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# THE LEVEL 13 ANALOGUE OF THE ROGERS-RAMANUJAN CONTINUED FRACTION AND ITS MODULARITY

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ABSTRACT. We prove the modularity of the level 13 analogue $r_{13}(\tau)$ of the Rogers-Ramanujan continued fraction. We establish some properties of $r_{13}(\tau)$ using the modular function theory. We first prove that $r_{13}(\tau)$ is a generator of the function field on $\Gamma_0(13)$. We then find modular equations of $r_{13}(\tau)$ of level $n$ for every positive integer $n$ by using affine models of modular curves; this is an extension of Cooper and Ye's results with levels $n = 2, 3$ and 7 to every level $n$. We further show that the value $r_{13}(\tau)$ is an algebraic unit for any $\tau \in K - \mathbb{Q}$, where $K$ is an imaginary quadratic field.

## 1. INTRODUCTION

Let $\mathfrak{H}$ be the complex upper half plane and $q := e^{2\pi i\tau}$ for $\tau \in \mathfrak{H}$. The *Rogers-Ramanujan continued fraction* $r(\tau)$ is defined by

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}.$$

It can be also written as an infinite product as follows:

$$r(\tau) = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)} \tag{1.1}$$

with the Jacobi symbol $\left(\frac{n}{N}\right)$. On the other hand, it has the following interesting properties:

$$\frac{1}{r(\tau)} - 1 - r(\tau) = q^{-\frac{1}{5}} \prod_{n=1}^{\infty} \frac{(1 - q^{\frac{n}{5}})}{(1 - q^{5n})} \tag{1.2}$$

and

$$\frac{1}{r^5(\tau)} - 11 - r^5(\tau) = q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^n)^6}{(1 - q^{5n})^6}. \tag{1.3}$$

The identity in (1.1) was proved in [9], and (1.2) and (1.3) were stated by Ramanujan [1, p. 85 and p. 267] and proved by Watson [11].

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Recently, Gee and Honesbeek studied the modularity of $r(\tau)$ and evaluated $r(\tau)$ for an imaginary quadratic quantity $\tau$ [6]. Cais and Conrad investigated the modular equation of $r(\tau)$ and its properties in the view of arithmetic models of modular curves [3].

For any positive integer $N > 2$ with $\left(\frac{-1}{N}\right) = 1$, we define the *level $N$ analogue of the Rogers-Ramanujan continued fraction* to be

$$r_N(\tau) := q^{\alpha_N} \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{N}\right)},$$

where

$$\alpha_N = \sum_{r=1}^{\left[\frac{N}{2}\right]} \frac{r(r-N)}{2N} \left(\frac{r}{N}\right)$$

and $[\cdot]$ denotes the floor function value. For example, $\alpha_5 = 1/5$, $\alpha_{10} = 3/5$ and $\alpha_{13} = 1$. We notice that $r_5(\tau)$ is exactly the same as the Rogers-Ramanujan continued fraction $r(\tau)$.

In Ramanujan's second notebook [1, Entry 8 (i)] he stated $r_{13}(\tau)$ satisfies the following identity:

$$\frac{1}{r_{13}(\tau)} - 3 - r_{13}(\tau) = q^{-1} \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{13n})^2}. \tag{1.4}$$

We prove the modularity of $r_{13}(\tau)$ (Theorem 1.1). We establish some properties of $r_{13}(\tau)$ using the modular function theory. We first prove that $r_{13}(\tau)$ is a generator of the function field on $\Gamma_0(13)$. We then find modular equations of $r_{13}(\tau)$ of level $n$ for every positive integer $n$ by using affine models of modular curves (Theorem 1.2); this is an extension of Cooper and Ye's results with levels $n = 2, 3$ and 7 to every level $n$. We further show that the value $r_{13}(\tau)$ is an algebraic unit for any $\tau \in K - \mathbb{Q}$, where $K$ is an imaginary quadratic field (Theorem 1.3).

The examples of the modular equations are given in Appendix, which includes Cooper and Ye's result [5]. Assuming that the value $r_{13}(\tau)$ is given, Cooper and Ye expressed $r_{13}(2^n\tau)$ and $r_{13}(3^n\tau)$ in terms of radicals for $n \in \mathbb{Z}$; they used the modular equations of $r_{13}(\tau)$ obtained from the identities in [2]. By Theorem 1.2 or modular equations in Appendix B, we obtain the values $r_{13}(r\tau)$ for any positive rational number $r$.

It was proved that $r_{13}(\sqrt{-n}/2)$ is a unit in [5, Theorem 6.2] for any positive integer $n$. Using a completely different approach, we prove that $r_{13}(\tau)$ is a modular unit, and therefore $r_{13}(z)$ is a unit for every $z \in K - \mathbb{Q}$.

We state our main results as follows.

**Theorem 1.1.** *The level 13 analogue of the Rogers-Ramanujan continued fraction $r_{13}(\tau)$ is a modular function on $\Gamma_1(13)$. Moreover, the field of modular functions on $\Gamma_0(13)$ is generated by $1/r_{13}(\tau) - r_{13}(\tau)$.*

**Theorem 1.2.** *For any positive integer $n$, there is a modular equation $\mathcal{F}_n(X, Y)$ of $r_{13}(\tau)$ of level $n$.*

**Theorem 1.3.** *Let $K$ be an imaginary quadratic field. Then $r_{13}(\tau)$ is a unit for every $\tau \in K - \mathbb{Q}$.*

This paper is organized as follows. Section 2 provides brief preliminaries about modular functions and Klein forms and states lemmas regarding the cusps of congruence subgroup $\Gamma_0(N)$. Furthermore, we describe a method to get an affine model which is used for the proof of our main theorems. In Section 3, we focus on the functions with modularity of level 13. These are stated in Ramanujan's second notebooks [1]. In Section 4, we prove Theorem 1.1 and 1.2, where we show the modularity of $r_{13}(\tau)$ and give the properties of modular equations of $1/r_{13}(\tau) - 3 - r_{13}(\tau)$. Then, we prove that $r_{13}(\tau)$ and $1/r_{13}(\tau)$ are algebraic for any imaginary quadratic quantity $\tau$; hence, $r_{13}(\tau)$ is a unit. Furthermore, we obtain the modular equation of $r_{13}(\tau)$ in Appendix B for levels $2, 3, 5, 7, 11, 13$ and $17$ using MAPLE program.

## 2. Preliminaries

We introduce some definitions and properties from the theory of modular functions. Let $\mathfrak{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ be the complex upper half plane, $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$, and $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ the full modular group. For any positive integer $N$, we have congruent subgroups $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$ of $\Gamma(1)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo $N$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, respectively.

A congruence subgroup $\Gamma$ acts on $\mathfrak{H}^*$ by linear fractional transformations as $\gamma(\tau) = (a\tau + b)/(c\tau + d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and the quotient space $\Gamma\backslash\mathfrak{H}^*$ becomes a compact Riemann surface with an appropriate complex structure. We identify $\gamma$ with its action on $\mathfrak{H}^*$. An element $s$ of $\mathbb{Q} \cup \{\infty\}$ is called a *cusp*, and two cusps $s_1, s_2$ are equivalent under $\Gamma$ if there exists $\gamma \in \Gamma$ such that $\gamma(s_1) = s_2$. The equivalence class of such $s$ is also called a *cusp*. In fact, there exist at most finitely many inequivalent cusps of $\Gamma$. Let $s$ be any cusp of $\Gamma$, and let $\rho$ be an element of $\Gamma(1)$ such that $\rho(s) = \infty$. We define the *width* of the cusp $s$ in $\Gamma\backslash\mathfrak{H}^*$ by the smallest positive integer $h$ satisfying $\rho^{-1}\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rho \in \{\pm 1\} \cdot \Gamma$. Then the width of the cusp $s$ depends only on the equivalence class of $s$ under $\Gamma$, and it is independent of the choice of $\rho$.

By a modular function on a congruence subgroup $\Gamma$ we mean a $\mathbb{C}$-valued function $f(\tau)$ of $\mathfrak{H}$ satisfying the following three conditions:

(1) $f(\tau)$ is meromorphic on $\mathfrak{H}$.
(2) $f(\tau)$ is invariant under $\Gamma$, i.e., $f \circ \gamma = f$ for all $\gamma \in \Gamma$.
(3) $f(\tau)$ is meromorphic at all cusps of $\Gamma$.

The precise meaning of the last condition is as follows. For a cusp $s$ for $\Gamma$, let $h$ be the width for $s$ and $\rho$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\rho(s) = \infty$. Since

$$(f \circ \rho^{-1})(\tau + h) = \left(f \circ \rho^{-1} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rho\right)(\rho^{-1}\tau) = (f \circ \rho^{-1})(\tau),$$

$f \circ \rho^{-1}$ has a Laurent series expansion in $q_h = e^{2\pi i\tau/h}$, namely for some integer $n_0$, $(f \circ \rho^{-1})(\tau) = \sum_{n \geq n_0} a_n q_h^n$ with $a_{n_0} \neq 0$. This integer $n_0$ is called the *order* of $f(\tau)$ at the cusp $s$ and denoted by $\mathrm{ord}_s f(\tau)$. If $\mathrm{ord}_s f(\tau)$ is positive (respectively, negative), then we say that $f(\tau)$ has a zero (respectively, a pole) at $s$. If a modular function $f(\tau)$ is holomorphic on $\mathfrak{H}$ and $\mathrm{ord}_s f(\tau)$ is non-negative for all cusps $s$, then we say that $f(\tau)$ is holomorphic on $\mathfrak{H}^*$. Since we may identify a modular function on $\Gamma$ with a meromorphic function on the compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$, any holomorphic modular function on $\Gamma$ is a constant.

Let $A_0(\Gamma)$ be the field of all modular functions on $\Gamma$, and let $A_0(\Gamma)_{\mathbb{Q}}$ be the subfield of $A_0(\Gamma)$ which consists of all modular functions $f(\tau)$ whose Fourier coefficients belong to $\mathbb{Q}$.

We may identify $A_0(\Gamma)$ with the field $\mathbb{C}(\Gamma\backslash\mathfrak{H}^*)$ of all meromorphic functions on the compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$, and if $f(\tau)\in A_0(\Gamma)$ is non-constant, then the field extension degree $[A_0(\Gamma):\mathbb{C}(f(\tau))]$ is finite and is equal to the total degree of poles of $f(\tau)$. Since we will consider the modular functions with neither zeros nor poles on $\mathfrak{H}$, the total degree of poles of $f(\tau)$ is $-\sum_s \mathrm{ord}_s f(\tau)$, where the summation runs over all the inequivalent cusps $s$ at which $f(\tau)$ has poles.

To recall the Klein form, which is a main tool of this paper, consider the *Weierstrass $\sigma$-function* by

$$\sigma(z;L):=z\prod_{\omega\in L-\{0\}}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}(\frac{z}{\omega})^2},$$

where $L$ is any lattice in $\mathbb{C}$ and $z\in\mathbb{C}$. This is holomorphic with only simple zeros at all points $z\in L$. The *Weierstrass $\zeta$-function* is also defined by

$$\zeta(z;L):=\frac{\sigma'(z;L)}{\sigma(z;L)}=\frac{1}{z}+\sum_{\omega\in L-\{0\}}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right)$$

by the logarithmic derivative of $\sigma(z;L)$. This is meromorphic with only simple poles at all points $z\in L$. We can see that $\sigma(\lambda z;\lambda L)=\lambda\sigma(z;L)$ and $\zeta(\lambda z;\lambda L)=\lambda^{-1}\zeta(z;L)$ for any $\lambda\in\mathbb{C}^\times$. In fact $\zeta'(z;L)$ is $-\wp(z;L)$ with *Weierstrass $\wp$-function* defined by

$$\wp(z;L):=\frac{1}{z^2}+\sum_{\omega\in L-\{0\}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right).$$

For any $\omega\in L$, $\wp(z+\omega;L)=\wp(z;L)$ and $\frac{d}{dz}[\zeta(z+\omega;L)-\zeta(z;L)]=0$. In other words, $\zeta(z+\omega;L)-\zeta(z;L)$ depends only on a lattice point $\omega\in L$ and not on $z\in\mathbb{C}$, so we may let $\eta(\omega;L)$ be $\zeta(z+\omega;L)-\zeta(z;L)$ for all $\omega\in L$. When we fix the basis $\omega_1,\omega_2$ for $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, for $z=a_1\omega_1+a_2\omega_2\in L$ with $a_1,a_2\in\mathbb{R}$, define the *Weierstrass $\eta$-function* by

$$\eta(z;L):=a_1\eta(\omega_1;L)+a_2\eta(\omega_2;L).$$

Note that $\eta(z;L)$ does not depend on the choice of the basis $\{\omega_1,\omega_2\}$, and it is well-defined. Moreover, $\eta(z;L)$ is $\mathbb{R}$-linear so that $\eta(rz;L)=r\eta(z;L)$ for any $r\in\mathbb{R}$.

We define the *Klein form* by

$$K(z;L)=e^{-\frac{\eta(z;L)z}{2}}\sigma(z;L).$$

For $\mathbf{a}=(a_1,a_2)\in\mathbb{R}^2$ and $\tau\in\mathfrak{H}$, we further define

$$K_{\mathbf{a}}(\tau)=K(a_1\tau+a_2;\mathbb{Z}\tau+\mathbb{Z})$$

as the *Klein form* by abuse of terminology. We observe that $K_{\mathbf{a}}(\tau)$ is holomorphic and nonvanishing on $\mathfrak{H}$ if $\mathbf{a}\in\mathbb{R}^2-\mathbb{Z}^2$ and homogeneous of degree 1, i.e., $K(\lambda z;\lambda L)=\lambda K(z;L)$.

The Klein form satisfies the following properties. Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{a}=(a_1,a_2)\in\mathbb{R}^2$.

(K0) $K_{-\mathbf{a}}(\tau)=-K_{\mathbf{a}}(\tau)$.
(K1) $K_{\mathbf{a}}(\gamma\tau)=(c\tau+d)^{-1}K_{\mathbf{a}\gamma}(\tau)$.
(K2) For $\mathbf{b}=(b_1,b_2)\in\mathbb{Z}^2$, we have that

$$K_{\mathbf{a}+\mathbf{b}}(\tau)=\varepsilon(\mathbf{a},\mathbf{b})K_{\mathbf{a}}(\tau),$$

where $\varepsilon(\mathbf{a},\mathbf{b})=(-1)^{b_1b_2+b_1+b_2}e^{\pi i(b_2a_1-b_1a_2)}$.

(K3) For $\mathbf{a} = (r/N, s/N) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ and $\gamma \in \Gamma(N)$ with an integer $N > 1$, we obtain that
$$K_{\mathbf{a}}(\gamma\tau) = \varepsilon_{\mathbf{a}}(\gamma)(c\tau + d)^{-1}K_{\mathbf{a}}(\tau),$$
where $\varepsilon_{\mathbf{a}}(\gamma) = -(-1)^{((a-1)r+cs+N)(br+(d-1)s+N)/N^2} e^{\pi i(br^2+(d-a)rs-cs^2)/N^2}$.

(K4) Let $\tau \in \mathfrak{H}$, $z = a_1\tau + a_2$ with $\mathbf{a} = (a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and further let $q = e^{2\pi i\tau}$ and $q_z = e^{2\pi iz} = e^{2\pi ia_2}e^{2\pi ia_1\tau}$. Then
$$K_{\mathbf{a}}(\tau) = -\frac{1}{2\pi i}e^{\pi ia_2(a_1-1)}q^{\frac{a_1(a_1-1)}{2}}(1 - q_z)\prod_{n=1}^{\infty}\frac{(1 - q^nq_z)(1 - q^nq_z^{-1})}{(1 - q^n)^2}$$
and $\mathrm{ord}_q K_{\mathbf{a}}(\tau) = \langle a_1\rangle(\langle a_1\rangle - 1)/2$, where $\langle a_1\rangle$ denotes the rational number such that $0 \le \langle a_1\rangle < 1$ and $a_1 - \langle a_1\rangle \in \mathbb{Z}$.

(K5) Let $f(\tau) = \prod_{\mathbf{a}} K_{\mathbf{a}}^{m(\mathbf{a})}(\tau)$ be a finite product of Klein forms with $m(\mathbf{a}) \in \mathbb{Z}$ and $\mathbf{a} = (r/N, s/N) = (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for an integer $N > 1$, and let $k = -\sum_{\mathbf{a}} m(\mathbf{a})$. Then $f(\tau)$ is a modular form of weight $k$ on $\Gamma(N)$ if and only if
$$\begin{cases} \sum_{\mathbf{a}} m(\mathbf{a})r^2 \equiv \sum_{\mathbf{a}} m(\mathbf{a})s^2 \equiv \sum_{\mathbf{a}} m(\mathbf{a})rs \equiv 0 \pmod N & \text{if } N \text{ is odd,} \\ \sum_{\mathbf{a}} m(\mathbf{a})r^2 \equiv \sum_{\mathbf{a}} m(\mathbf{a})s^2 \equiv 0 \pmod{2N}, \ \sum_{\mathbf{a}} m(\mathbf{a})rs \equiv 0 \pmod N & \text{if } N \text{ is even.} \end{cases}$$

For more details on Klein forms, we refer to [8].

The following lemma is some information on cusps of the congruence subgroup $\Gamma_0(N)$.

**Lemma 2.1.** *Let $a, c, a', c' \in \mathbb{Z}$ be such that $(a, c) = 1$ and $(a', c') = 1$. We understand that $\pm 1/0 = \infty$. We denote by $S_{\Gamma_0(N)}$ a set of all the inequivalent cusps of $\Gamma_0(N)$. Then $a/c$ and $a'/c'$ are equivalent under $\Gamma_0(N)$ if and only if there exist $\overline{s} \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $n \in \mathbb{Z}$ such that $\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \begin{pmatrix} \overline{s}^{-1}a + nc \\ \overline{s}c \end{pmatrix} \pmod N$. Furthermore, we can take $S_{\Gamma_0(N)}$ as the following set*
$$\begin{aligned} S_{\Gamma_0(N)} \quad = \quad & \Big\{\frac{a_{c,j}}{c} \in \mathbb{Q} :\ 0 < c \mid N, 0 < a_{c,j} \le N, (a_{c,j}, N) = 1, \\ & a_{c,j} = a_{c,j'} \Leftrightarrow a_{c,j} \equiv a_{c,j'} \pmod{\left(c, \frac{N}{c}\right)}\Big\} \end{aligned}$$
*and the width of the cusp $a/c$ in $\Gamma_0(N)\backslash\mathfrak{H}^*$ is $N/(N, c^2)$.*

*Proof.* See [4, Corollary 4 (1)]. □

Suppose that two modular functions $f_1(\tau)$ and $f_2(\tau)$ satisfy the relation $F(f_1(\tau), f_2(\tau)) = 0$, where $F(X, Y)$ is a two variable polynomial with complex coefficients. Ishida and Ishii proved the following lemma, and this lemma shows which coefficients of $F(X, Y)$ are zeros [7].

**Lemma 2.2.** *For any congruence subgroup $\Gamma'$, let $f_1(\tau), f_2(\tau)$ be nonconstants such that $\mathbb{C}(f_1(\tau), f_2(\tau)) = A_0(\Gamma')$ with the total degree $D_k$ of poles of $f_k(\tau)$ for $k = 1, 2$. Let*
$$F(X, Y) = \sum_{\substack{0 \le i \le D_2 \\ 0 \le j \le D_1}} C_{i,j}X^iY^j \in \mathbb{C}[X, Y]$$

*be such that $F(f_1(\tau), f_2(\tau)) = 0$. Let $S_{\Gamma'}$ be a set of all the inequivalent cusps of $\Gamma'$, and for $k = 1, 2$,*

$$S_{k,0} = \{s \in S_{\Gamma'} : f_k(\tau) \text{ has zeros at } s\}$$

*and*

$$S_{k,\infty} = \{s \in S_{\Gamma'} : f_k(\tau) \text{ has poles at } s\}.$$

*Further let*

$$a = -\sum_{s \in S_{1,\infty} \cap S_{2,0}} ord_s f_1(\tau) \quad \text{and} \quad b = \sum_{s \in S_{1,0} \cap S_{2,0}} ord_s f_1(\tau).$$

*We assume that $a$ (respectively, $b$) is zero if $S_{1,\infty} \cap S_{2,0}$ (respectively, $S_{1,0} \cap S_{2,0}$) is empty. Then we obtain the following assertions:*

(1) *$C_{D_2,a} \neq 0$. If further $S_{1,\infty} \subset S_{2,\infty} \cup S_{2,0}$, then $C_{D_2,j} = 0$ for any $j \neq a$.*
(2) *$C_{0,b} \neq 0$. If further $S_{1,0} \subset S_{2,\infty} \cup S_{2,0}$, then $C_{0,j} = 0$ for any $j \neq b$.*
(3) *$C_{i,D_1} = 0$ for any $i$ satisfying that $0 \leq i < |S_{1,0} \cap S_{2,\infty}|$ or $D_2 - |S_{1,\infty} \cap S_{2,\infty}| < i \leq D_2$.*
(4) *$C_{i,0} = 0$ for any $i$ satisfying that $0 \leq i < |S_{1,0} \cap S_{2,0}|$ or $D_2 - |S_{1,\infty} \cap S_{2,0}| < i \leq D_2$.*
(5) *Suppose further that there exist $r \in \mathbb{R}$ and $N, n_1, n_2 \in \mathbb{Z}$ with $N > 0$ such that $f_k(\tau + r) = \zeta_N^{n_k} f_k(\tau)$ for $k = 1, 2$, where $\zeta_N = e^{2\pi i/N}$. Then for $i, j$ satisfying $n_1 i + n_2 j \not\equiv n_1 D_2 + n_2 a \pmod{N}$, we have $C_{i,j} = 0$.*

*If we interchange the roles of $f_1$ and $f_2$, then we obtain further properties similar to (1) through (4).*

## 3. Modular functions of level 13

Ramanujan studied $r_{13}(\tau)$ with the following six complex valued functions $\mu_1(\tau), \cdots, \mu_6(\tau)$ on $\mathfrak{H}$:

$$\begin{aligned}
\mu_1(\tau) &:= q^{-\frac{7}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-9})(1-q^{13n-4})}{(1-q^{13n-11})(1-q^{13n-2})},\\
\mu_2(\tau) &:= q^{-\frac{6}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-7})(1-q^{13n-6})}{(1-q^{13n-10})(1-q^{13n-3})},\\
\mu_3(\tau) &:= q^{-\frac{5}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-11})(1-q^{13n-2})}{(1-q^{13n-12})(1-q^{13n-1})},\\
\mu_4(\tau) &:= q^{-\frac{2}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-8})(1-q^{13n-5})}{(1-q^{13n-9})(1-q^{13n-4})},\\
\mu_5(\tau) &:= q^{\frac{5}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-10})(1-q^{13n-3})}{(1-q^{13n-8})(1-q^{13n-5})},\\
\mu_6(\tau) &:= q^{\frac{15}{13}} \prod_{n=1}^{\infty} \frac{(1-q^{13n-12})(1-q^{13n-1})}{(1-q^{13n-7})(1-q^{13n-6})}.
\end{aligned}$$

**Lemma 3.1.** [1, Entry 8 (i)]

(1)
$$\mu_1(\tau)-\mu_2(\tau)-\mu_3(\tau)+\mu_4(\tau)-\mu_5(\tau)+\mu_6(\tau)=-1+q^{-\frac{7}{13}}\prod_{n=1}^{\infty}\frac{(1-q^{\frac{n}{13}})}{(1-q^{13n})},$$
(2)
$$\mu_1(\tau)\mu_2(\tau)-\mu_3(\tau)\mu_5(\tau)-\mu_4(\tau)\mu_6(\tau)=1+q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{13n})^2},$$
(3)
$$\frac{1}{\mu_1(\tau)\mu_2(\tau)}-\frac{1}{\mu_3(\tau)\mu_5(\tau)}-\frac{1}{\mu_4(\tau)\mu_6(\tau)}=-4-q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{13n})^2},$$
(4)
$$\mu_2(\tau)\mu_3(\tau)\mu_4(\tau)-\mu_1(\tau)\mu_5(\tau)\mu_6(\tau)=3+q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{13n})^2},$$
(5)
$$\mu_1(\tau)\mu_2(\tau)\mu_3(\tau)\mu_4(\tau)\mu_5(\tau)\mu_6(\tau)=1.$$

*Proof.* See [1, p.372-375]. □

*Remark* 3.2. (1) By comparing the infinite products, we get

(3.1) $$r_{13}(\tau)=\mu_1(\tau)\mu_5(\tau)\mu_6(\tau).$$

(2) The property in (4) of Lemma 3.1 is exactly the same as (1.4).

For simplicity, throughout this paper we denote $f(\tau)$ by
$$f(\tau):=q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^{13n})^2}.$$

**Proposition 3.3.** *Let $\zeta_N=e^{2\pi i/N}$. We can write the functions $\mu_j$ $(j=1,\cdots,6)$ by the following finite product of Klein forms:*

(1) $\mu_1(\tau)=\zeta_{13}^{7}\prod_{s=0}^{12}K_{(4/13,s/13)}(\tau)K^{-1}_{(2/13,s/13)}(\tau)$,
(2) $\mu_2(\tau)=\zeta_{13}^{4}\prod_{s=0}^{12}K_{(6/13,s/13)}(\tau)K^{-1}_{(3/13,s/13)}(\tau)$,
(3) $\mu_3(\tau)=\zeta_{13}^{10}\prod_{s=0}^{12}K_{(2/13,s/13)}(\tau)K^{-1}_{(1/13,s/13)}(\tau)$,
(4) $\mu_4(\tau)=\zeta_{13}^{10}\prod_{s=0}^{12}K_{(5/13,s/13)}(\tau)K^{-1}_{(4/13,s/13)}(\tau)$,
(5) $\mu_5(\tau)=\zeta_{13}^{6}\prod_{s=0}^{12}K_{(3/13,s/13)}(\tau)K^{-1}_{(5/13,s/13)}(\tau)$,
(6) $\mu_6(\tau)=\zeta_{13}^{2}\prod_{s=0}^{12}K_{(1/13,s/13)}(\tau)K^{-1}_{(6/13,s/13)}(\tau)$.

*Moreover, all $\mu_1(\tau),\ldots,\mu_6(\tau)$ are modular functions on $\Gamma(13)$.*

*Proof.* By (K4), the product of Klein forms has an infinite product. We have
$$\prod_{s=0}^{12}K_{(\frac{r}{13},\frac{s}{13})}(\tau)=(-2\pi i)^{-13}\zeta_{13}^{3r}q^{\frac{r(r-13)}{26}}\prod_{n=1}^{\infty}\frac{(1-q^{13n-r})(1-q^{13n-(13-r)})}{(1-q^n)^{26}}$$
and
$$\prod_{s=0}^{12}\frac{K_{(\frac{r_1}{13},\frac{s}{13})}(\tau)}{K_{(\frac{r_2}{13},\frac{s}{13})}(\tau)}=\zeta_{13}^{3(r_1-r_2)}q^{\frac{1}{26}(r_1-r_2)(r_1+r_2-13)}\prod_{n=1}^{\infty}\frac{(1-q^{13n-r_1})(1-q^{13n-(13-r_1)})}{(1-q^{13n-r_2})(1-q^{13n-(13-r_2)})}.$$

By the property (K5) of Klein forms, $\prod_{s=0}^{12} K_{(r_1/13,s/13)}(\tau)K^{-1}_{(r_2/13,s/13)}(\tau)$ is a modular function on $\Gamma(13)$. Comparing this with the definition of $\mu_j$, the result follows immediately. □

By the following lemma we can see the existence of an affine plane model defined over $\mathbb{Q}$, which is called the *modular equation* in this paper.

**Lemma 3.4.** *Let $n$ be a positive integer. Then we have*

$$\mathbb{Q}(f(\tau), f(n\tau)) = A_0(\Gamma_0(13n))_{\mathbb{Q}}.$$

*Proof.* By Theorem 1.1, $\mathbb{Q}(f(\tau)) = A_0(\Gamma_0(13))_{\mathbb{Q}}$. For any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, $f(\alpha\tau) = f(\tau)$ means that $\alpha \in \mathbb{Q}^\times \cdot \Gamma_0(13)$. For $\beta = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\Gamma_0(13) \cap \beta^{-1}\Gamma_0(13)\beta = \Gamma_0(13n).$$

So it is clear that $f(\tau), f(n\tau) \in A_0(\Gamma_0(13) \cap \Gamma_0(13n))_{\mathbb{Q}}$.

Thus it is enough to show that $\mathbb{Q}(f(\tau), f(n\tau)) \supset A_0(\Gamma_0(13n))_{\mathbb{Q}}$. Let $\Gamma'$ be the subgroup of $\Gamma(1)$ satisfying $\mathbb{Q}(f(\tau), f(n\tau)) = A_0(\Gamma')_{\mathbb{Q}}$, and let $\gamma$ be an element of $\Gamma'$. Since $\mathbb{Q}(f(\tau)) = A_0(\Gamma_0(13))_{\mathbb{Q}}$ and $f(\tau)$ is invariant under $\gamma$, we can see that $\gamma \in \Gamma_0(13)$. Since $f(n\tau)$ is also invariant under $\gamma$, $f(\beta\gamma\tau) = f(\beta\tau)$ and $f(\tau)$ is invariant under $\beta\gamma\beta^{-1}$. In other words, $\beta\gamma\beta^{-1} \in \Gamma_0(13)$ and $\gamma \in \beta^{-1}\Gamma_0(13)\beta$. Hence $\Gamma' \subset \Gamma_0(13) \cap \beta^{-1}\Gamma_0(13)\beta$ and $\mathbb{Q}(f(\tau), f(n\tau)) = A_0(\Gamma')_{\mathbb{Q}} \supset A_0(\Gamma_0(13) \cap \Gamma_0(13n))_{\mathbb{Q}}$. □

**Lemma 3.5.** *$f(\tau)$ has a simple pole at $\infty$ and has a simple zero at 0.*

*Proof.* By Lemma 2.1, the group $\Gamma_0(13)$ has two inequivalent cusps $\infty$ and 0 with width 1 and 13, respectively. Since $f(\tau) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{13n})^{-2} = q^{-1} + O(1)$, we can see that $f(\tau)$ has a simple pole at $\infty$. To observe the behavior at 0, we note

$$f \mid \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)(\tau) = q^{1/13} + O(1).$$

By considering the width 13 at 0, $\mathrm{ord}_0 f(\tau) = 1$ and we find that $f(\tau)$ has a simple zero at 0. □

Now we suggest an efficient algorithm for finding modular equation.

**Lemma 3.6.** *Let $a, c, a', c' \in \mathbb{Q}$ and $f(\tau) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{13n})^{-2}$. Then we obtain the following assertions:*

(1) *$f(\tau)$ has a pole at $a/c \in \mathbb{Q}\cup\{\infty\}$ with $(a,c) = 1$ if and only if $(a,c) = 1$ and $c \equiv 0 \pmod{13}$.*
(2) *$f(n\tau)$ has a pole at $a'/c' \in \mathbb{Q}\cup\{\infty\}$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a,c) = 1$, and $c \equiv 0 \pmod{13}$.*
(3) *$f(\tau)$ has a zero at $a/c \in \mathbb{Q}\cup\{\infty\}$ with $(a,c) = 1$ if and only if $(a,c) = 1$ and $(c, 13) = 1$.*
(4) *$f(n\tau)$ has a zero at $a'/c' \in \mathbb{Q}\cup\{\infty\}$ if and only if there exist $a, c \in \mathbb{Q}$ such that $a/c = na'/c'$, $(a,c) = 1$, and $(c, 13) = 1$.*

*Proof.* (1) By Lemma 2.1, $f(\tau)$ has a simple pole at $a/c$ if and only if there exists $\overline{s} \in (\mathbb{Z}/13\mathbb{Z})^\times$ such that $\begin{pmatrix} a \\ c \end{pmatrix} \equiv \begin{pmatrix} \overline{s} \\ 0 \end{pmatrix} \pmod{13}$. This is equivalent to the condition that $a, c \in \mathbb{Z}$ such that $(a, 13) = 1$ and $c \equiv 0 \pmod{13}$.

(3) We note that any $a/c \in \mathbb{Q}\cup\{\infty\}$ is equivalent to either $\infty$ or 0. If $f(\tau)$ has a zero at $a/c \in \mathbb{Q}\cup\{\infty\}$, $a/c$ is not equivalent to $\infty$, $(a,c) = 1$, and $(13, c) = 1$. If $a, c \in \mathbb{Z}$ satisfying

$(a,c)=1$ and $(13,c)=1$, then $a/c\in\mathbb{Q}$ is not equivalent to $\infty$ but equivalent to 0; this shows that $f(\tau)$ has a zero at such $a/c$.

(2) and (4) are obtained by (1) and (3), respectively. □

## 4. Proofs of main theorems

Now we prove Theorem 1.1 using Lemma 3.1 and Proposition 3.3 in the previous section.

***Proof of Theorem 1.1.*** Note that $\Gamma_1(13)=<\Gamma(13),\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)>$ and $\Gamma_0(13)=<\Gamma_1(13),\left(\begin{smallmatrix}2&1\\13&7\end{smallmatrix}\right)>$.

By Proposition 3.3 since all $\mu_1(\tau),\ldots,\mu_6(\tau)$ are modular functions on $\Gamma(13)$ and

$$r_{13}(\tau)=\mu_1(\tau)\mu_5(\tau)\mu_6(\tau)=\frac{1}{\mu_2(\tau)\mu_3(\tau)\mu_4(\tau)},$$

$r_{13}(\tau)$ is also a modular function on $\Gamma(13)$. In fact, $q=e^{2\pi i\tau}$ is invariant under the action $\tau\to\tau+1$, $r_{13}(\tau+1)=r_{13}(\tau)$; thus, $r_{13}(\tau)$ is a modular function on $\Gamma_1(13)$.

For the second statement, it is sufficient to prove that

$$A_0(\Gamma_0(13))_{\mathbb{Q}}=\mathbb{Q}(f(\tau))$$

with $f(\tau)=1/r_{13}(\tau)-3-r_{13}(\tau)$.

Let $\gamma=\left(\begin{smallmatrix}2&1\\13&7\end{smallmatrix}\right)$. Then the action of $\gamma$ on the product of Klein forms are

$$\begin{aligned}
\prod_{s=0}^{12}K_{(\frac{1}{13},\frac{s}{13})}(\gamma\tau) &= \zeta_{26}^{9}(13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{2}{13},\frac{s}{13})}(\tau),\\
\prod_{s=0}^{12}K_{(\frac{2}{13},\frac{s}{13})}(\gamma\tau) &= \zeta_{13}^{11}(13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{4}{13},\frac{s}{13})}(\tau),\\
\prod_{s=0}^{12}K_{(\frac{3}{13},\frac{s}{13})}(\gamma\tau) &= -(13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{6}{13},\frac{s}{13})}(\tau),\\
\prod_{s=0}^{12}K_{(\frac{4}{13},\frac{s}{13})}(\gamma\tau) &= (13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{5}{13},\frac{s}{13})}(\tau),\\
\prod_{s=0}^{12}K_{(\frac{5}{13},\frac{s}{13})}(\gamma\tau) &= \zeta_{26}^{23}(13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{3}{13},\frac{s}{13})}(\tau),\\
\prod_{s=0}^{12}K_{(\frac{6}{13},\frac{s}{13})}(\gamma\tau) &= \zeta_{13}^{12}(13\tau+7)^{-13}\prod_{s=0}^{12}K_{(\frac{1}{13},\frac{s}{13})}(\tau),
\end{aligned}$$

and

$$\begin{aligned}
\mu_1(\gamma\tau)&=\zeta_{13}^{12}\mu_4(\tau), & \mu_2(\gamma\tau)&=\zeta_{26}^{15}\mu_6(\tau),\\
\mu_3(\gamma\tau)&=\zeta_{26}^{19}\mu_1(\tau), & \mu_4(\gamma\tau)&=\zeta_{26}^{5}\mu_5(\tau),\\
\mu_5(\gamma\tau)&=\zeta_{13}^{10}\mu_2(\tau), & \mu_6(\gamma\tau)&=\zeta_{26}^{21}\mu_3(\tau).
\end{aligned}$$

Hence the actions of $\gamma$ on $r_{13}(\tau)$ and $f(\tau)$ are

$$r_{13}(\gamma\tau)=\mu_1(\gamma\tau)\mu_5(\gamma\tau)\mu_6(\gamma\tau)=-\mu_2(\tau)\mu_3(\tau)\mu_4(\tau)=-\frac{1}{r_{13}(\tau)}$$

and

$$f(\gamma\tau) = \frac{1}{r_{13}(\gamma\tau)} - 3 - r_{13}(\gamma\tau) = -r_{13}(\tau) - 3 + \frac{1}{r_{13}(\tau)} = f(\tau).$$

Hence $f(\tau) \in A_0(\Gamma_0(13))$. Moreover, by the fact that the genus of $\Gamma_0(13)$ is zero, we have $A_0(\Gamma_0(13)) = \mathbb{C}(f(\tau))$. Since the Fourier coefficients of $f$ belong to $\mathbb{Q}$, we get $A_0(\Gamma_0(13))_{\mathbb{Q}} = \mathbb{Q}(f(\tau)) = \mathbb{Q}(1/r_{13}(\tau) - r_{13}(\tau))$. □

Let $d_1$ (respectively, $d_n$) be the total degree of poles of $f(\tau)$ (respectively, $f(n\tau)$). Let $F_n(X,Y)$ be a polynomial such that

$$F_n(X,Y) = \sum_{\substack{0\le i\le d_n\\ 0\le j\le d_1}} C_{i,j}X^iY^j \in \mathbb{Q}[X,Y]$$

and $F_n(f(\tau), f(n\tau)) = 0$. From Lemma 2.2, for any prime $p \neq 13$, we can remove $4p+2$ coefficients of modular equations of level $p$.

**Theorem 4.1.** *Let $p$ be any prime $p \neq 13$ and $F_n(X,Y)$ be the polynomial satisfying $F_n(f(\tau), f(n\tau)) = 0$. Then*

$$F_p(X,Y) = \sum_{0\le i,j\le p+1} C_{i,j}X^iY^j \in \mathbb{Q}[X,Y]$$

*and*

(1) $C_{p+1,0} \neq 0$ *and* $C_{0,p+1} \neq 0$.
(2) *For* $j = 0, \ldots p$, $C_{0,j} = 0$ *and* $C_{j,0} = 0$.
(3) *For* $j = 1, \ldots p+1$, $C_{p+1,j} = 0$ *and* $C_{j,p+1} = 0$.

*Proof.* Note that $S_{\Gamma_0(13p)} = \{\infty, 0, 1/p, 1/13\}$ is the set of inequivalent cusps of $\Gamma_0(13p)$. Let $f_1(\tau) = f(\tau)$ and $f_2(\tau) = f(p\tau)$. Since $\mathbb{C}(f_1(\tau), f_2(\tau)) = A_0(\Gamma_0(13n))$, $f_1(\tau)$ and $f_2(\tau)$ have poles at $\infty, 1/13$ and zeros at $0, 1/p$ with

$$\mathrm{ord}_\infty f_1(\tau) = \mathrm{ord}_{1/13} f_2(\tau) = -1, \quad \text{and } \mathrm{ord}_{1/13} f_1(\tau) = \mathrm{ord}_\infty f_2(\tau) = -p.$$

Hence, we have $F_p(X,Y) = \sum_{0\le i,j\le p+1} C_{i,j}X^iY^j$. Additionally,

$$\mathrm{ord}_0 f_1(\tau) = \mathrm{ord}_{1/p} f_2(\tau) = p \text{ and } \mathrm{ord}_{1/p} f_1(\tau) = \mathrm{ord}_0 f_2(\tau) = 1.$$

From Lemma 2.2, the fact that $a = 0$ and $b = p+1$ implies that both $C_{p+1,a} = C_{p+1,0}$ and $C_{0,b} = C_{0,p+1}$ are nonzero. Moreover, $C_{p+1,j} = 0$ for $j \neq 0$ and $C_{0,j} = 0$ for $j \neq p+1$.

On the other hand, let $f_1(\tau) = f(p\tau)$ and $f_2(\tau) = f(\tau)$. Assume that $F'_p(X,Y) = \sum_{0\le i,j\le p+1} C'_{i,j}X^iY^j$ is a polynomial satisfying $F'_p(f(p\tau), f(\tau)) = 0$. Then $C'_{i,j} = C_{j,i}$, $a = 0$ and $b = p+1$ by Lemma 2.2. By using Lemma 2.2, we get $C'_{p+1,j} = C_{j,p+1} = 0$ for $j = 1, \ldots, p+1$ and $C'_{0,j} = C_{j,0} = 0$ for $j = 0, \ldots p$. □

We want to point out that the following proof of Theorem 1.2 presents a constructive way of finding the modular equation of $r_{13}(\tau)$.

***Proof of Theorem 1.2.*** If $\mathbb{C}(f_1(\tau), f_2(\tau))$ is the field of all modular functions on some congruence subgroup for which $f_1(\tau)$ and $f_2(\tau)$ are nonconstants, then $[\mathbb{C}(f_1(\tau), f_2(\tau)) : \mathbb{C}(f_i(\tau))]$ is the total degree $d_i$ of poles of $f_i(\tau)$ for $i = 1, 2$. Therefore, there exists a polynomial $\Phi(X,Y) \in \mathbb{C}[X,Y]$ such that $\Phi(f_1(\tau), Y)$ is a minimal polynomial of $f_2(\tau)$ over $\mathbb{C}(f_1(\tau))$ with degree $d_1$, and similarly so is $\Phi(X, f_2(\tau))$ with degree $d_2$. Let $f_1(\tau) = f(\tau) = q^{-1}\prod_{m=1}^{\infty}(1-q^m)^2(1-q^{13m})^{-2}$ and $f_2(\tau) = f(n\tau)$. Then by Lemma 2.2, for every positive integer $n$, we can consider a polynomial $F_n(X,Y) \in \mathbb{Q}[X,Y]$ such that $F_n(f(\tau), f(n\tau)) = 0$

with $\deg_X F_n(X,Y) = d_2$ and $\deg_Y F_n(X,Y) = d_1$. We thus get the modular equation $F_n(X,Y)$ of $f(\tau)$ for every positive integer level $n$. Since $f(\tau) = 1/r_{13}(\tau) - 3 - r_{13}(\tau)$,

$$\widehat{\mathcal{F}}_n(X,Y) := X^{d_2}Y^{d_1} \times F_n\left(\frac{1}{X} - 3 - X, \frac{1}{Y} - 3 - Y\right)$$

is a polynomial with $\widehat{\mathcal{F}}_n(r_{13}(\tau), r_{13}(n\tau)) = 0$. After factorizing the polynomial $\widehat{\mathcal{F}}_n(X,Y)$, one can choose exactly one irreducible factor $\mathcal{F}_n(X,Y)$ of $\widehat{\mathcal{F}}_n(X,Y)$ satisfying $\mathcal{F}_n\ (r_{13}(\tau), r_{13}(n\tau)) = 0$. In fact, this $\mathcal{F}_n(X,Y)$ is a modular equation of $r_{13}(\tau)$ of level $n$ for a positive integer $n$. □

We have to calculate the modular equation of level 13 separately since Theorem 4.1 does not cover the level 13 case. For a modular function $f_j(\tau)$ of level $N$, let $S_{j,0}$ (respectively, $S_{j,\infty}$) be the subset of the set $S_{\Gamma_0(N)}$ of inequivalent cusps of $\Gamma_0(N)$ satisfying $\mathrm{ord}_s f_j(\tau) > 0$ (respectively, $\mathrm{ord}_s f_j(\tau) < 0$) (as defined in Lemma 2.2). To find the modular equation of level 13, the subsets $S_{j,0}$ and $S_{j,\infty}$ play important roles. When the modular equation is of level 13, the cusps of congruence subgroup $\Gamma_0(169)$ and the behaviors of $f(\tau)$ and $f(13\tau)$ at the cusps are all different from the ones of $\Gamma_0(13p)$, $f(\tau)$ and $f(p\tau)$ as $p \neq 13$.

**Theorem 4.2** (**A modular equation of level 13**). *We explicitly obtain the modular equation $\mathcal{F}_{13}(X,Y)$ of $r_{13}(\tau)$ with level 13 as given in Appendix B.*

*Proof.* Let $S_{\Gamma_0(169)}$ be the set of inequivalent cusps.

Let $f_1(\tau) = f(\tau) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{13n})^{-2}$ and $f_2(\tau) = f(13\tau)$ in Lemma 2.2.

Then we may write

$$S_{\Gamma_0(169)} = \{\infty, 0, 1/13, 2/13, \ldots, 12/13\}.$$

Here the useful subsets $S_{j,\infty}$ and $S_{j,0}$ $(j = 1, 2)$ of $S_{\Gamma_0(169)}$ are

$$S_{1,\infty} = \left\{\infty, \frac{1}{13}, \ldots, \frac{12}{13}\right\}, S_{1,0} = \{0\} \text{ and}$$

$$S_{2,\infty} = \{\infty\}, S_{2,0} = \left\{0, \frac{1}{13}, \ldots, \frac{12}{13}\right\},$$

where

$$S_{j,\infty} := \left\{s \in S_{\Gamma_0(169)} : f_j(\tau) \text{ has a pole at } s\right\}$$

and

$$S_{j,0} := \left\{s \in S_{\Gamma_0(169)} : f_j(\tau) \text{ has a zero at } s\right\}.$$

Since $\mathrm{ord}_\infty f_1(\tau) = \mathrm{ord}_{l/13} f_1(\tau) = -1$ $(l = 1, \ldots, 12)$ and $\mathrm{ord}_\infty f_2(\tau) = -13$, the modular equation of level 13 is

$$F_{13}(X,Y) = \sum_{0,\leq i,j \leq 13} C_{i,j} X^i Y^j.$$

Since

$$-\sum_{s \in S_{1,\infty} \cap S_{2,0}} \mathrm{ord}_s f_1(\tau) = 12 \text{ and } \sum_{s \in S_{1,0} \cap S_{2,0}} \mathrm{ord}_s f_1(\tau) = 13,$$

we have the following:

(1) $C_{13,12}$ and $C_{0,13}$ are nonzero.
(2) $C_{13,j} = 0$ for all $j = 0, \ldots 11, 13$.
(3) $C_{0,j} = 0$ for all $j = 0, \ldots, 12$.

By switching the roles of $f_1$ and $f_2$, let $f_1(\tau) = f(13\tau)$ and $f_2(\tau) = f(\tau)$. Then

$$S_{1,\infty} = \{\infty\}, S_{1,0} = \left\{0, \frac{1}{13}, \ldots, \frac{12}{13}\right\}$$

$$S_{2,\infty} = \left\{\infty, \frac{1}{13}, \ldots, \frac{12}{13}\right\}, \text{ and } S_{2,0} = \{0\}.$$

A similar computation as above shows the following:

(1) $C_{0,13}$ and $C_{1,0}$ are nonzero.
(2) $C_{j,13} = 0$ for $j = 1, 2, \ldots, 13$.
(3) $C_{j,0} = 0$ for $j = 0, 2, \ldots, 13$.

Assume that $C_{0,13} = 1$. By substituting the $q$-expansion of $f(\tau)$ and $f(13\tau)$, we get the modular equation $F_{13}(X,Y)$ in Appendix A.

For obtaining $\mathcal{F}_{13}(X,Y)$, let

$$\widehat{\mathcal{F}}_{13}(X,Y) = X^{13}Y^{13}F_{13}(1/X - 3 - X, 1/Y - 3 - Y).$$

This polynomial has two irreducible factors $\mathcal{F}_{13}(X,Y)$ and $\mathcal{F}'_{13}(X,Y)$, where the lowest term of $\mathcal{F}'_{13}(X,Y)$ is 1. Since $\mathcal{F}'_{13}(r_{13}(\tau), r_{13}(13\tau)) = 1 + O(q)$, it cannot be zero. It thus follows that $\mathcal{F}_{13}(X,Y)$ is the modular equation of $r_{13}(\tau)$ with level 13. □

In Appendix A, the modular equations of $f(\tau)$ satisfy the congruence relation

$$(X^p - Y)(X - Y^p) \pmod{p}$$

for $p = 2, 3, 5, 7, 11$ and 17. This property is called *Kronecker's congruence*. We discuss some properties of modular equations including the Kronecker's congruence.

Consider $\Gamma = \Gamma_0(13)$. For any integer $a$ with $(a, 13) = 1$, we choose $\sigma_a \in \Gamma(1)$ so that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \pmod{13}$ and $\sigma_a \in \Gamma_0(13)$. For example, we may take $\sigma_a$ as

$$\sigma_{\pm 1} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_{\pm 2} = \pm\begin{pmatrix} 85 & 13 \\ 13 & 2 \end{pmatrix}, \sigma_{\pm 3} = \pm\begin{pmatrix} 113 & 26 \\ 13 & 3 \end{pmatrix},$$

$$\sigma_{\pm 4} = \pm\begin{pmatrix} 10 & 13 \\ 13 & 17 \end{pmatrix}, \sigma_{\pm 5} = \pm\begin{pmatrix} 34 & 13 \\ 13 & 5 \end{pmatrix}, \sigma_{\pm 6} = \pm\begin{pmatrix} -28 & -13 \\ 13 & 6 \end{pmatrix}.$$

For every integer $n$ with $(n, 13) = 1$, one has

$$\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma = \bigsqcup_{\substack{a>0 \\ a|n}} \bigsqcup_{0 \le b < \frac{n}{a}} \Gamma\sigma_a \begin{pmatrix} a & b \\ 0 & \frac{n}{a} \end{pmatrix}$$

with disjoint union and $|\Gamma\backslash\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \Gamma| = n\prod_{p|n}(1 + 1/p)$ by [10, Proposition 3.36].

Consider the polynomial

$$\Psi_n(X,\tau) := \prod_{0<a|n} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} (X - (f \circ \alpha_{a,b})(\tau))$$

with degree $n\prod_{p|n}(1 + 1/p)$, where $\alpha_{a,b} = \sigma_a \left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right)$.

The coefficients of $\Psi_n(X,\tau)$ are elementary symmetric functions of $f \circ \alpha_{a,b}$ and invariant under $\Gamma$. In other words, these are in $A_0(\Gamma) = \mathbb{C}(f(\tau))$ and $\Psi_n(X,\tau) \in \mathbb{C}(f(\tau))[X]$. Therefore, we may write $\Psi_n(X, f(\tau))$ instead of $\Psi_n(X,\tau)$. By observing $\alpha_{n,0} = \sigma_n \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have $(f \circ \alpha_{n,0})(\tau) = f(n\tau)$ and $\Psi_n(f(n\tau), f(\tau)) = 0$. When $f_1(\tau) = f(n\tau)$ and $f_2(\tau) = f(\tau)$, we define $S_{j,\infty}$ to be the set of cusps which are poles of $f_j(\tau)$. By $S_{j,0}$ we mean the set of cusps where $f_j(\tau)$ has zero. From Lemma 2.2, we recall that $a$ is a nonnegative integer determined by the order $\mathrm{ord}_s f_1(\tau)$ of $f_1(\tau)$ at the cusp $s$ in $S_{1,\infty} \cap S_{2,0}$. If we multiply $\Psi_n(X, f(\tau))$ by a suitable power of $f(\tau)$, we have a polynomial $F_n(X, f(\tau)) \in \mathbb{C}[X, f(\tau)]$ such that $F_n(f(n\tau), f(\tau)) = 0$. By Lemma 3.6 since $S_{1,\infty} \cap S_{2,0} = \phi$, we may take $a = 0$. We thus will regard $\Psi_n(X, f(\tau))$ as a polynomial of $X$ and $f(\tau)$ for proving the following theorem.

**Theorem 4.3.** *With the notation as above, for a positive integer $n$ relatively prime to 13, let $\Psi_n(X,Y)$ be a polynomial such that $\Psi_n(f(\tau), f(n\tau)) = 0$. Then we get the following assertions:*

1. $\Psi_n(X,Y) \in \mathbb{Z}[X,Y]$ *and* $\deg_X \Psi_n(X,Y) = \deg_Y \Psi_n(X,Y) = n\prod_{p|n}(1+1/p)$.
2. $\Psi_n(X,Y)$ *is irreducible both as a polynomial in $X$ over $\mathbb{C}(Y)$ and as a polynomial in $Y$ over $\mathbb{C}(X)$.*
3. $\Psi_n(X,Y) = \Psi_n(Y,X)$.
4. *If $n$ is not a square, $\Psi_n(X,X)$ is a polynomial of degree $> 1$ whose leading coefficient is $\pm 1$.*
5. *(Kronecker's congruence) Let $p$ be an odd prime. Then*
$$\Psi_p(X,Y) \equiv (X^p - Y)(X - Y^p) \pmod{p\mathbb{Z}[X,Y]}.$$

*Proof.* We write $\Gamma$ for $\Gamma_0(13)$. We note that $f(\tau) = q^{-1}\prod_{m=1}^{\infty}(1-q^m)^2(1-q^{13m})^{-2}$ has a Fourier expansion

$$f(\tau) = q^{-1} + \sum_{m=0}^{\infty} c_m q^m, \quad \text{where } c_m \in \mathbb{Z},$$

and let $\psi_k$ be an automorphism of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ such as $\psi_k(\zeta_n) = \zeta_n^k$ for $k$ relatively prime to $n$. The action of $\left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right)$ on $f$ is

$$\begin{aligned}\left(f \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}\right)(\tau) &= f\left(\frac{a\tau + b}{n/a}\right) = f\left(\frac{a^2\tau + ab}{n}\right) \\ &= \zeta_n^{-ab} q^{-\frac{a^2}{n}} + \sum_{m=0}^{\infty} c_m \zeta_n^{abm} q^{\frac{a^2 m}{n}}.\end{aligned}$$

Then $\psi_k$ induces an automorphism $\psi_k$ of $\mathbb{Q}(\zeta_n)((q^{1/n}))$ over $\mathbb{Q}(\zeta_n)$ :

$$\psi_k\left(f \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}(\tau)\right) = \zeta_n^{-abk} q^{-\frac{a^2}{n}} + \sum_{m=0}^{\infty} c_m \zeta_m^{abkm} q^{\frac{a^2 m}{n}}.$$

Choosing $0 \le b' < n/a$ such that $b' \equiv bk \pmod{n/a}$, we have $ab' \equiv abk \pmod{n}$ and

$$\begin{aligned}\psi_k(f \circ \alpha_{a,b}) &= \psi_k\left(f \circ \sigma_a \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}\right) = \psi_k\left(f \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}\right) \\ &= f \circ \begin{pmatrix} a & b' \\ 0 & n/a \end{pmatrix} = f \circ \sigma_a \begin{pmatrix} a & b' \\ 0 & n/a \end{pmatrix} = f \circ \alpha_{a,b'}.\end{aligned}$$

Hence

$$\psi_k(\Psi_n(X, f(\tau))) = \Psi_n(X, f(\tau))$$

and $\Psi_n(X, f(\tau)) \in \mathbb{Q}((q^{1/n}))[X]$. Furthermore, we have $\Psi_n(f(\tau/n), f(\tau)) = 0$ and $[\mathbb{C}(f(\tau/n), f(\tau)) : \mathbb{C}(f(\tau))] \le d$ since $(f \circ \alpha_{1,0})(\tau) = (f \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right))(\tau) = f(\tau/n)$ where $d = \prod_{p|n}(1+1/p)$.

For $a, b$, $\Gamma\alpha_{a,b} \subset \Gamma\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)\Gamma$, there exist $\gamma, \gamma'$ and $\gamma_{a,b}$ in $\Gamma$ such that

$$\gamma \begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix} \gamma_{a,b} = \gamma' \alpha_{a,b},$$

i.e., $\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)\gamma_{a,b}\alpha_{a,b}^{-1} \in \Gamma = \Gamma_0(13)$.

We consider an embedding $\xi_{a,b}$ of $\mathbb{C}(f(\tau/n), f(\tau))$ to the field of all meromorphic functions on $\mathfrak{H}$ containing $\mathbb{C}(f(\tau/n), f(\tau))$ over $\mathbb{C}(f(\tau))$ defined by

$$\xi_{a,b}(h) = h \circ \gamma_{a,b}.$$

In fact, $\xi_{a,b}(f) = f$ and

$$\begin{aligned}\xi_{a,b}\left(f(\tau/n)\right) &= \xi_{a,b}\left(f \circ \begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\right)(\tau)\\ &= \left(f \circ \begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\gamma_{a,b}\right)(\tau) = (f \circ \alpha_{a,b})(\tau).\end{aligned}$$

When $\alpha_{a,b} \neq \alpha_{a',b'}$, we have $f \circ \alpha_{a,b} \neq f \circ \alpha_{a',b'}$. This means that there exist distinct $d$ embeddings $\xi_{a,b}$ of $\mathbb{C}(f(\tau/n), f(\tau))$ over $\mathbb{C}(f(\tau))$ and

$$\left[\mathbb{C}\left(f\left(\frac{\tau}{n}\right), f(\tau)\right) : \mathbb{C}(f(\tau))\right] = d.$$

Therefore, $\Psi_n(X, f(\tau))$ is irreducible over $\mathbb{C}(f(\tau))$.

Let $F(X,Y)$ be the polynomial in Lemma 2.2. Let $f_1(\tau) = f(\tau)$ and $f_2(\tau) = f(n\tau)$, and let $d_1$ (respectively, $d_n$) be the total degree of $f(\tau)$ (respectively, $f(n\tau)$). Let $a = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s f(\tau)$. Then

$$F(X,Y) = C_{d_n,a}X^n + \sum_{\substack{0\le i<d_n\\ 0\le j\le d_1}} C_{i,j}X^iY^j.$$

Since $F(X, f(\tau))$ is the minimal polynomial of $f(\tau/n)$ over $\mathbb{C}(f(\tau))$ and $F(f(\tau/n), Y)$ is a minimal polynomial of $f(\tau)$ over $\mathbb{C}(f(\tau/n))$, we get

$$f(\tau)^a \Psi_n(X, f(\tau)) = \frac{F(X, f(\tau))}{C_{d_n,a}}.$$

In our case, $a = 0$, $F(X,Y) \in \mathbb{Z}[X,Y]$, and $\Psi_n(X,Y) \in \mathbb{Z}[X,Y]$; hence, (1) and (2) follow.

(3) We observe that $(f\circ\alpha_{n,0})(\tau) = f(n\tau)$. Since $\Psi_n(f(n\tau), f(\tau)) = 0$, $\Psi_n(f(\tau), f(\tau/n)) = 0$ and $f(\tau/n)$ is a root of $\Psi_n(f(\tau), X) = 0$. From that $\Psi_n(X, f(\tau)) \in \mathbb{Z}[X, f(\tau)]$ and $\Psi_n(X, f(\tau))$ is irreducible, there is a polynomial $g(X, f(\tau))$ such that

$$\Psi_n(f(\tau), X) = g(X, f(\tau))\Psi_n(X, f(\tau)).$$

By changing the places of variables and multiplying $g(X, f(\tau))$, we get

$$g(X, f(\tau))\Psi_n(X, f(\tau)) = g(X, f(\tau))g(f(\tau), X)\Psi_n(f(\tau), X),$$

which is just $\Psi_n(f(\tau), X)$; so, $g(X,Y)$ should be $\pm 1$.

Suppose that $g = -1$. Then $\Psi_n(f(\tau), X) = -\Psi_n(X, f(\tau))$ and by substituting $f(\tau)$ for $X$, we have $\Psi_n(f(\tau), f(\tau)) = -\Psi_n(f(\tau), f(\tau))$; so, $f(\tau)$ is a root of $\Psi_n(X, f(\tau)) = 0$ and the polynomial $X - f(\tau)$ divides $\Psi_n(X, f(\tau))$. However, $\Psi_n(X, f(\tau))$ is irreducible over $\mathbb{C}(f(\tau))$, so we get a contradiction. Therefore, $g = 1$ and $\Psi_n(f(\tau), X) = \Psi_n(X, f(\tau))$.

(4) Consider that

$$f(\tau) - (f \circ \alpha_{a,b})(\tau) = q^{-1} - \zeta_n^{-ab} q^{-\frac{a^2}{n}} + O(q^{\frac{1}{n}}). \tag{4.1}$$

If $n$ is not a square, the coefficient of (4.1) is a unit; hence, $\Psi_n(f(\tau), f(\tau))$ is a unit. Therefore, $\Psi_n(X, X)$ is a polynomial with constant term $\pm 1$.

(5) Let $p$ be an odd prime with $p \neq 13$. For $g(\tau), h(\tau) \in \mathbb{Z}[\zeta_p]((q^{1/p}))$, we write

$$g(\tau) \equiv h(\tau) \pmod{\alpha}$$

if $g(\tau) - h(\tau) \in \alpha\mathbb{Z}[\zeta_p]((q^{\frac{1}{n}}))$.

Consider the following for $f(\tau) = q^{-1} + \sum_{m=0}^{\infty} c_m q^m$:

$$\begin{aligned}
(f \circ \alpha_{1,b})(\tau) &= \zeta_p^{-b} q^{-\frac{1}{p}} + \sum_{m=0}^{\infty} c_m \zeta_p^{bm} q^{\frac{m}{p}} \\
&\equiv q^{-\frac{1}{p}} + \sum_{m=0}^{\infty} c_m q^{\frac{m}{p}} \pmod{1 - \zeta_p} \\
&= (f \circ \alpha_{1,0})(\tau), \\
(f \circ \alpha_{p,0})(\tau) &= q^{-p} + \sum_{m=0}^{\infty} c_m q^{pm} \\
&\equiv q^{-p} + \sum_{m=0}^{\infty} c_m^p q^{pm} \equiv f(\tau)^p \pmod{p}, \\
((f \circ \alpha_{1,0})(\tau))^p &= \left( q^{-\frac{1}{p}} + \sum_{m=0}^{\infty} c_f(m) q^{\frac{m}{p}} \right)^p \\
&\equiv q^{-1} + \sum_{m=0}^{\infty} c_f(m) q^m \pmod{1 - \zeta_p} \\
&= f(\tau).
\end{aligned}$$

We thus get

$$\begin{aligned}
\Psi_p(X, f(\tau)) &= \left[ \prod_{0 \leq b < p} (X - (f \circ \alpha_{1,b})(\tau)) \right] (X - (f \circ \alpha_{p,0})(\tau)) \\
&\equiv (X - (f \circ \alpha_{1,0})(\tau))^p (X - f(\tau)^p) \pmod{1 - \zeta_p} \\
&\equiv (X^p - (f \circ \alpha_{1,0})(\tau)^p)(X - f(\tau)^p) \pmod{1 - \zeta_p} \\
&\equiv (X^p - f(\tau))(X - f(\tau)^p) \pmod{1 - \zeta_p}
\end{aligned}$$

and

$$\sum_{\nu} \psi_\nu(f(\tau)) X^\nu := \Psi_p(X, f(\tau)) - (X^p - f(\tau))(X - f(\tau)^p) \in (1 - \zeta_p) \cdot \mathbb{Z}[X, f(\tau)].$$

By the fact that

$$\Psi_p(X, Y) - (X^p - Y)(X - Y^p) \in \mathbb{Z}[X, Y],$$

$\psi_\nu(f(\tau))$ belongs to $\mathbb{Z}$ and $(1 - \zeta_p)$ divides $\psi_\nu(f(\tau))$ in $\mathbb{Z}[\zeta_p]((q^{1/n}))$. Hence $\psi_\nu(f(\tau)) \in p\mathbb{Z}[f(\tau)]$ and (5) is proved. $\square$

To prove our last theorem, we need to find criteria for being modular units. Let $j(\tau)$ be the classical elliptic modular function. By definition a *modular unit* over $\mathbb{Z}$ is a modular function $h(\tau)$ of some level $N$ which is rational over $\mathbb{Q}(\zeta_N)$ such that $h(\tau)$ and $1/h(\tau)$ are integral over $\mathbb{Z}[j(\tau)]$.

**Lemma 4.4.** *Let $h(\tau)$ be a modular function of some level $N$ which is rational over $\mathbb{Q}(\zeta_N)$ for which $h(\tau)$ has neither zeros nor poles on $\mathfrak{H}$. If for every $\gamma \in SL_2(\mathbb{Z})$ the Fourier expansion of $h \circ \gamma$ has algebraic integer coefficients and the coefficient of the term of lowest degree is a unit, then $h(\tau)$ is a modular unit over $\mathbb{Z}$.*

*Proof.* One can refer to [8, Chapter 2, Lemma 2.1], which is proved by the theory of Shimura reciprocity law [10]. □

Let $h(\tau)$ be a modular unit over $\mathbb{Z}$ and $K$ be an imaginary quadratic field. Since it is well known that $j(\tau)$ is an algebraic integer for every $\tau \in K - \mathbb{Q}$, we can derive that for such $\tau$, $h(\tau)$ is an algebraic integer which is a unit. By observing this fact and the following elementary lemma, we derive the property of $r_{13}(\tau)$.

**Lemma 4.5.** *Let $p$ be a prime and $r, s \in \mathbb{Z}$ such that $(p, rs) = 1$. Then*

$$(1-\zeta_{p^n}^r)(1-\zeta_{p^n}^s)^{-1}$$

*is a unit of $\mathbb{Z}[\zeta_{p^n}]$.*

*Proof.* If $s \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, then $r \equiv st \pmod{p^n}$ for some $t \in \mathbb{Z}_{>0}$. So

$$\frac{1-\zeta_{p^n}^r}{1-\zeta_{p^n}^s} = \frac{1-\zeta_{p^n}^{st}}{1-\zeta_{p^n}^s} = 1 + \zeta_{p^n}^s + \cdots + \zeta_{p^n}^{s(t-1)} \in \mathbb{Z}[\zeta_{p^n}].$$

Similarly, $(1-\zeta_{p^n}^s)(1-\zeta_{p^n}^r)^{-1} \in \mathbb{Z}[\zeta_{p^n}]$. □

***Proof of Theorem 1.3.*** It is enough to prove that $r_{13}(\tau)$ is a modular unit over $\mathbb{Z}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. By (3.1) and Proposition 3.3, we get

$$r_{13}(\tau) = \zeta_{13}^2 \prod_{j=0}^{12} \frac{K_{\left(\frac{1}{13},\frac{j}{13}\right)}(\tau) K_{\left(\frac{3}{13},\frac{j}{13}\right)}(\tau) K_{\left(\frac{4}{13},\frac{j}{13}\right)}(\tau)}{K_{\left(\frac{2}{13},\frac{j}{13}\right)}(\tau) K_{\left(\frac{5}{13},\frac{j}{13}\right)}(\tau) K_{\left(\frac{6}{13},\frac{j}{13}\right)}(\tau)}.$$

By (K1) in Section 2, the action $\gamma$ on $r_{13}(\tau)$ is

$$r_{13}(\gamma\tau) = \zeta_{13}^2 \prod_{j=0}^{12} \frac{K_{\left(\frac{a+cj}{13},\frac{b+dj}{13}\right)}(\tau) K_{\left(\frac{3a+cj}{13},\frac{3b+dj}{13}\right)}(\tau) K_{\left(\frac{4a+cj}{13},\frac{4b+dj}{13}\right)}(\tau)}{K_{\left(\frac{2a+cj}{13},\frac{2b+dj}{13}\right)}(\tau) K_{\left(\frac{5a+cj}{13},\frac{5b+dj}{13}\right)}(\tau) K_{\left(\frac{6a+cj}{13},\frac{6b+dj}{13}\right)}(\tau)}.$$

If we replace the Klein forms by the $q$-products in (K4) and expand the products as a series, then the series is the Fourier expansion of $r_{13}(\gamma\tau)$. Since we want to prove that $r_{13}(\gamma\tau)$ has algebraic integer Fourier coefficients and the coefficient of the lowest degree term is a unit, we may assume that

$$0 \le (la+cj)/13 < 1, \text{ for } l = 1, \ldots, 6$$

by (K2). If we assume these, then the only term we should consider in (K4) is

$$1 - q_z = 1 - \zeta_{13}^{lb+dj} q^{\frac{la+cj}{13}}.$$

First, assume that $c$ is a multiple of 13. Then $a$ is relatively prime to 13 and $la+cj \equiv la \not\equiv 0 \pmod{13}$; thus, the exponent $(la+cj)/13$ of $q$ is not an integer for any $1 \le l \le 6$ and $1-q_z$

cannot be complex numbers, namely it has algebraic integer coefficients with the coefficient of the lowest degree term 1, and the series expansion of $r_{13}(\gamma\tau)$ has the desired properties.

Now assume that for given $c \in (\mathbb{Z}/13\mathbb{Z})^\times$, there exist unique $j_1, \ldots, j_6 \in \{0, \cdots, 12\}$ such that

$$la + c \cdot j_l \equiv 0 \pmod{13}, \text{ for each } l = 1, \ldots, 6.$$

Hence, the coefficient of the lowest degree term of $r_{13}(\gamma\tau)$ is

$$\frac{(1-\zeta_{13}^{b+d\cdot j_1})(1-\zeta_{13}^{3b+d\cdot j_3})(1-\zeta_{13}^{4b+d\cdot j_4})}{(1-\zeta_{13}^{2b+d\cdot j_2})(1-\zeta_{13}^{5b+d\cdot j_5})(1-\zeta_{13}^{6b+d\cdot j_6})} \tag{4.2}$$

up to a unit. Since

$$\left(\frac{l}{13}, \frac{j_l}{13}\right) = \left(\frac{la + c\cdot j_l}{13}, \frac{lb + d\cdot j_l}{13}\right)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \left(\frac{la + c\cdot j_l}{13}d - \frac{lb + d\cdot j_l}{13}c, *\right),$$

$l = (la + c\cdot j_l)d - (lb + d\cdot j_l)c \equiv -(lb + d\cdot j_l)c \pmod{13}$. Hence for each $l \not\equiv 0 \pmod{13}$, $\zeta_{13}^{lb+d\cdot j_l} \neq 1$ and (4.2) is a unit by Lemma 4.5. $\square$

**Corollary 4.6.** *Let $f(\tau) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{13n})^{-2}$ and $K$ be an imaginary quadratic field. Then $f(\tau)$ is an algebraic integer for every $\tau \in K - \mathbb{Q}$.*

*Proof.* By Theorem 1.3, for any $\tau \in K - \mathbb{Q}$, $1/r_{13}(\tau)$ and $r_{13}(\tau)$ are algebraic integers. Hence $f(\tau) = 1/r_{13}(\tau) - 3 - r_{13}(\tau)$ is also an algebraic integer. $\square$

## Appendix A. Modular equations $F_p(X,Y)$ of $f(\tau)$ of levels $p = 2, 3, 5, 7, 11, 13$ and 17

$$F_2(X,Y) = (X^2 - Y)(X - Y^2) - 4XY(X + Y + 3),$$

$$F_3(X,Y) = (X^3 - Y)(X - Y^3) - 3XY[(2XY + 26)(X + Y) + 5(X^2 + Y^2) + 11XY + 56],$$

$$\begin{aligned}F_5(X,Y) &= (X^5 - Y)(X - Y^5)\\ &-5XY[(2X^3Y^3 + 106X^2Y^2 + 1378XY + 4394)(X + Y) + (9X^2Y^2 + 250XY + 1521)(X^2 + Y^2)\\ &+(20XY + 260)(X^3 + Y^3) + 20(X^4 + Y^4) + 22X^3Y^3 + 548X^2Y^2 + 3718XY + 5712],\end{aligned}$$

$$\begin{aligned}F_7(X,Y) &= (X^7 - Y)(X - Y^7)\\ &-7XY[(2X^5Y^5 + 210X^4Y^4 + 6778X^3Y^3 + 88114X^2Y^2 + 461370XY + 742586)(X + Y)\\ &+(13X^4Y^4 + 848X^3Y^3 + 17764X^2Y^2 + 143312XY + 371293)(X^2 + Y^2)\\ &+(48X^3Y^3 + 2058X^2Y^2 + 26754XY + 105456)(X^3 + Y^3) + (105X^2Y^2 + 2807XY + 17745)\\ &(X^4 + Y^4) + (126XY + 1638)(X^5 + Y^5) + 65(X^6 + Y^6) + 30X^5Y^5 + 1566X^4Y^4 + 31400X^3Y^3\\ &+264654X^2Y^2 + 856830XY + 689544],\end{aligned}$$

$$\begin{aligned}F_{11}(X,Y) &= (X^{11} - Y)(X - Y^{11})\\ &-11XY[(2X^9Y^9 + 462X^8Y^8 + 32328X^7Y^7 + 1079056X^6Y^6 + 21825550X^5Y^5 + 283732150X^4Y^4\\ &+2370686032X^3Y^3 + 12003160104X^2Y^2 + 28989814854XY + 21208998746)(X + Y) + (21X^8Y^8\\ &+3004X^7Y^7 + 148381X^6Y^6 + 3871222X^5Y^5 + 63156789X^4Y^4 + 654236518X^3Y^3\end{aligned}$$

$$+4237909741X^2Y^2+14499734236XY+17130345141)(X^2+Y^2)+(136X^7Y^7+13314X^6Y^6$$
$$+494114X^5Y^5+10358244X^4Y^4+134657172X^3Y^3+1085568458X^2Y^2+4943395002XY$$
$$+8533798312)(X^3+Y^3)+(595X^6Y^6+41857X^5Y^5+1212811X^4Y^4+20350590X^3Y^3$$
$$+204965059X^2Y^2+1195477777XY+2871951355)(X^4+Y^4)+(1818X^5Y^5+94446X^4Y^4$$
$$+2169148X^3Y^3+28198924X^2Y^2+207497862XY+675010674)(X^5+Y^5)+(3883X^4Y^4$$
$$+151074X^3Y^3+2706539X^2Y^2+25531506XY+110902363)(X^6+Y^6)+(5632X^3Y^3$$
$$+164326X^2Y^2+2136238XY+12373504)(X^7+Y^7)+(5175X^2Y^2+109113XY+874575)(X^8$$
$$+Y^8)+(2590XY+33670)(X^9+Y^9)+481(X^{10}+Y^{10})+44X^9Y^9+4883X^8Y^8+222110X^7Y^7$$
$$+5629459X^6Y^6+91150852X^5Y^5+951378571X^4Y^4+6343683710X^3Y^3+23569308347X^2Y^2$$
$$+35892151724XY+12532590168],$$

$$F_{13}(X,Y)=Y^{13}-X^{13}Y^{12}$$
$$-13X[(2Y^{12}+Y^{11})X^{11}+(25Y^{12}+26Y^{11}+13Y^{10})X^{10}+(196Y^{12}+325Y^{11}+338Y^{10}$$
$$+169Y^9)X^9+(1064Y^{12}+2548Y^{11}+4225Y^{10}+4394Y^9+2197Y^8)X^8+(4180Y^{12}+13832Y^{11}$$
$$+33124Y^{10}+54925Y^9+57122Y^8+28561Y^7)X^7+(12086Y^{12}+54340Y^{11}+179816Y^{10}$$
$$+430612Y^9+714025Y^8+742586Y^7+371293Y^6)X^6+(25660Y^{12}+157118Y^{11}+706420Y^{10}$$
$$+2337608Y^9+5597956Y^8+9282325Y^7+9653618Y^6+4826809Y^5)X^5+(39182Y^{12}+333580Y^{11}$$
$$+2042534Y^{10}+9183460Y^9+30388904Y^8+72773428Y^7+120670225Y^6+125497034Y^5$$
$$+62748517Y^4)X^4+(41140Y^{12}+509366Y^{11}+4336540Y^{10}+26552942Y^9+119384980Y^8$$
$$+395055752Y^7+946054564Y^6+1568712925Y^5+1631461442Y^4+815730721Y^3)X^3+(27272Y^{12}$$
$$+534820Y^{11}+6621758Y^{10}+56375020Y^9+345188246Y^8+1552004740Y^7+5135724776Y^6$$
$$+12298709332Y^5+20393268025Y^4+21208998746Y^3+10604499373Y^2)X^2+(9604Y^{12}$$
$$+354536Y^{11}+6952660Y^{10}+86082854Y^9+732875260Y^8+4487447198Y^7+20176061620Y^6$$
$$+66764422088Y^5+159883221316Y^4+265112484325Y^3+275716983698Y^2+137858491849Y)X$$
$$+(1165Y^{12}+124852Y^{11}+4608968Y^{10}+90384580Y^9+1119077102Y^8+9527378380Y^7$$
$$+58336813574Y^6+262288801060Y^5+867937487144Y^4+2078481877108Y^3+3446462296225Y^2$$
$$+3584320788074Y+1792160394037)],$$

$$F_{17}(X,Y)=(X^{17}-Y)(X-Y^{17})$$
$$-17XY[(2X^{15}Y^{15}+1006X^{14}Y^{14}+149368X^{13}Y^{13}+9449296X^{12}Y^{12}+277283808X^{11}Y^{11}$$
$$+2807841744X^{10}Y^{10}-57040799630X^9Y^9-2270242197246X^8Y^8-29513148564198X^7Y^7$$
$$-125318636787110X^6Y^6+1042531984654992X^5Y^5+17399147740112736X^4Y^4$$
$$+100205053507291408X^3Y^3+267691413736518616X^2Y^2+304692357231806518XY$$
$$+102371786028181514)(X+Y)+(33X^{14}Y^{14}+10157X^{13}Y^{13}+1036338X^{12}Y^{12}$$
$$+46468884X^{11}Y^{11}+937380339X^{10}Y^{10}+2995082193X^9Y^9-267189542724X^8Y^8$$
$$-5871087945322X^7Y^7-45155032720356X^6Y^6+85542542514273X^5Y^5+4524555856708251X^4Y^4$$
$$+37906096249385364X^3Y^3+142867993725808962X^2Y^2+236638650589039517XY$$
$$+129933420728076537)(X^2+Y^2)+(348X^{13}Y^{13}+72900X^{12}Y^{12}+5330940X^{11}Y^{11}$$
$$+172095882X^{10}Y^{10}+2429452302X^9Y^9-443129954X^8Y^8-548090669746X^7Y^7$$

$$
\begin{aligned}
&-7125178706698X^6Y^6 - 973556508938X^5Y^5 + 902038633566486X^4Y^4 + 10798761377306994X^3Y^3\\
&+56531949887500620X^2Y^2 + 130648492725297300XY + 105400537094104044)(X^3+Y^3)\\
&+(2610X^{12}Y^{12} + 390234X^{11}Y^{11} + 20741415X^{10}Y^{10} + 491255172X^9Y^9 + 5806735314X^8Y^8\\
&+33465558288X^7Y^7 + 131762476550X^6Y^6 + 5655679350672X^5Y^5 + 165846167303154X^4Y^4\\
&+2371194885506148X^3Y^3 + 16919409412510215X^2Y^2 + 53797070708202666XY\\
&+60808002169675410)(X^4+Y^4) + (14688X^{11}Y^{11} + 1593248X^{10}Y^{10} + 61339470X^9Y^9\\
&+1140305022X^8Y^8 + 17187561070X^7Y^7 + 285078068466X^6Y^6 + 3706014890058X^5Y^5\\
&+37761071670790X^4Y^4 + 423387272533446X^3Y^3 + 3848960776065990X^2Y^2 + 16895597417033504XY\\
&+26323251867615456)(X^5+Y^5) + (63848X^{10}Y^{10} + 4988615X^9Y^9 + 136925586X^8Y^8\\
&+2406980596X^7Y^7 + 55738342810X^6Y^6 + 987969825467X^5Y^5 + 9419779934890X^4Y^4\\
&+68745772802356X^3Y^3 + 660913650835074X^2Y^2 + 4069366510741415XY + 8801988987574952)\\
&(X^6+Y^6) + (217580X^9Y^9 + 11851494X^8Y^8 + 225876162X^7Y^7 + 5041477534X^6Y^6\\
&+138821927106X^5Y^5 + 1804685052378X^4Y^4 + 11076126142198X^3Y^3 + 83866237817466X^2Y^2\\
&+743663672734398XY + 2307326973577340)(X^7+Y^7) + (584079X^8Y^8 + 20627250X^7Y^7\\
&+260091576X^6Y^6 + 9359618100X^5Y^5 + 213828174988X^4Y^4 + 1581775458900X^3Y^3\\
&+7428475502136X^2Y^2 + 99563795945250XY + 476451183790959)(X^8+Y^8) + (1230630X^7Y^7\\
&+23904234X^6Y^6 + 158815970X^5Y^5 + 11625462802X^4Y^4 + 151131016426X^3Y^3 + 348918686090X^2Y^2\\
&+8875474754562XY + 77220207475710)(X^9+Y^9) + (2010261X^6Y^6 + 12158085X^5Y^5\\
&-118403154X^4Y^4 + 4866872225X^3Y^3 - 20010133026X^2Y^2 + 347247065685XY + 9703145887149)\\
&(X^{10}+Y^{10}) + (2488512X^5Y^5 - 12961106X^4Y^4 - 503723122X^3Y^3 - 6548400586X^2Y^2\\
&-28475549882XY + 923967086016)(X^{11}+Y^{11}) + (2248129X^4Y^4 - 32004376X^3Y^3 - 684731010X^2Y^2\\
&-5408739544XY + 64208812369)(X^{12}+Y^{12}) + (1393390X^3Y^3 - 27203110X^2Y^2 - 353640430XY\\
&+3061277830)(X^{13}+Y^{13}) + (532563X^2Y^2 - 8091869XY + 90003147)(X^{14}+Y^{14}) + (101964XY\\
&+1325532(X^{15}+Y^{15}) + 5802(X^{16}+Y^{16}) + 64X^{15}Y^{15} + 15271X^{14}Y^{14} + 1414170X^{13}Y^{13}\\
&+60227704X^{12}Y^{12} + 1122003630X^{11}Y^{11} - 1373219807X^{10}Y^{10} - 511693728152X^9Y^9\\
&-10467989087570X^8Y^8 - 86476240057688X^7Y^7 - 39220530907727X^6Y^6 + 5415697219316670X^5Y^5\\
&+49129588408094584X^4Y^4 + 194955343418100330X^3Y^3 + 355785057905407351X^2Y^2\\
&+251992088684754496XY + 39142153481363520].
\end{aligned}
$$

## Appendix B. Modular equations $\mathcal{F}_p(X,Y)$ of $r_{13}(\tau)$ of levels $p = 2, 3, 5, 7, 11, 13$ and 17

$$\mathcal{F}_2(X,Y) = X^3Y^2 + X^2 + XY^3 - Y + 2XY(XY - X + Y + 1)$$

$$\mathcal{F}_3(X,Y) = (X^3 - Y)(X - Y^3) - 3XY[(XY-1)(X+Y) + (X^2+Y^2) + XY],$$

$$
\begin{aligned}
&\mathcal{F}_5(X,Y) = (XY^5+1)(X^5-Y)\\
&+5XY[X^3Y^3(X+2Y) - 3X^2Y^2(X^2-Y^2) + XY(2X^3+Y^3) - (X^4+Y^4) - 9X^2Y^2(X-Y)\\
&+6XY(X^2+Y^2) - (X^3+2Y^3) - 9XY(X-Y) + 3(X^2-Y^2) + X^4Y^4 + 6X^3Y^3 + 6XY\\
&-2X - Y + 1],
\end{aligned}
$$

$$\mathcal{F}_7(X,Y) = (XY^7+1)(X^7-Y)$$
$$-7XY[27X^2Y^2(XY-1)(X+Y) - X^5Y^5(X+3Y) + (3X+Y) + X^4Y^4(20X-7Y)$$
$$+XY(7X-20Y) - XY(22X^2Y^2+47XY+22)(X^2+Y^2) + 3X^4Y^4(X^2-2Y^2)$$
$$-3(2X^2-Y^2) + XY(20X^3-7Y^3) + X^2Y^2(7X^3-20Y^3) - 7(X^3Y^3-1)(X^3+Y^3)$$
$$-4XY(X^4+Y^4) - 3(X^4-2Y^4) + 3X^2Y^2(2X^4-Y^4) + (X^5+3Y^5) - XY(3X^5+Y^5)$$
$$+(X^6+Y^6) - X^6Y^6 - 4X^5Y^5 + 47X^4Y^4 + 56X^3Y^3 + 47X^2Y^2 - 4XY - 1],$$

$$\mathcal{F}_{11}(X,Y) = (XY^{11}+1)(X^{11}-Y)$$
$$+11XY[2470(X^5Y^5 - X^4Y^4)(X+Y) - 6XY(4X-Y) - 6X^8Y^8(X-4Y) - X^2Y^2(79X$$
$$-147Y) - X^7Y^7(147X-79Y) + 3X^6Y^6(691X+1439Y) - 3X^3Y^3(1439X+691Y) + 2X^3Y^3$$
$$(2717X^2Y^2 - 4187XY + 2717)(X^2+Y^2) + 2XY(52X^2+7Y^2) + 2X^7Y^7(7X^2+52Y^2)$$
$$+2(8X^2-Y^2) + 2X^2Y^2(481X^2+284Y^2) - 2X^6Y^6(284X^2-481Y^2) + 2070X^2Y^2(X^3Y^3$$
$$-1)(X^3+Y^3) - 3X^4Y^4(1439X^3+691Y^3) + 3X^3Y^3(691X^3+1439Y^3) - 2(18X^3+5Y^3)$$
$$+2X^7Y^7(5X^3+18Y^3) - X^3Y^3(307X^3-267Y^3) - XY(267X^3-307Y^3) + 2XY(233X^4Y^4$$
$$+235X^3Y^3+233)(X^4+Y^4) + 8(7X^4-5Y^4) - 8X^6Y^6(5X^4-7Y^4) - 2X^4Y^4(481X^4$$
$$-284Y^4) + 2X^2Y^2(284X^4-481X^4) + 60(X^5Y^5-1)(X^5+Y^5) + X^2Y^2(147X^5-79Y^5)$$
$$+X^3Y^3(79X^5-147Y^5) - XY(307X^5-267Y^5) - X^4Y^4(267X^5-307Y^5) + 10X^2Y^2(X^6$$
$$+Y^6) + 2X^3Y^3(52X^6+7Y^6) + 2XY(7X^6+52Y^6) - 8X^4Y^4(7X^6-5Y^6) + 8(5X^6-7Y^6)$$
$$+2X^3Y^3(18X^7+5Y^7) - 2(5X^7+18Y^7) - 6X^2Y^2(4X^7-Y^7) - 6XY(X^7-4Y^7) + 2XY$$
$$(X^8+Y^8) - 2X^2Y^2(8X^8-Y^8) + 2(X^8-8Y^8) - (X^{10}+Y^{10}) + X^{10}Y^{10} + 5X^9Y^{10} + 2X^9Y^9$$
$$-10X^8Y^8 + 470X^7Y^7 + 8374X^6Y^6 + 5X^{10}Y - 9592X^5Y^5 - 5Y^9 + 8374X^4Y^4 + 470X^3Y^3$$
$$-10X^2Y^2 + 2XY - 5X + 1],$$

$$\mathcal{F}_{13}(X,Y) =$$
$$(Y^{12} + 18Y^{11} + 129Y^{10} + 450Y^9 + 690Y^8 + 18Y^7 - 911Y^6 - 18Y^5 + 690Y^4 - 450Y^3 + 129Y^2$$
$$-18Y+1)X^{13} + (13Y^{12} + 221Y^{11} + 1495Y^{10} + 4940Y^9 + 7345Y^8 + 1079Y^7 - 7267Y^6$$
$$-1040Y^5 + 4355Y^4 - 2015Y^3 + 377Y^2 - 26Y)X^{12} + (78Y^{12} + 1183Y^{11} + 7163Y^{10} + 21541Y^9$$
$$+30940Y^8 + 10075Y^7 - 21099Y^6 - 9841Y^5 + 13000Y^4 - 3991Y^3 + 455Y^2 - 13Y)X^{11}$$
$$+(299Y^{12} + 3731Y^{11} + 18837Y^{10} + 50245Y^9 + 74126Y^8 + 41548Y^7 - 34307Y^6 - 25610Y^5$$
$$+21411Y^4 - 6292Y^3 + 1066Y^2 - 91Y)X^{10} + (819Y^{12} + 7553Y^{11} + 29900Y^{10} + 77181Y^9$$
$$+134303Y^8 + 105092Y^7 + 2249Y^6 - 22529Y^5 + 12012Y^4 - 897Y^3 - 2847Y^2 + 676Y)X^9$$
$$+(1638Y^{12} + 9230Y^{11} + 29458Y^{10} + 91260Y^9 + 169312Y^8 + 222261Y^7 + 181649Y^6$$
$$+45617Y^5 + 35594Y^4 - 39Y^3 - 1053Y^2 - 1729Y)X^8 + (2405Y^{12} + 4134Y^{11} + 31707Y^{10}$$
$$+42328Y^9 + 138658Y^8 + 191854Y^7 + 260013Y^6 + 201695Y^5 + 58539Y^4 + 47060Y^3 + 4186Y^2$$
$$+2535Y)X^7 + (2535Y^{12} - 4186Y^{11} + 47060Y^{10} - 58539Y^9 + 201695Y^8 - 260013Y^7$$

$$+191854Y^6 - 138658Y^5 + 42328Y^4 - 31707Y^3 + 4134Y^2 - 2405Y)X^6 + (1729Y^{12}$$
$$-1053Y^{11} + 39Y^{10} + 35594Y^9 - 45617Y^8 + 181649Y^7 - 222261Y^6 + 169312Y^5$$
$$-91260Y^4 + 29458Y^3 - 9230Y^2 + 1638Y)X^5 + (676Y^{12} + 2847Y^{11} - 897Y^{10} - 12012Y^9$$
$$-22529Y^8 - 2249Y^7 + 105092Y^6 - 134303Y^5 + 77181Y^4 - 29900Y^3 + 7553Y^2 - 819Y)X^4$$
$$+(91Y^{12} + 1066Y^{11} + 6292Y^{10} + 21411Y^9 + 25610Y^8 - 34307Y^7 - 41548Y^6 + 74126Y^5$$
$$-50245Y^4 + 18837Y^3 - 3731Y^2 + 299Y)X^3 - (13Y^{12} + 455Y^{11} + 3991Y^{10} + 13000Y^9$$
$$+9841Y^8 - 21099Y^7 - 10075Y^6 + 30940Y^5 - 21541Y^4 + 7163Y^3 - 1183Y^2 + 78Y)X^2$$
$$+(26Y^{12} + 377Y^{11} + 2015Y^{10} + 4355Y^9 + 1040Y^8 - 7267Y^7 - 1079Y^6 + 7345Y^5$$
$$-4940Y^4 + 1495Y^3 - 221Y^2 + 13Y)X - Y^{13} - 18Y^{12} - 129Y^{11} - 450Y^{10} - 690Y^9$$
$$-18Y^8 + 911Y^7 + 18Y^6 - 690Y^5 + 450Y^4 - 129Y^3 + 18Y^2 - Y,$$

$$\mathcal{F}_{17}(X,Y) = (X^{17} - Y)(X - Y^{17})$$
$$-17XY[(X^{15}Y^{15} + 162X^{14}Y^{14} + 7565X^{13}Y^{13} + 130796X^{12}Y^{12} + 1108735X^{11}Y^{11}$$
$$+4690617X^{10}Y^{10} + 5573662X^9Y^9 + 4972452X^8Y^8 - 4972452X^7Y^7 - 5573662X^6Y^6$$
$$-4690617X^5Y^5 - 1108735X^4Y^4 - 130796X^3Y^3 - 7565X^2Y^2 - 162XY - 1)(X + Y)$$
$$+(802X^{13}Y^{13} + 21656X^{12}Y^{12} + 177732X^{11}Y^{11} + 444265X^{10}Y^{10} - 587823X^9Y^9$$
$$-3035548X^8Y^8 - 5449182X^7Y^7 - 3035548X^6Y^6 - 587823X^5Y^5 + 444265X^4Y^4$$
$$+177732X^3Y^3 + 21656X^2Y^2 + 802XY + 8)(X^2 + Y^2) + (41X^{13}Y^{13} + 2572X^{12}Y^{12}$$
$$+31049X^{11}Y^{11} - 112484X^{10}Y^{10} - 1815352X^9Y^9 - 2938240X^8Y^8 - 1946882X^7Y^7$$
$$+1946882X^6Y^6 + 2938240X^5Y^5 + 1815352X^4Y^4 + 112484X^3Y^3 - 31049X^2Y^2 - 2572XY$$
$$-41)(X^3 + Y^3) + (152X^{12}Y^{12} + 5702X^{11}Y^{11} + 3313X^{10}Y^{10} - 671569X^9Y^9 - 2010796X^8Y^8$$
$$-394118X^7Y^7 - 1609356X^6Y^6 - 394118X^5Y^5 - 2010796X^4Y^4 - 671569X^3Y^3 + 3313X^2Y^2$$
$$+5702XY + 152)(X^4 + Y^4) + (429X^{11}Y^{11} + 8879X^{10}Y^{10} - 63934X^9Y^9 - 787172X^8Y^8$$
$$-329602X^7Y^7 - 1645574X^6Y^6 + 1645574X^5Y^5 + 329602X^4Y^4 + 787172X^3Y^3$$
$$+63934X^2Y^2 - 8879XY - 429)(X^5 + Y^5) + (943X^{10}Y^{10} + 9625X^9Y^9 - 104188X^8Y^8$$
$$-246330X^7Y^7 + 197325X^6Y^6 + 1147150X^5Y^5 + 197325X^4Y^4 - 246330X^3Y^3 - 104188X^2Y^2$$
$$+9625XY + 943)(X^6 + Y^6) + (1628X^9Y^9 + 8736X^8Y^8 - 34032X^7Y^7 + 333791X^6Y^6$$
$$-107541X^5Y^5 + 107541X^4Y^4 - 333791X^3Y^3 + 34032X^2Y^2 - 8736XY - 1628)(X^7 + Y^7)$$
$$+(2188X^8Y^8 + 11038X^7Y^7 + 86419X^6Y^6 - 69205X^5Y^5 + 352004X^4Y^4 - 69205X^3Y^3$$
$$+86419X^2Y^2 + 11038XY + 2188)(X^8 + Y^8) + (2236X^7Y^7 + 14714X^6Y^6 + 22005X^5Y^5$$
$$+93983X^4Y^4 - 93983X^3Y^3 - 22005X^2Y^2 - 14714XY - 2236)(X^9 + Y^9) + (1666X^6Y^6$$
$$+8178X^5Y^5 + 1029X^4Y^4 - 658X^3Y^3 + 1029X^2Y^2 + 8178XY + 1666)(X^{10} + Y^{10})$$
$$+(822X^5Y^5 + 990X^4Y^4 + 7728X^3Y^3 - 7728X^2Y^2 - 990XY - 822)(X^{11} + Y^{11})$$
$$+(234X^4Y^4 + 48X^3Y^3 - 2948X^2Y^2 + 48XY + 234)(X^{12} + Y^{12}) + (36X^3Y^3 + 180X^2Y^2$$
$$-180XY - 36)(X^{13} + Y^{13}) - 213XY(X^{14} + Y^{14}) - 6(X^{16} + Y^{16}) + 19X^{15}Y^{15} + 1491X^{14}Y^{14}$$
$$+41816X^{13}Y^{13} + 530775X^{12}Y^{12} + 3722685X^{11}Y^{11} + 10393859X^{10}Y^{10} + 10751728X^9Y^9$$

$$+14168096X^8Y^8+10751728X^7Y^7+10393859X^6Y^6+3722685X^5Y^5+530775X^4Y^4$$
$$+41816X^3Y^3+1491X^2Y^2+19XY].$$

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