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The p-adic representation of the Weil–Deligne group associated to an abelian variety

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Let A be an abelian variety defined over a number field $F \subset \mathbf{C}$ and let G_A be the Mumford–Tate group of $A_{/\mathbf{C}}$. After replacing F by a finite extension, we can assume that, for every prime number ℓ, the action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}}, \mathbf{Q}_\ell)$ factors through a map $\rho_\ell \colon \Gamma_F \to G_A(\mathbf{Q}_\ell)$. Fix a valuation v of F and let p be the residue characteristic at v. For any prime number $\ell \neq p$, the representation ρ_ℓ gives rise to a representation $'W_{F_v} \to G_{A/\mathbf{Q}_\ell}$ of the Weil–Deligne group. In the case where A has semistable reduction at v it was shown in a previous paper that, with some restrictions, these representations form a compatible system of $\mathbf{Q}$-rational representations with values in G_A.
The p-adic representation ρ_p defines a representation of the Weil–Deligne group $'W_{F_v} \to G^\iota_{A/F_{v,0}}$, where $F_{v,0}$ is the maximal unramified extension of $\mathbf{Q}_p$ contained in F_v and G^ι_A is an inner form of G_A over $F_{v,0}$. It is proved, under the same conditions as in the previous theorem, that, as a representation with values in G_A, this representation is $\mathbf{Q}$-rational and that it is compatible with the above system of representations $'W_{F_v} \to G_{A/\mathbf{Q}_\ell}$.

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0. Introduction

This paper is dedicated to the comparison of the étale and the log-crystalline cohomologies of an abelian variety over a number field. More precisely, it treats the way in which the action of the Galois group of the base field on the étale cohomology is reflected in the crystalline theory. We start with a succinct overview of relevant conjectures and results, referring to the introduction of [Noo13] for a more detailed discussion concerning the system of representations afforded by the étale cohomology.

Let X be a proper and smooth variety over a finite extension F_v of $\mathbf{Q}_p$. For any prime number ℓ and any i, the absolute Galois group $\Gamma_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$ acts on the étale cohomology group $\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_\ell)$. If $\ell \neq p$, the corresponding representation of Γ_{F_v} gives rise to an ℓ-adic representation of the Weil–Deligne group $'W_{F_v}$ of F_v. For general X, it is conjectured that these representations are $\mathbf{Q}$-rational and that, for fixed i and variable $\ell \neq p$, they form a compatible system of representations of $'W_{F_v}$, see [Del73] and [Fon94b, 2.4]. In the case where X has good reduction, the étale cohomology of $X_{\bar{F}_v}$ is isomorpic to the étale cohomology of the special fibre of a proper and smooth model of X over the valuation ring of F_v and hence the inertia subgroup of Γ_{F_v} acts trivially. In this case, the conjecture comes down to the fact that the characteristic polynomial of the Frobenius element has rational coefficients and that it is independent of ℓ. This is a consequence of the Weil conjectures proved by Deligne.

Now assume that $X = A$ is an abelian variety, not necessarily with good reduction. As $\mathrm{H}^i_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell) \cong \wedge^i \mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ for every i, it is harmless to assume that $i = 1$. In this case, it is well known that the above conjecture is true, cf. [Del73, Exemple 8.10]. There is a more precise result in the case where A can be defined over a number field $F \subset F_v$ for which we also fix an embedding $F \subset \mathbf{C}$. In this case, the Mumford–Tate group G_A of A is defined. It is a linear algebraic group and $G_{A/\mathbf{Q}_\ell}$ acts on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ for every ℓ. Up to a finite extension of F, the representation of Γ_{F_v} on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ factors through $G_A(\mathbf{Q}_\ell)$ for all ℓ. For $\ell \neq p$, it follows that the associated representation of $'W_{F_v}$ factors through $G_{A/\mathbf{Q}_\ell}$. Under the hypothesis that A has semistable reduction and with a number of other restrictions, it is shown in [Noo13] that these representations of $'W_{F_v}$ with values in $G_{A/\mathbf{Q}_\ell}$ are $\mathbf{Q}$-rational and that they are pairwise conjugate for the action of a group containing G_A with finite index. The precise statement is recalled in Theorem 3.3. Note that the fact that A has semistable reduction at v implies that the inertia group $I_{F_v} \subset W_{F_v} \subset {'W_{F_v}}$ acts trivially on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ so that the representation of $'W_{F_v}$ is determined by the action of the monodromy operator N and the image the Frobenius element. The case of an abelian variety with good reduction, for which the monodromy N is also trivial, had previously been treated in [Noo09]. A more general motivic conjecture is proposed by Serre [Ser94, §12].

The main Theorem 3.8 of this paper extends the compatibility result for the ℓ-adic representations to the p-adic representation. In order to get an idea of the statement, it is useful to return briefly to the general case of a variety X/F_v with good reduction. Contrary to the its action on the ℓ-adic cohomology, the action of Γ_{F_v} on the

$\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_p)$ is ramified, so the compatibility statement for the ℓ-adic representations can not extend to the p-adic representation as it stands. In order to include a p-adic component in the compatible system one may consider the crystalline cohomology. Katz and Messing [KM74] prove that $\mathrm{H}^i_{\text{cris}}(X_k/W(k)) \otimes F_{v,0}$, endowed with the $F_{v,0}$-linear Frobenius map, is compatible with the system of $\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_\ell)$ for $\ell \neq p$. Here k is the residue field of F_v and $W(k)$ the ring of Witt vectors with coefficients in k. The field $F_{v,0}$ is the maximal unramified extension of $\mathbf{Q}_p$ contained in F_v, it coincides with the fraction field of $W(k)$. We write X_k/k for the special fibre of a smooth and proper model of X over the valuation ring of F_v. The $F_{v,0}$-linear Frobenius map is the fth power of the crystalline Frobenius, for f such that $|k| = p^f$.

We again consider an abelian variety A over a number field F, with good reduction at a valuation v. As before, the Mumford–Tate group G_A is assumed to be connected. An inner form G^ι_A of the Mumford–Tate group G_A acts on $\mathrm{H}^1_{\text{cris}}(A_k/W(k)) \otimes F_{v,0}$ and the fth power of the crystalline Frobenius belongs to this group. Under some extra conditions, it is proved in [Noo09, §4] that the $F_{v,0}$-linear crystalline Frobenius completes the compatible system of Frobenius elements in the $G_A(\mathbf{Q}_\ell)$ provided by the ℓ-adic étale cohomology groups for $\ell \neq p$. As the crystalline cohomology and the p-adic étale cohomology are linked by Fontaine's functor D_{cris}, this yields an equivalent statement for the action of Frobenius on $D_{\text{cris}}(\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_p))$.

For any proper and smooth variety X/F_v, Fontaine [Fon94b] constructs a p-adic representation of $'W_{F_v}$ on the P_0-module $\widehat{D}_{\text{pst}}(\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_p))$. Here P_0 is the fraction field of the Witt ring $W(\bar{k})$. He conjectures that this representation fits in the compatible system of ℓ-adic representations of $'W_{F_v}$ defined by the $\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_\ell)$, see [Fon94b, 2.4.3], conjecture C_{WD}. The conjecture is true if X is a curve or an abelian variety.

In this paper we will consider the case of an abelian variety A/F with semistable reduction at v and the $F_{v,0}$-vector space $D_{\text{st}}(\mathrm{H}^i_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_p))$. This space again carries a representation of the Weil–Deligne group $'W_{F_v}$. It is linked to the P_0-vector space considered by Fontaine by

$$\widehat{D}_{\text{pst}}(\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_p)) = D_{\text{st}}(\mathrm{H}^i_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_p)) \otimes_{F_{v,0}} P_0.$$

The hypothesis that A has semistable reduction at v implies that, as in the case of an ℓ-adic representation, the inertia group $I_{F_v} \subset {}'W_{F_v}$ acts trivially. As before, we may replace F by a finite extension such that the p-adic representation of $'W_{F_v}$ factors through an inner form $G^\iota_{A/F_{v,0}}$ of the Mumford–Tate group G_A so we obtain a representation $'W_{F_v} \to G^\iota_{A/F_{v,0}}$. Under some extra conditions, it is shown in Theorem 3.8 that this is a $\mathbf{Q}$-rational representation of $'W_{F_v}$ with values in G^ι_A and that, together with the representations of $'W_{F_v} \to G_{A/\mathbf{Q}_\ell}$ for $\ell \neq p$, it gives rise to a compatible system of representations of $'W_{F_v}$ with values in G_A.

1. Semistable reduction of abelian varieties with L-action

1.1 Abelian varieties over local fields. We place ourselves in the situation considered in [CI99, Part I, §2]. So let p be a prime number, F a finite extension of $\mathbf{Q}_p$ and let $F_0 \subset F$ be the maximal unramified extension of $\mathbf{Q}_p$ contained in F. Let σ be the automorphism of F_0 inducing the absolute Frobenius map $x \mapsto x^p$ on the residue field k. Assume that A is an abelian variety over F with semistable reduction. We fix a number field $L \subset \mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$. In this section, we will review the results of Coleman and Iovita from [CI99] comparing the p-adic étale cohomology of A and its p-adic de Rham cohomology, with F_0-structure and endowed with the Frobenius and the monodromy operators. In particular, we discuss the compatibility of the isomorphism with the natural L-action on both sides.

As pointed out in [CI99], the p-adic uniformisation realises the rigid analytic group associated to A as a quotient G/Y, where Y is a F-group scheme which is étale locally isomorphic to $\mathbf{Z}^r$ and G is a semiabelian variety over F with good reduction. The latter condition means that G is an extension

$$1 \to T \to G \to B \to 1,$$

where T is a torus and B an abelian variety, both admitting good reduction over F. We will write B_k for the special fibre of the smooth and proper model of B over the valuation ring of F. By functoriality, we have $L \subset \mathrm{End}^0(Y)$ and $L \subset \mathrm{End}^0(G)$, whence also $L \subset \mathrm{End}^0(T)$ and $L \subset \mathrm{End}^0(B)$, cf. [Noo13, 1.1]. The groups Y and $Y^\star = \mathrm{Hom}(T, \mathbf{G}_m)$ are free $\mathbf{Z}$-modules of finite rank endowed with a continuous action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$. Moreover, $Y \otimes_{\mathbf{Z}} \mathbf{Q}$ and $Y^\star \otimes_{\mathbf{Z}} \mathbf{Q}$ are endowed with L-vector space structures, the actions of Γ_F on both spaces being L-linear.

1.2 Remarks. As pointed out, without proof, in [Noo13, Remark 3.7.1], the action of Γ_F on both Y and $Y^\star$ is unramified. This can be seen as follows. For the action on $Y^\star$, it follows from the fact that the torus T splits over a finite unramified extension of F, cf. [DG70, Exposé X, Corollaire 4.5]. By [BL91, Theorem 1.2], the lattice Y is constant over the same extension. This also shows that Y and $Y^\star$ become constant over a finite extension of F, which is of course a general fact for finite dimensional $\mathbf{Q}$-linear representations of Γ_F.

In the terminology of [Ray94, 4.2], in particular theorem 4.2.2 of that paper, the data considered in 1.1 give rise to a strict 1-motive $M' = [Y \to G]$ and a morphism of rigid 1-motives

$$M'_{\mathrm{rig}} \to [0 \to A]_{\mathrm{rig}}$$

which is an isomorphism in the derived category $D^b_{\mathrm{rig}}(\mathrm{fppf})$. In [Noo13, 3.8], the situation is discussed from this point of view, see also Remark 1.9 and [MP15].

1.3 Formula (2.1) of [CI99, Part I] shows that the de Rham cohomology of A is the central term in the exact sequence

$$0 \to \mathrm{Hom}(Y, F) \to \mathrm{H}^1_{\mathrm{dR}}(A) \to \mathrm{H}^1_{\mathrm{dR}}(G) \to 0 \tag{1.3.$*$}$$

of F-vector spaces with L-action. In [CI99, subsection I.2.2], the log-crystalline cohomology of A is calculated by introducing an F_0-structure on $\mathrm{H}^1_{\mathrm{dR}}(A)$, under the assumption that Y and $Y^\star$ are constant. When this condition holds, the putative F_0-structure is obtained by splitting the exact sequence (1.3.$*$) and defining F_0-structures on the outer terms. We will briefly review the argument, and show that the construction is compatible with the L-action.

A class in $\mathrm{H}^1_{\mathrm{dR}}(A)$ corresponds to an invariant differential ω on the universal vector extension of G. In [CI99, Theorem I.2.1] it is shown that ω admits a unique primitive λ_ω which is a group homomorphism and whose restriction to T satisfies a supplementary condition. The map sending ω to $\lambda_{\omega|Y}$ defines the splitting we are looking for. The uniqueness of λ_ω implies that it is L-linear. The F_0-structure on $\mathrm{Hom}(Y, F)$ is given by $\mathrm{Hom}(Y, F_0)$, this space inherits an L-action from $\mathrm{Hom}(Y, F)$, given by the L-vector space structure on $Y \otimes_{\mathbf{Z}} \mathbf{Q}$.

In order to define the F_0-structure on $\mathrm{H}^1_{\mathrm{dR}}(G)$ one splits the exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{dR}}(B) \to \mathrm{H}^1_{\mathrm{dR}}(G) \to \mathrm{H}^1_{\mathrm{dR}}(T) \to 0 \tag{1.3.$\dagger$}$$

and uses the natural F_0-structures on the outer terms. Concretely these are given by $\mathrm{H}_{\mathrm{cris}}(B_k/F_0) \subset \mathrm{H}^1_{\mathrm{dR}}(B)$ and $\mathrm{Hom}(Y^\star, F_0) \subset \mathrm{Hom}(Y^\star, F) = \mathrm{H}^1_{\mathrm{dR}}(T)$. The splitting of (1.3.$\dagger$) is defined by identifying $\mathrm{H}^1_{\mathrm{dR}}(T)$ with the subspace of $\mathrm{H}^1_{\mathrm{dR}}(B)$ where the Frobenius operator acts as multiplication by p. As the Frobenius is L-linear, the sequence (1.3.$\dagger$) is L-linearly split and L acts compatibly on all F_0-vector spaces we constructed. For the details, in particular the constructions of the splittings, see [CI99].

In general, dropping the assumption that Y and $Y^\star$ are constant, we argue as follows. As pointed out in Remark 1.2, Y and $Y^\star$ become constant over a finite unramified Galois extension $\widetilde{F}$ of F. The maximal unramified extension $\widetilde{F}_0 \subset \widetilde{F}$ of $\mathbf{Q}_p$ is Galois over $\mathbf{Q}_p$ so it is stable under $\mathrm{Gal}(\widetilde{F}/F)$ and

$$F_0 = \widetilde{F}_0^{\mathrm{Gal}(\widetilde{F}/F)}.$$

Applying the above constructions to $A_{\widetilde{F}}$, we obtain sequences corresponding to (1.3.$*$) and (1.3.$\dagger$) for the cohomology over $\widetilde{F}$, both admitting an L-equivariant splitting. All constructions are compatible with the L-action and commute with the $\mathrm{Gal}(\widetilde{F}/F)$-action so the splittings descend to the sequences of $\mathrm{Gal}(\widetilde{F}/F)$-invariants. Taking $\mathrm{Gal}(\widetilde{F}/F)$-invariants in the $\widetilde{F}_0$-valued cohomology groups defined above, we obtain the promised F_0-structures. We write

$$\mathrm{Hom}'(Y, F_0) = \mathrm{Hom}_{\mathrm{Gal}(\widetilde{F}/F)}(Y, \widetilde{F}_0),$$

for the F_0-structure on $\mathrm{Hom}(Y, \widetilde{F}_0)$. This is the F_0-vector space of $\mathrm{Gal}(\widetilde{F}/F)$-equivariant maps. Here the action of the Galois group on Y is deduced from the $\widetilde{F}$-isomorphism $Y \cong \mathbf{Z}^r$ and its action on $\widetilde{F}_0 \subset \widetilde{F}$ is the restriction of the action on $\widetilde{F}$. Similarly, the F_0-structure on $\mathrm{H}^1_{\mathrm{dR}}(T)$ is obtained by taking Galois invariants in $Y^\star \otimes_{\mathbf{Z}} \widetilde{F}_0$. The L-actions on $\mathrm{Hom}'(Y, F_0)$ and on $\mathrm{H}^1_{\mathrm{dR}}(T)$ are induced by the ones on $Y \otimes_{\mathbf{Z}} \mathbf{Q}$ and on $Y^\star \otimes_{\mathbf{Z}} \mathbf{Q}$. The F_0-structure on $\mathrm{H}^1_{\mathrm{dR}}(B)$ is given by $\mathrm{H}^1_{\mathrm{cris}}(B/F_0)$.

We will write $\mathrm{H}^1_{\mathrm{HK}}(A)$, resp. $\mathrm{H}^1_{\mathrm{HK}}(G)$ etc. for these F_0-vector spaces. By construction, the split exact sequences (1.3.$*$) and (1.3.†) have natural counterparts for the $\mathrm{H}^1_{\mathrm{HK}}(\,\cdot\,)$, where in (1.3.$*$) one has to replace $\mathrm{Hom}(Y, F)$ by $\mathrm{Hom}'(Y, F_0)$ as defined above. Taking into account that $Y \otimes_{\mathbf{Z}} F' = (Y \otimes_{\mathbf{Z}} F_0) \otimes_{F_0} F'$ and $\mathrm{Hom}_{\mathbf{Z}}(Y^\star, F') = \mathrm{Hom}_{F_0}(Y^\star \otimes {}_{\mathbf{Z}} F_0, F')$ for any extension F'/F_0, the following lemma, applied to $X = Y \otimes_{\mathbf{Q}} F_0$ and to $X = Y^\star \otimes_{\mathbf{Q}} F_0$ respectively, implies that $\mathrm{Hom}'(Y, F_0)$ and $\mathrm{H}^1_{\mathrm{HK}}(T)$ are free $L \otimes_{\mathbf{Q}} F_0$-modules and that the dimensions are given by $\dim_{F_0} \mathrm{Hom}'(Y, F_0) = \mathrm{rk}_{\mathbf{Z}} Y$ and $\dim_{F_0} \mathrm{H}^1_{\mathrm{HK}}(T) = \mathrm{rk}_{\mathbf{Z}} Y^\star = \dim T$.

As $\mathrm{H}^1_{\mathrm{cris}}(B/F_0)$ is a free $L \otimes_{\mathbf{Q}} F_0$-module as well, cf. [Noo09, §4], the same is true for $\mathrm{H}^1_{\mathrm{HK}}(G)$ and $\mathrm{H}^1_{\mathrm{HK}}(A)$. We have $\dim_{F_0} \mathrm{H}^1_{\mathrm{HK}}(A) = 2 \dim A$.

1.4 Lemma. *For L, F and F_0 as above, let X be a free $L \otimes_{\mathbf{Q}} F_0$-module of finite rank d, endowed with a continuous, unramified, $L \otimes_{\mathbf{Q}} F_0$-linear action of Γ_F. This action factors through $\mathrm{Gal}(\widetilde{F}/F)$ for some finite, unramified Galois extension $\widetilde{F}$ of F. Let $\widetilde{F}_0$ be the maximal unramified extension of $\mathbf{Q}_p$ contained in $\widetilde{F}$. Then*

$$\left(X \otimes_{F_0} \widetilde{F}_0\right)^{\mathrm{Gal}(\widetilde{F}/F)}$$

is a free $L \otimes_{\mathbf{Q}} F_0$-module of rank d.

Proof. We sketch a proof of this classical fact. Write $G = \mathrm{Gal}(\widetilde{F}/F)$ and identify G with $\mathrm{Gal}(\widetilde{F}_0/F_0)$ via the restriction map $\mathrm{Gal}(\widetilde{F}/F) \to \mathrm{Gal}(\widetilde{F}_0/F_0)$.

First assume that $L = \mathbf{Q}$ so that $d = \dim_{F_0} X$. The normal base theorem implies that $\widetilde{F}_0$, considered as F_0-linear representation of G, is isomorphic to the regular representation of G. It follows that, as $\mathbf{C}$-linear representations, each irreducible factor of $X^\vee \otimes_{F_0} \mathbf{C}$ occurs in $\widetilde{F}_0 \otimes_{F_0} \mathbf{C}$ with multiplicity equal to its dimension. This implies that $\dim_{\mathbf{C}}((X \otimes_{F_0} \widetilde{F}_0) \otimes_{F_0} \mathbf{C})^{\mathrm{Gal}(\widetilde{F}/F)} = \dim X$. The case $L = \mathbf{Q}$ of the lemma follows from this. One may also use an argument based on the vanishing of the Galois cohomology group $\mathrm{H}^1(\Gamma_F, \mathrm{GL}(d, \bar{F}))$.

In the general case, $(X \otimes_{F_0} \widetilde{F}_0)^G$ is clearly an $L \otimes_{\mathbf{Q}} F_0$-module. To see that it is free of rank d, write $L \otimes_{\mathbf{Q}} F_0 = \oplus L_i$ as a direct sum of finite extension fields of F_0 and decompose $X = \oplus X_i$ accordingly. Then $(X \otimes_{F_0} \widetilde{F}_0)^G = \oplus (X_i \otimes_{F_0} \widetilde{F}_0)^G$ as $L \otimes_{\mathbf{Q}} F_0$-modules and by the above argument

$$[L_i : F_0] \dim_{L_i} (X_i \otimes_{F_0} \widetilde{F}_0)^G = \dim_{F_0} (X_i \otimes_{F_0} \widetilde{F}_0)^G = \dim_{F_0} X_i = d[L_i : F_0],$$

so each $(X_i \otimes_{F_0} \widetilde{F}_0)^G$ has L_i-dimension d, whence the conclusion. □

1.5 The monodromy and Frobenius operators. The torus T and its character group $Y^\star$ being as above, the descriptions of the monodromy paring given in [Ray94, 4.3] and in [CI99, I.2.1] (where Y is denoted Γ) show that the parings on $\mu\colon Y \times Y^\star \to \mathbf{Z}$ defined in both references coincide. Giving μ is equivalent to giving either the map $N\colon Y \otimes \mathbf{Q} \to (Y^\star \otimes \mathbf{Q})^\vee$ from [Noo13, (1.4∗)] or its dual

$$N^\vee\colon (Y^\star \otimes \mathbf{Q}) \to (Y \otimes \mathbf{Q})^\vee = \mathrm{Hom}(Y, \mathbf{Q}). \tag{1.5.∗}$$

By functoriality, the monodromy paring is both L- and Γ_F-equivariant, hence the maps N and $N^\vee$ are L-linear and commute with the action of Γ_F.

With $\widetilde{F}$ as above, $N^\vee$ induces an $L \otimes F_0$-linear map

$$N^\vee_{F_0}\colon \mathrm{H}^1_{\mathrm{HK}}(T) = \left(Y^\star \otimes_{\mathbf{Z}} \widetilde{F}_0\right)^{\mathrm{Gal}(\widetilde{F}/F)} \to \mathrm{Hom}'(Y, F_0) = \mathrm{Hom}_{\mathrm{Gal}(\widetilde{F}/F)}(Y, \widetilde{F}_0).$$

The monodromy operator N_{HK} on $\mathrm{H}^1_{\mathrm{HK}}(A)$ is defined as the composite

$$\mathrm{H}^1_{\mathrm{HK}}(A) \longrightarrow \mathrm{H}^1_{\mathrm{HK}}(G) \longrightarrow \mathrm{H}^1_{\mathrm{HK}}(T) \xrightarrow{N^\vee_{F_0}} \mathrm{Hom}'(Y, F_0) \longrightarrow \mathrm{H}^1_{\mathrm{HK}}(A).$$

This is the construction of [CI99, 2.1] in case T is split over F. In the general case, $N^\vee_{F_0}$ is obtained by constructing $N^\vee_{\widetilde{F}_0}$ after base change to $\widetilde{F}$ and then taking $\mathrm{Gal}(\widetilde{F}/F)$-invariants in the resulting $\widetilde{F}_0$-cohomology groups.

To define the σ-linear Frobenius operator $\varphi\colon \mathrm{H}^1_{\mathrm{HK}}(A) \to \mathrm{H}^1_{\mathrm{HK}}(A)$, one uses the split exact HK-versions of the exact sequences (1.3.∗) and (1.3.†) and the canonical σ-linear operators on $\mathrm{Hom}(Y, F_0)$, and on the groups $\mathrm{H}^1_{\mathrm{HK}}(B)$ and $\mathrm{H}^1_{\mathrm{HK}}(T)$. On $\mathrm{H}^1_{\mathrm{HK}}(B) = \mathrm{H}^1_{\mathrm{cris}}(B/F_0)$ this is the crystalline Frobenius φ of the abelian variety B. To define φ on $\mathrm{Hom}'(Y, F_0)$, we first extend scalars to $\widetilde{F}$, making Y constant, and define $\varphi\colon \mathrm{Hom}(Y, \widetilde{F}_0) \to \mathrm{Hom}(Y, \widetilde{F}_0)$ by composition of homomorphisms $Y \to \widetilde{F}_0$ with $\sigma\colon \widetilde{F}_0 \to \widetilde{F}_0$. This map commutes with the action of $\mathrm{Gal}(\widetilde{F}/F)$ and hence it defines a σ-linear map on $\mathrm{Hom}'(Y, F_0)$. Finally, for $\mathrm{H}^1_{\mathrm{HK}}(T) = (Y^\star \otimes_{\mathbf{Z}} \widetilde{F}_0)^{\mathrm{Gal}(\widetilde{F}/F)}$, one defines $\varphi = p \otimes \sigma$ on $Y^\star \otimes_{\mathbf{Z}} \widetilde{F}_0$ and passes to the $\mathrm{Gal}(\widetilde{F}/F)$-invariants. The maps give rise to σ-linear Frobenius maps, still denoted, φ on $\mathrm{H}^1_{\mathrm{HK}}(G)$ and on $\mathrm{H}^1_{\mathrm{HK}}(A)$. All these maps are L-linear. By construction, φ and N_{HK} satisfy $N_{\mathrm{HK}}\varphi = p\varphi N_{\mathrm{HK}}$.

1.6 Proposition. *Let $f = [F_0 : \mathbf{Q}_p] = [k : \mathbf{F}_p]$ and let $\Phi \in \Gamma_F$ be an arithmetic Frobenius element. It defines an $L \otimes_{\mathbf{Q}} F_0$-linear endomorphism of $(Y \otimes \mathbf{Q})^\vee \otimes_{\mathbf{Q}} F_0$ by its action on $(Y \otimes \mathbf{Q})^\vee$. On the other hand, φ^{-f} defines a $L \otimes_{\mathbf{Q}} F_0$-linear endomorphism of $\mathrm{Hom}'(Y, F_0)$. There exists an $L \otimes_{\mathbf{Q}} F_0$-linear isomorphism*

$$(Y \otimes \mathbf{Q})^\vee \otimes_{\mathbf{Q}} F_0 \cong \mathrm{Hom}'(Y, F_0)$$

transforming Φ to φ^{-f}. There is an $L \otimes_{\mathbf{Q}} F_0$-linear isomorphism $Y^\star \otimes F_0 \cong \mathrm{H}^1_{\mathrm{HK}}(T)$ transforming the action of Φ on $Y^\star \otimes F_0$ deduced from that on $Y^\star$ into φ^{-f}. These isomorphisms can be chosen as to be compatible with $N^\vee$ and $N^\vee_{F_0}$.

Proof. Let $\widetilde{F}$ and $G = \mathrm{Gal}(\widetilde{F}/F) = \mathrm{Gal}(\widetilde{F}_0/F_0)$ be as in the proof of Lemma 1.4. The action of Γ_F factors through G so we consider Φ as an element of G. The effect of φ^{-f} on $\mathrm{Hom}(Y, \widetilde{F}_0) = (Y \otimes \mathbf{Q})^\vee \otimes \widetilde{F}_0$ is given by $g \mapsto \Phi^{-1} \circ g$. The restriction of this action to $\mathrm{Hom}'(Y, F_0)$ coincides with the restriction of the endomorphism of $(Y \otimes \mathbf{Q})^\vee \otimes \widetilde{F}_0$ defined by $g \otimes \lambda \mapsto (g \circ \Phi^{-1}) \otimes \lambda$, that is to say the action of Φ on $Y \otimes \mathbf{Q}$.

Consider the inclusions

$$(Y \otimes \mathbf{Q})^\vee \subset \mathrm{Hom}(Y, \widetilde{F}_0) \supset \mathrm{Hom}'(Y, F_0).$$

These have the property that the image of any L-free, respectively $L \otimes F_0$-free, family in the outer spaces is $L \otimes \widetilde{F}_0$-free in the middle space. It follows from the Lemma 1.4 that there exist $L \otimes_{\mathbf{Q}} \widetilde{F}_0$-linear identifications of $(Y \otimes \mathbf{Q})^\vee \otimes_{\mathbf{Q}} \widetilde{F}_0$ and of $\mathrm{Hom}'(Y, F_0) \otimes_{F_0} \widetilde{F}_0$ with $\mathrm{Hom}(Y, \widetilde{F}_0)$. By the above, the action of Φ on $(Y \otimes \mathbf{Q})^\vee$ transforms to the action of φ^{-f} on $\mathrm{Hom}(Y, \widetilde{F}_0)$ and on $\mathrm{Hom}'(Y, F_0)$. This proves the existence of the desired isomorphism after $\otimes_{F_0} \widetilde{F}_0$. It follows that there already is an isomorphism with the desired properties on the level of the underlying $L \otimes_{\mathbf{Q}} F_0$ modules.

By [Noo13, Lemma 3.9] or [CI99, Proposition 4.5], $N^\vee$ is an isomorphism in this case so the statements concerning $\mathrm{H}^1_{\mathrm{HK}}(T)$ and the possibility to choose the isomorphisms to be compatible with $N^\vee$ and $N^\vee_{F_0}$ follow immediately. □

1.7 The *L*-vector space *V*. The aim of this subsection is to construct a graded L-vector space $V = V^0 \oplus V^1 \oplus V^2$ endowed with operators Φ and N such that $V \otimes F_0 \cong \mathrm{H}^1_{\mathrm{HK}}(A)$, compatibly with the L-action and carrying N, Φ to $N_{\mathrm{HK}}, \varphi^{-f}$.

Let $V^0 = (Y \otimes_{\mathbf{Z}} \mathbf{Q})^\vee$ and $V^2 = Y^\star \otimes_{\mathbf{Z}} \mathbf{Q}$, endowed with the action of L from 1.1. The action of the Frobenius $\Phi \in \Gamma_F$ is defined by the action of Γ_F on Y where V^0 is concerned and by q times its action on $Y^\star$ for V^2. Here $q = |k|$. Let $N^\vee \colon V^2 \to V^0$ be as in (1.5.∗), it is L-linear and satisfies $N\Phi = q\Phi N$.

To define the L-vector space V^1, consider $\mathrm{H}^1_{\mathrm{cris}}(B_k/F_0)$, where B/F is as in 1.1. It is an abelian variety with good reduction and B_k the special fibre of a smooth model over the valuation ring of F. The field L naturally acts on $\mathrm{H}^1_{\mathrm{cris}}(B_k/F_0)$, making it an $L \otimes_{\mathbf{Q}} F_0$-module. Following [Noo06, Lemma 6.13] and [Noo09, Proposition 4.2], we describe its structure. It is well known, see [Wat69, Chapter 2] and [Mum70, §19, Theorem 4], that for every endomorphism α of B_k the characteristic polynomial of α acting on $\mathrm{H}^1_{\mathrm{cris}}(B_k/F_0)$ has coefficients in $\mathbf{Q}$ and is equal to the characteristic polynomial of α acting on $\mathrm{H}^1_{\text{ét}}(B_{\bar{k}}, \mathbf{Q}_\ell)$, for any prime number $\ell \neq p$.

Let $\pi \colon B_k \to B_k$ be the Frobenius endomorphism of the abelian variety B_k. As π is semisimple and lies in the centre of $\mathrm{End}^0(B_k)$, the subalgebra $L(\pi)$ is a product of number fields $L_i = \mathbf{Q}(\alpha_i)$ and applying the above remark to each α_i, we conclude that there exists an $L(\pi)$-module V^1 such that

$$\mathrm{H}^1_{\mathrm{cris}}(B_k/F_0) \cong V^1 \otimes_{\mathbf{Q}} F_0 \tag{1.7.∗}$$

as $L(\pi)\otimes_{\mathbf{Q}} F_0$-modules. Moreover $\mathrm{H}^1_{\text{ét}}(B_{\bar{k}},\mathbf{Q}_\ell)\cong V^1\otimes_{\mathbf{Q}}\mathbf{Q}_\ell$ as $L(\pi)\otimes_{\mathbf{Q}}\mathbf{Q}_\ell$-modules, for every $\ell\neq p$ as above. We consider V^1 as an L-vector space and make Φ act as π^{-1}. By [Noo09, §4], this map corresponds to the crystalline Frobenius under the isomorphism (1.7.$*$).

We have now defined $V=V^0\oplus V^1\oplus V^2$ endowed with L-linear operators N and Φ. As $(\mathrm{H}^1_{\mathrm{HK}}(A),\varphi)$ is a direct sum by construction, this proves the first part of the Proposition 1.8 below.

As for the second statement, we refer to [Noo13, §1], taking into account that that paper concerns the Tate modules which are dual to the $\mathrm{H}^1_{\text{ét}}$. For any prime number $\ell\neq p$, giving the action on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}},\mathbf{Q}_\ell)$ of the inertia subgroup of Γ_F is equivalent to giving a nilpotent endomorphism N_ℓ of $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}},\mathbf{Q}_\ell)$, see [Noo13, 1.6]. The Frobenius element $\Phi\in\Gamma_F$ acts on the étale cohomology so $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}},\mathbf{Q}_\ell)$ is endowed with operators N_ℓ and Φ. The second statement of the proposition follows from [Noo13, 1.8] and the fact that $N\colon V^2\to V^0$ is an isomorphism.

1.8 Proposition. *With the above notations and definitions, there exists an $L\otimes_{\mathbf{Q}}F_0$-linear isomorphism $\mathrm{H}^1_{\mathrm{HK}}(A)\cong V\otimes_{\mathbf{Q}}F_0$ taking N_{HK} to $N\otimes 1$ and φ^{-f} to Φ. For each prime number $\ell\neq p$ there is an $L\otimes_{\mathbf{Q}}\mathbf{Q}_\ell$-linear isomorphism $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}},\mathbf{Q}_\ell)\cong V\otimes\mathbf{Q}_\ell$ taking N_ℓ to $N\otimes 1$ and compatible with Φ.*

1.9 Remark. As an alternative to [CI99], one may use [MP15], especially 1.3 and Appendix A.1, to define the log-crystalline cohomology of A. The construction of Madapusi-Pera works over a more general base than what is needed here. In the case considered in Proposition 1.8, his argument is closer to the approach of [Noo13]. It amounts to replacing A by the strict 1-motive M constructed by Raynaud, as in the second remark of 1.2, and by constructing the log-crystalline cohomology of M.

The comparison theorem of Coleman and Iovita relating the log-crystalline and p-adic étale cohomologies of A plays an essential role in the proof of Corollary 2.5. Madapusi-Pera [MP15, Proposition 1.4.10] establishes a similar result, again in a more general setting.

2. Representations of the Weil–Deligne group

2.1 The action of the Weil–Deligne group. As before, F is a finite extension of $\mathbf{Q}_p$. We adopt the notations of [Noo13, 2.1], which in turn follow [Del73, §8] and [Fon94b]. So W_F is the Weil group of F, i.e. the set of elements $w\in\Gamma_F$ inducing an integral, say $\alpha(w)$th, power of the Frobenius automorphism of the residue field extension $\bar{k}/k$. This defines a surjective morphism $\alpha\colon W_F\to\mathbf{Z}$. Note that it differs by a factor f from the morphism from [Fon94b], for f defined by $q=p^f=|k|$, where k is the residue field of F. For the purpose of this paper, it is sufficient to describe the linear representations of $'W_F$. If E is a field of characteristic 0 and H is a linear algebraic group over E, then a representation of $'W_F$ with values in H is a pair (ρ',N) where $\rho'\colon W_F\to H(E)$ is a homomorphism, trivial on some open subgroup of the inertia group I_F, and $N \in \mathrm{Lie}(H)$ is a nilpotent element such that $\mathrm{Ad}(\rho'(w))N = q^{\alpha(w)}N$ for all $w \in W_F$.

As in section 1, let A/F be an abelian variety with semistable reduction. The (φ, N)-module $(\mathrm{H}^1_{\mathrm{HK}}(A), \varphi, N_{\mathrm{HK}})$ gives rise to an F_0-linear representation $(\rho'_{F_0}, N_{\mathrm{HK}})$ of $'W_F$ by setting $\rho'_{F_0}(w) = \varphi^{-f\alpha(w)}$, for $w \in W_F$.

Similarly, the triple (V, Φ, N) constructed in 1.7 defines a $\mathbf{Q}$-linear representation $(\rho'_{\mathbf{Q}}, N)$ of $'W_F$ by $\rho'_{\mathbf{Q}}(w) = \Phi^{-\alpha(w)}$, for $w \in W_F$. This representation takes values in the group $H = \mathrm{Res}_{L/\mathbf{Q}}\mathrm{GL}_{/L}(V)$ of L-linear automorphisms of V. By the Proposition 1.8, there is an identification $\mathrm{H}^1_{\mathrm{HK}}(A) \cong V \otimes F_0$, compatible with L action, describing $(\rho'_{F_0}, N_{\mathrm{HK}})$ as the extension of scalars to F_0 of $(\rho'_{\mathbf{Q}}, N)$. In particular, the representation of $'W_F$ associated to $\mathrm{H}^1_{\mathrm{HK}}(A)$ takes values in $H_{/F_0}$.

By [Noo13, 2.2, 2.4], for each $\ell \neq p$ the Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_\ell)$ gives rise to a representation (ρ'_ℓ, N_ℓ) of $'W_F$ with values in H. These representations are also base extensions of $(\rho'_{\mathbf{Q}}, N_{\mathbf{Q}})$. This establishes the following corollary.

2.2 Corollary. *The system of representations with values in H formed by $(\rho'_{F_0}, N_{\mathrm{HK}})$ and the (ρ'_ℓ, N_ℓ), for $\ell \neq p$ prime, is defined over $\mathbf{Q}$ and forms a compatible system of representations of $'W_F$ with values in H.*

2.3 Some terminology. Let E be a field of characteristic 0 and $E' \supset E$ an extension. Fix a linear algebraic group H over E. A representation (ρ', N) of $'W_F$ with values in $H_{/E'}$ is *defined over E* if, for every algebraically closed overfield $\Omega \supset E'$ and every $\tau \in \mathrm{Aut}_E(\Omega)$, the representations $(\rho' \otimes_E \Omega, N_\Omega)$ and $(\tau(\rho' \otimes_E \Omega), \tau(N_\Omega))$ are $H(\Omega)$-conjugate. If E_i is a family of extensions of E and if, for each index i, we have a representation (ρ'_i, N_i) of $'W_F$ with values in $H_{/E_i}$, then the system of (ρ'_i, N_i) is a *compatible system of representations of $'W_F$* if for every pair (i,j) of indices and every algebraically closed field $\Omega \supset E_i, E_j$, there is a $h \in H(\Omega)$ such that $\rho'_i \otimes_{E_i} \Omega = h(\rho'_j \otimes_{E_j} \Omega)h^{-1}$ and $N_i \otimes_{E_i} 1 = \mathrm{Ad}(h)(N_j \otimes_{E_j} 1)$ in $\mathrm{Lie}(H) \otimes \Omega$.

As in [Noo13, 3.4], we also consider a more general situation where $H^\natural$ is a linear algebraic group over E, acting on H. In this case, we define the notions of a representation of $'W_F$ with values in H *defined over E modulo $H^\natural$* and of a *compatible system of representations of $'W_v$ modulo $H^\natural$* by replacing $H(\Omega)$-conjugacy by $H^\natural(\Omega)$-conjugacy in the above definitions.

2.4 The p-adic representation. Using the semistable comparison theorem, the statement of the corollary can be rephrased in terms of the p-adic étale cohomology of A as follows. Let B_{st} be Fontaine's period ring, cf. [Fon94a, §3] and put

$$D_{\mathrm{st}}(A) = D_{\mathrm{st}}\left(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)\right) = \left(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}\right)^{\Gamma_F}. \tag{2.4.*}$$

As $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ is a semi-stable Galois representation, as B_{st} is endowed with operators φ and N commuting with the Γ_F-action and as $B_{\mathrm{st}}^{\Gamma_F} = F_0$, this is a (φ, N)-module of rank

$2\dim(A)$ over F_0, i.e. a F_0-vector space of dimension $2\dim(A)$ endowed with operators φ and N such that φ is σ-linear and N is F_0-linear and satisfying $N\varphi = p\varphi N$. Since L acts $\mathbf{Q}_p$-linearly on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ and since this action commutes with the Γ_F-action, $D_{\mathrm{st}}(A)$ carries an L-action commuting with all these structures. By the procedure of 2.1, this gives rise to an F_0-linear representation $(\rho'_{F_0}, N_{\mathrm{st}})$ of $'W_v$ on $D_{\mathrm{st}}(A)$, with values in $H_{/F_0}$. Here we have identified $H_{/F_0}$ with the group of $L \otimes_{\mathbf{Q}} F_0$-linear automorphisms of $D_{\mathrm{st}}(A)$. As any two such identifications are conjugate under $H_{/F_0}$, the following statement is independent of this choice.

2.5 Corollary. *The above representation $(\rho'_{F_0}, N_{\mathrm{st}})$ and the (ρ'_ℓ, N_ℓ), for ℓ running through the prime numbers $\ell \neq p$, are defined over $\mathbf{Q}$ and form a compatible system, as representations of $'W_F$ with values in H.*

Proof. In [CI99, Part II, Theorem 7.13] Coleman and Iovita establish an isomorphism $D_{\mathrm{st}}(A) \cong \mathrm{H}^1_{\mathrm{HK}}(A)$, compatible with the Frobenius and monodromy operators. Note that this is an isomorphism of F_0-vector spaces, but that for our purposes the corresponding isomorphism over F, proved in theorem II.5.4 of [CI99], or even over $\bar{F}$ would be sufficient. We need to show that this isomorphism is compatible with the L-action. Taking into account that the semistable representations $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ and $V_p(A) = T_pA \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ are canonically dual, the above isomorphism is induced by the p-adic integration pairing

$$V_p(A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \times \mathrm{H}^1_{\mathrm{dR}}(A) \longrightarrow B^+_{\mathrm{dR}}$$

defined by Colmez in [Col92], so it suffices to prove that this pairing is L-equivariant. This follows from the functoriality of the antiderivative used to define the integration pairing, see [Col92, Proposition 4.1(iii)]. □

2.6 Remark. As endomorphisms of the appropriate cohomology groups, the monodromy operators appearing in the Corollaries 2.2 and 2.5 are nilpotent of echelon 2.

2.7 Remark. This approach differs from the point of view adopted in [Fon94b] for treating potentially semistable representations. For any potentially semistable $\mathbf{Q}_p$-linear representation V_p of Γ_F, Fontaine considers

$$\widehat{D}_{\mathrm{pst}}(V_p) = \varinjlim \left(V_p \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}\right)^H,$$

where the limit is taken over the open subgroups H of the inertia group $I \subset \Gamma_F$. The resulting space $\widehat{D}_{\mathrm{pst}}(V_p)$ is a P_0-vector space of dimension $\dim_{\mathbf{Q}_p} V_p$, where P_0 is the fraction field of the Witt ring $W(\bar{k})$. It is endowed with a σ-linear Frobenius map φ, a monodromy operator N and a semi-linear action of Γ_F. The operators φ and N satisfy the usual condition $N\varphi = p\varphi N$. The action of Γ_F is trivial on an open subgroup of the inertia group and hence defines an action of the Weil group W_F on $\widehat{D}_{\mathrm{pst}}(V_p)$. This operation commutes with the other structures.

If V_p is semistable, and not just potentially semistable, then the F_0 vector space $D_{\mathrm{st}}(V_p)$ defined as in (2.4.∗) is related to the above P_0-vector space by $\widehat{D}_{\mathrm{pst}}(V_p) = D_{\mathrm{st}}(V_p) \otimes_{F_0} P_0$. The operators N and φ on $\widehat{D}_{\mathrm{pst}}(V_p)$ correspond, on $D_{\mathrm{st}}(V_p) \otimes_{F_0} P_0$, to the P_0-linear extensions of the operator N considered in 2.4 and to the map defined by $x \otimes \lambda \mapsto \varphi(x) \otimes \sigma(\lambda)$ respectively. The action of the Weil group W_F on $\widehat{D}_{\mathrm{pst}}(V_p)$ corresponds to its action on the second factor P_0 of the tensor product $D_{\mathrm{st}}(V_p) \otimes_{F_0} P_0$. Note that the inertia group acts trivially on $\widehat{D}_{\mathrm{pst}}(V_p)$ if V_p is semistable.

According to [Fon94b, 1.3.5], this defines a P_0-linear representation (ρ_0, N) of the Weil–Deligne group $'W_F$ on $\widehat{D}_{\mathrm{pst}}(V_p)$ by setting $\rho_0(w) = w\varphi^{-f\alpha(w)}$, for α as in 2.1. The above discussion shows that, under the above comparison isomorphism, this representation of $'W_F$ corresponds to the P_0-linear extension to $D_{\mathrm{st}}(V_p) \otimes_{F_0} P_0$ of the representation of $'W_F$ on $D_{\mathrm{st}}(V_p)$ defined as in 2.4.

3. Representations with values in the Mumford–Tate group

3.1 Abelian varieties over number fields. Changing notation, we fix a number field $F \subset \mathbf{C}$ and an abelian variety A/F. We write $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$, where $\bar{F}$ is the algebraic closure of F in $\mathbf{C}$. Let G_A be the Mumford–Tate group of A. To define G_A, let $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$ be the tannakian category of absolute Hodge motives generated by the motive of A. Then G_A is the automorphism group of the fibre functor on $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$ defined by the Betti cohomology $A \mapsto \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. It is a linear algebraic group acting on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. We will assume that F is sufficiently large for G_A to be connected or equivalently that all Hodge classes on all powers of $A(\mathbf{C})$ are defined over F. This can always be achieved by replacing F by a finite extension.

For every prime number ℓ there is a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_\ell) \cong \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$$

so $G_{A/\mathbf{Q}_\ell}$ acts on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_\ell)$. The main theorem of [Del82] implies that the action of Γ_F on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_\ell)$ factors through a representation $\rho_\ell\colon \Gamma_F \to G_A(\mathbf{Q}_\ell)$. A less concise exposition of these matters can be found in [Noo09, 1.2].

Let $\bar{v}$ be a valuation of $\bar{F}$, with residue characteristic p, and let v be its restriction to F. Write F_v for the completion of F at v and identify $\Gamma_{F_v} = D_{\bar{v}}$, where $D_{\bar{v}} \subset \Gamma_F$ is the decomposition group of $\bar{v}$. As pointed out in 2.1, it follows from [Noo13, 2.2, 2.4] that, for $\ell \neq p$, giving the restriction $\rho_\ell|_{D_{\bar{v}}}$ is equivalent to giving a representation (ρ'_ℓ, N_ℓ) of the Weil–Deligne group $'W_v = {}'W_{F_v}$ with values in $G_{A/\mathbf{Q}_\ell}$. In particular $N_\ell \in \mathfrak{g}_A \otimes \mathbf{Q}_\ell$, where $\mathfrak{g}_A = \mathrm{Lie}(G_A)$. The system of (ρ'_ℓ, N_ℓ) for $\ell \neq p$ was studied in [Noo13]. Before citing the result, we recall some terminology.

3.2 Some definitions and notations. To state the main result we first recall that, if Ω is an algebraically closed field and g is a semisimple endomorphism of a finite dimensional

Ω-vector space V, then g is *neat* if the Zariski closure of the subgroup of $\mathrm{GL}_V(\Omega)$ generated by g is connected. The element $g \in G(\Omega)$ is *weakly neat* if 1 is the only root of unity among the quotients $\lambda\mu^{-1}$ of eigenvalues λ and μ of g. As suggested by the terminology, any neat element is weakly neat.

Secondly, let the linear algebraic group $G_A^{\natural\mathrm{ad}}$ be as in [Noo13, 3.3]. It is defined as follows. The derived group $G^{\mathrm{der}}_{A/\overline{\mathbf{Q}}}$ is the almost direct product of almost simple subgroups $G_i \subset G_{A/\overline{\mathbf{Q}}}$, for i in some index set I. Let $J \subset I$ be the set of indices i such that $G_i \cong \mathrm{SO}(2k_i)_{/\overline{\mathbf{Q}}}$ for some $k_i \geq 4$. For each $i \in J$ put $G'_i = \mathrm{O}(2k_i) \supset G_i$ and define

$$G_A^{\natural\mathrm{ad}} = \prod_{i\in J} G_i'^{\mathrm{ad}} \times \prod_{i\in I\setminus J} G_i^{\mathrm{ad}} \supset G^{\mathrm{ad}}_{A/\overline{\mathbf{Q}}}.$$

The adjoint actions of $G^{\mathrm{ad}}_{A/\overline{\mathbf{Q}}}$ on $G_{A/\overline{\mathbf{Q}}}$ and on $\mathfrak{g}_A$ extend to actions of $G_A^{\natural\mathrm{ad}}$. The group $G^{\mathrm{ad}}_{A/\overline{\mathbf{Q}}}$ acts trivially on the centres of $G_{A/\overline{\mathbf{Q}}}$ and of $\mathfrak{g}_A \otimes \overline{\mathbf{Q}}$. Applying the terminology of 2.3 to these groups, we will consider representations with values in G_A defined over $\mathbf{Q}$ modulo $G_A^{\natural\mathrm{ad}}$, as well as compatible systems modulo $G_A^{\natural\mathrm{ad}}$ of representations of $'W_v$ with values in G_A.

Let A/F and v be as above and let $\Phi_v \in D_v$ be an arithmetic Frobenius element. Assume that A has semistable reduction at v and that for some, hence any, prime number $\ell \neq p$, the image $\rho'_\ell(\Phi_v)$ is weakly neat.

3.3 Theorem *([Noo13, Theorem 3.6]). For each $\ell \neq p$, the representation (ρ'_ℓ, N_ℓ) is defined over $\mathbf{Q}$ modulo the action $G_A^{\natural\mathrm{ad}}$. The set $\{(\rho'_\ell, N_\ell) \mid \ell \neq p\}$ is a compatible system of representations of $'W_v$ with values in G_A modulo the action of $G_A^{\natural\mathrm{ad}}$.*

3.4 The p-adic representation of $'W_v$. Adapting the notations of 1.1, write $F_{v,0} \subset F_v$ for the maximal unramified extension of $\mathbf{Q}_p$ contained in F_v and let $\sigma\colon F_{v,0} \to F_{v,0}$ be the absolute Frobenius, that is the map inducing $x \mapsto x^p$ on the residue field k_v. Let $|k_v| = p^f$.

Consider the representation ρ_p of D_v on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$. In order to associate a p-adic representation of $'W_v$, one proceeds as in 2.1 and 2.4, but noting that F is now a global field and that the completion F_v plays the role of the base field F from sections 1 and 2. So let $D_{\mathrm{st}}(A)$ be as in (2.4.∗), replacing the group Γ_F from that formula by $D_v = \Gamma_{F_v}$. As in 2.1 and 2.4, this gives rise to a $F_{v,0}$-linear representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})$ of $'W_v$ on $D_{\mathrm{st}}(A)$.

3.5 Proposition. *The functor defined by $A \mapsto D_{\mathrm{st}}(A)$ defines an $F_{v,0}$-linear fibre functor on the tannakian category $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. The representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})$ of $'W_v$ on $D_{\mathrm{st}}(A)$ takes values in the automorphism group $G_{A,\mathrm{st}}$ of this fibre functor.*

Proof. The functor $A \mapsto \mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ defined by étale cohomology with coefficients in $\mathbf{Q}_p$ defines a fibre functor on $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. The group Γ_F maps to the automorphism group of this functor so we obtain a functor to the category of (continuous, finite dimensional) $\mathbf{Q}_p$-linear representations of $D_v = \Gamma_{F_v}$. By [CI99, Theorem 2.6(a)], $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ is a semistable representation of D_v and hence the same is true for the p-adic étale realisation of any object of $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. It thus follows from [Fon94c], in particular propositions 1.5.2 and 5.1.2, that $(\,\cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{D_v}$ defines a fibre functor on the image category of $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$ under the fibre functor defined by the p-adic étale cohomology. The functor considered in the proposition, as composite of these two fibre functors, is also a fibre functor.

For any $\mathbf{Q}_p$-linear representation V_p of D_v, a D_v-invariant in V_p defines a φ- and N-invariant element in $(V_p \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{D_v}$ so ${}'W_v$ acts on the composite fibre functor. □

3.6 Fitting the p-adic representation into the system. It follows from the general theory of tannakian categories, in particular from [DM82, Theorem 3.2], that the fibre functor on $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$ defined by D_{st} is $\bar{F}_v$-isomorphic to the one defined by $\mathrm{H}^1_{\mathrm{B}}$, where $\bar{F}_v$ is an algebraic closure of $F_{v,0}$. We fix an $\bar{F}_v$-isomorphism ι between these fibre functors. Conjugating by ι, the representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})$ of ${}'W_v$ gives rise to a representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^\iota$ of ${}'W_v$ with values in $G_{A/\bar{F}_v}$.

3.7 Remark. Replacing ι by another isomorphism ι', possibly modifying $\bar{F}_v$ as well, results in replacing $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^\iota$ by a conjugate under $G_A(\bar{F}_v)$. The Theorem 3.8 for $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^{\iota'}$ is therefore equivalent to the result for $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^\iota$. Also note that the existence of ι implies that $G_{A,\mathrm{st}}$ is an inner form of G_A.

The automorphism group of the ℓ-adic étale fibre functor is canonically identified with $G_{A/\mathbf{Q}_\ell}$ because $F \subset \mathbf{C}$ and because $\bar{F}$ is defined as the algebraic closure of F in $\mathbf{C}$, see also the closing Remarks 4.4.

3.8 Theorem. *Assume that A has semistable reduction at v, let $\Phi_v \in W_{F_v}$ be an arithmetic Frobenius element and suppose that either $\rho'_{F_{v,0}}(\Phi_v)$ or one of the $\rho'_\ell(\Phi_v)$, for $\ell \neq p$ is weakly neat. Then the representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^\iota$ is defined over $\mathbf{Q}$ modulo the action $G_A^{\natural\mathrm{ad}}$ and, together with the representations $\{(\rho'_\ell, N_\ell) \mid \ell \neq p\}$, it forms a compatible system of representations of ${}'W_v$ with values in G_A modulo the action of $G_A^{\natural\mathrm{ad}}$.*

3.9 Remarks. In the statement of the theorem, the hypothesis of weak neatness is required for one of the representations of the system. If it holds, then the conclusion of the theorem implies that it holds for all the representations of the system. This fact actually already follows from Corollary 2.5.

The main theorem 3.6 of [Noo13], states that the (ρ'_ℓ, N_ℓ) form a compatible system, with values in G_A and defined over $\mathbf{Q}$, modulo the action of $G_A^{\natural\mathrm{ad}}$. We thus only have to prove that $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^\iota$ is also defined over $\mathbf{Q}$ and compatible with any of the (ρ'_ℓ, N_ℓ), modulo the action of $G_A^{\natural\mathrm{ad}}$.

4. The proof of Theorem 3.8

4.1 Tractable abelian varieties. Following the strategy of [Noo13], we first prove the theorem for tractable abelian varieties. The precise definition of this notion can be found in [Noo09, 2.3] and in [Noo13, 4.1]. Summarising, an abelian variety $A/\mathbf{C}$ is *strictly tractable* if it satisfies the following conditions and if these conditions do not hold for any proper abelian subvariety of A. Firstly, G_A^{der} is $\mathbf{Q}$-simple, of the form $\mathrm{Res}_{K/\mathbf{Q}} G^s$ for an almost simple group G^s of classical type over an totally real number field K. Secondly, the representation of G_A^{der} on $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ is the restriction of scalars of a representation V^s of G^s of a particular type, either a multiple of the standard representation (if G^s is of type A_n, C_n or D_n) or a multiple of a spin representation (if G^s is of type B_n or D_n). Note that if G^s is of type D_n, this allows two kinds of representations. In the case where G^s is of type D_n and V^s is a spinorial representation it is required that in each character space for the action of the centre of $G_{A/\overline{\mathbf{Q}}}$ on $V \otimes \overline{\mathbf{Q}}$, both semi-spin representations of the corresponding factor of $G^{\mathrm{der}}_{A/\overline{\mathbf{Q}}}$ occur with the same multiplicity.

The variety A is *tractable* if it is isogenous to a product $\prod_{i=1}^m A_i$ of strictly tractable abelian varieties A_i and if $G_A^{\mathrm{der}} \cong \prod_{i=1}^m G_{A_i}^{\mathrm{der}}$. If $F \subset \mathbf{C}$ is a subfield, an abelian variety A/F is (strictly) tractable if $A_{/\mathbf{C}}$ is and if G_A is connected.

For a tractable abelian variety A, let $L \subset \mathrm{End}(A) \otimes \mathbf{Q}$ be the algebra defined in the proof of théorème 2.4 of [Noo09], see also [Noo13, 4.3]. It is characterised by the fact that it decomposes $L \otimes \overline{\mathbf{Q}} = \overline{\mathbf{Q}}^d$, where d is the number of isotypic components of the representation of $G_{A/\overline{\mathbf{Q}}}$ on $V \otimes \overline{\mathbf{Q}}$ and with each factor $\overline{\mathbf{Q}}$ acting by scalar multiplication on exactly on of these components. Lemma 4.4 of [Noo13] states that, if A is strictly tractable, then either L is a number field or $L = L' \times L'$ for a number field L'. It follows that, for A tractable, L is a product of number fields. By construction, the action of L on V commutes with the action of G_A, so $L \subset \mathrm{End}(A) \otimes \mathbf{Q}$ as stated.

Finally, let $H \subset \mathrm{GL}(V)$ be the linear algebraic group of L-linear automorphisms of V as in 2.1. Endomorphisms of A define absolute Hodge classes on $A \times A$, so $G_A \subset H$.

4.2 Proposition. *Let A be a tractable abelian variety defined over a number field F and let v be a valuation of F with residue characteristic p. There exists a finite extension F' of F such that the Theorem 3.8 holds for $A_{/F'}$.*

Proof. Let $\bar{F}_v$ and ι be as in 3.6, let $\Omega \supset \bar{F}_v$ be an algebraically closed field and let (ρ'_1, N_1) be the base extension to Ω of the representation $(\rho'_{F_{v,0}}, N_{\mathrm{st}})^{\iota}$. Assume that (ρ'_2, N_2) is a second representations of $'W_v$ with values in $G_{A/\Omega}$, either a conjugate $\tau(\rho_1, N_1)$ of (ρ_1, N_1) for some $\tau \in \mathrm{Aut}(\Omega)$, or the base extension to Ω of a (ρ'_ℓ, N_ℓ) for some prime number $\ell \neq p$ such that $\mathbf{Q}_\ell \subset \Omega$. We need to show that (ρ'_1, N_1) and (ρ'_2, N_2) are conjugate under $G_A^{\natural\mathrm{ad}}(\Omega)$.

First assume that A is strictly tractable. Put $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$, let $L \subset \mathrm{End}(A) \otimes \mathbf{Q}$ and $H \subset \mathrm{GL}(V)$ be as in 4.1. Explicitly, $H \cong \mathrm{Res}_{L/\mathbf{Q}} \mathrm{GL}_{d'/L}$ if L is a field and $H \cong \left(\mathrm{Res}_{L'/\mathbf{Q}} \mathrm{GL}_{d'/L'}\right)^2$ if $L = L' \times L'$ for a number field L'. Here $d' = \mathrm{rk}_L V$. It follows from Corollary 2.5 that (ρ'_1, N_1) and (ρ'_2, N_2), considered as representations of $'W_v$ with values in $H_{/\Omega}$, are conjugate by an element of $H(\Omega)$.

The arguments of [Noo13], sections 5 and 6, show that (ρ'_1, N_1) and (ρ'_2, N_2) are $G_A^{\natural\mathrm{ad}}(\Omega)$-conjugate, as representations of $'W_v$ with values in $G_{A/\Omega}$. The verification carried out in [Noo13] is a rather tedious case-by-case argument using the classification of classical linear algebraic groups and their representations. It depends on the facts that the monodromy is nilpotent of echelon 2, that the eigenvalues of Frobenius have complex absolute values 1, $q^{-1/2}$ and q^{-1} and that the monodromy operator defines an isomorphism between the 1 and q^{-1} weight spaces of Frobenius. Here q is the cardinality of the residue field of F at v. These facts are also valid for the p-adic representations and it follows that (ρ'_1, N_1) and (ρ'_2, N_2) are $G_A^{\natural\mathrm{ad}}(\Omega)$-conjugate. This establishes the proposition for strictly tractable abelian varieties. Note that no extension of the base field is needed in this case.

If A is only tractable then, proceeding as in the proof of [Noo13, Proposition 7.1], we can find a finite extension F'/F and an isogeny $A_{/F'} \sim \prod_{i=1}^m A_i$, where the A_i/F' are strictly tractable and such that $G_A^{\mathrm{der}} \cong \prod_{i=1}^m G_{A_i}^{\mathrm{der}}$. As moreover $\mathfrak{g}_A^{\mathrm{ss}} \cong \oplus_{i=1}^m \mathfrak{g}_{A_i}^{\mathrm{ss}}$ and $G_A^\natural \cong \prod_{i=1}^m G_{A_i}^\natural$, the general case of the proposition, over F', follows from the strictly tractable case. $\square$

4.3 Reduction to the tractable case. We use the method of [Noo13], section 7. Except for the neatness condition, we place ourselves in the situation of Theorem 3.8, so F is a number field, v is a valuation of F, restriction of the valuation $\bar{v}$ on $\bar{F}$ and A/F is an abelian variety. The Mumford–Tate group G_A is assumed to be connected.

It follows from [Noo06], especially section 2 and corollary 3.2, there exists a tractable abelian variety $B_{/\bar{F}}$ such that $B_{/\mathbf{C}}$ is a weak Mumford–Tate lift of $A_{/\mathbf{C}}$. By [Noo09, §3] there is an abelian variety of CM-type $C_{/\bar{F}}$ such that $h^1(A_{/\bar{F}})$ belongs to $\langle h^1(B_{/\bar{F}}), h^1(C_{/\bar{F}}), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. Replacing F by a finite extension F', we can assume that B and C admit models $B_{/F'}$ and $C_{/F'}$ over F' and that $h^1(A_{/F'})$ belongs to $\langle h^1(B), h^1(C), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. The restriction of $\bar{v}$ gives rise to a well determined valuation v' on F'. As in Proposition 3.5, let $(\rho'_{A,F_{v,0}}, N_{A,\mathrm{st}})$ be the representation of $'W_{v'}$ with values $G_{A,\mathrm{st}}$ attached to $A_{/F'}$. It is the restriction of the representation over the original field F. We similarly consider the representations $(\rho'_{B,F_{v,0}}, N_{B,\mathrm{st}})$ and $(\rho'_{C,F_{v,0}}, N_{C,\mathrm{st}})$ associated to $B_{/F'}$ and $C_{F'}$, with values in $G_{B,\mathrm{st}}$ and in $G_{C,\mathrm{st}}$ respectively. Composing the fibre functor defined by the p-adic étale cohomology with D_{st}, we obtain an $F_{v,0}$-valued fibre functor on the appropriate categories of absolute Hodge motives. This fibre functor induces functors to the categories of $F_{v,0}$-linear representations of the groups $G_{A,\mathrm{st}}$, $G_{B,\mathrm{st}}$ and $G_{C,\mathrm{st}}$. This gives rise to the variant

$$\begin{tikzcd}
{}'W_{v'} = {}'W_{F'_{v'}} \arrow[rr, "{((\rho'_{B,F_{v,0}},N_{B,\mathrm{st}}),(\rho'_{C,F_{v,0}},N_{C,\mathrm{st}}))}"] \arrow[drr, "{(\rho'_{B\times C,F_{v,0}},N_{B\times C,\mathrm{st}})}"] \arrow[ddrr, "{(\rho'_{A,F_{v,0}},N_{A,\mathrm{st}})}"'] & & G_{B,\mathrm{st}} \times G_{C,\mathrm{st}} \\
 & & G_{B\times C,\mathrm{st}} \arrow[u, hook] \arrow[d] \\
 & & G_{A,\mathrm{st}}
\end{tikzcd} \tag{4.3.$*$}$$

of the commutative diagram (7.2∗) from [Noo13].

As in 3.6, let $\bar{F}_v$ be an algebraic closure of F_v and hence of $F_{v,0}$ and let ι be an $\bar{F}_v$-isomorphism between the fibre functors on $\langle h^1(B), h^1(C), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$ defined by D_{st} and by $\mathrm{H}^1_{\mathrm{B}}$. Via ι, identify the groups $G_{\star,\mathrm{st}/\bar{F}_v}$ with the corresponding Mumford–Tate groups $G_{\star/\bar{F}_v}$. Consider an algebraically closed field $\Omega \supset \bar{F}_v$ and let the representations (ρ'_1, N_1) and (ρ'_2, N_2) of ${}'W_{v'}$ with values in $G_{A/\Omega}$ be as in the beginning of the proof of Proposition 4.2. To avoid confusion, we will henceforth denote these representations by $(\rho'_{A,i}, N_{A,i})$ for $i = 1, 2$. In order to prove the Theorem 3.8 over the extension $F' \supset F$, the task at hand is to show that these representations are conjugate under $G_A^{\natural\mathrm{ad}}(\Omega)$. In the course of the following arguments we will allow further finite extensions of F', which amounts to restricting the representations to subgroups of ${}'W_v$ of finite index.

For $\star = B, C$ or $B \times C$ we have corresponding representations $(\rho'_{\star,i}, N_{\star,i})$ with values in $G_{\star/\Omega}$. For $i = 1, 2$ we obtain commutative diagrams by replacing the $G_{\star,\mathrm{st}}$ by the $G_{\star/\Omega}$ and the $(\rho'_{\star,F_{v,0}}, N_{\star,\mathrm{st}})$ by the $(\rho'_{\star,i}, N_{\star,i})$ in the diagram (4.3.∗). The abelian variety C is of CM-type so it has potentially good reduction and after replacing F' by a finite extension we may assume that it has good reduction. Its Mumford–Tate group is commutative and the monodromy is trivial so $(\rho'_{C,1}, N_{C,2})$ and $(\rho'_{C,2}, N_{C,2})$ coincide by [Noo09, Corollaires 2.2, 4.4]. Where the tractable abelian variety B is concerned, the Proposition 4.2 implies that, again after a finite extension of F', the representations $(\rho'_{B,1}, N_{B,2})$ and $(\rho'_{B,2}, N_{B,2})$ are conjugate under $G_B^{\natural\mathrm{ad}}(\Omega) = G_{B\times C}^{\natural\mathrm{ad}}(\Omega)$. This proves the Theorem 3.8 over a finite extension of F, cf. also [Noo13, §7], proof of theorem 3.6.

It remains to establish the theorem over the original field F. By their definition in 2.1, the representations $\rho'_{A,i}$ are trivial on the inertia subgroup so they are determined by the image of the Frobenius element $\Phi \in W_{F_v}$. As the theorem holds over F', there exist $e \in \mathbf{N}$ and $g \in G_A^{\natural\mathrm{ad}}(\Omega)$ such that $N_{A,2} = \mathrm{Ad}(g)(N_{A,1})$ and $\rho'_{A,2}(\Phi^e) = g\rho'_{A,1}(\Phi^e)g^{-1}$. On the other hand, the Corollary 2.5, with $L = \mathbf{Q}$, implies that $\rho'_{A,2}(\Phi)$ and $g\rho'_{A,1}(\Phi)g^{-1}$ have the same characteristic polynomial. It now follows from [Noo13, Lemma 7.4], that $g\rho'_{A,1}(\Phi)g^{-1} = \rho'_{A,2}(\Phi)$, so that the

$(\rho'_{A,i}, N_{A,i})$ are conjugate under $G_A^{\natural\mathrm{ad}}$. This concludes the proof of the main Theorem 3.8.

4.4 Closing remarks. In the arguments above, and even to state the results, a number of choices where made. We briefly comment on their influence.

Firstly, in the sections 3 and 4 concerning abelian varieties over number fields, the base field is assumed to be a subfield of $\mathbf{C}$. The embedding determines the Betti cohomology $\mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ and hence the corresponding fibre functor on the category of absolute Hodge motives over F. The Mumford–Tate group G_A of A is defined as the automorphism group of this fibre functor. Another choice of embedding $F \hookrightarrow \mathbf{C}$ defines another fibre functor, which is $\overline{\mathbf{Q}}$-isomorphic to the previous one. As $\bar{F}$ is the algebraic closure of F in $\mathbf{C}$, this also modifies the embedding $F \subset \bar{F}$ so it affects the étale cohomology as well. The automorphism group $\widetilde{G}_A$ of the new fibre functor is an inner form of G_A and the representations $\Gamma_F \to G_A(\mathbf{Q}_\ell)$ and $\Gamma_F \to \widetilde{G}_A(\mathbf{Q}_\ell)$ are conjugate under $G_A(\overline{\mathbf{Q}}_\ell)$. The main Theorem 3.8 for the system of Galois representations with values in $\widetilde{G}_A$ associated to A is equivalent to the statement for the system of representations with values in G_A.

In the second place, the construction of the p-adic representation of the Weil–Deligne group F_v is based on the application the functor D_{st} to the p-adic étale cohomology of A. The main theorem can thus be interpreted as a version of Fontaine's conjecture $C_{\mathrm{WD,\,faible}}$ for absolute Hodge motives. The representation of $'W_v$ considered here is constructed using the fibre functor on the category of absolute Hodge motives defined by $D_{\mathrm{st}} \circ \mathrm{H}^1_{\text{ét}}(\cdot_{\bar{F}}, \mathbf{Q}_p)$. One may also construct a fibre functor using the log-crystalline or 'Hyodo–Kato' cohomology groups $\mathrm{H}^1_{\mathrm{HK}}(\cdot_{F_v})$ introduced in section 1, show that this functor defines a representation of the Weil–Deligne group and prove the theorem for that representation. Note that by means of Corollary 2.5, the proof of Theorem 3.8 makes essential use of the isomorphism $\mathrm{H}^1_{\mathrm{HK}}(A_{F_v}) \cong D_{\mathrm{st}}(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}_{\bar{v}}}, \mathbf{Q}_p))$ of Coleman and Iovita but that it depends only on the compatibility of this isomorphism with the endomorphisms of A. We neither use the fact $\mathrm{H}^1_{\mathrm{HK}}(\cdot)$ defines a fibre functor nor the fact that the comparison isomorphism is an isomorphism of fibre functors on the category of abelian absolute Hodge motives.

Let us indicate why $\mathrm{H}^1_{\mathrm{HK}}(\cdot_{F_v})$ indeed defines a fibre functor and why $'W_v$ acts on it by automorphisms. Let $\mathrm{T}_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ some Tate twisted tensor construction in $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and let $\gamma_{\mathrm{B}} \in \mathrm{T}_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ be a Hodge class, by [Del82] it is an absolute Hodge class. We need to show that it defines an element of the corresponding twisted tensor construction $\mathrm{T}_{\mathrm{HK}}(A_{F_v})$ in $\mathrm{H}^1_{\mathrm{HK}}(A_{F_v})$ and that this element is invariant under N_{HK} and φ. By construction, we have an isomorphism $\mathrm{H}^1_{\mathrm{HK}}(A_{F_v}) \otimes_{F_{v,0}} F_v \cong \mathrm{H}^1_{\mathrm{dR}}(A/F) \otimes_F F_v$. Via the comparison the isomorphism $\mathrm{H}^1_{\mathrm{dR}}(A/F) \otimes_F \mathbf{C} \cong \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$, the class γ_{B} defines an element of $\mathrm{T}_{\mathrm{dR}}(A/F) \otimes_F \mathbf{C}$. The fact that γ_{B} is an absolute Hodge class implies that this element is of the form $\gamma_{\mathrm{dR}} \otimes 1$ for some $\gamma_{\mathrm{dR}} \in \mathrm{T}_{\mathrm{dR}}(A/F)$.

To show that $\gamma_{\mathrm{dR}} \otimes 1 \in \mathrm{T}_{\mathrm{dR}}(A/F) \otimes_F F_v$ belongs to $\mathrm{T}_{\mathrm{HK}}(A_{F_v})$, consider $\gamma_p \in \mathrm{T}_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$, the image of γ_{B} under the isomorphism induced by

$$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p) = \mathrm{H}^1_{\text{ét}}(A_{\mathbf{C}}, \mathbf{Q}_p) \cong \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{Q}_p.$$

By [Bla94, Theorem 0.3], the classes $\gamma_{\mathrm{dR}} \otimes 1$ and $\gamma_p \otimes 1$ correspond under

$$\mathrm{H}^1_{\mathrm{dR}}(A/F) \otimes_F B_{\mathrm{dR}} \cong \mathrm{H}^1_{\text{ét}}(A_{\bar{F}_{\bar{v}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}. \tag{4.4.$*$}$$

It follows from [CI99, Part II, Theorem 7.13] that the isomorphism (4.4.$*$) identifies the subspace $\mathrm{H}^1_{\mathrm{dR}}(A/F) \otimes_F B_{\mathrm{st}} \subset \mathrm{H}^1_{\mathrm{dR}}(A/F) \otimes_F B_{\mathrm{dR}}$ with the subspace

$$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}_{\bar{v}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \subset \mathrm{H}^1_{\text{ét}}(A_{\bar{F}_{\bar{v}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}$$

and that $\mathrm{H}^1_{\mathrm{HK}}(A_{F_v})$ is the space of Γ_{F_v}-invariants. As the class $\gamma_p \in \mathrm{T}_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ is Γ_{F_v}-invariant, it follows that $\gamma_{\mathrm{dR}} \otimes 1 \in \mathrm{T}_{\mathrm{HK}}(A_{F_v})$ as claimed. This implies that $\mathrm{H}^1_{\mathrm{HK}}(\cdot\,_{F_v})$ defines a fibre functor on the category $\langle h^1(A), \mathbf{Q}(1)\rangle_{\mathrm{AH}}$. Moreover, $\gamma_p \otimes 1 \in \mathrm{T}_{\text{ét}}(A_{\bar{F}_{\bar{v}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}}$ is φ- and N-invariant, for the operators φ and N defined by φ and N_{st} on B_{st}, so $\gamma_{\mathrm{dR}} \otimes 1$ is invariant under φ and N_{HK} and hence $'W_v$ acts on the fibre functor defined by $\mathrm{H}^1_{\mathrm{HK}}(\cdot\,_{F_v})$.

These arguments also show that γ_{dR} and γ_p correspond under the comparison isomorphism, so the isomorphism $\mathrm{H}^1_{\mathrm{HK}}(\cdot\,_{F_v}) \cong D_{\mathrm{st}} \circ \mathrm{H}^1_{\text{ét}}(\cdot\,_{\bar{F}}, \mathbf{Q}_p)$ is an isomorphism of fibre functors. It follows that the main theorem also holds for the representation of $'W_v$ on $\mathrm{H}^1_{\mathrm{HK}}(A_{F_v})$. Alternatively, the rest of the proof of the main theorem, for the representation of the Weil–Deligne group defined by this functor, carries through exactly as the proof for the functor $D_{\mathrm{st}} \circ \mathrm{H}^1_{\text{ét}}(\cdot\,_{\bar{F}}, \mathbf{Q}_p)$.

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