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Congruences between Siegel modular forms II

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ABSTRACT

Using a p-adic monodromy theorem on the affine ordinary locus in the minimally compactified moduli scheme modulo powers of a prime p of abelian varieties, we extend Katz's results on congruence and p-adic properties of elliptic modular forms to Siegel modular forms of higher degree.

1. Introduction

The aim of this paper is to show that a strong purity, i.e., the *affineness* in the moduli theory of abelian varieties modulo powers p^m of a prime p is useful to study congruence and p-adic properties of Siegel modular forms. More precisely, we study the p-adic monodromy on the affine ordinary locus in the minimally compactified moduli scheme modulo p^m. Using this tool, as a continuation of [3], we can extend Katz's results [5,6] on the description of

- the kernel of the Fourier expansion map modulo p^m on the ring of elliptic modular forms,
- elliptic modular forms on $\Gamma_0(p)$ as p-adic elliptic modular forms

to the Siegel modular case of degree $g > 1$. Note that his results are obtained by considering modular forms as automorphic sections on the Igusa tower over the affine ordinary locus in the moduli scheme, and that the affineness is easily seen in the elliptic modular case.

In this paper, for any $g > 1$, we show that the determinant of the p-adic monodromy representation on the ordinary locus in the moduli scheme modulo p^m factors through a surjective 1-dimensional p-adic representation on the affine ordinary locus in the minimal compactification. From this result, we can provide an Igusa tower consisting of affine schemes over truncated Witt

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rings whose automorphic sections are controlled by Siegel modular forms. The existence of such a tower plays a key role extending Katz's results to the Siegel modular case, especially we describe

- the kernel of the Fourier expansion map modulo p^m on the ring of Siegel modular forms (cf. Theorem 2),
- Siegel modular forms on $\Gamma_0(p)$ as p-adic Siegel modular forms (cf. Theorem 3).

We only indicate the proof of this extension since the readers familiar with Katz's results can easily complete this. Note that results of Böcherer and Nagaoka [1,8,9] on lifts of generalized Hasse invariants are necessary to prove and improve some of our results.

2. Moduli spaces and modular forms

2.1. Review of moduli spaces and modular forms

We review results of Chai and Faltings [2] on moduli spaces and modular forms. For positive integers g and N, let ζ_N be a primitive N-th root of 1, and let $\mathcal{A}_{g,N}$ be the moduli stack over $\mathbb{Z}[1/N, \zeta_N]$ which classifies principally polarized abelian schemes of relative dimension g with symplectic level N structure. Then the associated complex orbifold $\mathcal{A}_{g,N}(\mathbb{C})$ is of dimension $g(g+1)/2$, and is represented as the quotient space $\mathcal{H}_g/\Gamma_{g,N}$ of the Siegel upper half-space $\mathcal{H}_g$ of degree g by the integral symplectic group $\Gamma_{g,N} = \mathrm{Ker}(Sp_g(\mathbb{Z}) \to Sp_g(\mathbb{Z}/N\mathbb{Z}))$ of degree g and level N. There exists the universal abelian scheme $\mathcal{X}$ with 0-section s over $\mathcal{A}_{g,N}$, and the *Hodge line bundle* ω is defined as $\det(s^*(\Omega_{\mathcal{X}/\mathcal{A}_{g,N}}))$ which corresponds to the automorphic factor over $\mathcal{A}_{g,N}(\mathbb{C})$. In [2, Chapter IV], Chai and Faltings constructed a smooth compactification $\overline{\mathcal{A}}_{g,N}$ of $\mathcal{A}_{g,N}$ associated with a good cone decomposition of the set of positive semi-definite symmetric bilinear forms on $\mathbb{R}^g$, and a semi-abelian scheme $\mathcal{G}$ with 0-section s over $\overline{\mathcal{A}}_{g,N}$ extending $\mathcal{X} \to \mathcal{A}_{g,N}$. Then $\overline{\omega} = \det(s^*(\Omega_{\mathcal{G}/\overline{\mathcal{A}}_{g,N}}))$ gives an extension of ω to $\overline{\mathcal{A}}_{g,N}$, and

$$\mathcal{A}^*_{g,N} = \mathrm{Proj}\left(\bigoplus_{h\geqslant 0} H^0\left(\overline{\mathcal{A}}_{g,N}, \overline{\omega}^{\otimes h}\right)\right)$$

is a projective scheme over $\mathbb{Z}[1/N, \zeta_N]$ called *Satake's minimal compactification*.

Assume that $N \geqslant 3$. Then $\mathcal{A}_{g,N}$ becomes the fine moduli scheme. Further, $\mathcal{A}^*_{g,N}$ contains $\mathcal{A}_{g,N}$, and its complement has a natural stratification by locally closed subschemes, each of which is isomorphic to $\mathcal{A}_{i,N}$ $(0 \leqslant i \leqslant g-1)$. There is a natural morphism $\overline{\mathcal{A}}_{g,N} \to \mathcal{A}^*_{g,N}$ (which is an isomorphism if $g = 1$) extending the identity map on $\mathcal{A}_{g,N}$ such that $\overline{\omega}$ is the pullback by this morphism of the tautological line bundle ω^* on $\mathcal{A}^*_{g,N}$.

Following [2, Chapter V], for any $\mathbb{Z}[1/N, \zeta_N]$-algebra R, we define the R-module $\mathcal{M}_{g,h,N}(R)$ of *Siegel modular forms over R of degree g, weight h and level N* by

$$\mathcal{M}_{g,h,N}(R) = H^0\left(\overline{\mathcal{A}}_{g,N}, \overline{\omega}^{\otimes h} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R\right),$$

and the *ring $\mathcal{M}^*_{g,N}(R)$ of Siegel modular forms over R of degree g and level N* by

$$\mathcal{M}^*_{g,N}(R) = \bigoplus_{h\geqslant 0} \mathcal{M}_{g,h,N}(R)$$

which is a graded R-algebra. Then by Koecher's principle,

$$\mathcal{M}_{g,h,N}(R) = H^0\left(\mathcal{A}_{g,N}, \omega^{\otimes h} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R\right)$$

if $g > 1$, and by Serre's GAGA and Hartogs' theorem, $\mathcal{M}_{g,h,N}(\mathbb{C})$ becomes the space of holomorphic functions on $\mathcal{H}_g$ with the automorphic condition of weight h for $\Gamma_{g,N}$ (and the cusp condition if $g = 1$).

Let q_{ij} $(1 \leqslant i, j \leqslant g)$ be variables with symmetry $q_{ij} = q_{ji}$, and put

$$A_{g,N} = \mathbb{Z}\big[1/N, \zeta_N, q_{ij}^{\pm 1/N}\ (i \neq j)\big]\big[\big[q_{11}^{1/N}, \dots, q_{gg}^{1/N}\big]\big].$$

Then for each 0-dimensional cusp on $\mathcal{A}_{g,N}^*$, there exists the associated Mumford's semi-abelian scheme (cf. [7]) formally represented as

$$\mathbb{G}_m^g / \big\langle (q_{ij})_{1 \leqslant i \leqslant g} \bigm| 1 \leqslant j \leqslant g \big\rangle$$

over $A_{g,N}$ with principal polarization and symplectic level N structure. By evaluating Siegel modular forms on this semi-abelian scheme, we have an R-linear ring homomorphism

$$F_R : \mathcal{M}_{g,N}^*(R) \longrightarrow A_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R.$$

This is called the *Fourier* (q-)*expansion map* and satisfies the following (cf. [2, Chapter V]):

- F_R is functorial for R,
- $F_{\mathbb{C}}$ becomes the classical Fourier expansion,
- F_R is injective on each $\mathcal{M}_{g,h,N}(R)$, and further for $f \in \mathcal{M}_{g,h,N}(R)$ and a sub-$\mathbb{Z}[1/N, \zeta_N]$-algebra R' of R,

$$F_R(f) \in A_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R' \quad \Longleftrightarrow \quad f \in \mathcal{M}_{g,h,N}\big(R'\big).$$

These statements are known as the *q-expansion principle*, and follow from the irreducibility of geometric fibers of $\overline{\mathcal{A}}_{g,N}$.

2.2. Generalized Hasse invariants

We review a Siegel modular form h_{p-1} over $\mathbb{F}_p$ given by the *generalized Hasse invariant* (cf. [9]). This is obtained as the image of 1 under the homomorphism $\mathcal{O}_{\mathcal{A}_{g,1}\otimes\mathbb{F}_p} \to \omega^{\otimes(p-1)}$ which comes from the bundle map $\omega \to \omega^{(p)} = \omega^{\otimes p}$ associated with the Verschiebung. Then h_{p-1} is an element of $\mathcal{M}_{g,p-1,1}(\mathbb{F}_p)$ satisfying that $F_{\mathbb{F}_p}(h_{p-1}) = 1$, and such an element is unique by the q-expansion principle. Further, the divisor of h_{p-1} is the complement of the *ordinary locus* which consists of principally polarized ordinary abelian varieties in characteristic p.

It is shown by Nagaoka [8,9] and by Böcherer and Nagaoka [1] that when p satisfies the condition:

(BN) $p \geqslant g + 3$, or $p \equiv 1 \bmod (4)$, or p is a regular prime $\geqslant g/2 + 3$,

there exists an element H_{p-1} of $\mathcal{M}_{g,p-1,1}(\mathbb{Q} \cap \mathbb{Z}_p)$ such that $F_{\mathbb{Z}_p}(H_{p-1}) \equiv 1 \bmod (p)$ which is equivalent to that H_{p-1} is a lift of h_{p-1}, i.e., $H_{p-1} \bmod (p) = h_{p-1}$ by the q-expansion principle.

2.3. Moduli spaces and modular forms modulo p^m

Assume that $N \geqslant 3$, let k be a perfect field of positive characteristic p containing $1/N$ and ζ_N (hence p is prime to N), and denote by $W = W(k)$ the ring of Witt vectors over k which also contains $1/N$ and ζ_N. For an integer $m \geqslant 1$, let $W_m = W/p^m W$ be the ring of Witt vectors of length m over k, and put

$$\mathcal{A}_m = \mathcal{A}_{g,N} \otimes W_m, \qquad \overline{\mathcal{A}}_m = \overline{\mathcal{A}}_{g,N} \otimes W_m, \qquad \mathcal{A}_m^* = \mathcal{A}_{g,N}^* \otimes W_m.$$

Proposition 1.

(1) *There are canonical isomorphisms*

$$\mathcal{M}_{g,h,N}(W) \cong H^0\bigl(\mathcal{A}^*_{g,N} \otimes W, (\omega^*)^{\otimes h}\bigr), \qquad \mathcal{M}_{g,h,N}(W_m) \cong H^0\bigl(\mathcal{A}^*_m, (\omega^*)^{\otimes h}\bigr).$$

(2) *The natural homomorphisms $\mathcal{M}_{g,h,N}(W) \to \mathcal{M}_{g,h,N}(W_m)$ give rise to an isomorphism*

$$\mathcal{M}_{g,h,N}(W) \cong \varprojlim \mathcal{M}_{g,h,N}(W_m).$$

(3) *For a sufficiently large positive integer h, $\mathcal{M}_{g,h,N}(W) \otimes W_m \cong \mathcal{M}_{g,h,N}(W_m)$.*

Proof. These statements are shown by Katz [5,6] when $g = 1$, and hence we assume that $g > 1$. First, we prove (1). Since the construction of Satake's minimal compactification works over the complete discrete valuation ring W in the same way to [2, Chapter V],

$$\mathcal{A}^*_{g,N} \otimes W = \mathrm{Proj}\biggl(\bigoplus_{h \geqslant 0} H^0\bigl(\overline{\mathcal{A}}_{g,N}, \overline{\omega}^{\otimes h} \otimes W\bigr)\biggr)$$

is normal, flat over W and $\mathrm{codim}_W(\mathcal{A}^*_{g,N} \otimes W - \mathcal{A}_{g,N} \otimes W) = g > 1$. This together with Koecher's principle imply that

$$\mathcal{M}_{g,h,N}(W) = H^0\bigl(\mathcal{A}_{g,N} \otimes W, \omega^{\otimes h}\bigr) = H^0\bigl(\mathcal{A}^*_{g,N} \otimes W, (\omega^*)^{\otimes h}\bigr).$$

Fix $m \geqslant 1$. Since $\mathcal{A}_m$ is Zariski dense in $\overline{\mathcal{A}}_m$, and $\overline{\mathcal{A}}_m$ is smooth over W_m with geometrically irreducible special fiber, the restriction maps

$$H^0\bigl(\overline{\mathcal{A}}_m, p^l \overline{\omega}^{\otimes h}/p^{l+1}\bigr) \to H^0\bigl(\mathcal{A}_m, p^l \omega^{\otimes h}/p^{l+1}\bigr) \quad (0 \leqslant l \leqslant m-1)$$

are injective, and hence the restriction map $H^0(\overline{\mathcal{A}}_m, \overline{\omega}^{\otimes h}) \to H^0(\mathcal{A}_m, \omega^{\otimes h})$ is also injective. Since $\mathcal{A}^*_m = \mathcal{A}^*_{g,N} \otimes W_m$ is flat over W_m, any nonzero meromorphic local section s of $\mathcal{O}_{\mathcal{A}^*_m}$ is represented as $p^l f/f'$, where $0 \leqslant l \leqslant m-1$ and f, f' are regular local sections of $\mathcal{O}_{\mathcal{A}^*_m}$ such that $f, f' \notin p\mathcal{O}_{\mathcal{A}^*_m}$. If s is regular on $\mathcal{A}_m$, then $f/f' \bmod (p)$ is regular on $\mathcal{A}^*_m \otimes k$, and hence s is regular on $\mathcal{A}^*_m$ since $\mathcal{A}^*_m \otimes k$ is geometrically normal (cf. [2, Chapter V, 2.7]) and $\mathcal{A}^*_m - \mathcal{A}_m$ is a union of finite copies of $\mathcal{A}_{i,N} \otimes W_m$ $(0 \leqslant i \leqslant g-1)$ whose relative codimension is $g > 1$. Therefore, local sections of $\omega^{\otimes h}$ are uniquely extended to those of $(\omega^*)^{\otimes h}$, and hence the restriction map $H^0(\mathcal{A}^*_m, (\omega^*)^{\otimes h}) \to H^0(\mathcal{A}_m, \omega^{\otimes h})$ is an isomorphism. The homomorphism $H^0(\mathcal{A}^*_m, (\omega^*)^{\otimes h}) \to H^0(\overline{\mathcal{A}}_m, \overline{\omega}^{\otimes h})$ induced from the morphism $\overline{\mathcal{A}}_m \to \mathcal{A}^*_m$ is compatible with the above two restriction maps, and hence it is an isomorphism.

Second, we prove (2) and (3). By (1), the exact sequence

$$0 \to (\omega^*)^{\otimes h} \xrightarrow{p^m} (\omega^*)^{\otimes h} \to (\omega^*)^{\otimes h} \otimes W_m \to 0$$

of sheaves on the flat W-scheme $\mathcal{A}^*_{g,N} \otimes W$ gives the exact sequence

$$0 \to \mathcal{M}_{g,h,N}(W) \xrightarrow{p^m} \mathcal{M}_{g,h,N}(W) \to \mathcal{M}_{g,h,N}(W_m) \to H^1\bigl(\mathcal{A}^*_{g,N} \otimes W, (\omega^*)^{\otimes h}\bigr)[p^m],$$

where $G[p^m] = \mathrm{Ker}(p^m : G \to G)$ for an additive group G. Therefore, (2) and (3) follow from that $H^1(\mathcal{A}^*_{g,N} \otimes W, (\omega^*)^{\otimes h})$ is a finitely generated W-module and becomes $\{0\}$ for $h \gg 0$ since $\mathcal{A}^*_{g,N} \otimes W$ is proper over W and ω^* is ample on $\mathcal{A}^*_{g,N} \otimes W$. This completes the proof. □

3. p-Adic monodromy and Igusa towers

3.1. p-Adic monodromy

Assume that $N \geqslant 3$. Let S_m (resp. $\bar{S}_m$, S_m^*) be the ordinary locus, i.e., the open subspace of $\mathcal{A}_m$ (resp. $\overline{\mathcal{A}}_m$, $\mathcal{A}_m^*$) on which the generalized Hasse invariant h_{p-1} is invertible. Then there is a natural morphism $\varphi : \bar{S}_m \to S_m^*$ compatible with the inclusions $\bar{\iota} : S_m \hookrightarrow \bar{S}_m$ and $\iota^* : S_m \hookrightarrow S_m^*$, and hence we have a commutative diagram of homomorphisms

$$\begin{tikzcd} & \pi_1(S_m) \arrow[dl] \arrow[dr] & \\ \pi_1(\bar{S}_m) \arrow[rr] & & \pi_1(S_m^*) \end{tikzcd}$$

For each integer $n \geqslant 0$, denote by $(\mathcal{G}[p^n]^\circ)^\vee$ the Cartier dual of the connected component containing 1 of the group scheme $\mathcal{G}[p^n]$ defined as the kernel of $p^n : \mathcal{G} \to \mathcal{G}$. Then $(\mathcal{G}[p^n]^\circ)^\vee$ is an étale sheaf which is a free $(\mathbb{Z}/p^n\mathbb{Z})$-module of rank g in the étale topology on $\bar{S}_m$, and hence we have the associated monodromy representation

$$\bar{\rho}_{m,n} : \pi_1(\bar{S}_m) \to GL_g\big(\mathbb{Z}/p^n\mathbb{Z}\big).$$

Therefore, combining the natural homomorphism $\pi_1(S_m) \to \pi_1(\bar{S}_m)$, we have the representation $\rho_{m,n} : \pi_1(S_m) \to GL_g(\mathbb{Z}/p^n\mathbb{Z})$ which is associated with the étale quotient $\mathcal{A}[p^n]^{\mathrm{et}}$ of $\mathcal{A}[p^n]$. Then by a result of Chai and Faltings (cf. [2, Chapter V, 7.2]), $\rho_{m,n}$, and hence $\bar{\rho}_{m,n}$ are surjective. Therefore, the representations

$$\begin{cases} \bar{\chi}_{m,n} = \det(\bar{\rho}_{m,n}) : \pi_1(\bar{S}_m) \to \big(\mathbb{Z}/p^n\mathbb{Z}\big)^\times, \\ \chi_{m,n} = \det(\rho_{m,n}) : \pi_1(S_m) \to \big(\mathbb{Z}/p^n\mathbb{Z}\big)^\times \end{cases}$$

obtained by taking the determinant in $GL_g(\mathbb{Z}/p^n\mathbb{Z})$ are also surjective. Note that this fact can be directly deduced from the surjectivity shown by Igusa [4] of p-adic monodromy in the elliptic modular case.

Proposition 2.

(1) *S_m^* is an affine and flat W_m-scheme whose special fiber is integral.*
(2) *There exists a unique system*

$$\big\{\chi_{m,n}^* : \pi_1\big(S_m^*\big) \to \big(\mathbb{Z}/p^n\mathbb{Z}\big)^\times\big\}_{m,n}$$

of representations satisfying that $\chi_{m,m}^*$ corresponds to an étale sheaf $\mathcal{E}_{m,m}$ on S_m^* such that

$$\mathcal{E}_{m,m} \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m^*} \cong \omega^*\big|_{S_m^*},$$

and that $\chi_{m,n}^* = \chi_{1,n}^*$ via the natural identification $\pi_1(S_m^*) = \pi_1(S_1^*)$. Further, $\{\chi_{m,n}^*\}$ satisfies that $\chi_{m,n}^* = \chi_{m,n+1}^* \bmod (p^n)$ and that $\bar{\chi}_{m,n}$, $\chi_{m,n}$ factor through $\chi_{m,n}^*$ via the natural homomorphisms

$$\pi_1(S_m) \to \pi_1\big(S_m^*\big), \qquad \pi_1(\bar{S}_m) \to \pi_1\big(S_m^*\big)$$

respectively.

Proof. These statements are shown by Katz [5,6] when $g = 1$, and hence we assume that $g > 1$. First, we prove (1). As seen in the proof of Proposition 1, $\mathcal{A}^*_{g,N} \otimes W$ is flat over W and geometrically normal. Hence the open subscheme S^*_m of $\mathcal{A}^*_m = \mathcal{A}^*_{g,N} \otimes W_m$ is flat over W_m with integral special fiber. By Proposition 1(3), one can take a sufficiently large positive integer c such that there is a basis of $\mathcal{M}_{g,c(p-1),N}(W)$ over W which contains a lift of h^c_{p-1} and gives rise to a closed immersion of $\mathcal{A}^*_m$ into a projective space $\mathbb{P}^d_{W_m}$. Therefore, S^*_m is a closed subscheme of the affine subspace $\mathbb{A}^d_{W_m}$ of $\mathbb{P}^d_{W_m}$ on which h^c_{p-1} (and hence h_{p-1}) is invertible. Therefore, S^*_m becomes an affine scheme.

Second, we prove (2). Each local isomorphism $(\mathbb{Z}/p^m\mathbb{Z})^g \xrightarrow{\sim} (\mathcal{G}[p^m]^\circ)^\vee$ in the étale topology on $\bar{S}_m$ corresponds to

$$\iota : \mathcal{G}\big[p^m\big]^\circ \xrightarrow{\sim} (\boldsymbol{\mu}_{p^m})^g \hookrightarrow \mathbb{G}_m^g = \operatorname{Spec}\big(\mathbb{Z}\big[X_1^{\pm 1}, \ldots, X_g^{\pm 1}\big]\big)$$

by the Cartier duality, and then $\iota^*(dX_i/X_i)$ $(i = 1, \ldots, g)$ are uniquely extended to a basis of 1-forms on $\mathcal{G}$. Hence we have

$$\big(\mathcal{G}\big[p^m\big]^\circ\big)^\vee \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{\bar{S}_m} \cong s^*(\Omega_{\mathcal{G}/\bar{S}_m})$$

which gives isomorphisms

$$\det\big(\big(\mathcal{G}\big[p^m\big]^\circ\big)^\vee\big) \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{\bar{S}_m} \cong \bar{\omega}|_{\bar{S}_m}$$

and

$$\det\big(\mathcal{X}\big[p^m\big]^{\mathrm{et}}\big) \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m} \cong \omega|_{S_m}.$$

The correspondence $\mathcal{G} \mapsto \mathcal{G}/\mathcal{G}[p]^\circ$ gives rise to a morphism $\phi : \bar{S}_m \to \bar{S}_m$ which is the p-th power map on $\bar{S}_m \otimes k = \bar{S}_1$ such that under the above isomorphism, $m \otimes f \mapsto m \otimes \phi(f)$ gives $F : \phi^*(\bar{\omega}|_{\bar{S}_m}) \xrightarrow{\sim} \bar{\omega}|_{\bar{S}_m}$. Further, the abelian part of each fiber $(\mathcal{G}/\mathcal{G}[p]^\circ)_s$ becomes $A_s/A_s[p]^\circ$ which is represented by only the abelian part A_s of $\mathcal{G}_s$. Therefore, ϕ factors through a morphism $S^*_m \to S^*_m$ which we denote by the same symbol, and hence we have $F : \phi^*(\omega^*|_{S^*_m}) \xrightarrow{\sim} \omega^*|_{S^*_m}$.

Then we recall Katz's p-adic monodromy theorem [5, Proposition 4.1.1] which states the following: Let X be a flat W_m-scheme with integral special fiber, and let $\phi : X \to X$ be a morphism such that $\phi|_{X\otimes k}$ is the q-th power map (q is a power of p such that $\mathbb{F}_q \subset k$). Then

$$\mathcal{E} \mapsto (\mathcal{F} = \mathcal{E} \otimes_{W_m(\mathbb{F}_q)} \mathcal{O}_X,\ F = \mathrm{id}_{\mathcal{E}} \otimes \phi)$$

is a fully faithful functor from the category of étale sheaves $\mathcal{E}$ on X as free $W_m(\mathbb{F}_q)$-modules of finite rank to that of locally free sheaves $\mathcal{F}$ on X of finite rank with isomorphism $F : \phi^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. Further, this functor gives an equivalence between these categories if X is affine. By applying this theorem to $q = p$, $X = S^*_m$ and $\mathcal{F} = \omega^*|_{S^*_m}$, there exists uniquely an étale sheaf $\mathcal{E}_{m,m}$ on S^*_m as an invertible $(\mathbb{Z}/p^m\mathbb{Z})$-module such that $\mathcal{E}_{m,m} \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S^*_m} \cong \omega^*|_{S^*_m}$. Let $\chi^*_{m,m}$ be the corresponding representation of $\pi_1(S^*_m)$, and put

$$\chi^*_{m,n} : \pi_1\big(S^*_m\big) = \pi_1\big(S^*_n\big) \xrightarrow{\chi^*_{n,n}} \big(\mathbb{Z}/p^n\mathbb{Z}\big)^\times.$$

Then $\{\chi^*_{m,n}\}_{m,n}$ is uniquely determined and satisfies the desired properties since

$$\big(\omega^*\big|_{S^*_{m+1}}\big)\big|_{S^*_m} = \omega^*\big|_{S^*_m}, \qquad \iota^*\big(\omega^*\big|_{S^*_m}\big) = \omega|_{S_m}, \qquad \varphi^*\big(\omega^*\big|_{S^*_m}\big) = \bar{\omega}|_{\bar{S}_m},$$

and Katz's functor is fully faithful for $X = S_m, \bar{S}_m$. This completes the proof. □

3.2. Igusa towers

Let

$$T_{m,n} \to S_m, \qquad \overline{T}_{m,n} \to \overline{S}_m, \qquad T^*_{m,n} \to S^*_m$$

be the étale and finite coverings defined by $\mathrm{Ker}(\chi_{m,n})$, $\mathrm{Ker}(\overline{\chi}_{m,n})$, $\mathrm{Ker}(\chi^*_{m,n})$ respectively. Then by Proposition 2(2), the above $\bar{\iota} : S_m \hookrightarrow \overline{S}_m$, $\iota^* : S_m \hookrightarrow S^*_m$ and $\varphi : \overline{S}_m \to S^*_m$ lift to natural inclusions $\bar{\iota} : T_{m,n} \hookrightarrow \overline{T}_{m,n}$, $\iota^* : T_{m,n} \hookrightarrow T^*_{m,n}$ and a morphism $\varphi : \overline{T}_{m,n} \to T^*_{m,n}$ such that $\varphi \circ \bar{\iota} = \iota^*$. The systems of coverings $\{T_{m,n}\}$, $\{\overline{T}_{m,n}\}$ and $\{T^*_{m,n}\}$ are called *Igusa towers.*

Proposition 3. *Assume that $g > 1$.*

(1) *$T^*_{m,n}$ is an affine and flat W_m-scheme whose special fiber is integral.*
(2) *Let $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{F}^*$ be invertible sheaves on $T_{m,n}$, $\overline{T}_{m,n}$, $T^*_{m,n}$ respectively such that $(\bar{\iota})^*(\overline{\mathcal{F}}) = \mathcal{F}$, $(\iota^*)^*(\mathcal{F}^*) = \mathcal{F}$, $\varphi^*(\mathcal{F}^*) = \overline{\mathcal{F}}$. Then there are compatible isomorphisms*

$$H^0(T_{m,n}, \mathcal{F}) \cong H^0(\overline{T}_{m,n}, \overline{\mathcal{F}}) \cong H^0\big(T^*_{m,n}, \mathcal{F}^*\big).$$

Proof. The assertion (1) follows from Proposition 2(1), and hence we prove (2). The open substack $\overline{S}_m \subset \overline{\mathcal{A}}_m$ and its étale covering $\overline{T}_{m,n}$ are smooth over W_m with integral special fiber, and $T_{m,n}$ is Zariski dense in $\overline{T}_{m,n}$. Hence as in the proof of Proposition 1(1), one can show that the restriction map $H^0(\overline{T}_{m,n}, \overline{\mathcal{F}}) \to H^0(T_{m,n}, \mathcal{F})$ is injective. We will prove that any local section of $\mathcal{F}$ is uniquely extended to that of $\mathcal{F}^*$, which implies that $H^0(T^*_{m,n}, \mathcal{F}^*) \to H^0(T_{m,n}, \mathcal{F})$ and $H^0(T^*_{m,n}, \mathcal{F}^*) \to H^0(\overline{T}_{m,n}, \overline{\mathcal{F}})$ are isomorphisms. We may assume that $\mathcal{F}$ and $\mathcal{F}^*$ are the structure sheaves on $T_{m,n}$ and $T^*_{m,n}$ respectively. As stated above, $\mathcal{A}^*_m \otimes k$ is geometrically normal, $\mathcal{A}^*_m - \mathcal{A}_m$ is a union of smooth schemes over W_m whose relative codimension is $g > 1$. Therefore, $T^*_{m,n}$ and $T^*_{m,n} - T_{m,n}$ have the same properties, and hence as in the proof of Proposition 1(1), one can show that any local section of $\mathcal{O}_{T_{m,n}}$ is uniquely extended to that of $\mathcal{O}_{T^*_{m,n}}$. This completes the proof. □

By Proposition 3, we have an Igusa tower $\{T^*_{m,n}\}_{m,n}$ consisting of affine schemes whose coordinate rings

$$V_{m,n} = H^0\big(T^*_{m,n}, \mathcal{O}_{T^*_{m,n}}\big) = H^0(T_{m,n}, \mathcal{O}_{T_{m,n}})$$

make a sequence of Artin–Schreier's extensions

$$V_{m,0} \subset V_{m,1} \subset V_{m,2} \subset \cdots,$$

and satisfy that $V_{m+1,n}/p^m V_{m+1,n} \cong V_{m,n}$. Further, by Propositions 2(2) and 3(2),

$$\begin{aligned} H^0\big(S_m, (\omega|_{S_m})^{\otimes h}\big) &\cong H^0\big(S^*_m, \big(\omega^*|_{S^*_m}\big)^{\otimes h}\big) \\ &\cong \big\{\phi \in V_{m,m} \,\big|\, [a]\phi = a^h\phi \ \big(a \in \big(\mathbb{Z}/p^m\mathbb{Z}\big)^\times\big)\big\}, \end{aligned}$$

where $[*]$ denotes the natural action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ on $V_{m,m}$. Therefore, as will be seen below, Katz's argument in [5,6] on the elliptic modular case can be applicable to our $V_{m,n}$.

4. Congruences and p-adic properties

4.1. Congruences

We prove congruence properties of Siegel modular forms extending results on elliptic modular forms given in [5,6,10–12]. First, we restate Theorem 1 in [3] in a more accurate form, especially noting that for any prime p, congruences between Siegel modular forms deduce congruences between their weights.

Theorem 1. *Assume that two Siegel modular forms $f_i \in \mathcal{M}_{g,h_i,N}(W_m)$ $(i = 1, 2)$ have the same Fourier expansion over W_m which are not congruent to 0 modulo p at (at least) one 0-dimensional cusp. Then we have:*

(1) *The weights h_i of f_i satisfy the congruence $h_1 \equiv h_2$ modulo the exponent $e(p^m)$ of $(\mathbb{Z}/p^m\mathbb{Z})^\times$.*
(2) *When p satisfies the condition (BN) in Section 2.2 and $H_{p-1} \in \mathcal{M}_{g,p-1,1}(\mathbb{Q} \cap \mathbb{Z}_p)$ is a lift of h_{p-1},*

$$f_i = f_j \cdot \left(H_{p-1}^{(h_i-h_j)/(p-1)} \bmod \left(p^m\right)\right)$$

if $h_i \geqslant h_j$. Therefore, f_1 and f_2 have the same Fourier expansion over W_m at any cusp.
(3) *When $m = 1$, the same statement to (2) holds for any prime p replacing $H_{p-1} \bmod (p)$ with h_{p-1}.*

From (3) of this theorem, we have the following extension of a result of Katz [6, Theorem 2.2] to the Siegel modular case by applying his argument to the affine morphism $S_1^* \hookrightarrow \mathcal{A}_1^*$.

Corollary. *The kernel of the Fourier expansion homomorphism $F_k : \mathcal{M}_{g,N}^*(k) \to A_{g,N} \otimes k$ is the ideal generated by $1 - h_{p-1}$.*

Second, we consider higher congruences between Siegel modular forms. Let

$$I_l = \left\{ \sum_h f_h \in \bigoplus_{h \geqslant 0} \mathcal{M}_{g,h,N}(W) \;\middle|\; F_W\left(\sum f_h\right) \overset{\text{def}}{=} \sum F_W(f_h) \equiv 0 \bmod \left(p^l\right) \right\}$$

be the non-graded ideal of $\mathcal{M}_{g,N}^*(W)$, put

$$D = \mathcal{M}_{g,N}^*(W) + (1/p) \cdot I_1 + \left(1/p^2\right) \cdot I_2 + \cdots = \bigcup_{l \geqslant 0} p^{-l} \cdot I_l,$$

and let $\mathbb{Z}_p^\times$ act on D as

$$[a]\left(\sum_h f_h\right) = \sum_h a^h f_h \quad \left(a \in \mathbb{Z}_p^\times,\ f_h \in \mathcal{M}_{g,h,N}(W)[1/p]\right).$$

Then we have the following extension of a result of Katz [6, Theorem 3.3] to the Siegel modular case by replacing his $V_{m,n}$ with our $V_{m,n}$.

Theorem 2. *Assume that $N \geqslant 3$, p satisfies the condition (BN), and let $H_{p-1} \in \mathcal{M}_{g,p-1,1}(\mathbb{Q} \cap \mathbb{Z}_p)$ be a lift of h_{p-1}. Then:*

(1) *For each integer $n \geqslant 1$, there exists an element d_n of D such that $d_1 = (1 - H_{p-1})/p$, and for any $k \geqslant 0$,*

$$\left[1 + p^{n+k}\right](d_n) \equiv d_n + p^k H_{p-1} \quad \bmod \left(p^{k+1} D\right).$$

(2) *$r_n = p^{(p^n-1)/(p-1)} \cdot d_n \in D$ belongs to $\mathcal{M}^*_{g,N}(W)$, and hence to $I_{(p^n-1)/(p-1)}$.*
(3) *The ideal I_n of $\mathcal{M}^*_{g,N}(W)$ is generated by $p^{a_0} \cdot r_1^{a_1} \cdots r_j^{a_j}$ such that*

$$a_0 + \sum_{i=1}^{j} a_i\big(p^i - 1\big)/(p-1) = n.$$

From (3) of this theorem, we have immediately:

Corollary. *Let the assumption and the notation be as in Theorem 2, and assume that $n \leqslant p$. Then $I_n = (p, 1 - H_{p-1})^n = (I_1)^n$.*

4.2. p-Adic properties

Let k be as above, and let K be the quotient field of the Witt ring W over k. Then for an element ϕ of $K[q_{ij}^{\pm 1/N}(i \neq j)][[q_{11}^{1/N}, \dots, q_{gg}^{1/N}]]$, $\mathrm{ord}_p(\phi)$ is defined as the infimum of the order at p of its coefficients in K. As in the elliptic modular case (cf. [11]), we call ϕ a *p-adic Siegel modular form over K of degree g and level N* if there are integers $h_m \geqslant 0$ and elements f_m of $\mathcal{M}_{g,h_m,N}(K)$ such that $\phi = \lim_{m\to\infty} F_K(f_m)$. This means that $\lim_{m\to\infty} \mathrm{ord}_p(\phi - F_K(f_m)) = \infty$ for the Fourier expansion map F_K associated with a 0-dimensional cusp on $\mathcal{A}^*_{g,N}$. Since $\mathcal{M}_{g,h_m,N}(K) = \mathcal{M}_{g,h_m,N}(W) \otimes_W K$, such a ϕ has coefficients in W if and only if $\phi = \lim_{m\to\infty} F_W(f_m)$ for $f_m \in \mathcal{M}_{g,h_m,N}(W)$, in which case ϕ is called a *p-adic Siegel modular form over W*. Then by Theorem 1, the *weight* $\lim_{m\to\infty} h_m$ of ϕ is well defined as an element of

$$\varprojlim \mathbb{Z}/(p-1)p^m\mathbb{Z} \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p,$$

which is the closure of $\mathbb{Z}$ in the ring $\mathrm{End}(\mathbb{Z}_p^\times)$ of continuous endomorphisms of $\mathbb{Z}_p^\times$ (cf. [3, Theorem 2]).

Applying Katz's argument in [6, Appendix I] to our $V_{m,n}$ and their inductive and projective limits, one can see that each p-adic Siegel modular form over W of degree g, weight h and level N is a rule ϕ associating isomorphism classes of $(A, \lambda, \alpha, \tau)$ over W-algebras R in which p is nilpotent with elements $\phi(A, \lambda, \alpha, \tau)$ of R. Here A are abelian schemes of relative dimension g over R with principal polarization λ, symplectic level N structure α and isomorphism $\tau : A[p^\infty]^\circ \xrightarrow{\sim} \mathbb{G}_m^g[p^\infty]$. Further, ϕ satisfies the conditions:

- the commutativity with base extensions,
- the holomorphy at any cusp,
- $\phi(A, \lambda, \alpha, a^{-1}\tau) = a^h \cdot \phi(A, \lambda, \alpha, \tau)$ $(a \in \mathbb{Z}_p^\times)$.

Therefore, we have the following extension of a result of Katz [5, Theorem 3.2] by applying his argument to the Siegel modular case.

Theorem 3. *Assume that $N \geqslant 3$, and let f be a Siegel modular form over K of degree g, weight h and level N on*

$$\Gamma_0(p) = \left\{ X \in Sp_g(\mathbb{Z}) \;\middle|\; X \equiv \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \bmod (p) \right\}.$$

Then the Fourier expansion of f at each 0-dimensional unramified cusp becomes a p-adic Siegel modular form over K of degree g, weight h and level N.

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