $2^{n-1}b^l \equiv \pm 1 (\mathrm{mod}\ p)$, a contradiction.

(ii) If $u \neq \pm 1, \pm b^l$, then q divides pvn. Since $\gcd(u, v) = \gcd(p, b) = 1$, then q divides n, a contradiction.

(iii) Assume $u = \pm b^l$. Put $\alpha = \frac{v\sqrt{p}+b^l i}{2}$. Then $(\alpha + \overline{\alpha})^2 = v^2 p$, $\alpha\overline{\alpha} = \frac{1}{4}(v^2 p + b^{2l})$, and $\alpha/\overline{\alpha}$ is not a root of unity. Hence $(\alpha, \overline{\alpha})$ is a Lehmer pair. Note that $\frac{\alpha^n - \overline{\alpha}^n}{\alpha - \overline{\alpha}} = \pm 1$. On

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the other hand, using [7] we obtain that $\frac{\alpha^n-\overline{\alpha}^n}{\alpha-\overline{\alpha}}$ has primitive divisors for $n=11$ and all primes $n>13$, and hence our equation has no solution for $n=11$ and for primes $n>13$. Let us consider the cases $n\in\{3,5,7,13\}$ separately. Let us stress that the data in [7] are given for equivalence classes of n-detective Lehmer pairs: two Lehmer pairs (α_1,β_1), (α_2,β_2) are equivalent (we write $(\alpha_1,\beta_1)\sim(\alpha_2,\beta_2)$) if $\alpha_1/\alpha_2=\beta_1/\beta_2\in\{\pm1,\pm i\}$.

$\underline{n=3}$. According to [7], we have two possibilities: (a) $\frac{v\sqrt{p}+ui}{2}\sim\frac{\sqrt{1+\lambda}+\sqrt{1-3\lambda}}{2}$, $\lambda\neq1$, or (b) $\frac{v\sqrt{p}+ui}{2}\sim\frac{\sqrt{3^k+\lambda}+\sqrt{3^k-3\lambda}}{2}$, $k>0$, $3\nmid\lambda$.

In the case (a) we have four subcases: (i) $1+\lambda=v^2p$ and $1-3\lambda=-u^2$ or (ii) $1+\lambda=-v^2p$ and $1-3\lambda=u^2$ or (iii) $1+\lambda=-u^2$ and $1-3\lambda=v^2p$ or (iv) $1+\lambda=u^2$ and $1-3\lambda=-v^2p$.

In the subcase (i) we obtain a contradiction reducing the second equation modulo 3.

In the subcase (ii) we obtain relation $u^2=3pv^2+4$. If $p\neq7$, then reducing this equation modulo 8, we obtain $1\equiv5 \pmod 8$, a contradiction. Now the case $p=7$ leads to Pell-type equation

$$u^2-21v^2=4. \tag{4}$$

Using the assumption u and v are odd for our equation, any solution to (4) is given by

$$\frac{u_t+v_t\sqrt{21}}{2}=\left(\frac{u_0+v_0\sqrt{21}}{2}\right)^t,$$

where $(u_0,v_0)=(5,1)$ is minimal solution and $3\nmid t$. Thus an infinite family of solutions of equation (1) is given by

$$(x_t,y_t,b_t^l,n)=\left(\frac{3u_t^2v_t-7v_t^3}{4},\frac{7v_t^2+u_t^2}{4},\frac{u_t^3-21u_tv_t^2}{4},3\right)$$

(see [12, Proposition 6.3.16] for the details about the equation (4)).

In the subcase (iii), note that $4+3u^2=v^2p$, and hence $7\equiv3 \pmod 8$ if $p\neq7$, a contradiction. If $p=7$, then we need to consider the Diophantine equation

$$7v^2-3u^2=4. \tag{5}$$

Such an equation has 3 infinite families of solutions $(v,u)\in\{(s+3r,s+7r),(-s+3r,s-7r),(4s+18r,6s+28r)\}$, where $s^2-21r^2=1$. But since u and v are odd, one gets 2 infinite families of solutions $(v,u)\in\{(s+3r,s+7r),(-s+3r,s-7r)\}$ for the equation (5). Any solution to the equation $s^2-21r^2=1$ is given by

$$s_t+r_t\sqrt{21}=(s_0+r_0\sqrt{21})^t$$

where $(s_0,r_0)=(55,12)$ is minimal solution. Thus 2 infinite families of solutions of equation (1) are given by

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$$x_t = -3r_t s_t^2 + 63r_t^3 + 21s_t r_t^2 - s_t^3$$
$$y_t = 2s_t^2 + 14s_t r_t + 28r_t^2$$
$$b_t^l = -5s_t^3 - 63s_t^3 r_t - 231s_t r_t^2 - 245r_t^3$$

or

$$x_t = -3s_t^2 r_t + s_t^3 - 21s_t r_t^2 + 63r_t^3$$
$$y_t = 2s_t^2 - 14s_t r_t + 28r_t^2$$
$$b_t^l = -5s_t^3 + 63s_t^2 r_t - 231s_t r_t^2 + 245r_t^3$$

with $n = 3$ (see [2, Theorems 4.5.1, 4.5.2] for details about the equation (5)).

In the subcase (iv) note that $4 + v^2 p = 3u^2$, and hence $7 \equiv 3 (\mathrm{mod}\ 8)$ if $p \neq 7$, a contradiction. If $p = 7$, then reducing $4 + 7v^2 = 3u^2$ modulo 7 we obtain $\square = -\square$, a contradiction.

In the case (b) we have four subcases: (i) $3^k + \lambda = v^2 p$ and $3^k - 3\lambda = -u^2$ or (ii) $3^k + \lambda = -v^2 p$ and $3^k - 3\lambda = u^2$ or (iii) $3^k + \lambda = -u^2$ and $3^k - 3\lambda = v^2 p$ or (iv) $3^k + \lambda = u^2$ and $3^k - 3\lambda = -v^2 p$.

In the subcase (i) note that $3v^2 p = u^2 + 4 \cdot 3^k$, hence necessarily $k = 1$ (otherwise $3 | \gcd(u, v)$). Therefore $v^2 p = 3t^2 + 4$, where $u = 3t$. If $p \neq 7$, then reducing this equation modulo 8, we obtain $3 \equiv 7 (\mathrm{mod}\ 8)$, a contradiction. If $p = 7$, then we need to consider the Diophantine equation

$$7v^2 - 3t^2 = 4. \tag{6}$$

As in subcase (iii) of part (a) Such an equation has 3 infinite families of solutions $(v, u) \in \{(s + 3r, 3(s + 7r)), (-s + 3r, 3(s - 7r)), (4s + 18r, 3(6s + 28r))\}$, where $s^2 - 21r^2 = 1$. But since u and v are odd, one obtains 2 infinite families of solutions $(v, u) \in \{(s + 3r, 3(s + 7r)), (-s + 3r, 3(s - 7r))$ for the equation (6). Any solution to the equation $s^2 - 21r^2 = 1$ is given by

$$s_t + r_t\sqrt{21} = (s_0 + r_0\sqrt{21})^t$$

where $(s_0, r_0) = (55, 12)$ is minimal solution. Thus 2 infinite families of solutions of equation (1) are given by

$$x_t = 567s_t r_t^2 + 99r_t s_t^2 + 5s_t^3 + 945r_t^3$$
$$y_t = 4s_t^2 + 42s_t r_t + 126r_t^2$$
$$b_t^l = -9s_t^3 - 63s_t^2 r_t + 189s_t r_t^2 + 1323r_t^3$$

or

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$$x_t = -567s_tr_t^2 + 99r_ts_t^2 - 5s_t^3 + 945r_t^3$$
$$y_t = 4s_t^2 - 42s_tr_t + 126r_t^2$$
$$b_t^l = -9s_t^3 + 63s_t^2r_t + 189s_tr_t^2 - 1323r_t^3$$

with $n = 3$.

In the subcase (ii) note that $4 \cdot 3^k + 3pv^2 = u^2$, hence necessarily $k = 1$ (otherwise $3 | \gcd(u, v)$). Therefore $4 = 3t^2 - pv^2$, where $u = 3t$. If $p \neq 7$, then reducing this equation modulo 8, we obtain $4 \equiv 0 (\mathrm{mod}\ 8)$, a contradiction. Now reducing $4 = 3t^2 - 7v^2$ modulo 7, we obtain $\Box = -\Box$, a contradiction again.

In the subcase (iii) note that $3t^2p = u^2 + 4$, where $v = 3t$. Now reducing modulo 3, we obtain a contradiction.

In the subcase (iv) note that $u^2 = 3pt^2 + 4$, where $v = 3t$. If $p \neq 7$, then reducing this equation modulo 8, we obtain $1 \equiv 5 (\mathrm{mod}\ 8)$, a contradiction. Now the case $p = 7$ leads to Pell-type equation

$$u^2 - 21t^2 = 4. \tag{7}$$

Since u and v are odd, one gets that t is odd for the equation (7). So, any solution to (7) is given by

$$\frac{u_m + t_m\sqrt{21}}{2} = \left(\frac{u_0 + t_0\sqrt{21}}{2}\right)^m,$$

where $(u_0, t_0) = (5, 1)$ is minimal solution and $3 \nmid m$. Thus an infinite family of solutions of equation (1) is given by

$$(x_m, y_m, b_m^l, n) = \left(\frac{3u_m^2v_m - 7v_m^3}{4}, \frac{7v_m^2 + u_m^2}{4}, \frac{u_m^3 - 21u_mv_m^2}{4}, 3\right),$$

with $v_m = 3t_m$.

<u>$n = 5$</u>. According to [7], we have two possibilities to consider: (a) $\frac{v\sqrt{p}+ui}{2} \sim \frac{\sqrt{F_{k-2\epsilon}}+\sqrt{F_{k-2\epsilon}-4F_k}}{2}$, $k \geq 3$, $\epsilon = \pm 1$, or (b) $\frac{v\sqrt{p}+ui}{2} \sim \frac{\sqrt{L_{k-2\epsilon}}+\sqrt{L_{k-2\epsilon}-4F_k}}{2}$, $k \neq 1$, $\epsilon = \pm 1$. Here F_m and L_m denote m-th Fibonacci and Lucas number respectively.

In the case (a) we have four subcases: (i) $v^2p = F_{k-2\epsilon} - 4F_k$ and $-u^2 = F_{k-2\epsilon}$ or (ii) $-v^2p = F_{k-2\epsilon} - 4F_k$ and $u^2 = F_{k-2\epsilon}$ or (iii) $v^2p = F_{k-2\epsilon}$ and $-u^2 = F_{k-2\epsilon} - 4F_k$ or (iv) $-v^2p = F_{k-2\epsilon}$ and $u^2 = F_{k-2\epsilon} - 4F_k$.

In the subcase (i) we obtain $F_{k-2\epsilon} = -u^2 < 0$, a contradiction.

In the subcase (ii), due to the fundamental work by Ljunggren [17] [19] (see also [11, Section 2]) we can find all solutions to the equation $u^2 = F_{k-2\epsilon}$ ($k \geq 3$, $\epsilon = \pm 1$). Ljunggren has proved that the only squares in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, and $F_{12} = 144$.

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The case $k-2\epsilon=1$ gives $k=3$, $\epsilon=1$, $u^2=1$, hence using the first equation from (ii) we obtain $-v^2p=-7$, i.e. $p=7$, $v^2=1$. This case gives the solution $(p,x,y,b^l,n)=(7,\pm1,2,\pm11,5)$ which contradicts with $2^{n-1}b^l\not\equiv\pm(\text{mod } p)$.

The case $k-2\epsilon=2$, gives $k=4$, $\epsilon=1$, $u^2=1$, hence using the first equation from (ii) we obtain $-v^2p=-11$, i.e. $p=11$, $v^2=1$. This case gives the solution $(p,x,y,b^l,n)=(11,\pm1,3,\pm31,5)$, which is impossible since $2^{n-1}b^l\equiv\pm1 \pmod p$.

The case $k-2\epsilon=12$, gives $k=14$, $\epsilon=1$, $u^2=144$ or $k=10$, $\epsilon=-1$, $u^2=144$. The first possibility gives $-v^2p=-2^2\cdot11\cdot31$, i.e. $p=11\cdot31$, $v^2=4$. The second possibility gives $-v^2p=-2^2\cdot19$, i.e. $p=19$, $v^2=4$. This case gives no solution satisfying $\gcd(x,b)=1$ (note that both u and v are even).

In the subcase (iii), let us note that $u^2=-(F_{k-2\epsilon}-4F_k)=F_k+F_{k+2\epsilon}$. Now a short look at the paper by Luca and Patel (see [20, Theorem 1], and their calculations in Section 5) shows that $k=4$, $\epsilon=-1$ is the only possibility. But then $F_6=8=v^2p$, a contradiction.

In the subcase (iv) we obtain $F_{k-2\epsilon}=-v^2p<0$, a contradiction.

In the case (b) we have four subcases: (i) $v^2p=L_{k-2\epsilon}-4L_k$ and $-u^2=L_{k-2\epsilon}$ or (ii) $-v^2p=L_{k-2\epsilon}-4L_k$ and $u^2=L_{k-2\epsilon}$ or (iii) $v^2p=L_{k-2\epsilon}$ and $-u^2=L_{k-2\epsilon}-4L_k$ or (iv) $-v^2p=L_{k-2\epsilon}$ and $u^2=L_{k-2\epsilon}-4L_k$.

In the subcase (i) we obtain $L_{k-2\epsilon}=-u^2<0$, a contradiction.

In the subcase (ii) we can find all solutions to the equation $u^2=L_{k-2\epsilon}$, $(k\neq1, \epsilon=\pm1)$. By the work by Cohn [13] we know all solutions: $L_1=1^2$ and $L_3=2^2$.

The case $k-2\epsilon=1$, gives $k=3$, $\epsilon=1$, $u^2=1$, hence using the first equation from (ii) we obtain $-v^2p=-13$, i.e. $p=13$, $v^2=1$. The case $k-2\epsilon=3$, gives $k=5$, $\epsilon=1$, $u^2=4$, hence using the first equation from (ii) we obtain $-v^2p=-40$, i.e. $p=10$, $v^2=4$. None of these two cases lead to solution of our Diophantine equation (13 is congruent to 1 modulo 4, while 10 is even).

In the subcase (iii), let us note that $L_k+L_{k-2\epsilon}=5F_{k-\epsilon}$. Therefore we need to determine all k such that $5F_{k-\epsilon}$ is a square. Again, the paper by Bugeaud, Mignotte and Siksek [10] shows that the only possibility is $5F_5=5^2$. But then $v^2p=L_4=7$ in case $\epsilon=1$ or $v^2p=L_6=18$ in case $\epsilon=-1$. In the first case we obtain $p=7$, but then $u^2=65$, a contradiction. The second case gives a contradiction by trivial observation.

In the subcase (iv) we obtain $L_{k-2\epsilon}=-v^2p<0$, a contradiction.

$\underline{n=7}$. According to [7] we have six equivalence classes of 7-defective Lehmer pairs. Two of them, $(\frac{1-\sqrt{-7}}{2},\frac{1+\sqrt{-7}}{2})$ and $(\frac{1-\sqrt{-19}}{2},\frac{1+\sqrt{-19}}{2})$, come from our Lehmer pairs, giving $(p,x,y,b^l,n)\in\{(7,\pm7,2,\pm13,7),(19,\pm1,5,\pm559,7)\}$, which are impossible since $2^{n-1}b^l\equiv\pm1 \pmod p$.

$\underline{n=13}$. The unique 13-detective equivalence class $(\frac{1-\sqrt{-7}}{2},\frac{1+\sqrt{-7}}{2})$ leads to the solution $(p,x,y,b^l,n)=(7,\pm1,2,\pm181,13)$, which contradicts with $2^{n-1}b^l\not\equiv\pm1(\text{mod } p)$.

Proof of Theorem 2. In this case, thanks to [6, Lemma 1], we can follow the same lines as in the proof of Theorem 1 for $n>3$.

3. Method via Galois representations and modular forms

We will consider the Diophantine equation $ax^2 + b^{2k} = 4y^n$, for $a \in \{3, 7, 11, 15\}$ in positive integers x, y, k, n, $\gcd(x, y) = 1$, $n \geq 7$ a prime dividing k. We will apply the Bennett-Skinner strategy [4], in particular we will use the results we need from [4]. We can compute systems of Hecke eigenvalues for conjugacy classes of newforms using MAGMA (or use Stein's Modular Forms Database provided the level is ≤ 7248).

Remarks. (a) If $a \equiv 3 \bmod 8$, then y is necessarily odd: if y is even, then reducing modulo 8 we obtain that the left hand side is congruent to 4 modulo 8, while the right hand side is congruent to 0 modulo 8, a contradiction.

(b) If $a \equiv 7 \bmod 8$, then y is necessarily even.

(i) The Diophantine equation $3x^2 + b^{2k} = 4y^n$ has no solution (x, y, k, n), $xy \neq 1$, $n \geq 7$ prime dividing k.

We will consider a more general Diophantine equation $X^n + 4Y^n = 3Z^2$ ($n \geq 7$ a prime) and use [4]. We are in case (iii) of [4, p.26], hence $\alpha \in \{1, 2\}$. From Lemma 3.2 it follows, that we need to consider the newforms of weight 2 and levels $N \in \{36, 72\}$.

a) There is only one newform of weight 2 and level 36, corresponding to an elliptic curve E of conductor 36 with complex multiplication by $\mathbb{Q}(\sqrt{-3})$. Here we will apply [4, Subsection 4.4], to prove that $ab = \pm 1$. Assume (a.a) that $ab \neq \pm 1$. Then (using Prop. 4.6 (b)) holds, hence if $n = 7$ or 13, n splits in $K = \mathbb{Q}(\sqrt{-3})$ and $E(K)$ is infinite for all elliptic curves of conductor $2n$. One checks that both primes 7 and 13 split in K. Now using Cremona's online tables we check, that all elliptic curves of conductor $126 = 2 \times 7 \times 3^2$ have rank zero, and all elliptic curves of conductor $234 = 2 \times 13 \times 3^2$, which are quadratic twists by 3 of quadratic curves of conductor 26, have rank zero too.

b) There is only one newform of weight 2 and level 72, corresponding to isogeny class of elliptic curves of conductor 72, with j-invariant $u/3^v$, with $v > 0$ and u some non-zero integer prime to 3. To eliminate such an elliptic curve we use [4, Prop. 4.4].

(ii) The Diophantine equation $7x^2 + b^{2k} = 4y^n$ has no solution (x, y, k, n), $n \geq 7$ prime dividing k.

We need to consider the newforms of weight 2 and level $N = 98$.

There are two Galois conjugacy classes of forms of weight 2 and level 98. We will use numbering as in Stein's tables: we have $a_3(f_2) = \pm\sqrt{2}$ and we can use [4, Prop. 4.3] to eliminate f_2. On the other hand, the form f_1 corresponds to an elliptic curve of conductor 98, with j-invariant $u/7^v$, with $v > 0$ and u some non-zero integer prime to 7. To eliminate such elliptic curves we use [4, Prop. 4.4].

(iii) The Diophantine equation $11x^2 + b^{2k} = 4y^n$ has no solution (x, y, k, n), $n \geq 7$, $n \neq 11, 13$ prime dividing k.

We need to consider the newforms of weight 2 and levels $N \in \{484, 968\}$.

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a) There are five Galois conjugacy classes of forms of weight 2 and level 484. We have $a_3(f_1) = 1$, $a_3(f_4) = \frac{1 \pm \sqrt{33}}{2}$, and we can use [4, Prop. 4.3] to eliminate f_1 and f_4. To eliminate f_2 and f_3 we need to consider coefficients a_3 and a_5: we have $a_3(f_2) = a_3(f_3) = \frac{-3 \pm \sqrt{5}}{2}$ and $a_5(f_2) = a_5(f_3) = \frac{-1 \pm \sqrt{5}}{2}$ (we cannot avoid $n = 29$ when considering only a_3). Finally, $a_7(f_5) = \pm 2\sqrt{3}$, and we can use [4, Prop. 4.3] to eliminate f_5 when $n \geq 7$ and $n \neq 13$.

b) There are fourteen Galois conjugacy classes of forms of weight 2 and level 968. We have $a_3(f_1) = -3$, $a_5(f_2) = a_5(f_3) = 3$, $a_3(f_4) = a_3(f_5) = 1$, $a_3(f_6) = a_3(f_7) = \pm 2\sqrt{5}$, $a_3(f_{10}) = \frac{1 \pm \sqrt{17}}{2}$, and we can easily use [4, Prop. 4.3] to eliminate $f_1, ..., f_7$, and f_{10}. Now considering a_3 and a_{13} for both newforms f_8, f_9, and a_3 and a_5 for both newforms f_{13}, f_{14}, and using [4, Prop. 4.3], we can eliminate these four forms when $n \geq 7$ and $n \neq 11$.

(iv) The Diophantine equation $15x^2 + b^{2k} = 4y^n$ has no solution (x, y, k, n), $n \geq 7$ prime dividing k.

We need to consider the newforms of weight 2 and level $N = 450$.

There are seven Galois conjugacy classes of forms of weight 2 and level 450. We have $a_{11}(f_3) = a_{11}(f_7) = 3$, and we can use [4, Prop. 4.3] to eliminate f_3 and f_7. The forms f_2 and f_6 correspond to elliptic curves of conductor 450 (named C and A respectively in Cremona's tables), with j-invariants $u/3^v$, with $v > 0$ and u some non-zero integer prime to 3; the forms f_1 and f_5 correspond to elliptic curves of conductor 450 (named F and E respectively in Cremona's tables), with j-invariants $u/5^v$, with $v > 0$ and u some non-zero integer prime to 5; the form f_4 corresponds to elliptic curve G of conductor 450, with j-invariant $u/(3^v 5^w)$, with $v, w > 0$ and u some non-zero integer prime to 15. To eliminate all these elliptic curves we use [4, Prop. 4.4].

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