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A formula for the action of Hecke operators on half-integral weight Siegel modular forms and applications

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ABSTRACT

We introduce an alternate set of generators for the Hecke algebra, and give an explicit formula for the action of these operators on Fourier coefficients. Using this, we compute the eigenvalues of Hecke operators acting on average Siegel theta series with half-integral weight (provided the prime associated to the operators does not divide the level of the theta series), and show that some of the Hecke operators annihilate theta series. Next, we bound the eigenvalues of these operators in terms of bounds on Fourier coefficients. Then we show that the half-integral weight Kitaoka subspace is stable under all Hecke operators. Finally, we observe that an obvious isomorphism between Siegel modular forms of weight $k + 1/2$ and “even” Jacobi modular forms of weight $k + 1$ is Hecke-invariant (here the level and character are arbitrary).

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0. Introduction

Modular forms are of great interest in modern number theory, one reason being that their Fourier coefficients carry number theoretic information. A theta series attached to a positive definite quadratic form Q is an example of this: Given $L = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_m$ scaled so that $Q(L) \subseteq 2\mathbb{Z}$, set

$$\theta(L; \tau) = \sum_{\ell \in L} e^{\pi i Q(\ell)\tau},$$

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where τ lies in the complex upper half-plane; it is well-known that $\theta(L; \tau)$ is a modular form of weight $m/2$. As a Fourier series, $\theta(L; \tau) = \sum_{t \in \mathbb{Z}} r(L, 2t) e^{2\pi i t \tau}$ where the Fourier coefficients are the representation numbers

$$r(L, 2t) = \#\{\ell \in L : Q(\ell) = 2t\}.$$

For each prime p we have a Hecke operator $T(p)$ acting on the space of modular forms; Hecke operators help us study Fourier coefficients of modular forms, as the space of modular forms has a basis of Hecke-eigenforms, and the Fourier coefficients of a Hecke-eigenform satisfy arithmetic relations.

Siegel was interested in generalised representation numbers that tell us the number of rank n sublattices Λ of L on which Q restricts to T , T any other quadratic form. For this he introduced generalised theta series, giving us the first examples of Siegel modular forms. These modular forms have Fourier series supported on symmetric, $n \times n$, even integral matrices (so the diagonal entries of these matrices are even). We again have Hecke operators, but for each prime p , we now have operators $T(p)$ and $T_j(p^2)$ ($1 \leq j \leq n$) generating the local Hecke algebra. For integral weight, the action of $T(p)$ on Fourier coefficients is given in [9], and the action of $T_j(p^2)$ on Fourier coefficients is given in [6]. In this paper we turn our attention to the half-integral weight case.

Hecke operators on half-integral weight Siegel modular forms, particularly Siegel theta series, have been studied before, notably by Zhuravlev using a representation theoretic approach. In [17], the author defines and explains properties of Hecke rings over a universal cover of the symplectic group; to help accomplish this, he uses a theta lift. In [18], the author considers the twisted Dirichlet series

$$\sum_M \lambda(\det M) c({}^t M R M) (\det M)^{-(s+k/2-1)}$$

where $c(T)$ is the T th coefficient of a weight $k/2$, degree n Siegel modular form F (here k is odd), λ is a completely multiplicative function (with moderate growth), R is fixed with $\det R \neq 0$, and M runs over a set of $n \times n$ integer matrices with positive determinant. When F is an eigenform for all the Hecke operators attached to primes not dividing the level of F , he uses the spherical map (which takes the Hecke ring into a ring of symmetric polynomials) to aid in establishing a factorisation of this Dirichlet series. The factorisation is a product over the primes p not dividing the level of F ; for each such p , the corresponding factor is a product of “Euler factors” that encode the eigenvalues, “Euler factors” that encode the generating relations in the local Hecke algebra, and a polynomial in the R th coefficients of the images of F under certain elements in the local Hecke algebra. The author also shows that this series has analytic continuation. In [19] the author uses a generalised theta lift to take an automorphic form on an (adelised) orthogonal group to an automorphic form on the universal cover of the (adelised) symplectic group; this latter form corresponds to a half-integral weight, vector valued theta series with a spherical harmonic. In [20] the author develops formulas for generalised Brandt matrices that give the action of the Hecke operators on these (generalised) theta series with spherical harmonics. One of his applications is to express, in terms of the Brandt matrices, some representation numbers of quadratic forms by quadratic forms. Finally, in [21], the author gives a recounting of this theory in a unified manner. At the end of the paper he specialises to half-integral weight, vector valued theta functions without spherical harmonics; he explicitly describes the Euler factors in his factorisation of the corresponding Dirichlet series as considered in [18] (although here he includes a twist by a character in the Dirichlet series). Further, he establishes conditions for linear dependence of theta series.

In this paper we take a different tack: Instead of using representation theory, we modify classical methods to explicitly determine the action of Hecke operators on the Fourier coefficients of a Siegel modular form of half-integral weight, and then we use elementary techniques to obtain various applications. We begin by considering the standard generators of the Hecke algebra; using the algorithm developed in [6], we obtain an explicit set of matrices giving the action of these Hecke operators. Then elementary techniques allow us to give a completely explicit formula for the action of these Hecke operators on Fourier coefficients (see Proposition 2.1 and Theorem 2.4). However, the formulas

for this action involve generalised “twisted” Gauss sums; while the values of these Gauss sums are explicitly known (proved in [11], repeated in Proposition 1.4), they are sufficiently complicated so that computations with these operators are cumbersome. Thus we introduce an alternate set of generators $\tilde{T}_j(p^2)$ for the Hecke algebra; taking a very direct approach, we obtain an explicit formula for the action of these operators on Fourier coefficients (Theorem 3.3). From this we rather easily obtain several applications: First, we extend [14,15] to reprove Siegel’s result that, with L an odd rank lattice equipped with a positive definite quadratic form, the average theta series $\theta^{(n)}(\text{gen } L)$ is an eigenform for the Hecke operators attached to primes not dividing the level. Further, we explicitly compute the eigenvalues of the average theta series (Theorem 4.3), and we identify Hecke operators that annihilate the (unaveraged) theta series (Theorem 4.5). Next, we bound the eigenvalues of Hecke operators in terms of bounds on Fourier coefficients (Theorem 5.1). Then we give a quick proof that the “Kitaoka space” of half-integral weight is invariant under all Hecke operators, where the Kitaoka space consists of those forms whose Fourier coefficients depend only on the genus of the parameter (Theorem 6.2). Finally, we observe that the formula for the Fourier coefficients of $f|\tilde{T}_j(p^2)$ is virtually identical to the formula in [16] for the Fourier coefficients of $F|\tilde{T}_j(p^2)$ where F is a Jacobi modular form of index 1; from this we see that with $\theta^{(n)}(\tau, Z)$ the classical Jacobi theta series, $f(\tau) \mapsto f(\tau)\theta^{(n)}(\tau, Z)$ is a Hecke-invariant isomorphism from the space of modular forms of level N , character χ , weight $k + 1/2$ onto the space of “even” Jacobi modular forms of level N , character χ' , weight $k + 1$, and index 1 (Theorem 7.4). (We say a Jacobi modular form is even if it is supported only on pairs (T, R) where R , which multiplies Z in the exponential, is even. The relation between the characters χ and χ' is stated and explained in Theorem 7.4.)

Note that Ibukiyama has a result related to the last application of our formula. In [7], without explicitly knowing the action of Hecke operators on Fourier coefficients, Ibukiyama shows that there is a Hecke-invariant isomorphism between a Kohnen-type subspace $\mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(4), 1)$ and $J_{k+1,1}(\Gamma_0^{(n,1)}(1), 1)$. Presuming there is a way to define a Kohnen-type subspace $\mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(4N), \chi)$ for arbitrary level $N \in \mathbb{Z}_+$ and character χ , and that there is a Hecke-invariant isomorphism $\nu: \mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(4N), \chi) \rightarrow J_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$, we should have the following: Let η denote our isomorphism from $J_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(4N), \chi')$ onto $\mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(4N), \chi)$, and let B_4 be the “shift operator” mapping $f(\tau)$ to $f(4\tau)$. Then B_4 should map $\mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(4N), \chi)$ into $\mathcal{M}_{k+1/2}^+(\Gamma_0^{(n)}(16N), \chi)$, and $\nu \circ B_4 \circ \eta$ should be the identity map.

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1. Preliminaries

Here we review the basic definitions, terminology, and results we rely on in the rest of the paper. To read more about the basic theory of Siegel modular forms, see for instance [1,2,4]; to read more about the basic theory of quadratic forms, see for instance [5,10].

Given odd $m \in \mathbb{Z}_+$ and a lattice $L = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m$ equipped with a positive definite quadratic form Q , and given $n \in \mathbb{Z}$ with $n > 1$, Siegel’s generalised theta series is

$$\theta^{(n)}(L; \tau) = \sum_{G \in \mathbb{Z}^{m,n}} e\{ {}^t G Q G \tau \}$$

where $\tau \in \mathcal{H}_{(n)} = \{X + iY: X, Y \in \mathbb{R}_{\text{sym}}^{n,n}, Y > 0\}$ and $e\{*\} = \exp(\pi i \text{Tr}(*))$; here we have associated Q with a symmetric $m \times m$ matrix relative to the given basis for L . We assume that Q has been scaled to be even integral, meaning that ${}^t \underline{x} Q \underline{x} \in 2\mathbb{Z}$ for $\underline{x} \in \mathbb{Z}^{m,1}$ (and hence as a matrix, Q is integral with even diagonal entries). The level of L is the smallest positive integer N so that NQ^{-1} is even integral. (Note that since L has odd rank, $4|N$; see Theorem 8.9 of [5].) As stated precisely in Theorem 1.2, and proved, for instance, in Section 1 of [1], $\theta^{(n)}(L; \tau)$ transforms under the congruence subgroup

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : C \equiv 0 \pmod{N} \right\}$$

(here $Sp_n(\mathbb{Z})$ is the symplectic group over \mathbb{Z} ; in our notation, the elements of $Sp_n(\mathbb{Z})$ are $2n \times 2n$ matrices). Note that elements $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbb{Q})$ act on Siegel’s upper half-space $\mathcal{H}_{(n)}$ by $\gamma\tau = (A\tau + B)(C\tau + D)^{-1}$.

We will need the following result, which is proved, for instance, in Lemma 1.3.15 of [1].

Theorem 1.1 (Inversion Formula). *Let L be a rank m lattice, m odd. Define the dual of L to be*

$$L^\# = \{w \in \mathbb{Q}L : B_Q(w, L) \subseteq \mathbb{Z}\},$$

where B_Q denotes the symmetric bilinear form associated to Q via the relation $Q(x + y) = Q(x) + Q(y) + 2B_Q(x, y)$. Also, for $G_0 \in \mathbb{Q}^{m,n}$, define the inhomogeneous theta series

$$\theta^{(n)}(L, G_0; \tau) = \sum_{G \in \mathbb{Z}^{m,n}} e\{Q[G + G_0]\tau\},$$

where $Q[E] = {}^tE Q E$. Then

$$\theta^{(n)}(L, G_0; \tau) = (\det Q)^{-n/2} (\det(-\tau))^{-m/2} \sum_{G \in \mathbb{Z}^{m,n}} e\{-Q^{-1}[G]\tau^{-1} - 2{}^tG G_0\}.$$

Here $(\det(-i\tau))^{1/2}$ is taken to be positive when $\tau = iY$, $Y > 0$; in general, the sign is found by analytic continuation.

One can use this to derive the Transformation Formula (below), either as done in Chapter 1 of [1], or alternatively, by adapting Eichler’s argument (where n was 1), beginning with the identity

$$(A\tau + B)(C\tau + D)^{-1} = {}^tD^{-1} {}^tB + {}^tD^{-1} \tau (D\tau + D)^{-1}$$

(valid when $\det D \neq 0$). Eichler’s approach yields the character as a generalised Gauss sum, which can then be evaluated using fairly standard techniques (see [3]).

Theorem 1.2 (Transformation Formula). *For L a rank $m = 2k + 1$ lattice of level N and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$,*

$$\theta^{(n)}(L; \gamma\tau) = \det(C\tau + D)^{m/2} \left(\frac{(-1)^k 2 \det Q}{|\det D|} \right) \operatorname{sgn}(\det D)^k \theta^{(n)}(L; \tau).$$

Here, for $\det D \neq 0$, $\lim_{\lambda \rightarrow 0^+} \det(Ci\lambda I + D)^{1/2} = (\det D)^{1/2}$, and in general the value of $\det(C\tau + D)^{1/2}$ is found by analytic continuation.

In particular, this theorem applies to the basic Siegel theta series

$$\theta^{(n)}(\tau) = \theta^{(n)}((2); \tau) = \sum_{G \in \mathbb{Z}^{1,n}} e\{2{}^tG G \tau\}$$

with $\gamma \in \Gamma_0^{(n)}(4)$. Thus we can simplify our notation by introducing automorphy factors and the slash operator, as follows.

Given $\gamma \in \text{GSp}_n^+(\mathbb{Q})$, an automorphy factor for γ is an analytic function $\varphi : \mathcal{H}_{(n)} \rightarrow \mathbb{C}$ so that $|\varphi(\gamma, \tau)|^2 = |\det(C_\gamma \tau + D_\gamma)| / \sqrt{|\det \gamma|}$. Given $\gamma_1, \gamma_2 \in \text{GSp}_n^+(\mathbb{Q})$ with corresponding automorphy factors φ_1, φ_2 , we define the product of the pairs $[\gamma_1, \varphi_1(\tau)], [\gamma_2, \varphi_2(\tau)]$ by

$$[\gamma_1, \varphi_1(\tau)][\gamma_2, \varphi_2(\tau)] = [\gamma_1 \gamma_2, \varphi_1(\gamma_2 \tau) \varphi_2(\tau)].$$

Correspondingly, we define a weight $m/2$ action of such a pair $[\gamma, \varphi(\tau)]$ on $f : \mathcal{H}_{(n)} \rightarrow \mathbb{C}$ by

$$f(\tau)|[\gamma, \varphi(\tau)] = f(\tau)|_{m/2}[\gamma, \varphi(\tau)] = \varphi(\tau)^{-m} f(\gamma \tau).$$

The default automorphy factor for $\gamma \in \Gamma_0^{(n)}(4)$ is

$$\frac{\theta^{(n)}(\gamma \tau)}{\theta^{(n)}(\tau)},$$

and we write $\tilde{\gamma}$ to denote $[\gamma, \theta^{(n)}(\gamma \tau) / \theta^{(n)}(\tau)]$.

Definition. Given $k, n, N \in \mathbb{Z}_+$ with $4|N$ and χ a Dirichlet character modulo N , a Siegel modular form of degree n , weight $k + 1/2$, level N and character χ is an analytic function $f : \mathcal{H}_{(n)} \rightarrow \mathbb{C}$ so that

$$f|_{k+1/2} \tilde{\gamma} = \chi(\det D_\gamma) f$$

for all $\gamma \in \Gamma_0(N)$. Here D_γ denotes the lower right $n \times n$ block of γ ; often we simply write $\chi(\gamma)$ to denote $\chi(D_\gamma)$. We write $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ to denote the space of all such modular forms.

Remark. Suppose L, Q, N are as in Theorem 1.2. Note that $(\theta^{(n)}(\tau))^m = \theta^{(n)}(L_0; \tau)$ where L_0 is a rank m lattice with quadratic form given by $2I_m$. Thus for a lattice L of odd rank m and level N , and $\gamma \in \Gamma_0^{(n)}(N)$, Theorem 1.2 gives us

$$\theta^{(n)}(L; \tau)|\tilde{\gamma} = \chi(\det D_\gamma) \theta^{(n)}(L; \tau)$$

where, for $d \neq 0$, $\chi(d) = (\frac{2 \det Q}{|d|})$.

Suppose $f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$. Then for any $B \in \mathbb{Z}_{\text{sym}}^{n,n}$, we have $\begin{pmatrix} I & B \\ & I \end{pmatrix} \in \Gamma_0^{(n)}(N)$, and hence $f(\tau + B) = f(\tau)$. Since f is also analytic, f has a Fourier series expansion

$$f(\tau) = \sum_T c(T) e\{T\tau\}$$

where T varies over $n \times n$, even integral matrices T with $T \geq 0$ (meaning T is positive semi-definite). Further, for any $G \in GL_n(\mathbb{Z})$, we have $\gamma = \begin{pmatrix} G^{-1} & \\ & I_G \end{pmatrix} \in \text{Sp}_n(\mathbb{Z})$ and $\theta^{(n)}(\gamma \tau) = \theta^{(n)}(\tau)$; so $f(G^{-1} \tau {}^t G^{-1}) = \chi(\det G) f(\tau)$, and consequently, $c({}^t G T G) = \chi(\det G) c(T)$. Hence we can write

$$f(\tau) = \sum_{\text{cls } \Lambda} c(\Lambda) e^*\{\Lambda \tau\}$$

where $\text{cls } \Lambda$ varies over isometry classes of rank n lattices equipped with even integral, positive semi-definite quadratic form (oriented when $\chi(-1) = -1$), $c(\Lambda) = c(T)$ where T is a matrix

for the quadratic form on Λ , and $e^*\{\Lambda\tau\} = \sum_G e\{{}^tGTG\tau\}$ with $G \in O(T) \setminus GL_n(\mathbb{Z})$ if $\chi(-1) = 1$, $G \in O^+(T) \setminus SL_n(\mathbb{Z})$ if $\chi(-1) = -1$. (Here $O(T)$ is the orthogonal group of T , $O^+(T) = O(T) \cap SL_n(\mathbb{Z})$.)

Definitions. A pair of $n \times n$ integer matrices (C, D) is called a coprime symmetric pair if $C{}^tD$ is symmetric, and for $G \in GL_n(\mathbb{Q})$, $G(C, D)$ is integral only if $G \in GL_n(\mathbb{Z})$. (Note that if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$, then (C, D) is a coprime symmetric pair, as is $({}^tB, {}^tD)$. Conversely, if $({}^tB, {}^tD)$ is a coprime symmetric pair, then there is some $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$.) For $({}^tB, {}^tD)$ a coprime symmetric pair of $n \times n$ integral matrices with $\det D \neq 0$, we define a generalised Gauss sum

$$\mathcal{G}_B(D) = \sum_{G \in \mathbb{Z}^{1,n} / \mathbb{Z}^{1,n} {}^tD} e\{2{}^tGGBD^{-1}\}.$$

Thus for b, d coprime integers with $d \neq 0$, $\mathcal{G}_b(d)$ is the classical Gauss sum.

Remark. When following Eichler’s approach to prove the Transformation Formula, we encounter the Gauss sum

$$\mathcal{G}_B(D; \mathbb{Q}) = \sum_{G \in \mathbb{Z}^{m,n} / \mathbb{Z}^{m,n} {}^tD} e\{{}^tGQGBD^{-1}\},$$

which we can evaluate using the theory of quadratic forms over finite fields.

Proposition 1.3. Suppose p is an odd prime, $D = \text{diag}\{I_{r_0}, pI_{r_1}, p^2I_{r_2}, I_{r_3}\}$ and Y is the integral matrix

$$Y = \begin{pmatrix} Y_0 & pY_2 & 0 & Y_3 \\ {}^tY_2 & Y_1 & 0 & \\ 0 & 0 & I_{r_2} & \\ {}^tY_3 & & & \end{pmatrix}$$

where, for $i = 0, 1$, Y_i is $r_i \times r_i$ and symmetric. Then

$$\mathcal{G}_Y(D) = p^{r_2} \left(\frac{\det Y_1}{p} \right) \mathcal{G}_1(p)^{r_1}$$

where $\mathcal{G}_1(p)$ is the classical Gauss sum.

Proof. With Y, D as above, we have

$$\mathcal{G}_Y(D) = \sum_{G_1 \in \mathbb{Z}^{1,r_1}} e\{2{}^tG_1G_1Y_1/p\} \sum_{G_2 \in \mathbb{Z}^{1,r_2}} e\{2{}^tG_2G_2/p^2\}.$$

It is easy to see that

$$\sum_{G_2} e\{2{}^tG_2G_2/p^2\} = \left(\sum_{g \in \mathbb{Z}/p\mathbb{Z}} e\{2g^2/p^2\} \right)^{r_2} = \mathcal{G}_1(p^2)^{r_2} = p^{r_2}.$$

Then, since Y_1 is symmetric and p is odd, the quadratic form given by Y_1 can be diagonalised over \mathbb{F}_p . Since $SL_r(\mathbb{Z})$ maps onto $SL_r(\mathbb{F}_p)$ (see, for instance, p. 21 of [12]), we can find $E \in SL_{r_1}(\mathbb{Z})$ so that $EY_1{}^tE \equiv \text{diag}\{u_1, \dots, u_{r_1}\} \pmod{p}$, and thus

$$\sum_{G_1} e\{2^t G_1 G_1 Y_1 / p\} = \sum_{G_1} e\{2^t (G_1 E^{-1})(G_1 E^{-1})U / p\}$$

where $U = \text{diag}\{u_1, \dots, u_{r_1}\}$. Since $G_1 E^{-1}$ varies over $\mathbb{Z}^{1,r_1}/p\mathbb{Z}^{1,r_1}$ as G_1 does, we have

$$\sum_{G_1} e\{2^t G_1 G_1 Y_1 / p\} = \prod_{i=1}^{r_1} \mathcal{G}_{u_i}(p) = \left(\frac{u_1 \cdots u_{r_1}}{p}\right) (\mathcal{G}_1(p))^{r_1}.$$

Finally, note that $\det Y_1 = \det U$, so $(\frac{u_1 \cdots u_{r_1}}{p}) = (\frac{\det Y_1}{p})$. \square

Next we introduce some notation for various types of representation numbers. Let p be a prime, $\mathbb{F} = \mathbb{F}_p$; let $T \in \mathbb{F}_{\text{sym}}^{d,d}$, $S \in \mathbb{F}_{\text{sym}}^{d,a}$, $a \leq d$. We set

$$r(T, S) = \#\{C \in \mathbb{F}^{d,a} : {}^t CTC = S\},$$

$$r^*(T, S) = \#\{C \in \mathbb{F}^{d,a} : {}^t CTC = S, \text{rank}_{\mathbb{F}} C = a\}.$$

Thus $r(T, S)$ is the number of times T represents S , and $r^*(T, S)$ is the number of times T primitively represents S . We use $O(T)$ to denote the orthogonal group of T , i.e.

$$O(T) = \{G \in GL_d(\mathbb{F}) : {}^t GTG = T\},$$

and set $o(T) = \#O(T)$. When $Y = {}^t GTG$ for some $G \in GL_d(\mathbb{F})$, we write $Y \sim T$.

When V is a dimension d vector space over \mathbb{F} with a quadratic form given by a symmetric matrix T (relative to some basis), we call V a quadratic space over \mathbb{F} , and we write $V \simeq T$. We say V is regular if $\det T \neq 0$. With W another quadratic space over \mathbb{F} with dimension $a \leq r$ and quadratic form given by a symmetric matrix S , we set

$$R^*(V, W) = \frac{r^*(T, S)}{o(S)};$$

so $R^*(V, W)$ is the number of dimension a subspaces W' of V isometric to W , meaning that relative to some basis for W' , the quadratic form on W' is given by S . If $\dim V < \dim W$, then $R^*(V, W) = 0$.

Now suppose the prime p is odd. We use \mathbb{H} to denote the dimension 2 quadratic space over \mathbb{F}_p with quadratic form given by the matrix $\langle 1, -1 \rangle = \text{diag}\{1, -1\}$; we also write $\mathbb{H} \simeq \langle 1, -1 \rangle$. Similarly, \mathbb{A} denotes the dimension 2 quadratic space over \mathbb{F}_p so that $\mathbb{A} \simeq \langle 1, -\omega \rangle$, where $(\frac{\omega}{p}) = -1$. Given any dimension d quadratic space V over \mathbb{F}_p , V splits as $V_0 \perp R$, where V_0 is regular and $R = \text{rad } V \simeq \langle 0 \rangle^s$, some $s \geq 0$ (where $\langle 0 \rangle^s$ denotes the $s \times s$ matrix of zeros). Also, the isometry class of V_0 is determined by the dimension of V_0 and the value of the Legendre symbol $(\frac{\text{disc } V_0}{p})$ where $\text{disc } V_0$ is the determinant of a matrix for the quadratic form on V_0 (so $\text{disc } V_0$ is well-defined up to squares in \mathbb{F}_p^\times). We have

$$V_0 \simeq \begin{cases} \mathbb{H}^c & \text{if } \dim V_0 = 2c, (\frac{\text{disc } V_0}{p}) = (\frac{-1}{p})^c, \\ \mathbb{H}^{c-1} \perp \mathbb{A} & \text{if } \dim V_0 = 2c, (\frac{\text{disc } V_0}{p}) \neq (\frac{-1}{p})^c, \\ \mathbb{H}^c \perp \langle 1 \rangle & \text{if } \dim V_0 = 2c + 1, (\frac{\text{disc } V_0}{p}) = (\frac{-1}{p})^c, \\ \mathbb{H}^c \perp \langle \omega \rangle & \text{if } \dim V_0 = 2c + 1, (\frac{\text{disc } V_0}{p}) \neq (\frac{-1}{p})^c. \end{cases}$$

(Note that while V_0 is not uniquely determined by V , the isometry class of V_0 is.) We say V is totally isotropic if $V = \text{rad } V$, i.e. $V \simeq \langle 0 \rangle^s$.

We now define another generalised Gauss sum that we will later encounter.

Definition. Suppose p is an odd prime and V is a dimension d quadratic space over $\mathbb{F} = \mathbb{F}_p$ with quadratic form given by T modulo p , where T is an even integral matrix over \mathbb{Z} . Then we define the twisted Gauss sum $\mathcal{G}^*(V)$ by

$$\mathcal{G}^*(V) = \sum_{Y \in \mathbb{F}_{\text{sym}}^{d,d}} \left(\frac{\det Y}{p} \right) e\{YT/p\}.$$

We define the normalised twisted Gauss sum $\tilde{\mathcal{G}}(V)$ by

$$\tilde{\mathcal{G}}(V) = p^{-d} (\mathcal{G}_1(p))^d \mathcal{G}^*(V).$$

When $V = \{0\}$, we agree that $\tilde{\mathcal{G}}(V) = 1$.

By Theorem 1.3 of [11], we have the following.

Proposition 1.4. *With p an odd prime, we have:*

$$\begin{aligned} \tilde{\mathcal{G}}(\mathbb{H}^c \perp \langle 0 \rangle^s) &= \tilde{\mathcal{G}}(\mathbb{H}^{c-1} \perp \mathbb{A} \perp \langle 0 \rangle^s) \\ &= \begin{cases} (-1)^c p^{(c+x)^2 - (c+x)} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x, \\ 0 & \text{if } s = 2x + 1; \end{cases} \end{aligned}$$

and with $\eta \in \mathbb{F}^\times$,

$$\tilde{\mathcal{G}}(\mathbb{H}^c \perp \langle 2\eta \rangle \perp \langle 0 \rangle^s) = \begin{cases} (-1)^c \left(\frac{-\eta}{p}\right) p^{(c+x)^2 + x} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x, \\ (-1)^c p^{(c+x)^2 - (c+x)} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x - 1. \end{cases}$$

There are several elementary functions we use throughout: For fixed prime p and $m, r \in \mathbb{Z}$ with $r > 0$,

$$\delta(m, r) = \delta_p(m, r) = \sum_{i=0}^{r-1} (p^{m-i} + 1),$$

$$\mu(m, r) = \mu_p(m, r) = \sum_{i=0}^{r-1} (p^{m-i} - 1),$$

$$\delta(m, 0) = \mu(m, 0) = 1.$$

When we have $m \geq r \geq 0$

$$\beta(m, r) = \beta_p(m, r) = \frac{\mu(m, r)}{\mu(r, r)},$$

which is the number of dimension r subspaces of a dimension m space over \mathbb{F}_p . We will sometimes write, for instance, $\delta\mu(m, r)$ for $\delta(m, r) \cdot \mu(m, r)$.

We will frequently use the fact that $\text{Tr}(AB) = \text{Tr}(BA)$, and hence $e\{AB\} = e\{BA\}$.

2. The standard generators of the Hecke algebra

We begin this section by defining the standard generators of the Hecke algebra for half-integral weight Siegel modular forms; then we analyse their action on Fourier coefficients.

Fix N so that $4|N$, and set $\tilde{\Gamma} = \{\tilde{\gamma}: \gamma \in \Gamma_0^{(n)}(N)\}$; let $\tilde{\delta} = \left[\begin{pmatrix} pI & \\ & I \end{pmatrix}, p^{-n/2} \right]$. Similar to the case of integral weight, we define

$$F|T(p) = \sum_{\tilde{\gamma}} \bar{\chi}(\gamma) F|\tilde{\delta}^{-1}\tilde{\gamma}$$

where $\tilde{\gamma}$ runs over a complete set of representatives for $(\tilde{\Gamma} \cap \tilde{\delta}\tilde{\Gamma}\tilde{\delta}^{-1}) \backslash \tilde{\Gamma}$. For $1 \leq j \leq n$, set

$$X_j = \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}, \quad \delta_j = \begin{pmatrix} X_j & \\ & X_j^{-1} \end{pmatrix}, \quad \text{and} \quad \tilde{\delta}_j = [\delta_j, p^{-j/2}].$$

For $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$, define

$$F|T_j(p^2) = \sum_{\tilde{\gamma}} \bar{\chi}(\gamma) F|\tilde{\delta}_j^{-1}\tilde{\gamma}$$

where $\tilde{\gamma}$ runs over a complete set of representatives for $(\tilde{\Gamma} \cap \tilde{\delta}_j\tilde{\Gamma}\tilde{\delta}_j^{-1}) \backslash \tilde{\Gamma}$.

Proposition 2.1. For $F \in \mathcal{M}_{k+1/2}(N, \chi)$ and p prime, $F|T(p) = 0$.

Proof. With $\ell \in 2\mathbb{Z}_+$, write $\theta^{(n)}(\ell; \tau)$ for $\theta^{(n)}(L; \tau)$ where $L = \mathbb{Z}x \simeq (\ell)$. So $\theta^{(n)}(\tau) = \theta^{(n)}((2); \tau)$.

Take $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma = \Gamma_0^{(n)}(N)$ so that $\gamma' = \delta\gamma\delta^{-1} \in \Gamma$. (So $p|C$, and $p \nmid \det D$.) By Theorem 1.2,

$$\theta^{(n)}(\gamma\delta^{-1}\tau) = \det(C\tau/p + D)^{1/2} \left(\frac{4}{|\det D|} \right) \theta^{(n)}(\tau/p),$$

and

$$\begin{aligned} \theta^{(n)}(\gamma'\tau) &= \theta^{(n)}((2p); \gamma\tau/p) \\ &= \det(C\tau/p + D)^{1/2} \left(\frac{4p}{|\det D|} \right) \theta^{(n)}((2p); \tau/p) \\ &= \det(C\tau/p + D)^{1/2} \left(\frac{p}{|\det D|} \right) \theta^{(n)}(\tau). \end{aligned}$$

Thus

$$\tilde{\delta}\tilde{\gamma}\tilde{\delta}^{-1} = \left[\gamma', \left(\frac{p}{|\det D|} \right) \theta^{(n)}(\gamma'\tau) / \theta^{(n)}(\tau) \right],$$

and hence with $\Gamma' = \delta\Gamma\delta^{-1}$, $[(\widetilde{\Gamma} \cap \widetilde{\Gamma}') : \widetilde{\Gamma} \cap \widetilde{\delta\Gamma\delta^{-1}}] \leq 2$. To show this index is 2, choose a prime $q \nmid N$ so that $(\frac{p}{q}) = -1$. Choose $a, b \in \mathbb{Z}$ so that $aq - Nbp = 1$, and set

$$\gamma_0 = \begin{pmatrix} a & & b & \\ & I_{n-1} & & 0_{n-1} \\ Np & & q & \\ & 0_{n-1} & & I_{n-1} \end{pmatrix} \in \Gamma.$$

So

$$\gamma'_0 = \delta\gamma_0\delta^{-1} = \begin{pmatrix} a & & pb & \\ & I_{n-1} & & 0_{n-1} \\ N & & q & \\ & 0_{n-1} & & I_{n-1} \end{pmatrix} \in \Gamma,$$

but $\widetilde{\gamma}'_0 \neq \widetilde{\delta\gamma_0\delta^{-1}}$. Therefore 1, $\widetilde{\gamma}'_0$ are coset representatives for $(\widetilde{\Gamma} \cap \widetilde{\delta\Gamma\delta^{-1}}) \setminus (\widetilde{\Gamma} \cap \widetilde{\Gamma}')$, and

$$f|\delta^{-1}\widetilde{\gamma}'_0 = -f|\widetilde{\gamma_0\delta^{-1}} = -\chi(\gamma_0)f|\delta^{-1}.$$

So with $\widetilde{\gamma}$ running over a set of coset representatives for $(\widetilde{\Gamma} \cap \widetilde{\Gamma}') \setminus \widetilde{\Gamma}$, and noting that $\chi(\gamma'_0) = \chi(\gamma_0)$, we have

$$\begin{aligned} f|T(p) &= \sum_{\widetilde{\gamma}} \overline{\chi}(\gamma)f|\delta^{-1}\widetilde{\gamma} + \sum_{\widetilde{\gamma}} \overline{\chi}(\gamma'_0)f|\delta^{-1}\widetilde{\gamma}'_0 \\ &= \left(\sum_{\widetilde{\gamma}} \overline{\chi}(\gamma)f|\delta^{-1}\widetilde{\gamma} \right) - \left(\sum_{\widetilde{\gamma}} \overline{\chi}(\gamma)f|\delta^{-1}\widetilde{\gamma} \right) \\ &= 0, \end{aligned}$$

proving the proposition. \square

We use a similar argument to prove the following.

Lemma 2.2. Let $\Gamma = \Gamma_0(N)$ and let $\delta_j, \widetilde{\delta}_j$ be as above. Set $\Gamma'_j = \delta_j\Gamma\delta_j^{-1}$; then for $p \nmid N$,

$$\widetilde{\Gamma'_j} \cap \widetilde{\Gamma} = \widetilde{\delta}_j\widetilde{\Gamma}\widetilde{\delta}_j^{-1} \cap \widetilde{\Gamma}.$$

Proof. Say $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ so that $\gamma' = \delta_j\gamma\delta_j^{-1} \in \Gamma$. Then by Theorem 1.2,

$$\begin{aligned} \theta^{(n)}(\gamma'\tau)/\theta^{(n)}(\tau) &= \det(X_j^{-1}CX_j^{-1}\tau + X_j^{-1}DX_j)^{1/2} \\ &= \det(CX_j^{-1}\tau X_j^{-1} + D)^{1/2} \\ &= \theta^{(n)}(\gamma\delta_j^{-1}\tau)/\theta^{(n)}(\delta_j^{-1}\tau), \end{aligned}$$

and hence $\widetilde{\gamma}' = \widetilde{\delta}_j\widetilde{\gamma}\widetilde{\delta}_j^{-1}$. \square

Theorem 2.3. Take p a prime and $F \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$.

(a) If $p \nmid N$, then

$$F|T_j(p^2) = \sum_{\Omega, \Lambda'_1, Y} \chi(\det D) F|\tilde{\delta}_j^{-1} \left[\begin{pmatrix} D & {}^tY \\ & D^{-1} \end{pmatrix}, \frac{\mathcal{G}_Y(D)}{(\det D)} \right] \left(\widetilde{\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}} \right)$$

where Ω, Λ'_1 vary subject to $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, and $\bar{\Lambda}'_1$ varies over all codimension $n - j$ subspaces of $\Lambda \cap \Omega/p(\Lambda + \Omega)$. Here

$$D = D(\Omega) = \text{diag}\{I_{n_0}, pI_{j-r}, p^2I_{n_2}, I_{n-j}\}$$

with $r = n_0 + n_2$, and $G = G(\Omega, \Lambda'_1) \in SL_n(\mathbb{Z})$ so that

$$\Omega = \Lambda G D^{-1} X_j, \quad \Lambda'_1 = \Lambda G \begin{pmatrix} 0_{r_0} & & \\ & I_{j-r} & \\ & & 0 \end{pmatrix}; \quad {}^tY = \begin{pmatrix} Y_0 & Y_2 & 0 & Y_3 \\ p {}^tY_2 & Y_1 & 0 & \\ 0 & 0 & I & \\ {}^tY_3 & & & \end{pmatrix}$$

with Y_0 symmetric, $n_0 \times n_0$, varying modulo p^2 , Y_1 symmetric, $(n - r) \times (n - r)$ varying modulo p with the restriction that $p \nmid \det Y_1$, Y_2 $n_0 \times (j - r)$, varying modulo p , Y_3 $n_0 \times (n - j)$, varying modulo p .

(b) If $p|N$, then

$$F|T_j(p^2) = \sum_{\Omega, Y} F|\tilde{\delta}_j^{-1} \left(\widetilde{\begin{pmatrix} I & Y \\ & I \end{pmatrix}} \right) \left(\widetilde{\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}} \right)$$

where Ω varies subject to $p\Lambda \subseteq \Omega \subseteq \Lambda$, $[\Lambda : \Omega] = p^j$, $G = G(\Omega) \in SL_n(\mathbb{Z})$ so that $\Omega = \Lambda G X_j$ and

$$Y = \begin{pmatrix} Y_0 & Y_3 \\ {}^tY_3 & 0 \end{pmatrix}$$

with Y_0 symmetric, $j \times j$, varying modulo p^2 , Y_3 $j \times (n - j)$, varying modulo p .

Proof. By Lemma 2.2, a set of coset representatives for the action of the half-integral weight Hecke operator $T_j(p^2)$ is $\{\tilde{\gamma}\}$ where $\{\gamma\}$ is a set of coset representatives for the integral weight Hecke operator $T_j(p^2)$, and a set of representatives for this was given in Section 6 of [6]; note that the matrices G presented there can be chosen from $SL_n(\mathbb{Z})$. Thus a set of coset representatives corresponding to $T_j(p^2)$ on $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ is

$$\left\{ \left(\widetilde{\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix}} \right) \left(\widetilde{\begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}} \right) \right\}$$

where $\begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix} \in \Gamma = \Gamma_0^{(n)}(N)$ with $U = \text{diag}\{U_0, 0_{n-j}\}$, $W = \text{diag}\{W_0, I_{n-j}\}$, and $G = G(\Omega, \Lambda'_1)$, $D = D(\Omega)$, Y vary as claimed. (From the definition, we know that for $\gamma_1, \gamma_2 \in \Gamma$, $\widetilde{\gamma_1 \gamma_2} = \tilde{\gamma}_1 \tilde{\gamma}_2$.) Note that $DW \equiv I \pmod{N}$, so $\bar{\chi}(\det W) = \chi(\det D)$.

Suppose first $p \nmid N$. Set $X = X_j$, $X' = XUD^{-1}X$,

$$\gamma = \begin{pmatrix} D & {}^tY \\ U & W \end{pmatrix}, \quad \beta = \begin{pmatrix} D & {}^tY \\ & D^{-1} \end{pmatrix}, \quad \gamma' = \delta_j^{-1}\gamma\beta^{-1}\delta_j = \begin{pmatrix} I & \\ X' & I \end{pmatrix};$$

since $XU = UX$, X' is integral and divisible by N , and so $\gamma' \in \Gamma_0^{(n)}(N)$. We will show that $\tilde{\gamma}' = \tilde{\delta}_j^{-1}\tilde{\gamma}[\beta^{-1}, (\det D)(\mathcal{G}_Y(D))^{-1}]\tilde{\delta}_j$, and hence

$$F|\tilde{\delta}_j^{-1}\tilde{\gamma} = F|\tilde{\gamma}'\tilde{\delta}_j^{-1}[\beta, (\det D)^{-1}\mathcal{G}_Y(D)] = F|\tilde{\delta}_j^{-1}[\beta, (\det D)^{-1}\mathcal{G}_Y(D)].$$

We have

$$\tilde{\delta}_j^{-1}\tilde{\gamma}\left[\beta^{-1}, \frac{\det D}{\mathcal{G}_Y(D)}\right]\tilde{\delta}_j = \left[\gamma', \frac{\det D}{\mathcal{G}_Y(D)} \cdot \frac{\theta^{(n)}(\gamma\beta^{-1}\delta_j\tau)}{\theta^{(n)}(\beta^{-1}\delta_j\tau)}\right], \quad \beta^{-1} = \begin{pmatrix} D^{-1} & -Y \\ & D \end{pmatrix}.$$

Also, since DU must be symmetric, DUD^{-1} is integral and divisible by 4. Therefore, using the Inversion Formula (Theorem 1.1), we have

$$\begin{aligned} \theta^{(n)}(\gamma\beta^{-1}\delta_j\tau) &= \frac{1}{\sqrt{2^n}}(\det X)^{-1}(\det(-i\tau(X'\tau + I)^{-1}))^{-1/2} \\ &\quad \cdot \sum_{g \in \mathbb{Z}^{1,n}} e\left\{-\frac{1}{2}{}^tgg(UD^{-1}X\tau + X^{-1})\tau^{-1}X^{-1}\right\} \\ &= \frac{1}{\sqrt{2^n}}(\det X)^{-1}(\det(-i\tau(X'\tau + I)^{-1}))^{-1/2} \\ &\quad \cdot \sum_{g_0(D)} e\left\{-\frac{1}{2}{}^tg_0g_0UD^{-1}\right\}\theta^{(n)}\left(\left(\frac{1}{2}\right), g_0D^{-1}; -DX^{-1}\tau^{-1}X^{-1}D\right) \\ &= (\det X)^{-1}(\det(-iD^{-1}X\tau XD^{-1}))^{1/2}(\det(-i\tau(XUD^{-1}X\tau + I)^{-1}))^{-1/2} \\ &\quad \cdot \sum_{g_0(D)} e\left\{-\frac{1}{2}{}^tg_0g_0UD^{-1}\right\} \sum_{g \in \mathbb{Z}^{1,n}} e\{2{}^tggD^{-1}X\tau XD^{-1} - 2D^{-1}{}^tg_0g\}. \end{aligned}$$

(By $g_0(D)$ we really mean $g \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}D$.) Since γ, γ^{-1} are symplectic, we know $UY = WD - I$, $YU = {}^tWD - I$, and $D^{-1}{}^tU = UD^{-1}$ with $4|U$; thus

$$e\{2{}^t(g_0U/2 + g)(g_0U/2 + g)YD^{-1}\} = e\left\{-\frac{1}{2}{}^tg_0g_0UD^{-1} - 2D^{-1}{}^tg_0g\right\}e\{2{}^tggYD^{-1}\}.$$

Thus, remembering that $\det D, \det X > 0$,

$$\begin{aligned} \theta^{(n)}(\gamma\beta^{-1}\delta_j\tau) &= (\det D)^{-1}(\det(-i\tau))^{1/2}(\det(-i\tau(XUD^{-1}X\tau + I)^{-1}))^{-1/2} \\ &\quad \cdot \sum_{\substack{g, g_0 \in \mathbb{Z}^{1,n} \\ g_0(D)}} e\{2{}^t(g_0U/2 + g)(g_0U/2 + g)YD^{-1}\} \\ &\quad \cdot e\{2{}^tgg(D^{-1}X\tau XD^{-1} - YD^{-1})\}. \end{aligned}$$

We know that $(D, {}^tU)$ is a symmetric pair with $U = \text{diag}\{U_0, 0_{n-j}\}$, so

$$U = \begin{pmatrix} W_0 & W_3 & W_5 \\ p {}^tW_3 & W_1 & W_4 \\ p^2 {}^tW_5 & p {}^tW_4 & W_2 \end{pmatrix}$$

with the diagonal blocks symmetric and of sizes $n_0 \times n_0$, $(j - r) \times (j - r)$, and $n_2 \times n_2$. Since we also have that $(D, {}^tU)$ is a coprime pair, $p \nmid \det \begin{pmatrix} W_1 & W_4 \\ p {}^tW_4 & W_2 \end{pmatrix}$; thus for fixed $g \in \mathbb{Z}^{1,n}$, $g_0U/2 + g$ varies over $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}D$ as g_0 does (recall that $4|U$). Thus the sum on g_0 is independent of the choice of g , and so the sum on g_0 is $\mathcal{G}_Y(D)$. Hence

$$\begin{aligned} \theta^{(n)}(\gamma\beta^{-1}\delta_j\tau) &= (\det D)^{-1}(\det(-i\tau))^{1/2}(\det(-i\tau(XUD^{-1}X\tau + I)^{-1}))^{-1/2} \\ &\quad \cdot \mathcal{G}_Y(D)\theta^{(n)}(\beta^{-1}\delta_j\tau). \end{aligned}$$

Next, recall that $XU = UX$ with $N|X'$, and hence for $g \in \mathbb{Z}^{1,n}$, $4|{}^tggX'$. Thus, using Theorem 1.1, we have

$$\begin{aligned} \theta^{(n)}(\gamma'\tau) &= \frac{1}{\sqrt{2}^n}(\det(-i\tau(X'\tau + I)^{-1}))^{-1/2} \sum_{g \in \mathbb{Z}^{1,n}} e\left\{-\frac{1}{2}{}^tgg\tau^{-1}\right\} \\ &= (\det(-i\tau(X'\tau + I)^{-1}))^{-1/2}(\det(-i\tau))^{1/2}\theta^{(n)}(\tau). \end{aligned}$$

This means

$$\frac{\theta^{(n)}(\gamma'\tau)}{\theta^{(n)}(\tau)} = \frac{\mathcal{G}_Y(D)}{\det D} \frac{\theta^{(n)}(\gamma\beta^{-1}\delta_j\tau)}{\theta^{(n)}(\beta^{-1}\delta_j\tau)},$$

completing the proof in the case that $p \nmid N$.

In the case $p|N$, the argument is much simpler, as the coset representatives for $(\Gamma'_j \cap \Gamma) \backslash \Gamma$ are those representatives as above where $D = I$. Since $\Omega = \Lambda GX$, we have $p\Lambda \subseteq \Omega \subseteq \Lambda$ with $[\Lambda : \Omega] = p^j$. \square

We complete this section by evaluating the action of $T_j(p^2)$ on the Fourier coefficients of a half-integral weight Siegel modular form. These involve the normalised twisted Gauss sums, as defined in Section 1 and whose values are given in Proposition 1.4.

Theorem 2.4. *Take $f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ where $4|N$, and let p be a prime. Let χ' be the character modulo N defined by*

$$\chi'(d) = \chi(d) \left(\frac{(-1)^{k+1}}{|d|} \right) (\text{sgn } d)^{k+1}.$$

(a) *Suppose $p \nmid N$. Given*

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2, \quad \Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2$$

with $n_i = \text{rank } \Lambda_i$, $r = n_0 + n_2$, set

$$A_j(\Lambda, \Omega) = \chi'(p^{j-r})p^{j/2+k(n_2-n_0)+n_0(n-n_2)} \sum_{\substack{\text{cls } U \\ \dim U = j-r}} R^*(\Lambda_1/p\Lambda_1, U)\tilde{\mathcal{G}}(U)$$

if Λ, Ω are even integral, and set $A_j(\Lambda, \Omega) = 0$ otherwise. Then the Λ th coefficient of $f|T_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} A_j(\Lambda, \Omega)c(\Omega).$$

(b) Suppose $p|N$. Then the Λ th coefficient of $f|T_j(p^2)$ is

$$p^{j(n-k+1/2)} \sum_{\substack{p\Lambda \subseteq \Omega \subseteq \Lambda \\ [\Lambda:\Omega]=p^j}} c(\Omega).$$

Proof. Suppose first that $p \nmid N$; then $p \neq 2$ and

$$f(\tau)|T_j(p^2) = p^{-j(k+1/2)} \sum_{D, Y, G} \chi(\det D)(\det D)^{2k+1} \mathcal{G}_Y(D)^{-2k-1} \cdot \sum_T c(T)e\{TX^{-1}DG^{-1}\tau^tG^{-1}DX^{-1}\}e\{TX^{-1}tYDX^{-1}\}$$

where D, Y, G vary as in Theorem 2.3, and T varies over all $n \times n$ even integral, positive semi-definite matrices.

Fix T, G and $D = \text{diag}\{I_{n_0}, pI_{j-r}, p^2I_{n_2}, I_{n-j}\}$, and let Y vary. As described in Theorem 2.3, we have

$$tY = \begin{pmatrix} Y_1 & Y_2 & 0 & Y_3 \\ p^tY_2 & Y_1 & 0 & \\ 0 & 0 & I & \\ tY_3 & & & \end{pmatrix}; \quad \text{write } T = \begin{pmatrix} T_0 & T_2 & * & T_3 \\ tT_2 & T_1 & * & * \\ * & * & * & * \\ tT_3 & * & * & * \end{pmatrix}$$

with T_i the size of Y_i . By Proposition 1.3, $\mathcal{G}_Y(D) = p^{n_2}(\frac{\det Y_1}{p})\mathcal{G}_1(p)^{j-r}$; so

$$\begin{aligned} &\sum_Y \mathcal{G}_Y(D)^{-2k-1} e\{TX^{-1}tYDX^{-1}\} \\ &= p^{-n_2(2k+1)} \mathcal{G}_1(p)^{(r-j)(2k+1)} \sum_{Y_0(p^2)} e\{T_0Y_0/p^2\} \cdot \sum_{Y_2(p)} e\{2T_2Y_2/p\} \\ &\quad \cdot \sum_{Y_3(p)} e\{2T_3Y_3/p\} \cdot \sum_{Y_1(p)} \left(\frac{\det Y_1}{p}\right) e\{T_1Y_1/p\}. \end{aligned}$$

If $T_0 \equiv 0 \pmod{p^2}$, $T_2 \equiv 0 \pmod{p}$, $T_3 \equiv 0 \pmod{p}$ then the sum on Y is

$$\left(\frac{-1}{p}\right)^{(j-r)(k+1)} p^{-n_2(2k+1)+(r-j)k+n_0(n-n_2+1)} \tilde{\mathcal{G}}(T_1 \pmod{p}),$$

and otherwise the sum on Y is 0 (here we used that $(\mathcal{G}_1(p))^2 = (\frac{-1}{p})p$). Therefore,

$$f(\tau)|T_j(p^2) = \sum_{D,G} \chi'(\det D)p^{j/2+k(n_2-n_0)+n_0(n-n_2)} \cdot \sum_T \tilde{\mathcal{G}}(T_1 \bmod p)c(T)e\{T[X^{-1}DG^{-1}]\tau\}.$$

$T[X^{-1}D]$ integral

Let $\Lambda = \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_n$ be equipped with the quadratic form $T[X^{-1}DG^{-1}]$ (relative to the given basis); then $\Omega = \Lambda GD^{-1}X \simeq T$ relative to $(x_1 \dots x_n)GD^{-1}X$. Also, relative to these bases we have splittings

$$\Lambda = \Lambda_0 \oplus \Lambda'_1 \oplus \Lambda_2 \oplus \Lambda''_1, \quad \Omega = p\Lambda_0 \oplus \Lambda'_1 \oplus \frac{1}{p}\Lambda_2 \oplus \Lambda''_1$$

where $n_0 = \text{rank } \Lambda_0$, $j - r = \text{rank } \Lambda'_1$, $n_2 = \text{rank } \Lambda_2$, and $\Lambda'_1 \simeq T_1$. From [HW] we know that with $\Lambda_1 = \Lambda'_1 \oplus \Lambda''_1$ and Ω fixed, $G = G(\Omega, \Lambda'_1)$ varies to vary Λ'_1 over all dimension $j - r$ subspaces of $\Lambda_1/p\Lambda_1 \approx \Lambda \cap \Omega/p(\Lambda + \Omega)$. Thus

$$\sum_{\Lambda'_1} \tilde{\mathcal{G}}(T_1 \bmod p) = \sum_{\substack{\text{cls } U \\ \dim U = j-r}} R^*(\Lambda_1/p\Lambda_1, U)\tilde{\mathcal{G}}(U).$$

This proves the theorem in the case $p \nmid N$.

In the case $p|N$, the analysis is simpler, as D is always I (and hence $n_0 = r = j$, $n_2 = 0$); following the above reasoning yields the theorem in this case. \square

3. An alternate set of generators for the Hecke algebra

In pursuit of a more utile formula for the action of Hecke operators on Fourier coefficients, we introduce an alternate set of generators. When we studied the action of Hecke operators on integral weight Siegel modular forms in [6], in the case $p \nmid N$ we encountered incomplete character sums, which we completed by replacing $T_j(p^2)$ with

$$\tilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{\ell=0}^j \chi(p^{j-\ell})\beta(n - \ell, j - \ell)T_\ell(p^2).$$

Here, with half-integral weight, the incomplete character sums are twisted, giving us the generalised twisted Gauss sums. In Theorem 3.3 we show that by replacing $T_j(p^2)$ with

$$\tilde{T}_j(p^2) = p^{j(k-n)} \sum_{\ell=0}^j p^{-\ell/2} \chi'(p^{j-\ell})\beta(n - \ell, j - \ell)T_\ell(p^2),$$

we eliminate these Gauss sums from the formula for the action on Fourier coefficients.

To ready ourselves to prove Theorem 3.3, we first establish the following relationship between $\tilde{\mathcal{G}}(W)$ and a weighted sum of representation numbers.

Lemma 3.1. For p an odd prime, $\mathbb{F} = \mathbb{F}_p$, and W a quadratic space over \mathbb{F} with dimension $m \geq 0$,

$$\tilde{\mathcal{G}}(W) = \sum_{a=0}^m (-1)^{m+a} p^{m(m-1)/2+a(a-m)} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a).$$

Proof. Since $R^*(\langle 2 \rangle, \{0\}) = 1$, the lemma holds for $m = 0$. So suppose $m \geq 1$. Then standard theory tells us that with $\mathbb{F}v \simeq \langle 2 \rangle$, $W \perp \mathbb{F}v$ splits as $W_0 \perp R$ where $R = \text{rad}(W \perp \mathbb{F}v) \simeq \langle 0 \rangle^s$, some $s \in \mathbb{Z}_{\geq 0}$, with W_0 regular. Any totally isotropic subspace U of $W \perp \mathbb{F}v$ splits as $U_0 \perp U_1$ where $U_1 = U \cap R$. Given a dimension t subspace U_1 of R and $a \geq t$, the number of distinct totally isotropic, dimension a subspaces U of $W \perp \mathbb{F}v$ with $U \cap R = U_1$ is

$$p^{(s-t)(a-t)} R^*(W_0 \perp \langle 2 \rangle, \langle 0 \rangle^{a-t}).$$

Since $\beta(s, t)$ is the number of dimension t subspaces of R ,

$$R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a) = \sum_{t=0}^s \beta(s, t) p^{(s-t)(a-t)} R^*(W_0 \perp \langle 2 \rangle, \langle 0 \rangle^{a-t}).$$

For $s, t, c, q \in \mathbb{Z}$ with $s, t, c \geq 0$, set

$$S_t(c, q) = (-1)^c p^{c(c-t+q)} \sum_{\ell=0}^c (-1)^\ell p^{\ell(\ell-2c+t-q)} \beta(c, \ell) \delta(c+q, \ell),$$

$$X_s(c, q) = \sum_{t=0}^s (-1)^t p^{t(t-c-s-q)} \beta(s, t) S_t(c, q).$$

Using, for instance, Theorems 2.59 and 2.60 of [5], we know

$$R^*(W_0 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) = \begin{cases} \beta(c, \ell) \delta(c-1, \ell) & \text{if } W_0 \perp \langle 2 \rangle \simeq \mathbb{H}^c, \\ \beta(c-1, \ell) \delta(c, \ell) & \text{if } W_0 \perp \langle 2 \rangle \simeq \mathbb{H}^{c-1} \perp \mathbb{A}, \\ \beta(c, \ell) \delta(c, \ell) & \text{if } W_0 \perp \langle 2 \rangle \simeq \mathbb{H}^c \perp \langle \eta \rangle. \end{cases}$$

(Here $\eta \in \mathbb{F}^\times$.) Hence, with $\omega \in \mathbb{F}^\times$ so that $(\frac{\omega}{p}) = -1$,

$$\sum_{a=0}^m (-1)^a p^{a(a-m)} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a) = \begin{cases} (-1)^c p^{c(1-c)} X_s(c, -1) & \text{if } W \simeq \mathbb{H}^{c-1} \perp \langle -2 \rangle \perp \langle 0 \rangle^s, \\ (-1)^{c-1} p^{c(1-c)} X_s(c-1, 1) & \text{if } W \simeq \mathbb{H}^{c-1} \perp \langle -2\omega \rangle \perp \langle 0 \rangle^s, \\ (-1)^c p^{-c^2} X_s(c, 0) & \text{if } W \simeq W_0 \perp \langle 0 \rangle^s, \dim W_0 = 2c. \end{cases}$$

When $c > 0$, replacing ℓ by $c - \ell$ and using the identities $\beta(c, \ell) = \beta(c, c - \ell)$ and $\beta(c, \ell) = p^\ell \beta(c - 1, \ell) + \beta(c - 1, \ell - 1)$, and then replacing $\ell - 1$ by ℓ we get

$$S_0(c, q) = S_0(c - 1, q + 1) = S_0(0, q + c) = 1.$$

Clearly $S_t(0, q) = 1$; using the definitions of β and δ , when $c > 0$ we have

$$S_t(c, q) + (p^c - 1)(p^{c+q} + 1)S_t(c - 1, q) = (-1)^c p^{c(c-t+q)} \sum_{\ell=0}^c (-1)^\ell p^{\ell(\ell-2c+t-q)} \frac{\mu(c, \ell) \delta(c+q, \ell)}{\mu(\ell, \ell)} + (-1)^c p^{c(c-t+q)-2c-q+t+1} \sum_{\ell=1}^c (-1)^\ell p^{(\ell-1)(\ell+1-2c+t-q)} \frac{\mu(c, \ell) \delta(c+q, \ell)}{\mu(\ell - 1, \ell - 1)}$$

$$\begin{aligned}
 &= (-1)^c p^{c(c-t+q)} \left[1 + \sum_{\ell=1}^c (-1)^\ell p^{\ell(\ell+1-2c+t-q)} \beta(c, \ell) \delta(c+q, \ell) \right] \\
 &= p^c S_{t+1}(c, q).
 \end{aligned}$$

Taking $S_t(c, q) = 0$ when $c < 0$, the above relation also holds for $c = 0$.

Using that $\beta(s+1, t) = p^t \beta(s, t) + \beta(s, t-1)$, and the recursion relation for $S_t(c, q)$, we have

$$\begin{aligned}
 X_{s+1}(c, q) &= \sum_{t=0}^s (-1)^t p^{t(t-c-s-q)} \beta(s, t) S_t(c, q) + \sum_{t=1}^{s+1} (-1)^t p^{t(t-c-s-q-1)} \beta(s, t-1) S_t(c, q) \\
 &= X_s(c, q) - p^{-2c-s-q} X'_s(c, q) - p^{-2c-s-q} (p^c - 1)(p^{c+q} + 1) X_s(c-1, q)
 \end{aligned}$$

where

$$X'_s(c, q) = \sum_{t=0}^s (-1)^t p^{t(t-c-s-q+1)} \beta(s, t) S_t(c, q).$$

Similarly, using that $\beta(s+1, t) = \beta(s, t) + p^{s+1-t} \beta(s, t-1)$,

$$X'_{s+1}(c, q) = X_s(c, q) - p^{-2c+1-q} X'_s(c, q) - p^{-2c+1-q} (p^c - 1)(p^{c+q} + 1) X_s(c-1, q).$$

Using induction on x , the recursion relations for X and X' , and the fact that $S_0(c, q) = 1$, we get the following.

$$\begin{aligned}
 X_s(c, -1) &= \begin{cases} p^{2x-2cs-x^2} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x, \\ -p^{-c-2cx-x^2} \prod_{i=1}^{x+1} (p^{2i-1} - 1) & \text{if } s = 2x + 1, \end{cases} \\
 X'_s(c, -1) &= \begin{cases} 1 & \text{if } s = 0, \\ X_{s-1}(c, -1) + (p^{c-1} + 1) X_s(c-1, -1) & \text{if } s = 2x > 0, \\ (p^{c-1} + 1) X_s(c-1, -1) & \text{if } s = 2x + 1; \end{cases} \\
 X_s(c-1, 1) &= \begin{cases} p^{2x-2cs-x^2} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x, \\ p^{-c-2cx-x^2} \prod_{i=1}^{x+1} (p^{2i-1} - 1) & \text{if } s = 2x + 1, \end{cases} \\
 X'_s(c-1, 1) &= \begin{cases} 1 & \text{if } s = 0, \\ X_{s-1}(c-1, 1) + (p^{c-1} - 1) X_s(c-2, 1) & \text{if } s = 2x > 0, \\ -(p^{c-1} - 1) X_s(c-2, 1) & \text{if } s = 2x + 1; \end{cases} \\
 X_s(c, 0) &= \begin{cases} p^{-2cx-x^2} \prod_{i=1}^x (p^{2i-1} - 1) & \text{if } s = 2x, \\ 0 & \text{if } s = 2x + 1; \end{cases} \\
 X'_s(c, 0) &= \begin{cases} 1 & \text{if } s = 0, \\ X_s(c-1, 0) & \text{if } s = 2x > 0, \\ -p^{-2cx-x^2} \prod_{i=1}^{x+1} (p^{2i-1} - 1) & \text{if } s = 2x + 1. \end{cases}
 \end{aligned}$$

The lemma now follows from these identities and Proposition 1.4. \square

Using this, we also establish the following.

Lemma 3.2. Suppose W is a dimension $m \geq 0$ quadratic space over $\mathbb{F} = \mathbb{F}_p$, p an odd prime.

(a) For $0 \leq a \leq m$,

$$R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a) = R^*(W, \langle 0 \rangle^a) + 2R^*(W, \langle 0 \rangle^{a-1} \perp \langle -2 \rangle).$$

(b) With $\text{cls } U$ denoting the isometry class of a space U ,

$$\sum_{q=0}^m \sum_{\substack{\text{cls } U \\ \dim U=q}} R^*(W, U) \tilde{\mathcal{G}}(U) = p^{m(m-1)/2} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^m).$$

Proof. (a) Fix a , $0 \leq a \leq m$. If $a = 0$, then the claim is that

$$R^*(W \perp \langle 2 \rangle, \{0\}) = R^*(W, \{0\}),$$

which holds since there is only 1 dimension 0 subspace of any space. So suppose $1 \leq a \leq m$. Let U' be a dimension a totally isotropic subspace of $W \perp \mathbb{F}v$ where $\mathbb{F}v \simeq \langle 2 \rangle$, and let U be the projection of U' onto W . Then either $U = U' \simeq \langle 0 \rangle^a$, or $U \simeq \langle 0 \rangle^{a-1} \perp \langle -2 \rangle$; also, there are exactly 2 totally isotropic subspaces of $W \perp \mathbb{F}v$ that project onto a given subspace $U \simeq \langle 0 \rangle^{a-1} \perp \langle -2 \rangle$ of W , and from this the claim in (a) follows.

(b) Here we argue by induction on m . For $m = 0$, the claim is trivially true. So suppose $m \geq 1$. Using the identity $\beta(m, a) = \beta(m - 1, a) + p^{m-a} \beta(m - 1, a - 1)$, and then replacing $a - 1$ by a , we get

$$\sum_{a=0}^m (-1)^{m+a} p^{(m-a)(m-a-1)/2} \beta(m, a) = 0.$$

Thus for $a \leq q \leq m$, replacing m and a by $m - q$ and $a - q$ in the above identity, we get

$$\sum_{a=q}^{m-1} (-1)^{m+a+1} p^{(m-a)(m-a-1)/2} \beta(m - q, a - q) = 1.$$

With W a dimension m space over \mathbb{F} , the number of ways to extend a dimension q subspace U of W to a dimension a subspace Y of W is $\beta(m - q, a - q)$. Thus with the preceding expression for 1, using the induction hypothesis and (a) of this lemma, we have

$$\begin{aligned} & \sum_{\substack{\text{cls } U \\ \dim U < m}} R^*(W, U) \tilde{\mathcal{G}}(U) \\ &= \sum_{a=0}^{m-1} (-1)^{m+a+1} p^{(m-a)(m-a-1)/2} \sum_{\substack{\text{cls } Y \\ \dim Y=a}} R^*(W, Y) \sum_{\substack{\text{cls } U \\ \dim U \leq a}} R^*(Y, U) \tilde{\mathcal{G}}(U) \\ &= \sum_{a=0}^{m-1} (-1)^{m+a+1} p^{m(m-1)/2+a(a-m)} R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^a). \end{aligned}$$

Now add $\tilde{\mathcal{G}}(W) = R^*(W, W) \tilde{\mathcal{G}}(W)$ to both sides of this equation; using Lemma 3.1 yields the result. \square

Now it is easy to prove our main formula.

Theorem 3.3. Take $f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ where $4|N$, and let p be a prime such that $p \nmid N$; let χ' be defined as in Theorem 2.4. Given

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2, \quad \Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2$$

with $n_i = \text{rank } \Lambda_i, r = n_0 + n_2$, set

$$E_j(\Lambda, \Omega) = j(k - n) + k(n_2 - n_0) + n_0(n - n_2) + (j - r)(j - r - 1)/2;$$

set

$$\tilde{A}_j(\Lambda, \Omega) = \chi'(p^{j-r})p^{E_j(\Lambda, \Omega)}R^*(\Lambda_1/p\Lambda_1 \perp \langle 2 \rangle, (0)^{j-r})$$

if Λ, Ω are even integral, and set $\tilde{A}_j(\Lambda, \Omega) = 0$ otherwise. Then the Λ th coefficient of $f|\tilde{T}_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \tilde{A}_j(\Lambda, \Omega)c(\Omega).$$

Proof. By Theorem 2.4, the Λ th coefficient of $f|\tilde{T}_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \tilde{A}_j(\Lambda, \Omega)c(\Omega)$$

where

$$\tilde{A}_j(\Lambda, \Omega) = \chi'(p^{j-r})p^{E'_j(\Lambda, \Omega)} \sum_{\ell=0}^j \sum_{\substack{\text{cls } U \\ \dim U = \ell - r}} \beta(n - \ell, j - \ell)R^*(V, U)\tilde{\mathcal{G}}(U)$$

with $E'_j(\Lambda, \Omega) = j(k - n) + k(n_2 - n_0) + n_0(n - n_2)$ and $V = \Lambda_1/p\Lambda_1$. The number of ways to extend a dimension $\ell - r$ subspace U of V to a dimension $j - r$ subspace W of V is

$$\beta((n - r) - (\ell - r), (j - r) - (\ell - r)) = \beta(n - \ell, j - \ell);$$

thus

$$\tilde{A}_j(\Lambda, \Omega) = \chi'(p^{j-r})p^{E'_j(\Lambda, \Omega)} \sum_{\substack{\text{cls } W \\ \dim W = j - r}} R^*(V, W) \sum_{\substack{\text{cls } U \\ \dim U \leq j - r}} R^*(W, U)\tilde{\mathcal{G}}(U).$$

By Lemma 3.2, we get

$$\begin{aligned} \tilde{A}_j(\Lambda, \Omega) &= \chi'(p^{j-r})p^{E_j(\Lambda, \Omega)} \sum_{\substack{\text{cls } W \\ \dim W = j-r}} R^*(V, W)R^*(W \perp \langle 2 \rangle, \langle 0 \rangle^{j-r}) \\ &= \chi'(p^{j-r})p^{E_j(\Lambda, \Omega)} (R^*(V, \langle 0 \rangle^{j-r}) + 2R^*(V, \langle 0 \rangle^{j-r-1} \perp \langle -2 \rangle)) \\ &= \chi'(p^{j-r})p^{E_j(\Lambda, \Omega)} R^*(V \perp \langle 2 \rangle, \langle 0 \rangle^{j-r}), \end{aligned}$$

proving the theorem. \square

4. Hecke operators on Siegel theta series of weight $k + 1/2$

Throughout this section, we assume that L is a rank $2k + 1$ lattice with an even integral, positive definite quadratic form Q ; we fix $n \leq 2k + 1$. As in the case when rank L is even [14,15], we prove a generalised Eichler Commutation Relation, and from this show the average theta series is an eigenform for $\tilde{T}_j(p^2)$ ($p \nmid$ level L), computing the eigenvalues. As in Theorem 1.1, we use B_Q to denote the symmetric bilinear form associated to Q , N the level of L , χ the character of $\theta^{(n)}(L)$, and χ' the character defined by $\chi'(d) = \chi(d) \left(\frac{-1}{|d|}\right)^{k+1} (\text{sgn } d)^{k+1}$. With $L = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_{2k+1}$, the matrix for Q relative to this basis for L is given by $Q = (B_Q(v_h, v_i))$. Thus for $C \in \mathbb{Z}^{2k+1, n}$ and with $(x_1 \dots x_n) = (v_1 \dots v_{2k+1})C$, ${}^t C Q C = (B_Q(x_h, x_i))$ is the matrix for the quadratic form Q restricted to the (external) direct sum $\Lambda = \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_n$. Hence

$$\theta^{(n)}(L; \tau) = \sum_{x_1, \dots, x_n \in L} e\{(B_Q(x_h, x_i))\tau\}.$$

Note that as a sublattice of L , we may have $d = \text{rank}(\mathbb{Z}x_1 + \dots + \mathbb{Z}x_n) < n$. In such a case there exists some $G \in GL_n(\mathbb{Z})$ so that

$$(x_1 \dots x_n)G = (x'_1 \dots x'_d \ 0 \dots 0).$$

Still, we can consider Λ as a sublattice of L with “formal rank” n . Given a sublattice $\Lambda' = \mathbb{Z}x'_1 + \dots + \mathbb{Z}x'_d$ of L with $\text{rank } \Lambda' = d$ and $T' = (B_Q(x'_h, x'_i))$ (a $d \times d$ matrix),

$$\sum_{\substack{x_1, \dots, x_n \in L \\ \mathbb{Z}x_1 + \dots + \mathbb{Z}x_n = \Lambda'}} e\{(B_Q(x_h, x_i))\tau\} = \sum_G e\left\{{}^t G \begin{pmatrix} T' & \\ & 0_{n-d} \end{pmatrix} G\tau\right\}$$

where G varies over

$$\left\{ \begin{pmatrix} I_d & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{Z}) \right\} \setminus GL_n(\mathbb{Z}).$$

Thus with $x'_{d+1} = \dots = x'_n = 0$, $\Lambda = \mathbb{Z}x'_1 \oplus \dots \oplus \mathbb{Z}x'_n$ (the external direct sum), we define

$$e\{\Lambda\tau\} = \sum_G e\left\{{}^t G \begin{pmatrix} T' & \\ & 0_{n-d} \end{pmatrix} G\tau\right\}$$

where G varies as above. Then

$$\theta^{(n)}(L; \tau) = \sum_{\Lambda \subseteq L} e\{\Lambda\tau\}$$

where Λ varies over all distinct sublattices of L with formal rank n . (When $x_i, y_i \in L$, we say $\mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_n$ and $\mathbb{Z}y_1 \oplus \dots \oplus \mathbb{Z}y_n$ are distinct sublattices of L with formal rank n when $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_n \neq \mathbb{Z}y_1 + \dots + \mathbb{Z}y_n$.)

Remark. For $x_i \in L$, $\Lambda = \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_n$, and $\Lambda' = \mathbb{Z}x_1 + \dots + \mathbb{Z}x_n$, we have $e\{\Lambda\tau\} = o(\Lambda')e^*\{\Lambda\tau\}$ since, with $d = \text{rank } \Lambda'$ (as a sublattice of L),

$$O(\Lambda' \perp \langle 0 \rangle^{n-d}) = \left\{ \begin{pmatrix} E' & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{Z}) : E' \in O(\Lambda') \right\}.$$

Proposition 4.1. For p a prime not dividing N and $1 \leq j \leq n$, take $\Omega \subseteq \frac{1}{p}L$ so that Ω is even integral and has formal rank n ; decompose Ω as $\frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ where $\Omega_i \subseteq L$ and $\Omega_0 \oplus \Omega_1$ is primitive in L modulo p , meaning that the formal rank of $\Omega_0 \oplus \Omega_1$ is its rank in L , which is also the dimension of $\overline{\Omega_0 \oplus \Omega_1}$ in L/pL . Let r_i be the (formal) rank of Ω_i ; set

$$E(\ell, t, \Omega) = t(k - n) + t(t - 1)/2 + \ell(k - r_0 - r_1) + \ell(\ell - 1)/2,$$

and set

$$\begin{aligned} \tilde{c}_j(\Omega) &= \sum_{\ell, t} p^{E(\ell, t, \Omega)} R^*(\Omega_1/p\Omega_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \\ &\quad \cdot \delta(k - r_0 - \ell, t) \beta(r_2, t) \beta(n - r_0 - \ell - t, n - j); \end{aligned}$$

if $\chi'(p) = 1$, and

$$\begin{aligned} \tilde{c}_j(\Omega) &= \sum_{\ell, t} (-1)^\ell p^{E(\ell, t, \Omega)} R^*(\Omega_1/p\Omega_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \\ &\quad \cdot \mu(k - r_0 - \ell, t) \beta(r_2, t) \beta(n - r_0 - \ell - t, n - j) \end{aligned}$$

if $\chi'(p) = -1$. Then

$$\theta^{(n)}(L; \tau) | \tilde{T}_j(p^2) = \sum_{\Omega} \tilde{c}_j(\Omega) e\{\Omega\tau\}$$

where Ω varies over all even integral sublattices of $\frac{1}{p}L$ that have (formal) rank n .

Proof. The proof is virtually identical to that of Proposition 1.4 of [14] (see also Proposition 2.1 of [15]), so here we merely give an indication of how this is done.

By the definitions of $T_j(p^2)$ and $\tilde{T}_j(p^2)$, we have

$$\theta^{(n)}(L; \tau) | \tilde{T}_j(p^2) = \sum_{\substack{\Lambda \subseteq L \\ p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda}} \tilde{A}_j(\Omega, \Lambda) e\{\Omega\tau\}$$

where $\tilde{A}_j(\Omega, \Lambda)$ is defined in Theorem 3.3. (Note that at the end of the proof of Theorem 2.4 we made a change of variables that we do not make here.) Since $p \neq 2$ and $\Omega \subseteq \frac{1}{p}L$, Ω is even integral exactly when it is integral, so $\tilde{A}_j(\Omega, \Lambda) = 0$ when Ω is not integral. Interchanging the order of summation, we have

$$\theta^{(n)}(L; \tau) | \tilde{T}_j(p^2) = \sum_{\substack{\Omega \subseteq \frac{1}{p}L \\ \Omega \text{ integral}}} \sum_{p\Omega \subseteq \Lambda \subseteq (\frac{1}{p}\Omega \cap L)} \tilde{A}_j(\Omega, \Lambda) e\{\Omega \tau\}.$$

So one fixes Ω , and follows the procedure of [14] to construct all the Λ in the inner sum, keeping track of the data carried in $\tilde{A}_j(\Omega, \Lambda)$. \square

Proposition 4.2. *Let L be as above, and fix a prime $p \nmid N$; choose j so that $1 \leq j \leq n$ and $j \leq k$. We say a lattice K is a p^j -neighbour of L if $K \in \text{gen } L$ and*

$$L = L_0 \oplus L_1 \oplus L_2, \quad K = \frac{1}{p}L_0 \oplus L_1 \oplus pL_2$$

with $\text{rank } L_0 = \text{rank } L_2 = j$.

(a) *The number of p^j -neighbours of L is $p^{j(j-1)/2} \beta \delta(k, j)$.*

(b) *Take $\Omega \subseteq \frac{1}{p}L$ so that Ω is even integral and has formal rank n ; decompose Ω as in Proposition 4.1. Set*

$$b_j(\Omega) = p^{(j-r_0)(j-r_0-1)/2} \sum_{\ell=0}^{j-r_0} p^{\ell(k-j-r_1+\ell)} R^*(\Omega_1/p\Omega_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \cdot \delta(k-r_0-\ell, j-r_0-\ell) \beta(k-r_0-r_1, j-r_0-\ell)$$

if $\chi'(p) = 1$,

$$b_j(\Omega) = p^{(j-r_0)(j-r_0-1)/2} \sum_{\ell=0}^{j-r_0} (-1)^\ell p^{\ell(k-j-r_1+\ell)} R^*(\Omega_1/p\Omega_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \cdot \beta(k-r_0-\ell, j-r_0-\ell) \delta(k-r_0-r_1, j-r_0-\ell)$$

if $\chi'(p) = -1$. Then

$$\sum_{K_j} \theta^{(n)}(K_j; \tau) = \sum_{\Omega} b_j(\Omega) e\{\Omega\}$$

where K_j varies over all p^j -neighbours of L , and Ω varies over all even integral sublattices of $\frac{1}{p}L$ with (formal) rank n .

Proof. The proof is virtually identical to that of Proposition 1.5 of [14] (see also Proposition 2.2 of [15]), so here we merely give an indication of the proof.

To construct all p^j -neighbours K of L , we begin by choosing a j -dimensional totally isotropic subspace \bar{C} of L/pL . Thus $L = C \oplus D \oplus J$ where $\bar{C} \oplus \bar{D} \simeq \mathbb{H}^j$, $B_Q(C \oplus D, J) \equiv 0 \pmod{p}$. Set $K' = C \oplus pD \oplus pJ$ (the preimage of \bar{C}); then in K'/pK' (scaled by $1/p$), $\bar{C} \oplus p\bar{D} \simeq \mathbb{H}^j$ with $p\bar{D}$ totally isotropic. Thus we can refine \bar{C} to a totally isotropic subspace \bar{C}' ; set $K = \frac{1}{p}C' \oplus pD \oplus J$ (the preimage of $(\bar{C}')^\perp$). Given $\Omega \subseteq \frac{1}{p}L$ with Ω decomposed as described above, Ω lies in K exactly when $\bar{\Omega}_0 \oplus \bar{\Omega}_1 \subseteq \bar{C}'^\perp$ in L/pL , and $\bar{\Omega}_0 \subseteq \bar{C}'$ in K'/pK' , and we can easily count how often this is the case. \square

Next we have a generalised Eichler Commutation Relation.

Theorem 4.3. Let L be a lattice of rank $2k + 1$ equipped with a positive definite quadratic form Q of level N , and $1 \leq n \leq 2k + 1$; fix a prime p so that $p \nmid N$. Take j so that $1 \leq j \leq n$ and $j \leq k$; for $0 \leq q \leq j$, set

$$u_q(j) = (-1)^q p^{q(q-1)/2} \beta(n - j + q, q), \quad T'_j(p^2) = \sum_{q=0}^j u_q(j) \tilde{T}_{j-q}(p^2),$$

$$v_q(j) = \begin{cases} (-1)^q \beta(k - n + q - 1, q) \delta(k - j + q, q) & \text{if } \chi'(p) = 1, \\ (-1)^q \delta(k - n + q - 1, q) \beta(k - j + q, q) & \text{if } \chi'(p) = -1. \end{cases}$$

Then

$$\theta^{(n)}(L) | T'_j(p^2) = \sum_{q=0}^j v_q(j) \left(\sum_{K_{j-q}} \theta^{(n)}(K_{j-q}) \right)$$

where K_{j-q} runs over all p^{j-q} -neighbours of L (as defined in Proposition 6.3).

Proof. Fix $\Omega \subseteq \frac{1}{p}L$ as in Propositions 4.1 and 4.2; suppose that $\chi'(p) = 1$. Then with $\overline{\Omega}_1 = \Omega_1/p\Omega_1$,

$$\begin{aligned} \sum_{q=0}^j u_q(j) \tilde{c}_{j-q}(\Omega) &= \sum_{\ell, t} p^{E(\ell, t, \Omega)} R^*(\overline{\Omega}_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \delta(k - r_0 - \ell, t) \beta(r_2, t) \\ &\quad \cdot \sum_{q=0}^j u_q(j) \beta(n - r_0 - \ell - t, n - j + q) \end{aligned}$$

where $E(\ell, t, \Omega)$ is defined in Proposition 4.1. Using the identities

$$\begin{aligned} \beta(m + r, r + q) \beta(r + q, q) &= \frac{\mu(m + r, r) \mu(m, q) \mu(r + q, q)}{\mu(r + q, q) \mu(r, r) \mu(q, q)} \\ &= \beta(m + r, r) \beta(m, q) \end{aligned}$$

and $\beta(m, q) = p^q \beta(m - 1, q) + \beta(m - 1, q - 1)$, we find that when $m = j - r_0 - \ell - t \geq 1$, $\sum_{q=0}^m u_q(j) \beta(m, n - j + q) = 0$. Thus

$$\begin{aligned} \sum_q u_q(j) \tilde{c}_{j-q}(\Omega) &= \sum_\ell p^{E(\ell, j-r_0-\ell, \Omega)} R^*(\overline{\Omega}_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \\ &\quad \cdot \delta(k - r_0 - \ell, j - r_0 - \ell) \beta(r_2, j - r_0 - \ell). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_q v_q(j) b_{j-q}(\Omega) &= p^{(j-r_0)(j-r_0-1)/2} \sum_\ell p^{\ell(k-r_1+\ell-j)} R^*(\overline{\Omega}_1 \perp \langle 2 \rangle, \langle 0 \rangle^\ell) \\ &\quad \cdot \frac{\delta(k - r_0 - \ell, j - r_0 - \ell)}{\mu(j - r_0 - \ell, j - r_0 - \ell)} S(j - r_0 - \ell) \end{aligned}$$

where

$$S(m) = \sum_{q=0}^m (-1)^q p^{q(q+1)/2 - qm} \mu(k - r_0 - r_1, m - q) \mu(k - n - 1 + q, q) \beta(m, q).$$

Using that $n = r_0 + r_1 + r_2$ and that $\beta(m, q) = \beta(m - 1, q) + p^{m-q} \beta(m - 1, q - 1)$, we find that

$$S(m) = p^{k-n} (p^{r_2-m+1} - 1) S(m - 1) = p^{m(k-n)} \mu(r_2, m).$$

Thus

$$\sum_q v_q(j) b_{j-q}(\Omega) = \sum_q u_q(j) \tilde{c}_{j-q}(\Omega).$$

The case when $\chi'(p) = -1$ is virtually the same, and so the details are left to the reader. \square

In the next corollary, we average across the generalised Eichler Commutation Relation to show that $\theta^{(n)}(\text{gen } L)$ is a Hecke eigenform for primes $p \nmid N$, where

$$\theta^{(n)}(\text{gen } L) = \sum_{\text{cls } K \in \text{gen } L} \frac{1}{o(K)} \theta^{(n)}(K)$$

(here $\text{cls } K$ varies over all isometry classes within the genus of L).

Corollary 4.4. *With p a prime, $p \nmid N$, and $1 \leq j \leq n$ with $j \leq k$,*

$$\theta^{(n)}(\text{gen } L) | T'_j(p^2) = \lambda_j(p^2) \theta^{(n)}(\text{gen } L)$$

where

$$\lambda_j(p^2) = \begin{cases} p^{j(j-1)/2 + j(k-n)} \beta(n, j) \delta(k, j) & \text{if } \chi'(p) = 1, \\ p^{j(j-1)/2 + j(k-n)} \beta(n, j) \mu(k, j) & \text{if } \chi'(p) = -1. \end{cases}$$

Proof. First note that for $K \in \text{gen } L$, we have $\text{disc } K = \text{disc } L$, so K is a p^m -neighbour of L (as defined in Proposition 4.2) if and only if $pL \subseteq K \subseteq \frac{1}{p}L$, and either $\text{mult}_{\{L:K\}}(p) = m$ or $\text{mult}_{\{L:K\}}(1/p) = m$ where $\{L : K\}$ denotes the invariant factors of K in L . Classifying the p^m -neighbours into isometry classes, we see that the number of p^m -neighbours of L in $\text{cls } K \in \text{gen } L$ is

$$\frac{\#\{\text{isometries } \sigma : pL \subseteq \sigma K \subseteq \frac{1}{p}L, \text{mult}_{\{L:\sigma K\}}(p) = m\}}{o(K)}$$

(since $\sigma K = \sigma' K$ if and only if $\sigma^{-1} \sigma' \in O(K)$). Also, using Proposition 4.2(a),

$$\begin{aligned} & \sum_{\text{cls } L' \in \text{gen } L} \frac{\#\{\text{isometries } \sigma : pL' \subseteq \sigma K \subseteq \frac{1}{p}L', \text{mult}_{\{L':\sigma K\}}(p) = m\}}{o(L') o(K)} \\ &= \frac{1}{o(K)} \sum_{\text{cls } L'} \frac{\#\{\text{isometries } \sigma : pK \subseteq \sigma L' \subseteq \frac{1}{p}K, \text{mult}_{\{K:\sigma L'\}}(p) = m\}}{o(L')} \\ &= \frac{1}{o(K)} \#\{p^m\text{-neighbours of } K\} \end{aligned}$$

$$= \frac{1}{o(K)} p^{m(m-1)/2} \beta \delta(k, m).$$

Thus

$$\theta^{(n)}(\text{gen } L) | T'_j(p^2) = \lambda_j(p^2) \theta^{(n)}(\text{gen } L)$$

where

$$\lambda_j(p^2) = \sum_{q=0}^j v_q(j) p^{(j-q)(j-q-1)/2} \beta \delta(k, j - q).$$

When $\chi'(p) = 1$, $\lambda_j(p^2) = p^{j(j-1)/2} \frac{\delta(k, j)}{\mu(j, j)} S(j)$ where

$$S(j) = \sum_{q=0}^j (-1)^q p^{q(q+1)/2 - qj} \beta(j, q) \mu(k - n + q - 1, q) \mu(k, j - q).$$

Using the identity $\beta(j, q) = \beta(j - 1, q) + p^{j-q} \beta(j - 1, q - 1)$, we find that for $1 \leq d \leq j$,

$$S(j) = p^{d(k-n)} \mu(n - j + d, d) S(j - d) = p^{j(k-n)} \mu(n, j),$$

proving the corollary when $\chi'(p) = 1$.

The case $\chi'(p) = -1$ is virtually identical. \square

Just as in the integral weight case (Section 3 of [15]), we have the following.

Theorem 4.5. *When $1 \leq a \leq n - k$, p is a prime not dividing N , and $T'_j(p^2)$ is defined as in Theorem 4.3,*

$$\theta^{(n)}(L) | T'_{k+a}(p^2) = 0.$$

Proof. One argues exactly as in the integral weight case (see Section 3 [15]): First, using Proposition 4.1, one shows

$$\tilde{c}_{k+a}(\Omega) = \sum_{q=0}^k w_q(a) \tilde{c}_{k-q}(\Omega)$$

where $w_q(a) = (-1)^q p^{q(q+1)/2} \beta(a + q - 1, q) \beta(n - k + q, a + q)$. Hence

$$\theta^{(n)}(L) | \tilde{T}_{k+a}(p^2) = \theta^{(n)}(L) | \sum_{q=0}^k w_q(a) \tilde{T}_{k-q}(p^2).$$

Then one shows

$$\sum_{q=0}^r \beta(n - q, r - q) T'_q(p^2) = \tilde{T}_r(p^2).$$

Substituting and using induction on a yields the result. \square

5. Bounding Hecke eigenvalues

In [13] we used the formula from [6] to bound the eigenvalues of Hecke operators acting on integral weight Siegel forms. Using Theorem 3.3, we can do this for half-integral weight. We obtain the following bound.

Theorem 5.1. *Let f be a nonsingular Siegel modular form of degree n and weight $k + 1/2$ so that for some j , $1 \leq j \leq n$, $f|T_j(p^2) = \lambda_j(p^2)f$ for almost all primes p . Also suppose that the Fourier coefficients $c(\Lambda)$ of f satisfy the bound $|c(\Lambda)| \ll_f (\text{disc } \Lambda)^{k/2+1/4-\gamma}$. (Note that with $\gamma = 0$, this is the trivial bound for cusp forms.) Then*

$$|\lambda_j(p^2)| \ll_f p^M \quad (\text{as } p \mapsto \infty)$$

where $M = \frac{1}{4}(j + n - 2\gamma + \frac{1}{2})^2 + \frac{1}{6}(j - n + 2\gamma - 1)^2 + j(k - n)$. When $\gamma = 0$, we can take $M = \frac{1}{4}(j + n + \frac{1}{2})^2 + j(k - n)$.

Proof. The proof is somewhat similar to that of Proposition 4.1, and almost identical to that in [13]: Given Λ , we construct all the even integral Ω so that $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, keeping track of the relevant data for the formula of Theorem 3.3. Note that since we are bounding the Λ th coefficient of $f|T_j(p^2)$, we can replace $\chi'(p^{j-r})$ by 1 in this formula.

Since f is nonzero, we can choose Λ so that $c(\Lambda) \neq 0$ and $\text{disc } \Lambda \neq 0$. Also, assume $p \nmid 2 \text{disc } \Lambda$. We first partition the lattices Ω , $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, according to the invariant factors $\{\Lambda : \Omega\}$. Choosing $n_0, n_2 \geq 0$ so that $n_0 + n_2 \leq j$, we construct all integral Ω so that

$$\Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2 \quad \text{where } \Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2,$$

$n_i = \text{rank } \Lambda_i$. In this process, we compute $R^*(\Lambda_1/p\Lambda_1(2), \langle 0 \rangle^{j-r})$ where $r = n_0 + n_2$. (Note that since $p \nmid \text{disc } \Lambda$, $\Lambda/p\Lambda$ is a regular quadratic space; so when Ω is integral, we must have that, in $\Lambda/p\Lambda$, Λ_2 is totally isotropic and $\bar{\Lambda}_1$ is orthogonal to $\bar{\Lambda}_2$. Consequently we must have $n_2 \leq n_0$.)

First we choose $\bar{\Delta}_2$ to be a totally isotropic, dimension n_2 subspace of $\Lambda/p\Lambda$. Since $\Lambda/p\Lambda$ is regular, the lemma of Section 4 of [13] tells us that the number of choices for $\bar{\Delta}_2$ is bounded by

$$4^{n_2} p^{n_2(n-n_2)-n_2(n_2+1)/2}.$$

Note that $\Lambda/p\Lambda = (\bar{\Delta}_2 \oplus \bar{\Delta}_2') \perp J$ where $\bar{\Delta}_2 \oplus \bar{\Delta}_2' \simeq \mathbb{H}^{n_2}$ and J is regular; thus $\bar{\Delta}_2^\perp = \bar{\Delta}_2 \perp J$.

Next we extend $\bar{\Delta}_2$ to $\bar{\Delta}_2 \oplus W$ where $W \subseteq \bar{\Delta}_2^\perp$ and either $W \simeq \langle 0 \rangle^{j-r}$ or $W \simeq \langle 0 \rangle^{j-r-1} \perp \langle -2 \rangle$; when $W \simeq \langle 0 \rangle^{j-r-1} \perp \langle -2 \rangle$, we count W with multiplicity 2. So the number of such W , counted with the appropriate multiplicity, is $R^*(J \perp \langle 2 \rangle, \langle 0 \rangle^{j-r})$; by Section 4 of [13], this is bounded by

$$4^{j-r} p^{(j-r)(n-2n_2+1-j+r)-(j-r)(j-r+1)/2}.$$

Now we extend $\bar{\Delta}_2 \oplus W$ to $\bar{\Delta}_2 \oplus \bar{\Lambda}_1$ where $\bar{\Lambda}_1 \subseteq \bar{\Delta}_2^\perp$. The number of choices for $\bar{\Lambda}_1$ is $\beta(n - j + n_0 - n_2, n - j)$, which by Section 4 of [13] is bounded by

$$2^{n-j} p^{(n-j)(n_0-n_2)}.$$

Now let Δ be the preimage in Λ of $\overline{\Delta}_2$, and Ω' the preimage in Λ of $\overline{\Delta}_2 \oplus \overline{\Lambda}_1$. Thus

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Delta_2,$$

$$\Delta = p\Lambda_0 \oplus p\Lambda_1 \oplus \Delta_2,$$

$$\Omega' = p\Lambda_0 \oplus \Lambda_1 \oplus \Delta_2.$$

So $Q(\Delta_2) \equiv 0 \pmod{p}$, $B_Q(\Delta_2, \Lambda_1) \equiv 0 \pmod{p}$ where Q denotes the quadratic form on Λ , B_Q the corresponding symmetric bilinear form.

Our final step in constructing Ω is to refine our choice of Δ_2 so that $Q(\Delta_2) \equiv 0 \pmod{p^2}$. For this, we work in $\Delta/p\Delta$, with the quadratic form scaled by $1/p$. We extend $\overline{p\Omega'} = \overline{p\Lambda_1}$ to $\overline{p\Lambda_1} \oplus \overline{\Lambda_2}$ where $\overline{\Lambda_2}$ is totally isotropic of dimension n_2 and independent of $\overline{p\Lambda_1}$. As discussed in [13], the number of choices for $\overline{\Lambda_2}$ is bounded by $p^{n_2(n_0-n_2)}$. Now we take $p\Omega$ to be the preimage in Δ of $\overline{p\Lambda_1} \oplus \overline{\Lambda_2}$. Note that $\text{disc } \Omega = p^{2(n_0-n_2)} \text{disc } \Lambda$.

Thus, using Theorem 3.3 and the assumed bound on the Fourier coefficients of f , we see the Λ th coefficient of $f|\tilde{T}_j(p^2)$ is bounded by

$$\sum_{n_0+n_2 \leq j} 2^{n+j-2n_0} p^{E(n_0, n_2)} (\text{disc } \Lambda)^{k/2+1/4-\gamma}$$

where $E(n_0, n_2) = -n_0^2 + n_0(j+n-2\gamma+1/2) - \frac{3}{2}n_2^2 + n_2(j-n+2\gamma-1) + j(k-n)$. $E(n_0, n_2)$ has its maximum when $n_0 = \frac{1}{2}(j+n-2\gamma+1/2)$, $n_2 = \frac{1}{3}(j-n+2\gamma-1)$. Since n_0, n_2 are actually restricted to be non-negative, when $\gamma = 0$ we see that $E(n_0, n_2)$ has its maximum when $n_0 = \frac{1}{2}(j+n+1/2)$, $n_2 = 0$. With M the maximum value of $E(n_0, n_2)$, we have

$$|\lambda_j(p^2)c(\Lambda)| \ll_f (\text{disc } \Lambda)^{k/2+1/4-\gamma} p^M,$$

and hence $|\lambda_j(p^2)| \ll_{f, M} p^M$. \square

6. Hecke-stability of the Kitaoka space

In [8], Kitaoka identified a subspace of integral weight Siegel modular forms f where $c_f(\Lambda) = c_f(\Lambda')$ whenever $\Lambda' \in \text{gen } \Lambda$, and he showed that the space is stable under the Hecke operators associated to primes not dividing the level. Here we show that the half-integral weight Kitaoka subspace is invariant under all Hecke operators; note that our argument also works for the integral weight Kitaoka subspace (without the restriction that the prime does not divide the level).

Proposition 6.1. *For p a prime and Λ, Λ' lattices in the same genus, there is a bijective map*

$$\sigma : \left\{ \Omega : p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda \right\} \rightarrow \left\{ \Omega' : p\Lambda' \subseteq \Omega' \subseteq \frac{1}{p}\Lambda' \right\}$$

so that $\sigma(\Omega) \in \text{gen } \Omega$. Further, with $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$, we have

$$\{\Lambda : \Omega\} = \{\Lambda' : \sigma(\Omega)\} \quad \text{and} \quad \alpha_j(\Lambda, \Omega) = \alpha_j(\Lambda', \sigma(\Omega))$$

where $\{\Lambda : \Omega\}$ denotes the invariant factors of Ω in Λ . Hence σ also gives a bijection between

$$\{\Omega : p\Lambda \subseteq \Omega \subseteq \Lambda\} \quad \text{and} \quad \{\Omega' : p\Lambda' \subseteq \Omega' \subseteq \Lambda'\}.$$

Proof. Since Λ, Λ' are in the same genus, we can choose an isometry $\sigma_p : \mathbb{Q}_p \Lambda \rightarrow \mathbb{Q}_p \Lambda'$ so that $\sigma_p(\mathbb{Z}_p \Lambda) = \mathbb{Z}_p \Lambda'$. Then for $\Omega \subseteq \frac{1}{p} \Lambda$, and working in $\mathbb{Q}_p \Lambda'$, we set

$$\sigma(\Omega) = \sigma_p(\mathbb{Z}_p \Omega) \cap \frac{1}{p} \Lambda'.$$

(So $\sigma(\Lambda) = \Lambda'$.)

Now take Ω so that $p\Lambda \subseteq \Omega \subseteq \frac{1}{p} \Lambda$; thus

$$p\Lambda' = \sigma(p\Lambda) \subseteq \sigma(\Omega) \subseteq \sigma\left(\frac{1}{p} \Lambda\right) = \frac{1}{p} \Lambda'.$$

Also,

$$\sigma_p(\mathbb{Z}_p \Omega) \subseteq \sigma_p\left(\frac{1}{p} \mathbb{Z}_p \Lambda\right) = \frac{1}{p} \mathbb{Z}_p \Lambda',$$

so

$$\mathbb{Z}_p \sigma(\Omega) = \sigma_p(\mathbb{Z}_p \Omega) \cap \frac{1}{p} \mathbb{Z}_p \Lambda' = \sigma_p(\mathbb{Z}_p \Omega) \simeq \mathbb{Z}_p \Omega.$$

For all primes $q \neq p$, since $p\Lambda' \subseteq \sigma(\Omega) \subseteq \frac{1}{p} \Lambda'$, we have

$$\mathbb{Z}_q \Lambda' = p\mathbb{Z}_q \Lambda' \subseteq \mathbb{Z}_q \sigma(\Omega) \subseteq \frac{1}{p} \mathbb{Z}_q \Lambda' = \mathbb{Z}_q \Lambda';$$

thus $\mathbb{Z}_q \sigma(\Omega) = \mathbb{Z}_q \Lambda'$. Similarly, $\mathbb{Z}_q \Omega = \mathbb{Z}_q \Lambda$, so

$$\mathbb{Z}_q \sigma(\Omega) = \mathbb{Z}_q \Lambda' \simeq \mathbb{Z}_q \Lambda = \mathbb{Z}_q \Omega.$$

Hence $\Omega' \in \text{gen } \Omega$.

To see σ is a bijection as claimed, take $\Omega' \subseteq \frac{1}{p} \Lambda'$; working in $\mathbb{Q}_p \Lambda$, set

$$\sigma'(\Omega') = \sigma_p^{-1}(\mathbb{Z}_p \Omega') \cap \frac{1}{p} \Lambda.$$

Then one easily checks that $\sigma \circ \sigma'(\Omega') = \Omega'$, and for $\Omega \subseteq \frac{1}{p} \Lambda$, $\sigma' \circ \sigma(\Omega) = \Omega$.

Next, we show that for $p\Lambda \subseteq \Omega \subseteq \frac{1}{p} \Lambda$, we have $\{\Lambda : \Omega\} = \{\Lambda' : \sigma(\Omega)\}$. First recall that for all primes $q \neq p$, $\mathbb{Z}_q \Omega = \mathbb{Z}_q \Lambda$ and $\mathbb{Z}_q \sigma(\Omega) = \mathbb{Z}_q \Lambda'$. Hence

$$\begin{aligned} \{\Lambda' : \sigma(\Omega)\} &= \{\mathbb{Z}_p \Lambda' : \mathbb{Z}_p \sigma(\Omega)\} = \{\sigma_p(\mathbb{Z}_p \Lambda) : \sigma_p(\mathbb{Z}_p \Omega)\} \\ &= \{\mathbb{Z}_p \Lambda : \mathbb{Z}_p \Omega\} = \{\Lambda : \Omega\}. \end{aligned}$$

Finally, we show that with $p\Lambda \subseteq \Omega \subseteq \frac{1}{p} \Lambda$, $\alpha_j(\Lambda, \Omega) = \alpha_j(\Lambda', \sigma(\Omega))$. Note that since $\mathbb{Z}/p\mathbb{Z} \approx \mathbb{Z}_p/p\mathbb{Z}_p$, the $\mathbb{Z}/p\mathbb{Z}$ -space $(\Lambda \cap \Omega)/p(\Lambda + \Omega)$ and the $\mathbb{Z}_p/p\mathbb{Z}_p$ -space $(\mathbb{Z}_p \Lambda \cap \mathbb{Z}_p \Omega)/p(\mathbb{Z}_p \Lambda + \mathbb{Z}_p \Omega)$ can be viewed as isometric (over $\mathbb{Z}_p/p\mathbb{Z}_p$). Thus

$$\begin{aligned} \alpha_j(\Lambda, \Omega) &= \alpha_j(\mathbb{Z}_p \Lambda, \mathbb{Z}_p \Omega) \\ &= \alpha_j(\sigma_p(\mathbb{Z}_p \Lambda), \sigma_p(\mathbb{Z}_p \Omega)) = \alpha_j(\mathbb{Z}_p \Lambda', \mathbb{Z}_p \sigma(\Omega)) = \alpha_j(\Lambda', \sigma(\Omega)). \end{aligned}$$

This proves the proposition. \square

Note that when Λ, Λ' are oriented, Ω inherits its orientation from Λ , and $\sigma(\Omega)$ inherits its orientation from $\sigma(\Lambda) = \Lambda'$. Then the above proposition together with Theorem 3.4 immediately gives us the following.

Theorem 6.2. For $k, n, N \in \mathbb{Z}_+$ and χ a character modulo N , the subspace

$$\{f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi) : c_f(\Lambda) = c_f(\Lambda') \text{ when } \Lambda' \in \text{gen } \Lambda\}$$

is stable under the full Hecke-algebra.

7. A transparent Hecke-correspondence

In [16], we developed a formula for the action of Hecke operators on Jacobi modular forms; here we observe that this is almost identical to our formula in Theorem 3.4. From this we easily obtain a Hecke-correspondence. We begin by introducing notation and terminology for Jacobi modular forms that we will use in our correspondence.

A Jacobi modular form of index 1, weight $k + 1$, Siegel degree $n + 1$, level N , and character χ' is an analytic function $F : \mathbb{H}_{(n)} \times \mathbb{C}^{1,n} \rightarrow \mathbb{C}$ so that

$$\widehat{F} \left(\begin{matrix} \tau & {}^t Z \\ Z & \tau' \end{matrix} \right) = F(\tau, Z) e\{2\tau'\}$$

transforms like a degree $n + 1$, weight $k + 1$, level N , character χ' Siegel modular form under matrices

$$\underline{\gamma} \in \Gamma_0^{(n,1)}(N) = \left\{ \begin{pmatrix} A & 0 & B & * \\ * & 1 & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N) \right\}.$$

(Here τ' is a formal variable.) So with $\underline{\tau} = \begin{pmatrix} \tau & {}^t Z \\ Z & \tau' \end{pmatrix}$, $\underline{\gamma} \in \Gamma_0^{(n,1)}(N)$, we have $(F(\tau, Z)|\underline{\gamma})e\{2\tau'\} = \widehat{F}(\underline{\tau})|\underline{\gamma}$; also,

$$\widehat{F}(\underline{\tau}) = \sum_{\underline{T}} \widehat{c}(\underline{T}) e\{\underline{T}\underline{\tau}\} \quad \text{and} \quad F(\tau, Z) = \sum_{T,R} c(T, R) e\{T\tau + {}^t RZ\}$$

where T varies over $n \times n$ symmetric matrices and R varies over $1 \times n$ matrices so that $\underline{T} = \begin{pmatrix} T & {}^t R \\ R & 2 \end{pmatrix}$ is even integral and positive semi-definite, and $\widehat{c}(\underline{T}) = c(T, R)$. We say \widehat{F}, F are even if $2|R$ whenever \underline{T} is in the support of \widehat{F} . Let $J_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$ denote the space of Jacobi modular forms of weight $k + 1$, index 1, Siegel degree $n + 1$, level N , and character χ' . Let $J_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$ denote the subspace of even Jacobi modular forms in $J_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$. For

$$G \in GL_{n+1,1}^J(\mathbb{Z}) = \left\{ \underline{G} = \begin{pmatrix} G & 0 \\ R & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{Z}) \right\},$$

$c({}^t\mathbb{G}\mathbb{T}\mathbb{G}) = \chi'(\det \mathbb{G})(\det \mathbb{G})^{k+1}c(\mathbb{T})$. Consequently we can define \widehat{F} as a Fourier series supported on lattices as follows. Fix a rank 1 lattice $\Delta = \mathbb{Z}w \simeq (2)$, and for $\Lambda \oplus \Delta$ a rank $n+1$ lattice with quadratic form given by the matrix $\underline{\tau} = \begin{pmatrix} T & {}^tR \\ R & 2 \end{pmatrix}$, set $c_\Delta(\Lambda \oplus \Delta) = \widehat{c}(\underline{\tau})$. Let

$$e_\Delta^* \{(\Lambda \oplus \Delta)\underline{\tau}\} = \sum_{\mathbb{G}} e\{{}^t\mathbb{G}\mathbb{T}\mathbb{G}\underline{\tau}\}$$

where \mathbb{G} varies over $O(\underline{\tau}) \cap GL_{n+1,1}^J(\mathbb{Z}) \backslash GL_{n+1,1}^J(\mathbb{Z})$ (or, if $\chi'(-1)(-1)^{k+1} = -1$, over $O(\underline{\tau}) \cap SL_{n+1,1}^J(\mathbb{Z}) \backslash SL_{n+1,1}^J(\mathbb{Z})$ where $SL_{n+1,1}^J(\mathbb{Z}) = SL_{n+1}(\mathbb{Z}) \cap GL_{n+1,1}^J(\mathbb{Z})$). Then

$$\widehat{F}(\underline{\tau}) = \sum_{\Lambda \oplus \Delta} c_\Delta(\Lambda \oplus \Delta)e_\Delta^* \{(\Lambda \oplus \Delta)\underline{\tau}\}$$

where $\Lambda \oplus \Delta$ varies over all isometry classes of rank $n+1$, even integral, positive semi-definite lattices (with Δ fixed, and Λ oriented when $\chi'(-1)(-1)^{k+1} = -1$; note that the orientation of Δ is fixed). Note that with $G = \begin{pmatrix} I & 0 \\ -R & 1 \end{pmatrix}$, we have $\begin{pmatrix} G & \\ & {}^tG^{-1} \end{pmatrix} \in \Gamma_0^{(n,1)}(N)$ for any N , and so $c(T, 2R) = c(T - 2{}^tRR, 0)$. Thus when F is an even Jacobi form, the support of \widehat{F} is on lattices $\Lambda' \oplus \Delta \simeq \begin{pmatrix} T' & 2{}^tR \\ 2R & 2 \end{pmatrix}$, and we have $\Lambda' \oplus \Delta = \Lambda \perp \Delta$ with $\Lambda \perp \Delta \simeq \begin{pmatrix} T & \\ & 2 \end{pmatrix}$, $T = T' - 2{}^tRR$. We will see that the space of even Jacobi forms is stable under the Hecke operators $T_j(p^2)$ provided $p \neq 2$; to allow us to handle $p = 2$, we introduce a projection map onto the space of even Jacobi forms.

Proposition 7.1. For $F \in \mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$ with $4|N$, define $F|\psi$ by

$$F|\psi = 2^{-n} \cdot \sum_{Y(2)} F| \begin{pmatrix} I_n & & & {}^tY/2 \\ & 1 & Y/2 & \\ & & I_n & \\ & & & 1 \end{pmatrix}$$

where Y varies over $\mathbb{Z}^{1,n}$ modulo 2. Then ψ maps $\mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$ onto $\mathcal{J}_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$, and ψ acts as the identity map on $\mathcal{J}_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$.

Proof. Take $F \in \mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$. We first show $F|\psi \in \mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$. Take $Y \in \mathbb{Z}^{1,n}$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$, and set

$$\delta = \begin{pmatrix} I_n & & & {}^tY/2 \\ & 1 & Y/2 & \\ & & I_n & \\ & & & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} A & B & & \\ & 1 & & 0 \\ C & & D & \\ & 0 & & 1 \end{pmatrix}.$$

Then with $Y' = YD$ and

$$\delta' = \begin{pmatrix} I_n & & & {}^tY'/2 \\ & 1 & Y'/2 & \\ & & I_n & \\ & & & 1 \end{pmatrix},$$

we have $\delta\gamma_1 = \gamma'_1\delta'$ where $\gamma'_1 \in \Gamma_0^{(n,1)}(N)$ and $\chi'(\gamma'_1) = \chi'(\gamma_1)$. Also, since $4|N$, we know $2 \nmid \det D$; thus, as Y varies modulo 2, so does Y' . Hence $F|\psi|\gamma_1 = \chi'(\gamma_1)F|\psi$. Next, take $U, V \in \mathbb{Z}^{1,n}$, $w \in \mathbb{Z}$, and set

$$\gamma_2 = \begin{pmatrix} I & & & {}^tV \\ U & 1 & V & w \\ & & I & -{}^tU \\ & & & 1 \end{pmatrix}.$$

Then with δ as above, $\delta\gamma_2 = \gamma'_2\delta$ with $\gamma'_2 \in \Gamma_0^{(n,1)}(N)$ and $\chi'(\gamma_2) = \chi'(\gamma'_2)$; so $F|\psi|\gamma_2 = \chi'(\gamma_2)F|\psi$. Since matrices of the form γ_1, γ_2 generate $\Gamma_0^{(n,1)}(N)$, we have $F|\psi \in \mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$. Finally, applying ψ to the Fourier expansion for $F(\tau, Z)$, we see that $F|\psi \in \mathcal{J}_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$, and that ψ acts as the identity map on the even Jacobi modular forms. \square

Theorem 3.2 of [16] gives us the following theorem; note that in [16] we refer to the even integral quadratic form on Δ as the index of the Jacobi form, so index 1 in this paper corresponds to index (2) in [16].

Theorem 7.2. Take $F \in \mathcal{J}_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$, p prime, $1 \leq j \leq n$, and $\Delta \simeq (2)$.

(a) Suppose $p \nmid N$. For U a rank d lattice identified with an even integral quadratic form also denoted by U , let

$$\alpha'(U) = \sum_Y e\{UY/p\}$$

where Y varies over symmetric $d \times d$ matrices modulo p so that $p \nmid \det Y$; when $d = 0$, we take $\alpha'(U) = 1$. With $\Omega_1 \oplus \Delta$ an even integral lattice, let $R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, U)$ denote the number of subspaces of $\overline{\Omega}_1 \oplus \overline{\Delta} = \Omega_1/p\Omega_1 \oplus \Delta/p\Delta$ isometric to U and independent of $\overline{\Delta}$. Given even integral $\Lambda \oplus \Delta$ and $\Omega \oplus \Delta$ so that

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \Delta, \quad \Omega \oplus \Delta = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2 \oplus \Delta$$

with $n_i = \text{rank } \Lambda_i$, $r = n_0 + n_2$, set

$$A_{j,\Delta}^J(\Lambda, \Omega) = \chi'(p^{j-r})p^{(k+1)(n_2-n_0)+n_0(n-n_2+2)} \cdot \sum_{\substack{\text{cls } U \\ \dim U = j-r}} R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, U)\alpha'(U).$$

Then the $\Lambda \oplus \Delta$ th Fourier coefficient of $F|T_j^J(p^2)$ is

$$\sum_{\Omega} A_{j,\Delta}^J(\Lambda, \Omega)c_{\Delta}(\Omega \oplus \Delta)$$

where Ω varies so that $\Omega \oplus \Delta$ is even integral, and $p\Lambda' \subseteq \Omega \subseteq \frac{1}{p}\Lambda'$ for some Λ' that satisfies $\Lambda' \oplus \Delta = \Lambda \oplus \Delta$.

(b) If $p|N$ then the $\Lambda \oplus \Delta$ th Fourier coefficient of $F|T_j^J(p^2)$ is

$$p^{j(n+1-k)} \sum_{\Omega} c_{\Delta}(\Omega \oplus \Delta)$$

where $\Omega \oplus \Delta$ varies so that $p\Lambda \subseteq \Omega \subseteq \Lambda$ with $[\Lambda : \Omega] = p^j$.

To complete the incomplete character sum $\alpha'(U)$ when $p \nmid N$, we set

$$\tilde{T}_j^J(p^2) = p^{j(k-n-1)} \sum_{0 \leq \ell \leq j} \chi'(p^{j-\ell}) p^{j-\ell} \beta(n-\ell, j-\ell) T_{\ell}^J(p^2).$$

(The coefficients for this linear combination were not correct in [16], and were later corrected in an erratum.)

Corollary 7.3. Take $F \in J_{k+1,1}(\Gamma_0^{(n,1)}(N), \chi')$ and p a prime so that $p \nmid N$; using the notation of Theorem 7.2, set

$$\tilde{A}_{j,\Delta}^J(\Lambda, \Omega) = \chi'(p^{j-r}) p^{E_{j,\Delta}(\Lambda, \Omega)} R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, \langle 0 \rangle^{j-r})$$

where $E_{j,\Delta}(\Lambda, \Omega) = j(k-n) + k(n_2 - n_0) + n_0(n - n_2) + (j-r)(j-r-1)/2$. Then the $\Lambda \oplus \Delta$ th Fourier coefficient of $F|\tilde{T}_j^J(p^2)$ is

$$\sum_{\Omega} \tilde{A}_{j,\Delta}^J(\Lambda, \Omega) c_{\Delta}(\Omega \oplus \Delta)$$

where Ω varies as in Theorem 7.2.

Proof. To prove this, we need to show that

$$\begin{aligned} & \sum_{0 \leq \ell \leq j} p^{j-\ell} \beta(n-\ell, j-\ell) \sum_{\substack{\text{cls } U \\ \dim U = \ell-r}} R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, U) \alpha'(U) \\ &= \sum_{r \leq \ell \leq j} \sum_{\substack{\text{cls } W \\ \dim W = j-r}} R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, W) \alpha(W) \end{aligned}$$

where, identifying W with the even integral matrix for the quadratic form on W , $\alpha(W)$ is the complete character sum

$$\alpha(W) = \sum_{Y(p)} e\{WY/p\},$$

Y varying over all symmetric $(j-r) \times (j-r)$ matrices modulo p . (Note that $\alpha(W) = p^{(j-r)(j-r+1)/2}$ if $W \simeq \langle 0 \rangle^{j-r}$, and 0 otherwise.)

Given $r \leq \ell \leq j$ and a dimension $\ell-r$ subspace U of $\overline{\Omega}_1 \oplus \overline{\Delta}$ that is independent of $\overline{\Delta}$, $p^{j-\ell} \beta(n-\ell, j-\ell)$ is the number of ways to extend U to a dimension $j-r$ subspace W of $\overline{\Omega}_1 \oplus \overline{\Delta}$ that is independent of $\overline{\Delta}$. With such W , all subspaces of W are necessarily independent of $\overline{\Delta}$. Therefore, with $d = j-r$,

$$p^{j-\ell} \beta(n-\ell, j-\ell) R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, U) = \sum_{\substack{\text{cls } W \\ \dim W=d}} R_{\Delta}^*(\overline{\Omega}_1 \oplus \overline{\Delta}, W) R^*(W, U).$$

So we need to show that with W of dimension d ,

$$\alpha(W) = \sum_{a=0}^d \sum_{\substack{\text{cls } U \\ \dim U=a}} R^*(W, U) \alpha'(U).$$

First suppose $p \neq 2$; let $\mathbb{F} = \mathbb{F}_p$. Take $\omega \in \mathbb{F}^\times$ so that $(\frac{\omega}{p}) = -1$, and for $a \geq 1$, set $J_a = \text{diag}\{I_{a-1}, \omega\}$. Note that for $Y \in \mathbb{F}_{\text{sym}}^{d,d}$, Y can be diagonalised and so either $Y \sim I_a \perp \langle 0 \rangle^{d-a}$ or $Y \sim J_a \perp \langle 0 \rangle^{d-a}$, some a . In what follows we will sometimes write GL_d for $GL_d(\mathbb{F})$. Then (using notation from Section 1),

$$\begin{aligned} \alpha(W) - 1 &= \sum_{a=1}^d \left(\sum_{Y \sim I_a \perp \langle 0 \rangle^{d-a}} e\{WY/p\} + \sum_{Y \sim J_a \perp \langle 0 \rangle^{d-a}} e\{WY/p\} \right) \\ &= \sum_{a=1}^d \sum_{G \in GL_d} \frac{1}{o(I_a \perp \langle 0 \rangle^{d-a})} e \left\{ W {}^t G \begin{pmatrix} I_a & \\ & 0 \end{pmatrix} G/p \right\} \\ &\quad + \sum_{a=1}^d \sum_{G \in GL_d} \frac{1}{o(J_a \perp \langle 0 \rangle^{d-a})} e \left\{ W {}^t G \begin{pmatrix} J_a & \\ & 0 \end{pmatrix} G/p \right\}. \end{aligned}$$

We have

$$\begin{aligned} o(I_a \perp \langle 0 \rangle^{d-a}) &= r^*(I_a \perp \langle 0 \rangle^{d-a}, I_a) r^*(\langle 0 \rangle^{d-a}, \langle 0 \rangle^{d-a}) \\ &= p^{a(d-a)} o(I_a) \prod_{i=0}^{d-a-1} (p^{d-a} - p^i), \end{aligned}$$

and similarly, $o(J_a \perp \langle 0 \rangle^{d-a}) = p^{a(d-a)} o(J_a) \prod_{i=0}^{d-a-1} (p^{d-a} - p^i)$. Also, writing $GW {}^t G = \begin{pmatrix} U & * \\ * & * \end{pmatrix}$ with U an $a \times a$ matrix, we have

$$e \left\{ W {}^t G \begin{pmatrix} I_a & \\ & 0 \end{pmatrix} G/p \right\} = e \left\{ GW {}^t G \begin{pmatrix} I_a & \\ & 0 \end{pmatrix} /p \right\} = e\{U/p\}.$$

Given an $a \times a$ symmetric matrix U over \mathbb{F} , the number of $G \in GL_d(\mathbb{F})$ so that $GW {}^t G = \begin{pmatrix} U & * \\ * & * \end{pmatrix}$ is

$$r^*(W, U) p^{a(d-a)} \prod_{i=0}^{d-a-1} (p^{d-a} - p^i)$$

as $\prod_{i=0}^{d-1} (p^d - p^i)$ is the number of ways to extend a $d \times a$ matrix with rank a over \mathbb{F} to an element of $GL_d(\mathbb{F})$. Hence

$$\begin{aligned}
 \alpha(W) - 1 &= \sum_{a=1}^d \sum_{\substack{U \in \mathbb{F}_a^{\text{sym}} \\ \dim U = a}} r^*(W, U) \left(\frac{e\{U I_a/p\}}{o(I_a)} + \frac{e\{U J_a/p\}}{o(J_a)} \right) \\
 &= \sum_{a=1}^d \sum_{\substack{\text{cls } U \\ \dim U = a}} r^*(W, U) \sum_{G \in O(U) \setminus GL_a} \left(\frac{e\{G U^t G I_a/p\}}{o(I_a)} + \frac{e\{G U^t G J_a/p\}}{o(J_a)} \right) \\
 &= \sum_{a=1}^d \sum_{\substack{\text{cls } U \\ \dim U = a}} R^*(W, U) \sum_{G \in GL_a} \left(\frac{e\{U^t G I_a G/p\}}{o(I_a)} + \frac{e\{U^t G J_a G/p\}}{o(J_a)} \right) \\
 &= \sum_{a=1}^d \sum_{\substack{\text{cls } U \\ \dim U = a}} R^*(W, U) \alpha'(U).
 \end{aligned}$$

Hence $\alpha(W) = \sum_{a=0}^d R^*(W, U) \alpha'(U)$, as $R^*(W, \{0\}) \alpha'(\{0\}) = 1$.

When $p = 2$, the argument is virtually identical. The difference is that the symmetric, invertible $d \times d$ matrices over $\mathbb{Z}/2\mathbb{Z}$ lie in one $GL_d(\mathbb{Z}/2\mathbb{Z})$ -orbit if d is odd, and in two $GL_d(\mathbb{Z}/2\mathbb{Z})$ -orbits when d is even; for $d = 2c \geq 2$, the $GL_d(\mathbb{Z}/2\mathbb{Z})$ -orbits are represented by I_{2c} and $J_{2c} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \dots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. \square

Notice that the formulas in Theorem 3.4 and Corollary 7.3 are identical when the lattices Ω and Δ in Corollary 7.3 are orthogonal. This leads us to our final result.

Theorem 7.4. *Let f be a degree n Siegel modular form of weight $k + 1/2$, level N with $4|N$, and character χ . Let $\theta^{(n,1)}$ be the weight $1/2$, level 4 Jacobi modular form defined by*

$$\theta^{(n,1)}(\tau, Z) = \sum_{R \in \mathbb{Z}^{1,n}} e\{2^t R R \tau + 4^t R Z\}$$

where $\tau \in \mathbb{H}_{(n)}$ and $Z \in \mathbb{C}^{1,n}$. Then $f(\tau) \mapsto f(\tau) \cdot \theta^{(n,1)}(\tau, Z)$ is an isomorphism from $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ onto $\mathcal{J}_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$ where

$$\chi'(d) = \chi(d) \left(\frac{(-1)^{k+1}}{|d|} \right) (\text{sgn } d)^{k+1}.$$

Further, for p prime and $1 \leq j \leq n$, we have

$$(f | \tilde{T}_j(p^2)) \cdot \theta^{(n,1)} = (f \cdot \theta^{(n,1)}) | \tilde{T}_j^J(p^2)$$

if $p \nmid N$,

$$p^{j/2} \cdot (f | T_j(p^2)) \cdot \theta^{(n,1)} = (f \cdot \theta^{(n,1)}) | T_j^J(p^2)$$

if $p|N$ and $p \neq 2$, and

$$2^{j/2} \cdot (f | T_j(4)) \cdot \theta^{(n,1)} = (f \cdot \theta^{(n,1)}) | T_j^J(4) | \psi.$$

Remark. When $p \nmid N$, the definitions of $\tilde{T}_j(p^2)$ and $\tilde{T}_j^J(p^2)$ give us

$$p^{j/2} \cdot (f|T_j(p^2))\theta^{(n,1)} = (f\theta^{(n,1)})|T_j(p^2).$$

Proof of Theorem 7.4. We first explain why the character of $f(\tau)\theta^{(n,1)}(\tau, Z)$ is χ' . Recall that $\Gamma_0^{(n,1)}(N)$ is generated by matrices of the form

$$\underline{\gamma} = \begin{pmatrix} A & B & & \\ & 1 & & \\ C & & D & \\ & 0 & & 1 \end{pmatrix}, \quad \underline{\gamma}' = \begin{pmatrix} I & & & {}^tV \\ U & 1 & V & w \\ & & I & -{}^tU \\ & & & 1 \end{pmatrix}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$, $U, V \in \mathbb{Z}^{1,n}$, $w \in \mathbb{Z}$. By Theorem 1.2, for $\underline{\tau} = \begin{pmatrix} \tau & {}^tZ \\ \tau' & \end{pmatrix} \in \mathcal{H}_{(n+1)}$ with $\tau \in \mathcal{H}_{(n)}$, we have

$$\theta^{(n+1)}(\underline{\gamma}'\underline{\tau}) = \theta^{(n+1)}(\underline{\tau}) \quad \text{and} \quad \theta^{(n+1)}(\underline{\gamma}\underline{\tau}) = \frac{\theta^{(n)}(\gamma\tau)}{\theta^{(n)}(\tau)}\theta^{(n+1)}(\underline{\tau}).$$

Since $\theta^{(n,1)}(\tau, Z)$ is the 1st Fourier–Jacobi coefficient of $\theta^{(n+1)}(\underline{\tau})$, with $\widehat{\theta}^{(n,1)}(\underline{\tau}) = \theta^{(n,1)}(\tau, Z)e\{2\tau'\}$ (τ' a formal variable), we have

$$f(\tau)\widehat{\theta}^{(n,1)}(\tau, Z)|\underline{\gamma}' = f(\tau)\widehat{\theta}^{(n,1)}(\tau, Z),$$

and

$$\begin{aligned} f(\tau)\widehat{\theta}^{(n,1)}(\underline{\tau})|\underline{\gamma} &= (\det(C\tau + D))^{-(k+1)} f(\gamma\tau)\widehat{\theta}^{(n,1)}(\underline{\gamma}\underline{\tau}) \\ &= (\det(C\tau + D))^{-(k+1)} \left(\frac{\theta^{(n)}(\gamma\tau)}{\theta^{(n)}(\tau)}\right)^{2k+2} \chi(\det D) f(\tau)\widehat{\theta}^{(n,1)}(\underline{\tau}). \end{aligned}$$

Now, $(\theta^{(n)}(\tau))^{2k+2} = \theta^{(n)}(L; \tau)$ where L is a lattice with rank $2k + 2$ and quadratic form given by the matrix $2I_{2k+2}$. Hence $\theta^{(n)}(L; \tau)$ is a modular form of weight $k + 1$, level 4, and character φ defined by

$$\varphi(d) = \left(\frac{(-1)^{k+1}}{|d|}\right) (\text{sgn } d)^{k+1}.$$

Thus $\theta^{(n)}(L; \gamma\tau) = (\det(C\tau + D))^{k+1}\varphi(\det D)\theta^{(n)}(L; \tau)$, and hence

$$f(\tau)\widehat{\theta}^{(n,1)}(\underline{\tau})|\underline{\gamma} = \chi'(\det D) f(\tau)\widehat{\theta}^{(n,1)}(\underline{\tau}).$$

So $f \mapsto f\theta^{(n,1)}$ maps $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ into $J_{k+1,1}^{\text{even}}(\Gamma^{(n,1)}(N), \chi')$.

Now suppose $F \in J_{k+1,1}^{\text{even}}(\Gamma_0^{(n,1)}(N), \chi')$. Then

$$\begin{aligned} F(\tau, Z) &= \sum_{T', R} c(T', 2R) e\{T'\tau + 4{}^tRZ\} \\ &= \sum_{T, R} c(T, 0) e\{(T + 2{}^tRR)\tau + 4{}^tRZ\} \\ &= f(\tau)\theta^{(n,1)}(\tau, Z) \end{aligned}$$

where $f(\tau) = \sum_T c(T, 0)e\{T\tau\}$. Also, as a function of Z , $f(\tau)$ is the 0th Fourier coefficient of F ; consequently f is analytic. Then, reversing the derivation in the previous paragraph, we see $f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$.

It is clear that these maps between $\mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$ and $J_{k+1,1}^{\text{even}}(\Gamma^{(n,1)}(N), \chi')$ are inverses of each other, and hence we have a bijection between these spaces.

Now take p prime; suppose first that $p \nmid N$ (so in particular, $p \neq 2$). For $f \in \mathcal{M}_{k+1/2}(\Gamma_0^{(n)}(N), \chi)$, set $F = f\theta^{(n,1)}$. Then with $\Delta = \mathbb{Z}w \simeq (2)$, we have

$$F(\tau, Z)e\{2\tau'\} = \sum_{\Lambda} c_{\Delta}(\Lambda \perp \Delta)e_{\Delta}^*\{(\Lambda \perp \Delta)\underline{\tau}\}$$

where $c_{\Delta}(\Lambda \perp \Delta) = c_f(\Lambda)$. We first show $F|\tilde{T}_j^J(p^2)$ is even. As a Fourier series, $F(\tau, Z)|\tilde{T}_j^J(p^2)e\{2\tau'\}$ is a sum with exponentials

$$e\{\underline{\tau}[X_j^{-1}D\underline{G}^{-1}]\underline{\tau}\}$$

where $X_j = \text{diag}\{pI_j, I_{n+1-j}\}$, $D = \text{diag}\{I_{r_0}, pI_{r_1}, p^2I_{r_2}, I_{n+1-j}\}$ with $r_0 + r_1 + r_2 = j$, $\underline{G} \in GL_{n+1,1}^J(\mathbb{Z})$, $\underline{\tau} = \begin{pmatrix} T+2^t R R' & 2^t R \\ 2R & 2 \end{pmatrix}$ and $\underline{\tau}[X_j^{-1}D\underline{G}^{-1}]$ even integral. Consequently

$$\underline{\tau}[X_j^{-1}D\underline{G}^{-1}] = \begin{pmatrix} T' & 2^t R' \\ 2R' & 2 \end{pmatrix} \quad \text{where } R' \in \frac{1}{p}\mathbb{Z}^{1,n} \cap \frac{1}{2}\mathbb{Z}^{1,n} = \mathbb{Z}^{1,n}.$$

Thus with $\Lambda' \oplus \Delta \simeq \begin{pmatrix} T' & 2^t R' \\ 2R' & 2 \end{pmatrix}$, we have

$$\Lambda' \oplus \Delta = (\Lambda' \oplus \Delta) \begin{pmatrix} I & 0 \\ -R' & 1 \end{pmatrix} \simeq \begin{pmatrix} T' - 2^t R' R' & 0 \\ 0 & 2 \end{pmatrix}.$$

Hence

$$(F(\tau, Z)|\tilde{T}_j^J(p^2))e\{2\tau'\} = \sum_{\Lambda \perp \Delta} \tilde{c}_{\Delta}(\Lambda \perp \Delta)e_{\Delta}^*\{(\Lambda \perp \Delta)\underline{\tau}\};$$

by Theorem 7.2,

$$\tilde{c}_{\Delta}(\Lambda \perp \Delta) = \sum_{\Omega'} \tilde{A}_{j,\Delta}^J(\Lambda, \Omega')c_{\Delta}(\Omega' \oplus \Delta)$$

where Ω' varies so that $p\Lambda \perp \Delta \subseteq \Omega' \oplus \Delta \subseteq \frac{1}{p}(\Lambda \perp \Delta)$ and $p\Lambda' \subseteq \Omega' \subseteq \frac{1}{p}\Lambda'$ for some Λ' with $\Lambda' \oplus \Delta = \Lambda \perp \Delta$. (Recall that $c_{\Delta}(\Omega' \oplus \Delta) = 0$ unless $\Omega' \oplus \Delta$ is even integral.) Since $\Delta \simeq (2)$ and $p \neq 2$, any even integral sublattice of $\frac{1}{p}(\Lambda \perp \Delta)$ is actually a sublattice of $\frac{1}{p}\Lambda \perp \Delta$. So suppose

$$p\Lambda \perp \Delta \subseteq \Omega' \oplus \Delta \subseteq \frac{1}{p}\Lambda \perp \Delta.$$

Taking $\Omega = (\Omega' \oplus \Delta) \cap \frac{1}{p}\Lambda$, we have

$$\Omega \perp \Delta = \Omega \oplus \Delta = (\Omega' \oplus \Delta) \cap \left(\frac{1}{p}\Lambda \perp \Delta\right) = \Omega' \oplus \Delta.$$

Consequently

$$\begin{aligned}\tilde{c}_\Delta(\Lambda \perp \Delta) &= \sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \tilde{A}_{j,\Delta}^J(\Lambda, \Omega) c_\Delta(\Omega \perp \Delta) \\ &= \sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \tilde{A}_j(\Lambda, \Omega) c_f(\Omega),\end{aligned}$$

which is the Λ th coefficient of $f|\tilde{T}_j(p^2)$, and hence the $(\Lambda \perp \Delta)$ th coefficient of $(f(\tau)|\tilde{T}_j(p^2)) \cdot \theta^{(n,1)}(\tau, Z)$.

Now suppose $p|N$. When $p \neq 2$, the argument follows much as when $p \nmid N$, except the situation is simpler since D is always I . When $p = 2$, this argument breaks down since $\frac{1}{p}\mathbb{Z}^{1,n} \cap \frac{1}{2}\mathbb{Z}^{1,n} = \frac{1}{2}\mathbb{Z}^{1,n}$, not $\mathbb{Z}^{1,n}$. For this reason we need to follow $T_j^J(4)$ by ψ to get the desired equality. \square

References

- [1] A.N. Andrianov, Quadratic Forms and Hecke Operators, Grundlehren Math. Wiss., vol. 286, Springer-Verlag, 1987.
- [2] A.N. Andrianov, V.G. Zhuravlev, Modular Forms and Hecke Operators, Transl. Math. Monogr., vol. 145, Amer. Math. Soc., 1995.
- [3] M. Eichler, On theta functions of real algebraic number fields, Acta Arith. 33 (1977) 269–292.
- [4] E. Freitag, Siegelsche Modulformen, Grundlehren Math. Wiss., vol. 254, Springer-Verlag, 1983.
- [5] L. Gerstein, Basic Quadratic Forms, Grad. Stud. Math., vol. 90, Amer. Math. Soc., 2008.
- [6] J.L. Hafner, L.H. Walling, Explicit action of Hecke operators on Siegel modular forms, J. Number Theory 93 (2002) 34–57.
- [7] T. Ibukiyama, On Jacobi forms and Siegel modular forms of half integral weights, Comment. Math. Univ. St. Pauli 41 (1992) 109–124.
- [8] Y. Kitaoka, A space of Siegel modular forms closed under the action of Hecke operators, Proc. Japan Acad. Ser. A Math. Sci. 70 (1994) 194–197.
- [9] H. Maass, Die Primzahlen in der Theorie der Siegelschen Modulformen, Math. Ann. 124 (1951) 87–122.
- [10] O.T. O'Meara, Introduction to Quadratic Forms, Grundlehren Math. Wiss., vol. 117, Springer-Verlag, 1973.
- [11] H. Saito, A generalization of Gauss sums and its applications to Siegel modular forms and L -functions associated with the vector space of quadratic forms, J. Reine Angew. Math. 416 (1991) 91–142.
- [12] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press, 1971.
- [13] L.H. Walling, On bounding Hecke–Siegel eigenvalues, J. Number Theory 117 (2006) 387–396.
- [14] L.H. Walling, Action of Hecke operators on Siegel theta series I, Int. J. Number Theory 2 (2006) 169–186.
- [15] L.H. Walling, Action of Hecke operators on Siegel theta series II, Int. J. Number Theory 4 (2008) 981–1008.
- [16] L.H. Walling, Restricting Hecke–Siegel operators to Jacobi modular forms, J. Number Theory 129 (2009) 1709–1733.
- [17] V.G. Zhuravlev, Hecke rings for a covering of the symplectic group, Math. USSR Sb. 49 (1984) 379–399 (English translation).
- [18] V.G. Zhuravlev, Euler expansions of theta-transformations of Siegel modular forms of half integer weight and their analytic properties, Math. USSR Sb. 51 (1985) 169–190 (English translation).
- [19] V.G. Zhuravlev, Local duality of Hecke operators of the symplectic and orthogonal groups, J. Soviet Math. 59 (1992) 1156–1173 (English translation).
- [20] V.G. Zhuravlev, Generalized Eichler–Brandt matrices, Hecke operators, and vector-values theta series, St. Petersburg Math. J. 5 (1994) 545–576 (English translation).
- [21] V.G. Zhuravlev, Multiplicative arithmetic of theta series for odd quadratic forms, Izv. Math. 59 (1995) 517–578 (English translation).