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THE LEAST MODULUS FOR WHICH CONSECUTIVE POLYNOMIAL VALUES ARE DISTINCT

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ABSTRACT. Let $d \geq 4$ and $c \in (-d, d)$ be relatively prime integers. We show that for any sufficiently large integer n (in particular $n > 24310$ suffices for $4 \leq d \leq 36$), the smallest prime $p \equiv c \pmod{d}$ with $p \geq (2dn - c)/(d - 1)$ is the least positive integer m with $2r(d)k(dk - c) \pmod{m}$ ($k = 1, \dots, n$) pairwise distinct modulo m , where $r(d)$ is the radical of d . We also conjecture that for any integer $n > 4$ the least positive integer m such that $|\{k(k-1)/2 \pmod{m} : k = 1, \dots, n\}| = |\{k(k-1)/2 \pmod{m+2} : k = 1, \dots, n\}| = n$ is the least prime $p \geq 2n - 1$ with $p + 2$ also prime.

1. INTRODUCTION

To find nontrivial arithmetical functions taking only prime values is a fascinating topic in number theory. In 1947 W. H. Mills [M] showed that there exists a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$; unfortunately such a constant A cannot be effectively found.

For each integer $h > 1$ and sufficiently large integer n , it was determined in [BSW] the least positive integer m with $1^h, 2^h, \dots, n^h$ pairwise distinct modulo m , but such integers m are composite infinitely often. In a recent paper [S] the author proved that the smallest integer $m > 1$ such that $2k(k-1) \pmod{m}$ for $k = 1, \dots, n$ are pairwise distinct, is precisely the least prime greater than $2n - 2$, and that for $n \in \{4, 5, \dots\}$ the least positive integer m such that $18k(3k-1) \pmod{m}$ ($k = 1, \dots, n$) are pairwise distinct modulo m , is the least prime $p > 3n$ with $p \equiv 1 \pmod{3}$. When $d \in \{4, 5, 6, \dots\}$ and $c \in (-d, d)$ are relatively prime, it is natural to ask whether there is a similar result for primes in the arithmetic progression $\{c, c+d, c+2d, \dots\}$ since there are infinitely many such primes by Dirichlet's theorem.

Based on our computation we discover the following general result.

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Theorem 1.1. *Let $d \geq 4$ and $c \in (-d, d)$ be relatively prime integers. Let*

$$f_{d,c}(x) := 2r(d)x(dx - c), \quad (1.1)$$

where $r(d)$ is the radical of d (i.e., the product of all the distinct prime divisors of d). For $n \in \mathbb{Z}^+$ define $m_{d,c}(n)$ as the least positive integer m for which the integers $f_{d,c}(k)$ ($k = 1, \dots, n$) are pairwise distinct modulo m .

(i) If $n \in \mathbb{Z}^+$ is sufficiently large, then $m_{d,c}(n)$ is the least prime $p \equiv c \pmod{d}$ with $p \geq (2dn - c)/(d - 1)$.

(ii) When $4 \leq d \leq 36$ and $n > M_d$, the required result in the first part holds, where

$$\begin{aligned} M_4 &= 8, M_5 = 14, M_6 = 10, M_7 = 100, M_8 = 21, M_9 = 315, M_{10} = 53, \\ M_{11} &= 1067, M_{12} = 27, M_{13} = 1074, M_{14} = 122, M_{15} = 809, M_{16} = 329, \\ M_{17} &= 5115, M_{18} = 95, M_{19} = 5390, M_{20} = 755, M_{21} = 3672, M_{22} = 640, \\ M_{23} &= 11193, M_{24} = 220, M_{25} = 12810, M_{26} = 1207, M_{27} = 7087, \\ M_{28} &= 2036, M_{29} = 13250, M_{30} = 177, M_{31} = 24310, M_{32} = 3678, \\ M_{33} &= 12794, M_{34} = 5303, M_{35} = 15628, M_{36} = 551. \end{aligned}$$

Remark 1.1. To obtain the effective lower bounds M_d ($4 \leq d \leq 36$) in part (ii) of Theorem 1.1, we actually employ some computational results of O. Ramaré and R. Rumely [RR] on primes in arithmetic progressions. Define

$$\begin{aligned} c_4 &= -3, c_5 = -1, c_6 = 1, c_7 = -5, c_8 = 1, c_9 = 2, c_{10} = 3, \\ c_{11} &= -7, c_{12} = 5, c_{13} = -5, c_{14} = -5, c_{15} = -1, c_{16} = 11, \\ c_{17} &= 15, c_{18} = 1, c_{19} = 6, c_{20} = -9, c_{21} = 1, c_{22} = 5, \\ c_{23} &= 21, c_{24} = 1, c_{25} = 19, c_{26} = -3, c_{27} = 23, \\ c_{28} &= -9, c_{29} = -1, c_{30} = 17, c_{31} = 3, c_{32} = -1, \\ c_{33} &= -5, c_{34} = 15, c_{35} = 12, c_{36} = 23. \end{aligned}$$

Then, for every $d = 4, \dots, 36$, the number $m_{d,c_d}(M_d)$ is *not* the least prime $p \equiv c_d \pmod{d}$ with $p \geq (2dM_d - c_d)/(d - 1)$.

Theorem 1.1 with $d = 4, 5$ yields the following concrete consequence.

Corollary 1.1. (i) *For each integer $n \geq 6$, the least positive integer m such that $4k(4k - 1)$ (or $4k(4k + 1)$) for $k = 1, \dots, n$ are pairwise distinct modulo m , is the least prime $p \equiv 1 \pmod{4}$ with $p \geq (8n - 1)/3$ (resp., $p \equiv -1 \pmod{4}$) with $p \geq (8n + 1)/3$.*

(ii) *Let $C_1 = 8$, $C_2 = 10$, $C_{-1} = 15$ and $C_{-2} = 5$. For any $r \in \{\pm 1, \pm 2\}$ and integer $n \geq C_r$, the least positive integer m such that $10k(5k - r)$ for $k = 1, \dots, n$ are pairwise distinct modulo m , is the least prime $p \equiv r \pmod{5}$ with $p \geq (10n - r)/4$.*

As a supplement to Theorem 1.1, we are able to prove the following result for the cases $d = 2, 3$.

Theorem 1.2. For $d \in \{2, 3\}$ and integer $c \in (-d, d)$, let $S_{d,c}$ be the set of all primes $p \equiv c \pmod{d}$ and powers of d . Then

$$\begin{aligned} m_{2,1}(n) &= \min\{a \geq 4n - 1 : a \in S_{2,1}\} \text{ for } n \geq 5, \\ m_{2,-1}(n) &= \min\{a \geq 4n : a \in S_{2,-1}\} \text{ for } n \geq 7, \\ m_{3,1}(n) &= \min\{a \geq 3n : a \in S_{3,1}\} \text{ for } n \geq 4, \\ m_{3,-1}(n) &= \min\{a \geq 3n : a \in S_{3,-1}\} \text{ for } n \geq 5, \\ m_{3,2}(n) &= \min\{a \geq 3n - 1 : a \in S_{3,2}\} \text{ for } n \geq 3, \\ m_{3,-2}(n) &= \min\{a \geq 3n : a \in S_{3,-2}\} \text{ for } n \geq 8. \end{aligned}$$

Remark 1.2. As the proof of Theorem 1.2 is quite similar to and even easier than that of Theorem 1.1, we omit the details of the proof. Note that if $n > 1$ is a power of two with $4n - 1$ composite then $\min\{a \geq 4n - 1 : a \in S_{2,1}\} = 4n$ is a power of two. Also, if $n > 1$ is a power of three then $\min\{a \geq 3n - 1 : a \in S_{3,2}\} = 3n$ is a power of three.

To conclude this section, we pose some new conjectures.

Conjecture 1.1. For any $d \in \mathbb{Z}^+$ there is a positive integer n_d such that for any integer $n \geq n_d$ the least positive integer m satisfying

$$\left| \left\{ \binom{k}{2} \bmod m : k = 1, \dots, n \right\} \right| = \left| \left\{ \binom{k}{2} \bmod m + 2d : k = 1, \dots, n \right\} \right| = n$$

is the smallest prime $p \geq 2n - 1$ with $p + 2d$ also prime. Moreover, we may take

$$\begin{aligned} n_1 &= 5, \quad n_2 = n_3 = 6, \quad n_4 = 10, \quad n_5 = 9, \\ n_6 &= 8, \quad n_7 = 9, \quad n_8 = 18, \quad n_9 = 11, \quad n_{10} = 9. \end{aligned}$$

Remark 1.3. A well-known conjecture of de Polignac [P] asserts that for any positive integer d there are infinitely many prime pairs $\{p, q\}$ with $p - q = 2d$.

Conjecture 1.2. Let n be any positive integer and consider the least positive integer m such that

$$\left| \left\{ \binom{k}{2} \bmod m : k = 1, \dots, n \right\} \right| = \left| \left\{ \binom{k}{2} \bmod m + 1 : k = 1, \dots, n \right\} \right| = n.$$

Then, each of m and $m + 1$ is either a power of two (including $2^0 = 1$) or a prime times a power of two.

Conjecture 1.3. *Let n be any positive integer. Then the least positive integer m of the form $x^2 + x + 1$ (or $4x^2 + 1$) such that the coefficients $\binom{k}{2}$ ($k = 1, \dots, n$) are pairwise distinct modulo m , is the smallest prime $p \geq 2n - 1$ of the form $x^2 + x + 1$ (or $4x^2 + 1$)*

Remark 1.4. The conjecture that there are infinitely many primes of the form $x^2 + x + 1$ (or $4x^2 + 1$) is still open. We may also replace $\binom{k}{2}$ in Conjecture 1.3 by k^2 .

Conjecture 1.4. *For any integer $n > 2$, the smallest positive integer m such that the integers $6p_k(p_k - 1)$ ($k = 1, \dots, n$) are pairwise incongruent modulo m is precisely the least prime $p \geq p_n$ dividing none of the numbers $p_i + p_j - 1$ ($1 \leq i < j \leq n$), where p_k denotes the k -th prime.*

Remark 1.5. For any prime $p \geq p_n$ dividing none of the numbers $p_i + p_j - 1$ ($1 \leq i < j \leq n$), clearly $p_j(p_j - 1) - p_i(p_i - 1) = (p_j - p_i)(p_i + p_j - 1) \not\equiv 0 \pmod{p}$ for all $1 \leq i < j \leq n$.

We also have some other conjectures similar to Conjectures 1.1–1.4.

In the next section we provide some lemmas. Section 3 is devoted to our proof of Theorem 1.1.

2. SOME LEMMAS

Lemma 2.1. *Let c and $d > 1$ be relatively prime integers. For any $\varepsilon > 0$, if $n \in \mathbb{Z}^+$ is large enough, then there is a prime $p \equiv c \pmod{d}$ with*

$$\frac{d(2n - 1) - c}{d - 1} < p \leq \frac{d((2 + \varepsilon)n - 1) - c}{d - 1}.$$

Proof. This is an easy consequence of the Prime Number Theorem for arithmetic progressions (cf. (1.5) of [CP, p. 13] or Theorem 4.4.4 of [J, p. 175]) which states that

$$|\{p \leq x : p \text{ is a prime with } p \equiv c \pmod{d}\}| \sim \frac{x}{\varphi(d) \log x}$$

as $x \rightarrow +\infty$, where φ is Euler's totient function. $\square$

In view of (1.1), for any $c \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$ we have the useful identity

$$f_{d,c}(l) - f_{d,c}(k) = 2r(d)(l - k)(d(k + l) - c). \quad (2.1)$$

Lemma 2.2. *Let $d > 2$ and $c \in (-d, d)$ be relatively prime integers. Suppose that p is a prime not exceeding $(d((2 + \varepsilon)n - 1) - c)/(d - 1)$ where $n \geq 3d$ and $0 < \varepsilon \leq 2/(d - 2)$. Then*

$$\begin{aligned} & f_{d,c}(k) \text{ } (k = 1, \dots, n) \text{ are pairwise distinct modulo } p \\ \iff & p \equiv c \pmod{d} \text{ and } p > (d(2n - 1) - c)/(d - 1). \end{aligned} \quad (2.2)$$

Proof. If $p \mid 2d$, then $p \mid 2r(d)$ and hence $f_{d,c}(k) \equiv 0 \pmod{p}$ for all $k = 1, \dots, n$. Note that $(d(2n-1)-c)/(d-1) \geq (3d-c)/(d-1) \geq 2d/(d-1) > 2$. If $p \nmid d$ then $p \not\equiv c \pmod{d}$. So (2.2) holds in the case $p \mid 2d$.

From now on we assume that $p \nmid 2d$. Then $jp \equiv -c \pmod{d}$ for some $1 \leq j \leq d-1$.

Negating the right-hand side of (2.2), we suppose first that $p \not\equiv c \pmod{d}$ or $p \leq (d(2n-1)-c)/(d-1)$. Write $jp+c = dq$ with $q \in \mathbb{Z}$. If $p \not\equiv c \pmod{d}$, then $j \leq d-2$ and hence

$$\begin{aligned} q &\leq \frac{c}{d} + \frac{d-2}{d}p \leq \frac{c}{d} + \frac{d-2}{d} \cdot \frac{d((2+\varepsilon)n-1)-c}{d-1} \\ &\leq \frac{c-d(d-2)}{d(d-1)} + \frac{d-2}{d-1} \left(2 + \frac{2}{d-2}\right)n < 2n. \end{aligned}$$

If $p \equiv c \pmod{d}$ and $p \leq (d(2n-1)-c)/(d-1)$, then $j = d-1$ and $q \leq 2n-1$. When $q > 2$, we have $0 < k := \lfloor (q-1)/2 \rfloor < l := \lfloor (q+2)/2 \rfloor \leq n$, also

$$d(k+l)-c = dq - c = jp \equiv 0 \pmod{p}$$

and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{p}$ in view of (2.1). If $q \leq 2$, then $p \leq jp = dq - c \leq 2d - c < 3d \leq n$ and $f_{d,c}(p+1) \equiv f_{d,c}(1) \pmod{p}$.

We now assume the right-hand side of (2.2). Then $(d-1)p+c = dq$ for some integer $q \geq 2n$. In view of (2.1), we only need to show that $p \nmid (l-k)$ and $d(k+l) \not\equiv c \pmod{p}$ for any $1 \leq k < l \leq n$. Note that

$$0 < l-k < n \leq \frac{dq}{2d} = \frac{(d-1)p+c}{2d} < \frac{p+1}{2} \leq p$$

and also $d(k+l)-c \leq d(2n-1)-c < (d-1)p$. If $d(k+l) \equiv c \pmod{p}$, then for some $t = 1, \dots, d-2$ we have $d(k+l)-c = tp \equiv tc \not\equiv -c \pmod{d}$, which leads to a contradiction.

The proof of Lemma 2.2 is now complete. $\square$

Lemma 2.3. *Let $d > 2$ and $c \in (-d, d)$ be relatively prime integers, and let $n \geq 6d$ be an integer. Suppose that $m \in [n, (d((2+\varepsilon)n-1)-c)/(d-1)]$ is a power of two or twice an odd prime, where $0 < \varepsilon \leq 2/3$. Then, there are $1 \leq k < l \leq n$ such that $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$.*

Proof. Note that $m \geq n \geq 6d > 4$ and

$$\frac{m}{4} \leq \frac{d((2+\varepsilon)n-1)-c}{4(d-1)} < \frac{d(2+\varepsilon)}{4(d-1)}n \leq \frac{d(2+2/3)}{4(d-1)}n = \frac{8dn}{8d+4(d-3)} \leq n.$$

If d is even and m is a power of two, then for $k = 1$ and $l = m/4 + 1 \leq n$ we have $m \mid 2r(d)(l-k)$ and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$ by (2.1). If $m = 2p$ with p an odd prime dividing d , then $m \mid 2r(d)$ and hence $f_{d,c}(k) \equiv 0 \pmod{m}$ for all $k = 1, \dots, n$.

In the other cases, d and $m/2$ are relatively prime. Thus $jd \equiv c \pmod{m/2}$ for some $j = 1, \dots, m/2$. If $j \leq 2$, then

$$\frac{m}{2} \leq jd - c \leq 2d - c < 3d \leq \frac{n}{2}$$

which contradicts $m \geq n$. So $3 \leq j \leq m/2$ and hence

$$0 < k := \left\lfloor \frac{j-1}{2} \right\rfloor < l := \left\lfloor \frac{j+2}{2} \right\rfloor \leq \frac{m}{4} + 1 < n + 1.$$

Since $d(k+l) - c = jd - c \equiv 0 \pmod{m/2}$, by (2.1) we have $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$. This concludes the proof. $\square$

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Let $\varepsilon = 2/(\max\{11, d\} - 2)$. By Lemma 2.1, if $n \in \mathbb{Z}^+$ is large enough then there is at least a prime $p \equiv c \pmod{d}$ with

$$\frac{d(2n-1) - c}{d-1} < p \leq \frac{d((2+\varepsilon)n-1) - c}{d-1}. \quad (3.1)$$

(i) Choose an integer $N \geq \max\{6d, 243\}$ such that for any integer $n \geq N$ there is a prime $p \equiv c \pmod{d}$ satisfying (3.1). Fix an integer $n \geq N$ and let $m = m_{d,c}(n)$. Clearly $m \geq n$. By Lemma 2.2, $m \leq m'$ where m' denotes the least prime $p \equiv c \pmod{d}$ satisfying (3.1).

Assume that $m \neq m'$. We want to reach a contradiction. Clearly m is not a prime by Lemma 2.2. Note that $\varepsilon \leq 2/9$. In view of Lemma 2.3, m is neither a power of two nor twice an odd prime. So we have $m = pq$ for some odd prime p and integer $q > 2$. Observe that

$$\frac{m}{3} \leq \frac{d((2+\varepsilon)n-1) - c}{3(d-1)} < \frac{d(2+2/9)}{3(d-1)}n = \frac{20d}{27(d-1)}n \leq \frac{80}{81}n$$

and hence

$$\frac{m}{3} + 3 < \frac{80}{81}n + \frac{n}{81} = n. \quad (3.2)$$

If $p \mid d$, then for $k := 1$ and $l := q + 1 = m/p + 1 < m/3 + 3 < n$, we have $pq \mid r(d)(l-k)$ and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$ by (2.1).

Now suppose that $p \nmid d$. Then $2dk \equiv c - dq \pmod{p}$ for some $1 \leq k \leq p$. Clearly, $l := k + q \leq p + q = m/q + m/p$. Note that

$$(l-k)(d(l+k) - c) = q(d(2k+q) - c) \equiv 0 \pmod{pq}$$

and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$ by (2.1). If $\min\{p, q\} \leq 4$, then

$$l \leq p + q = \frac{m}{\min\{p, q\}} + \min\{p, q\} \leq \frac{m}{3} + 4 < n + 1$$

by (3.2). If $\min\{p, q\} \geq 5$, then

$$l \leq \frac{m}{q} + \frac{m}{p} \leq \max\left\{\frac{m}{6} + \frac{m}{7}, \frac{m}{5} + \frac{m}{8}\right\} < \frac{m}{3} < n$$

since $pq = m \geq n \geq 243 \geq 40$. So we get a contradiction as desired.

(ii) Now assume that $4 \leq d \leq 36$. By Table 1 of [RR, p. 419], we have

$$(1 - \varepsilon_d) \frac{x}{\varphi(d)} \leq \theta(x; c, d) \leq (1 + \varepsilon_d) \frac{x}{\varphi(d)} \quad \text{for all } x \geq 10^{10}, \quad (3.3)$$

where

$$\theta(x; c, d) := \sum_{\substack{p \leq x \\ p \equiv c \pmod{d}}} \log p \quad \text{with } p \text{ prime,}$$

and

$$\begin{aligned} \varepsilon_4 &= 0.002238, \varepsilon_5 = 0.002785, \varepsilon_6 = 0.002238, \varepsilon_7 = 0.003248, \varepsilon_8 = 0.002811, \\ \varepsilon_9 &= 0.003228, \varepsilon_{10} = 0.002785, \varepsilon_{11} = 0.004125, \varepsilon_{12} = 0.002781, \varepsilon_{13} = 0.004560, \\ \varepsilon_{14} &= 0.003248, \varepsilon_{15} = 0.008634, \varepsilon_{16} = 0.008994, \varepsilon_{17} = 0.010746, \varepsilon_{18} = 0.003228, \\ \varepsilon_{19} &= 0.011892, \varepsilon_{20} = 0.008501, \varepsilon_{21} = 0.009708, \varepsilon_{22} = 0.004125, \varepsilon_{23} = 0.012682, \\ \varepsilon_{24} &= 0.008173, \varepsilon_{25} = 0.012214, \varepsilon_{26} = 0.004560, \varepsilon_{27} = 0.011579, \varepsilon_{28} = 0.009908, \\ \varepsilon_{29} &= 0.014102, \varepsilon_{30} = 0.008634, \varepsilon_{31} = 0.014535, \varepsilon_{32} = 0.011103, \varepsilon_{33} = 0.011685, \\ \varepsilon_{34} &= 0.010746, \varepsilon_{35} = 0.012809, \varepsilon_{36} = 0.009544. \end{aligned}$$

As $\varepsilon = 2/(\max\{11, d\} - 2)$, we can easily verify that

$$\frac{\varepsilon}{2} - \frac{2}{10^{10}} > \frac{2\varepsilon_d}{1 - \varepsilon_d} = \frac{1 + \varepsilon_d}{1 - \varepsilon_d} - 1.$$

If $n \geq 10^{10}/2$, then

$$((2 + \varepsilon)n - 2) \frac{d}{d-1} \geq 2n \frac{d}{d-1} > 10^{10}$$

and

$$\frac{\varepsilon}{2} - \frac{1}{n} + 1 \geq \frac{\varepsilon}{2} - \frac{2}{10^{10}} + 1 > \frac{1 + \varepsilon_d}{1 - \varepsilon_d},$$

hence by (3.3) we have

$$\begin{aligned} & \frac{\theta(((2 + \varepsilon)n - 2)d/(d-1); c, d)}{\theta(2nd/(d-1); c, d)} \\ & \geq \frac{(1 - \varepsilon_d)((2 + \varepsilon)n - 2)d/(d-1)}{(1 + \varepsilon_d)2nd/(d-1)} = \frac{1 - \varepsilon_d}{1 + \varepsilon_d} \left(1 + \frac{\varepsilon}{2} - \frac{1}{n}\right) > 1 \end{aligned}$$

and thus there is a prime $p \equiv c \pmod{d}$ for which

$$\frac{2dn}{d-1} < p \leq \frac{((2+\varepsilon)n-2)d}{d-1}$$

and hence (3.1) holds.

Let N_d be the least positive integer such that for any $n = N_d, \dots, 10^{10}/2$ and any $a \in \mathbb{Z}$ relatively prime to d , the interval $(2dn/(d-1), ((2+\varepsilon)n-2)d/(d-1))$ contains a prime congruent to a modulo d . Via a computer we find that

$$\begin{aligned} N_4 &= 79, N_5 = 206, N_6 = 103, N_7 = 333, N_8 = 301, N_9 = 356, N_{10} = 232, \\ N_{11} &= 1079, N_{12} = 346, N_{13} = 1166, N_{14} = 806, N_{15} = 1310, N_{16} = 2183, \\ N_{17} &= 5153, N_{18} = 1135, N_{19} = 5402, N_{20} = 2388, N_{21} = 4059, N_{22} = 2934, \\ N_{23} &= 11246, N_{24} = 2480, N_{25} = 13144, N_{26} = 4775, N_{27} = 11646, \\ N_{28} &= 5314, N_{29} = 13478, N_{30} = 5215, N_{31} = 24334, N_{32} = 8964, \\ N_{33} &= 15044, N_{34} = 14748, N_{35} = 16896, N_{36} = 9847. \end{aligned}$$

For any integer $n \geq N_d$, there is a prime $p \equiv c \pmod{d}$ satisfying (3.1). Note that $6d \leq 6 \times 36 < 243$. So, for $n \geq N = \max\{N_d, 243\}$ we may apply part (i) to get the desired result. If $M_d < n \leq \max\{N_d, 243\}$, then we can easily verify the desired result via a computer.

In view of the above, we have completed the proof of Theorem 1.1. $\square$

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