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Wehler K3 surfaces with Picard number 3 and 4. Appendix to: "Orbits of points on certain K3 surfaces", by Arthur Baragar

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We show that Wehler K3 surfaces with Picard number three, which are the focus of the previous paper by Arthur Baragar, do indeed exist over the rational numbers.

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In this note we show that the Wehler K3 surfaces that are the focus of the previous paper [Bar11] do indeed exist over the rational numbers. In other words, we show that there exist smooth surfaces $V \subset \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^2_{\mathbb{Q}}$ given by a $(1,1)$-form and a $(2,2)$-form, whose Picard group has rank 3 and is generated by divisors D_1, D_2, and D_3, where D_i is the pull-back to V of a line in the i-th copy of $\mathbb{P}^2$ for $i = 1$ and 2, and D_3 is the product of a line in one of the two copies of $\mathbb{P}^2$ and a point in the other. We use the technique of [vL07], to which we refer for any details.

Consider the surface $\mathcal{V}$ in $\mathbb{P}^2_{\mathbb{Z}}(x,y,z) \times \mathbb{P}^2_{\mathbb{Z}}(q,r,s)$ given by $L = Q = 0$ with

$$L = qx + ry + sz,$$
$$Q = x^2Q_1 + xMq^2 + Q_2r^2 + Q_3s^2 + Q_4qr + Q_5qs + Q_6rs,$$

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where we use

$$\begin{aligned} Q_1 &= q(3r-s)+2s^2, \\ M &= 2x+2y+z, \\ Q_2 &= 3xz-y(x-2y+z), \\ Q_3 &= -xy-xz-2y^2+yz+z^2, \\ Q_4 &= xy+2xz+y^2+yz-z^2, \\ Q_5 &= 2xy+2xz-y^2+3yz+2z^2, \\ Q_6 &= -xy-2xz+3y^2-2yz+2z^2. \end{aligned}$$

As before, the projections of $\mathcal{V}$ to the two factors $\mathbb{P}^2$ give morphisms $\pi_1 : \mathcal{V} \to \mathbb{P}^2(x,y,z)$ and $\pi_2 : \mathcal{V} \to \mathbb{P}^2(q,r,s)$. For $i \in \{1,2\}$, let D_i denote the divisor class of $\pi_i^*(H)$ for any line H in $\mathbb{P}^2$ for $i = 1,2$.

Set $P_0 = R = [1:0:0] \in \mathbb{P}^2$ and $S = [0:1:0] \in \mathbb{P}^2$. Note that Q is written as the sum of seven terms, each of which is divisible by x, r, or s. The same holds for L (with three terms). This implies that $\mathcal{V}$ contains the line D_3 given by $x = r = s = 0$, which satisfies $\pi_2^{-1}(P_0) = D_3$. We also let D_3 denote the base extension of this line to several fields.

For $p \in \{2,3\}$, let $V_p \subset \mathbb{P}^2_{\mathbb{F}_p} \times \mathbb{P}^2_{\mathbb{F}_p}$ denote the reduction of $\mathcal{V}$ modulo p. Checking sufficiently many affine charts, it is easy to check that V_p is smooth for $p = 2$ and $p = 3$, which implies that V_p is a K3 surface; its Néron–Severi group $\mathrm{NS}\, V_p$ is isomorphic to its Picard group $\mathrm{Pic}\, V_p$.

Modulo 2 the quadric Q_1 is divisible by q, so the first two terms of Q are. Since $Q_2, \ldots, Q_6$ do not contain a term x^2, this implies that V_2 contains the line C_2 given by $y = z = q = 0$, which satisfies $\pi_1^{-1}(R) = C_2$.

Modulo 3 the quadric Q_2 is divisible by y. All terms of Q other than $Q_2 r^2$ are contained in the ideal generated by q and s. We conclude that V_3 contains the line C_3 given by $y = q = s = 0$, which satisfies $\pi_2^{-1}(S) = C_3$.

As before, abusing notation, we let D_3, C_2, C_3 denote both the curves they stand for and the divisor classes that these represent. For $p \in \{2,3\}$, let Λ_p denote the subgroup of $\mathrm{Pic}\, V_p$ generated by D_1, D_2, D_3, and C_p. The corresponding intersection matrices associated to these sequences of classes are

$$\begin{pmatrix} 2 & 4 & 1 & 0 \\ 4 & 2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 4 & 1 & 1 \\ 4 & 2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

for $p = 2$ and $p = 3$, respectively. The nondegeneracy of these matrices shows that Λ_2 and Λ_3 have rank 4 and discriminants -39 and -40 respectively. Write $\overline{V}_p = (V_p)_{\overline{\mathbb{F}}_p}$ for $p \in \{2,3\}$.

Proposition 0.1. *For $p = 2$ and $p = 3$ the rank of $\mathrm{Pic}\, \overline{V}_p$ equals 4.*

Proof. Let $l > 3$ be a prime. By [vL07, Proposition 2.2], there is an injection

$$(\mathrm{Pic}\, \overline{V}_p) \otimes \mathbb{Q}_l \hookrightarrow \mathrm{H}^2_{\mathrm{et}}\big(\overline{V}_p, \mathbb{Q}_l(1)\big)$$

of $\mathbb{Q}_l$-vectorspaces that respects the Galois action by $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. For $m \in \{0,1\}$, let $\Phi_p(m)$ denote the action on $\mathrm{H}^2_{\mathrm{et}}(\overline{V}_p, \mathbb{Q}_l(m))$ induced by the absolute Frobenius acting on V_p. Since some power of Frobenius acts as the identity on $\mathrm{Pic}\, \overline{V}_p$, it follows that the dimension of $(\mathrm{Pic}\, \overline{V}_p) \otimes \mathbb{Q}_l$ is bounded from above by the number of eigenvalues of $\Phi_p(1)$ that are roots of unity.

Table 1
Number of $\mathbb{F}_{p^n}$-points on V_p.

n	1	2	3	4	5	6	7	8	9
$p=2$	12	34	60	274	1132	4234	17204	66274	263868
$p=3$	17	95	830	7019	60467	536582	4795703	43059875	387558083

Since $\mathrm{H}^i_{\mathrm{et}}(\overline{V}_p, \mathbb{Q}_l)$ is 1-dimensional for $i=0$ and $i=4$ and 0-dimensional for $i=1$ and $i=3$, the Lefschetz fixed-point formula implies that we have

$$\#V_p(\mathbb{F}_{p^n}) = p^{2n} + 1 + \operatorname{tr} \Phi_p(0)^n \tag{1}$$

for all positive integers n, where $\operatorname{tr}\Phi_p(0)^n$ denotes the trace of the n-th power of Frobenius acting on $\mathrm{H}^2_{\mathrm{et}}(\overline{V}_p, \mathbb{Q}_l)$. The number of $\mathbb{F}_{p^n}$-points on V_p is given in Table 1 for $n \leqslant 9$.

Using (1) we find the traces $\operatorname{tr}\Phi_p(0)^n$ for $n \leqslant 9$. The eigenvalues of $\Phi_p(0)$ and $\Phi_p(1)$ differ by a factor p, which implies $\operatorname{tr}\Phi_p(1)^n = p^{-n}\operatorname{tr}\Phi_p(0)^n$. As Frobenius acts trivially on $\Lambda_p \otimes \mathbb{Q}_l \subset (\operatorname{Pic}\overline{V}_p) \otimes \mathbb{Q}_l$, the trace of the n-th power of Frobenius acting on the quotient

$$\mathrm{H}^2_{\mathrm{et}}\big(\overline{V}_p, \mathbb{Q}_l(1)\big)/(\Lambda_p \otimes \mathbb{Q}_l) \tag{2}$$

equals $\operatorname{tr}\Phi_p(1)^n - 4$. Using Newton's identity relating the traces of powers of a linear operator to the elementary symmetric polynomials in its eigenvalues, we find that the first ten terms of the characteristic polynomials f_p of Frobenius acting on the 18-dimensional quotient (2) are given by

$$f_2(t) = t^{18} + \frac{1}{2}t^{17} + \frac{3}{2}t^{15} + \frac{3}{2}t^{14} + \frac{1}{2}t^{13} + \frac{3}{2}t^{12} + \frac{3}{2}t^{11} + t^{10} + \frac{3}{2}t^9 + \cdots$$

for $p=2$ and

$$f_3(t) = t^{18} + \frac{5}{3}t^{17} + \frac{8}{3}t^{16} + 3t^{15} + \frac{8}{3}t^{14} + \frac{5}{3}t^{13} - \frac{5}{3}t^{11} - \frac{8}{3}t^{10} - \frac{10}{3}t^9 + \cdots$$

for $p=3$. These polynomials satisfy $f_p(t) = \pm t^{18} f_p(1/t)$, so they are palindromic or antipalindromic. As the middle coefficient of t^9 does not vanish, they are in fact palindromic, which determines f_2 and f_3 uniquely. Both f_2 and f_3 are irreducible modulo 101, so they are irreducible over $\mathbb{Q}$. As they are not integral, none of the roots of f_2 and f_3 are roots of unity, and the only eigenvalue of $\Phi_p(1)$ that is a root of unity is 1, with multiplicity 4. This proves the proposition. $\square$

Let V denote base extension of $\mathcal{V}$ to $\mathbb{Q}$, the generic fiber of $\mathcal{V}$ over $\operatorname{Spec}\mathbb{Z}$. Given that smoothness is an open condition and V_2 and V_3 are smooth, it follows that V is smooth as well. Set $\overline{V} = V_{\overline{\mathbb{Q}}}$.

Proposition 0.2. *The Picard group* $\operatorname{Pic}\overline{V}$ *is generated by* D_1, D_2*, and* D_3*.*

Proof. By [vL07, Proposition 2.2], there are intersection-number-preserving injections

$$(\operatorname{Pic}\overline{V}) \otimes \mathbb{Q}_l \hookrightarrow (\operatorname{Pic}\overline{V}_p) \otimes \mathbb{Q}_l$$

for $p=2$ and $p=3$. Since the inner product spaces $(\operatorname{Pic}\overline{V}_2) \otimes \mathbb{Q}_l$ and $(\operatorname{Pic}\overline{V}_3) \otimes \mathbb{Q}_l$ have different discriminants in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ (namely -39 and -40), it follows that the inner product space $(\operatorname{Pic}\overline{V}) \otimes \mathbb{Q}_l$ can not map isomorphically to both $(\operatorname{Pic}\overline{V}_2) \otimes \mathbb{Q}_l$ and $(\operatorname{Pic}\overline{V}_3) \otimes \mathbb{Q}_l$. Therefore $(\operatorname{Pic}\overline{V}) \otimes \mathbb{Q}_l$ has rank at most 3. The sublattice Λ of $\operatorname{Pic}\overline{V}$ generated by D_1, D_2 and D_3 has rank 3, so it has finite index in $\operatorname{Pic}\overline{V}$. The discriminant $\operatorname{disc}\Lambda = 22$ is squarefree, so the index is 1 and $\operatorname{Pic}\overline{V}$ equals Λ. $\square$

Remark 1. Let $\mathcal{W} \subset \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ be any Wehler K3 surface, thus given by a $(1,1)$-form and a $(2,2)$-form, that contains the line D_3 given by $x = r = s = 0$, and whose reductions modulo 2 and 3 equal V_2 and V_3 respectively. Let W denote the base extension of $\mathcal{W}$ to $\mathbb{Q}$. Then as was proved for V, the surface W is smooth and the Picard group $\operatorname{Pic}\overline{W}$ of $\overline{W} = W_{\overline{\mathbb{Q}}}$ is generated by D_1, D_2, and D_3. This shows that we have found a dense set of Wehler K3 surfaces, defined over $\mathbb{Q}$, with Picard number 3, in the corresponding component of the moduli space of K3 surfaces with Picard number at least 3. In fact, by lifting V_2 to any Wehler K3 surface over $\mathbb{Z}$ containing both D_3 and the line C_2 given by $y = z = q = 0$, we find that the same holds with 3 replaced by 4. By lifting V_3 to any Wehler K3 surface over $\mathbb{Z}$ containing both D_3 and the line C_3 given by $y = q = s = 0$, we obtain the same for a second component of the moduli space of K3 surfaces with Picard number at least 4.

References

[Bar11] Arthur Baragar, Orbits of points on certain K3 surfaces, J. Number Theory 131 (3) (2011) 578–599.

[vL07] Ronald van Luijk, K3 surfaces with Picard number one and infinitely many rational points, Algebra Number Theory 1 (1) (2007) 1–15, MR 2322921 (2008d:14058).