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Ramanujan-type congruences for overpartitions modulo 5

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ABSTRACT

Let $\bar{p}(n)$ denote the number of overpartitions of n . In this paper, we show that $\bar{p}(5n) \equiv (-1)^n \bar{p}(4 \cdot 5n) \pmod{5}$ for $n \geq 0$ and $\bar{p}(n) \equiv (-1)^n \bar{p}(4n) \pmod{8}$ for $n \geq 0$ by using the relation of the generating function of $\bar{p}(5n)$ modulo 5 found by Treneer and the 2-adic expansion of the generating function of $\bar{p}(n)$ due to Mahlborg. As a consequence, we deduce that $\bar{p}(4^k(40n + 35)) \equiv 0 \pmod{40}$ for $n, k \geq 0$. When $k = 0$, it was conjectured by Hirschhorn and Sellers, and confirmed by Chen and Xia. Furthermore, applying the Hecke operator on $\phi(q)^3$ and the fact that $\phi(q)^3$ is a Hecke eigenform, we obtain an infinite family of congruences $\bar{p}(4^k \cdot 5\ell^2 n) \equiv 0 \pmod{5}$, where $k \geq 0$ and ℓ is a prime such that $\ell \equiv 3 \pmod{5}$ and $(\frac{-n}{\ell}) = -1$. Moreover, we show that $\bar{p}(5^2 n) \equiv \bar{p}(5^4 n) \pmod{5}$ for $n \geq 0$. So we are led to the congruences $\bar{p}(4^k 5^{2i+3}(5n \pm 1)) \equiv 0 \pmod{5}$ for $n, k, i \geq 0$. In this way, we obtain various Ramanujan-type congruences for $\bar{p}(n)$ modulo 5 such as $\bar{p}(45(3n + 1)) \equiv 0 \pmod{5}$ and $\bar{p}(125(5n \pm 1)) \equiv 0 \pmod{5}$ for $n \geq 0$.

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1. Introduction

The objective of this paper is to use half-integral weight modular forms to derive three infinite families of congruences for overpartitions modulo 5.

Recall that a partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n . An overpartition of n is a partition of n where the first occurrence of each distinct part may be overlined. We denote the number of overpartitions of n by $\bar{p}(n)$. We set $\bar{p}(0) = 1$ and $\bar{p}(n) = 0$ if $n < 0$. For example, there are eight overpartitions of 3

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \bar{1} + 1 + 1.$$

Overpartitions arise in combinatorics [6], q -series [5], symmetric functions [2], representation theory [11], mathematical physics [8,7] and number theory [15,16]. They are also called standard MacMahon diagrams, joint partitions, jagged partitions or dotted partitions.

Cortee and Lovejoy [6] showed that the generating function of $\bar{p}(n)$ is given by

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}.$$

Recall that the generating function of $\bar{p}(n)$ can be expressed as

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{1}{\phi(-q)},$$

where $\phi(q)$ is Ramanujan's theta function as defined by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad (1.1)$$

see Berndt [2].

On the other hand, the generating function of $\bar{p}(n)$ has the following 2-adic expansion

$$\sum_{n \geq 0} \bar{p}(n)q^n = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} (-1)^{n+k} c_k(n)q^n, \quad (1.2)$$

where $c_k(n)$ denotes the number of representations of n as a sum of k squares of positive integers. The above 2-adic expansion (1.2) is useful to derive congruences for $\bar{p}(n)$ modulo powers of 2, see, for example, [12,13,18].

By employing dissection formulas, Fortin, Jacob and Mathieu [7], Hirschhorn and Sellers [9] independently derived various Ramanujan-type congruences for $\bar{p}(n)$, such as

$$\bar{p}(4n+3) \equiv 0 \pmod{8}. \quad (1.3)$$

Hirschhorn and Sellers [9] proposed the following conjectures

$$\bar{p}(27n+18) \equiv 0 \pmod{12}, \quad (1.4)$$

$$\bar{p}(40n+35) \equiv 0 \pmod{40}. \quad (1.5)$$

They also conjectured that if ℓ is prime and r is a quadratic nonresidue modulo ℓ then

$$\bar{p}(\ell n+r) \equiv \begin{cases} 0 \pmod{8} & \text{if } \ell \equiv \pm 1 \pmod{8}, \\ 0 \pmod{4} & \text{if } \ell \equiv \pm 3 \pmod{8}. \end{cases} \quad (1.6)$$

By using the 3-dissection formula for $\phi(-q)$, Hirschhorn and Sellers [10] proved (1.4) and obtained a family of congruences

$$\bar{p}(9^\alpha(27n+18)) \equiv 0 \pmod{12},$$

where $n, \alpha \geq 0$.

Employing the 2-dissection formulas of theta functions due to Ramanujan, Hirschhorn and Sellers [9], Chen and Xia [4] obtained a generating function of $\bar{p}(40n+35)$ modulo 5. Using the (p, k) -parametrization of theta functions given by Alaca, Alaca and Williams [1], they showed that

$$\bar{p}(40n+35) \equiv 0 \pmod{5}. \quad (1.7)$$

In view of (1.3), (1.7) implies Hirschhorn and Sellers' conjecture (1.5). Applying the 2-adic expansion (1.2), Kim [13] proved (1.6) and obtained congruence properties of $\bar{p}(n)$ modulo 8.

For powers of 2, Mahlburg [18] showed that $\bar{p}(n) \equiv 0 \pmod{64}$ holds for a set of integers of arithmetic density 1. Kim [12] showed that $\bar{p}(n) \equiv 0 \pmod{128}$ holds for a set of integers of arithmetic density 1. For the modulus 3, by using the fact that $\phi(q)^5$ is a Hecke eigenform in the half-integral weight modular form space $M_{\frac{5}{2}}(\tilde{\Gamma}_0(4))$, Lovejoy and Osburn [17] proved that

$$\bar{p}(3\ell^3 n) \equiv 0 \pmod{3},$$

where $\ell \equiv 2 \pmod{3}$ is an odd prime and $\ell \nmid n$. Moreover, by utilizing half-integral weight modular forms, Treener [21] showed that for a prime ℓ such that $\ell \equiv -1 \pmod{5}$,

$$\bar{p}(5\ell^3 n) \equiv 0 \pmod{5},$$

for all n coprime to ℓ .

In this paper, we establish the following two congruence relations for overpartitions modulo 5 and modulo 8 by using a relation of the generating function of $\bar{p}(5n)$ modulo 5 and applying the 2-adic expansion (1.2).

Theorem 1.1. *For $n \geq 0$, we have*

$$\bar{p}(5n) \equiv (-1)^n \bar{p}(4 \cdot 5n) \pmod{5}. \quad (1.8)$$

Theorem 1.2. *For $n \geq 0$, we have*

$$\bar{p}(n) \equiv (-1)^n \bar{p}(4n) \pmod{8}. \quad (1.9)$$

Combining the above two congruence relations with congruences (1.3) and (1.7), we arrive at a family of congruences modulo 40.

Corollary 1.3. *For $n, k \geq 0$, we have*

$$\bar{p}(4^k(40n + 35)) \equiv 0 \pmod{40}. \quad (1.10)$$

Based on the Hecke operator on $\phi(q)^3$ and the fact that $\phi(q)^3$ is a Hecke eigenform in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$, we obtain a family of congruences for overpartitions modulo 5.

Theorem 1.4. *Let $(\frac{\cdot}{\ell})$ denote the Legendre symbol. Assume that k is a nonnegative integer and ℓ is a prime with $\ell \equiv 3 \pmod{5}$. Then we have*

$$\bar{p}(4^k \cdot 5\ell^2 n) \equiv 0 \pmod{5},$$

where n is a nonnegative integer such that $(\frac{-n}{\ell}) = -1$.

Using the properties of the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ and the Hecke eigenform $\phi(q)^3$, we are led to another congruence relation for overpartitions modulo 5.

Theorem 1.5. *For $n \geq 0$, we have*

$$\bar{p}(5^2 n) \equiv \bar{p}(5^4 n) \pmod{5}. \quad (1.11)$$

Combining (1.8) and (1.11), we find the following family of congruences modulo 5.

Corollary 1.6. *For $n, k, i \geq 0$, we have*

$$\bar{p}(4^k 5^{2i+3}(5n \pm 1)) \equiv 0 \pmod{5}. \quad (1.12)$$

2. Preliminaries

To make this paper self-contained, we recall some definitions and notation on half-integral weight modular forms. For more details, see [3,21,14,19,20].

Let k be an odd positive integer and N be a positive integer with $4 \mid N$. We use $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ to denote the space of holomorphic modular forms on $\Gamma_0(N)$ of weight $\frac{k}{2}$.

Definition 2.1. Let

$$f(z) = \sum_{n \geq 0} a(n)q^n$$

be a modular form in $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$. For any odd prime $\ell \nmid N$, the action of the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ on $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ is given by

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) = \sum_{n \geq 0} \left(a(\ell^2 n) + \left(\frac{(-1)^{\frac{k-1}{2}} n}{\ell} \right) \ell^{\frac{k-3}{2}} a(n) + \ell^{k-2} a\left(\frac{n}{\ell^2}\right) \right) q^n, \quad (2.1)$$

where $a(\frac{n}{\ell^2}) = 0$ if n is not divisible by ℓ^2 .

The following proposition says that the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ maps the modular form space $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ into itself.

Proposition 2.2. Let ℓ be an odd prime and $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$, then

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N)).$$

A Hecke eigenform associated with the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ is defined as follows.

Definition 2.3. A half-integral weight modular form $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(4N))$ is called a Hecke eigenform for the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$, if for every prime $\ell \nmid 4N$ there exists a complex number $\lambda(\ell)$ for which

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) = \lambda(\ell)f(z).$$

For the space of half-integral weight modular forms on $\Gamma_0(4)$, we have the following dimension formula.

Proposition 2.4. We have

$$\dim M_{\frac{k}{2}}(\tilde{\Gamma}_0(4)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor.$$

By the above dimension formula, we see that $\dim M_{\frac{3}{2}}(\tilde{\Gamma}_0(4)) = 1$. From the fact that $\phi(q)^3 \in M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$, it is easy to deduce that

$$\phi(q)^3 \mid T_{\frac{3}{2},4}(\ell^2) = (\ell + 1)\phi(q)^3, \quad (2.2)$$

see, for example, [21, p. 18].

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2 by using a relation of the generating function of $\bar{p}(5n)$ modulo 5 and the 2-adic expansion (1.2) of $\bar{p}(n)$.

Proof of Theorem 1.1. Recall the following 2-dissection formula for $\phi(q)$,

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \quad (3.1)$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}},$$

see, for example, Hirschhorn and Sellers [9]. Replacing q by $-q$, (3.1) becomes

$$\phi(-q) = \phi(q^4) - 2q\psi(q^8). \quad (3.2)$$

We now consider the generating function of $\bar{p}(5n)$ modulo 5. The following relation is due to Treneer [21, p. 18],

$$\sum_{n \geq 0} \bar{p}(5n)q^n \equiv \phi(-q)^3 \pmod{5}. \quad (3.3)$$

Plugging (3.2) into (3.3) yields that

$$\sum_{n \geq 0} \bar{p}(5n)q^n \equiv \phi(q^4)^3 - q\phi(q^4)^2\psi(q^8) + 2q^2\phi(q^4)\psi(q^8)^2 - 3q^3\psi(q^8)^3 \pmod{5}. \quad (3.4)$$

Extracting the terms of q^{4n+i} for $i = 0, 1, 2, 3$ on both sides of (3.4) and setting q^4 to q , we obtain

$$\sum_{n \geq 0} \bar{p}(20n)q^n \equiv \phi(q)^3 \pmod{5}, \quad (3.5)$$

$$\sum_{n \geq 0} \bar{p}(20n + 5)q^n \equiv -\phi(q)^2\psi(q^2) \pmod{5}, \quad (3.6)$$

$$\sum_{n \geq 0} \bar{p}(20n + 10)q^n \equiv 2\phi(q)\psi(q^2)^2 \pmod{5}, \quad (3.7)$$

$$\sum_{n \geq 0} \bar{p}(20n + 15)q^n \equiv -3\psi(q^2)^3 \pmod{5}. \quad (3.8)$$

Substituting the 2-dissection formula (3.1) into (3.5), we find that

$$\sum_{n \geq 0} \bar{p}(20n)q^n \equiv \phi(q^4)^3 + q\phi(q^4)^2\psi(q^8) + 2q^2\phi(q^4)\psi(q^8)^2 + 3q^3\psi(q^8)^3 \pmod{5}. \quad (3.9)$$

Extracting the terms of q^{4n+i} for $i = 0, 1, 2, 3$ on both sides of (3.9) and setting q^4 to q , we obtain

$$\sum_{n \geq 0} \bar{p}(4 \cdot 20n)q^n \equiv \phi(q)^3 \pmod{5}, \quad (3.10)$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n + 5))q^n \equiv \phi(q)^2\psi(q^2) \pmod{5}, \quad (3.11)$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n + 10))q^n \equiv 2\phi(q)\psi(q^2)^2 \pmod{5}, \quad (3.12)$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n + 15))q^n \equiv 3\psi(q^2)^3 \pmod{5}. \quad (3.13)$$

Comparing Eqs. (3.5)–(3.8) with (3.10)–(3.13), we deduce that

$$\begin{aligned} \bar{p}(5 \cdot (4n)) &\equiv \bar{p}(4 \cdot 5 \cdot 4n) \pmod{5}, \\ \bar{p}(5 \cdot (4n + 1)) &\equiv -\bar{p}(4 \cdot 5 \cdot (4n + 1)) \pmod{5}, \\ \bar{p}(5 \cdot (4n + 2)) &\equiv \bar{p}(4 \cdot 5 \cdot (4n + 2)) \pmod{5}, \\ \bar{p}(5 \cdot (4n + 3)) &\equiv -\bar{p}(4 \cdot 5 \cdot (4n + 3)) \pmod{5}. \end{aligned}$$

So we conclude that

$$\bar{p}(5n) \equiv (-1)^n \bar{p}(4 \cdot 5n) \pmod{5}.$$

This completes the proof. \square

We note that extracting the terms of odd powers of q on both sides of (3.8) leads to the congruence $\bar{p}(40n + 35) \equiv 0 \pmod{5}$ due to Chen and Xia [4].

Next, we prove Theorem 1.2 by using the 2-adic expansion (1.2). Recall that $c_k(n)$ in (1.2) denotes the number of representations of n as a sum of k squares of positive integers. In particular, $c_1(n) = 1$ if n is a square; otherwise, $c_1(n) = 0$.

Proof of Theorem 1.2. It follows from (1.2) that

$$\bar{p}(n) \equiv (-1)^n (-2c_1(n) + 4c_2(n)) \pmod{8}, \quad (3.14)$$

where $n \geq 1$. Replacing n by $4n$ in (3.14), we get

$$\bar{p}(4n) \equiv -2c_1(4n) + 4c_2(4n) \pmod{8}. \quad (3.15)$$

Since $c_1(n) = c_1(4n)$ and $c_2(n) = c_2(4n)$, (3.15) can be rewritten as

$$\bar{p}(4n) \equiv -2c_1(n) + 4c_2(n) \pmod{8}. \quad (3.16)$$

Substituting (3.16) into (3.14), we arrive at

$$\bar{p}(n) \equiv (-1)^n \bar{p}(4n) \pmod{8},$$

as claimed. \square

From Theorems 1.1 and 1.2, it is easy to see that $\bar{p}(4 \cdot 5n) \equiv (-1)^n \bar{p}(5n) \pmod{40}$. Iteratively applying this congruence to (1.5), we are led to Corollary 1.3.

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by using the Hecke operator on $\phi(q)^3$ along with the fact that $\phi(q)^3$ is a Hecke eigenform in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$.

In view of Theorem 1.1, to prove Theorem 1.4, it suffices to consider the special case $k = 0$ that takes the following form.

Theorem 4.1. *Let ℓ be a prime with $\ell \equiv 3 \pmod{5}$. Then*

$$\bar{p}(5\ell^2 n) \equiv 0 \pmod{5} \quad (4.1)$$

holds for any nonnegative integer n with $(\frac{-n}{\ell}) = -1$.

Proof. Recall that $\phi(-q)^3$ is a modular form in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(16))$. Suppose that

$$\phi(-q)^3 = \sum_{n \geq 0} a(n) q^n \quad (4.2)$$

is the Fourier expansion of $\phi(-q)^3$.

Applying the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ to $\phi(-q)^3$ and using (2.1), we find that

$$\phi(-q)^3 \mid T_{\frac{3}{2},16}(\ell^2) = \sum_{n=0}^{\infty} \left(a(\ell^2 n) + \left(\frac{-n}{\ell} \right) a(n) + \ell a\left(\frac{n}{\ell^2} \right) \right) q^n, \quad (4.3)$$

where ℓ is an odd prime. Replacing q by $-q$ in (2.2), we see that $\phi(-q)^3$ is a Hecke eigenform in the space $M_{\frac{3}{2}}(\tilde{\Gamma}_0(16))$, and hence

$$\phi(-q)^3 \mid T_{\frac{3}{2},16}(\ell^2) = (\ell + 1)\phi(-q)^3. \quad (4.4)$$

Comparing the coefficients of q^n in (4.3) and (4.4), we deduce that

$$a(\ell^2 n) + \left(\frac{-n}{\ell}\right)a(n) + \ell a\left(\frac{n}{\ell^2}\right) = (\ell + 1)a(n). \quad (4.5)$$

Revoking the congruence (3.3), that is,

$$\phi(-q)^3 \equiv \sum_{n \geq 0} \bar{p}(5n)q^n \pmod{5}, \quad (4.6)$$

and comparing (4.2) with (4.6), we get

$$a(n) \equiv \bar{p}(5n) \pmod{5}. \quad (4.7)$$

Plugging (4.7) into (4.5), we deduce that

$$\bar{p}(5\ell^2 n) + \left(\frac{-n}{\ell}\right)\bar{p}(5n) + \ell \bar{p}\left(\frac{5n}{\ell^2}\right) \equiv (\ell + 1)\bar{p}(5n) \pmod{5}. \quad (4.8)$$

Since $\ell \equiv 3 \pmod{5}$ and $\left(\frac{-n}{\ell}\right) = -1$, we see that $\ell \nmid 5$ and $\ell \nmid n$, so that $\ell^2 \nmid 5n$ and $\bar{p}\left(\frac{5n}{\ell^2}\right) = 0$. Noting that $\left(\frac{-n}{\ell}\right) \equiv (\ell + 1) \equiv -1 \pmod{5}$, (4.8) becomes

$$\bar{p}(5\ell^2 n) \equiv 0 \pmod{5}.$$

This completes the proof. \square

We now give some special cases of Theorem 1.4. Setting $\ell = 3$ and $k = 0, 1$ in Theorem 1.4, respectively, we obtain the following congruences for $n \geq 0$,

$$\begin{aligned} \bar{p}(45(3n + 1)) &\equiv 0 \pmod{5}, \\ \bar{p}(180(3n + 1)) &\equiv 0 \pmod{5}. \end{aligned}$$

Setting $\ell = 13$, $k = 0$ in Theorem 1.4, we obtain the following congruences for $n \geq 0$,

$$\begin{aligned} \bar{p}(845(13n + 2)) &\equiv 0 \pmod{5}, \\ \bar{p}(845(13n + 5)) &\equiv 0 \pmod{5}, \\ \bar{p}(845(13n + 6)) &\equiv 0 \pmod{5}, \\ \bar{p}(845(13n + 7)) &\equiv 0 \pmod{5}, \end{aligned}$$

$$\begin{aligned}\bar{p}(845(13n+8)) &\equiv 0 \pmod{5}, \\ \bar{p}(845(13n+11)) &\equiv 0 \pmod{5}.\end{aligned}$$

5. Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5 by using the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ and the Hecke eigenform $\phi(-q)^3$.

Proof of Theorem 1.5. Setting $\ell = 5$ in the congruence relation (4.8), we find that

$$\bar{p}(5n) \equiv \bar{p}(5^3n) + \left(\frac{n}{5}\right)\bar{p}(5n) \pmod{5}. \quad (5.1)$$

By the definition of the Legendre symbol, we see that if $n \equiv 0 \pmod{5}$, then $(\frac{n}{5}) = 0$. Hence, by replacing n with $5n$ in congruence (5.1), we obtain that

$$\bar{p}(5^2n) \equiv \bar{p}(5^4n) \pmod{5}, \quad (5.2)$$

as claimed. \square

Furthermore, we note that if $n \equiv \pm 1 \pmod{5}$, then $(\frac{n}{5}) = 1$. Hence by setting n to $5n \pm 1$ in (5.1), we deduce that

$$\bar{p}(5^3(5n \pm 1)) \equiv 0 \pmod{5}. \quad (5.3)$$

By iteratively applying the congruence $\bar{p}(5n) \equiv (-1)^n \bar{p}(4 \cdot 5n) \pmod{5}$ given in Theorem 1.1 and congruence (5.2) to (5.3), we obtain that

$$\bar{p}(4^k 5^{2i+3}(5n \pm 1)) \equiv 0 \pmod{5}, \quad (5.4)$$

where $n, k, i \geq 0$. This proves Corollary 1.6.

For $n \geq 0$, setting $i = 0$ and $k = 0, 1$ in (5.4), we obtain the following special cases

$$\begin{aligned}\bar{p}(125(5n \pm 1)) &\equiv 0 \pmod{5}, \\ \bar{p}(500(5n \pm 1)) &\equiv 0 \pmod{5}.\end{aligned}$$

By replacing n by $5n \pm 2$ in (5.1) and iteratively using the congruence relation (5.2), we obtain the following relation.

Corollary 5.1. For $n, i \geq 0$, we have

$$\bar{p}(5(5n \pm 2)) \equiv 3\bar{p}(5^{2i+3}(5n \pm 2)) \pmod{5}.$$

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References

- [1] A. Alaca, S. Alaca, K.S. Williams, On the two-dimensional theta functions of the Borweins, *Acta Arith.* 124 (2006) 177–195.
- [2] B.C. Berndt, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, RI, 2006.
- [3] W.Y.C. Chen, D.K. Du, Q.H. Hou, L.H. Sun, Congruences of multipartition functions modulo powers of primes, *Ramanujan J.* 35 (1) (2014) 1–19.
- [4] W.Y.C. Chen, E.X.W. Xia, Proof of a conjecture of Hirschhorn and Sellers on overpartitions, *Acta Arith.* 163 (1) (2014) 59–69.
- [5] S. Corteel, P. Hitzenko, Multiplicity and number of parts in overpartitions, *Ann. Comb.* 8 (2004) 287–301.
- [6] S. Corteel, J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* 356 (2004) 1623–1635.
- [7] J.-F. Fortin, P. Jacob, P. Mathieu, Jagged partitions, *Ramanujan J.* 10 (2005) 215–235.
- [8] J.-F. Fortin, P. Jacob, P. Mathieu, Generating function for K -restricted jagged partitions, *Electron. J. Combin.* 12 (1) (2005) R12.
- [9] M.D. Hirschhorn, J.A. Sellers, Arithmetic relations for overpartitions, *J. Combin. Math. Combin. Comput.* 53 (2005) 65–73.
- [10] M.D. Hirschhorn, J.A. Sellers, An infinite family of overpartition congruences modulo 12, *Integers* 5 (2005) #A20.
- [11] S.-J. Kang, J.-H. Kwon, Crystal bases of the Fock space representations and string functions, *J. Algebra* 280 (2004) 313–349.
- [12] B. Kim, The overpartition function modulo 128, *Integers* 8 (2008) #A38.
- [13] B. Kim, A short note on the overpartition function, *Discrete Math.* 309 (2009) 2528–2532.
- [14] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer, New York, 1993.
- [15] J. Lovejoy, Overpartitions and real quadratic fields, *J. Number Theory* 106 (2004) 178–186.
- [16] J. Lovejoy, O. Mallet, Overpartition pairs and two classes of basic hypergeometric series, *Adv. Math.* 217 (2008) 386–418.
- [17] J. Lovejoy, R. Osburn, Quadratic forms and four partition functions modulo 3, *Integers* 11 (2011) #A4.
- [18] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Math.* 286 (2004) 263–267.
- [19] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -Series*, CBMS Reg. Conf. Ser. Math., vol. 102, AMS Press, Providence, RI, 2004.
- [20] G. Shimura, On modular forms of half-integral weight, *Ann. of Math.* 97 (1973) 440–481.
- [21] S. Treneer, Congruences for the coefficients of weakly holomorphic modular forms, *Proc. Lond. Math. Soc.* 93 (2006) 304–324.