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Bounding $\zeta(s)$ in the critical strip

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ABSTRACT

Assuming the Riemann Hypothesis, we make use of the recently discovered (Carneiro et al. (preprint) [1]) extremal majorants and minorants of prescribed exponential type for the function $\log(\frac{4+x^2}{(\alpha-1/2)^2+x^2})$ to find upper and lower bounds with explicit constants for $\log|\zeta(\alpha+it)|$ in the critical strip, extending the work of Chandee and Soundararajan (in press) [4].

1. Introduction

Littlewood showed in 1924 (see [8]) that the Riemann Hypothesis (RH) implies a strong form of the Lindelöf Hypothesis, namely, on RH, for large real numbers t there is a constant C such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right). \tag{1.1}$$

Over the years no improvement has been made on the order of magnitude of the upper bound (1.1). The advances have rather focused on reducing the value of the admissible constant C (see for instance the works by Ramachandra and Sankaranarayanan [12] and Soundararajan [13]) and extending the results to general L-functions (see the work of Chandee [3]). A similar situation occurs when bounding the argument function $S(t)=\frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$, where the argument is defined by continuous variation

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along the line segments joining $2, 2+it$ and $\frac{1}{2}+it$, taking the argument of $\zeta(s)$ at 2 to be zero. Under RH, Littlewood showed that

$$\big|S(t)\big| \ll \frac{\log t}{\log\log t}, \tag{1.2}$$

and this bound has not been improved except for the size of the implied constant.

Recently, the idea of using the theory of extremal functions of exponential type was proved useful in both contexts, resulting in improved constants (and best up-to-date) for the upper bounds (1.1) and (1.2). The method of Goldston and Gonek [6] uses the explicit formula together with the classical Beurling–Selberg majorants and minorants of characteristic functions of intervals, and leads to the bound

$$\big|S(t)\big| \leqslant \left(\frac{1}{2}+o(1)\right)\frac{\log t}{\log\log t}.$$

In [4], Chandee and Soundararajan recognized that the corresponding treatment for $|\zeta(\frac{1}{2}+it)|$, using the Hadamard's factorization and the explicit formula, would require the extremal minorant for the function $\log(\frac{4+x^2}{x^2})$, available in the framework of Carneiro and Vaaler [2]. This combination was successful and led to the following bound [4, Theorem 1]

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(\frac{\log 2}{2}\frac{\log t}{\log\log t}+O\left(\frac{\log t\log\log\log t}{(\log\log t)^2}\right)\right). \tag{1.3}$$

It was mentioned in that paper that a similar approach to bounding $|\zeta(\alpha+it)|$, for $\alpha\neq 1/2$, would require the solution of the Beurling–Selberg extremal problem for the function

$$f_\alpha(x) = \log\left(\frac{4+x^2}{(\alpha-1/2)^2+x^2}\right), \tag{1.4}$$

which was not available at that particular time.

Very recently, Carneiro, Littmann and Vaaler in [1] developed a new approach to the Beurling–Selberg extremal problem based on the solution for the Gaussian and tempered distribution arguments. With this method, they were able to extend the solution of this problem to a wide class of even functions, in particular, including the desired family (1.4).

The purpose of this paper should be clear at this point. Here we make use of the recently discovered extremal majorants and minorants for $f_\alpha(x)$ to find upper and lower bounds with explicit constants for $|\zeta(\alpha+it)|$ on the critical strip. Observe that majorants for $f_\alpha(x)$ exist when $\alpha\neq 1/2$ and this is what makes the lower bounds possible. For simplicity, we will focus on the off-critical-line case (although the methods here plainly apply to the case $\alpha=1/2$ with slightly different Fourier transform representations than those of [4]), assuming from now on that $\alpha=\alpha(t)$ is a real-valued function with $1/2<\alpha\leqslant 1$. Since $|\zeta(\alpha+it)|=|\zeta(\alpha-it)|$ we might as well assume that $t\geqslant 0$. Our main results are the following.

Theorem 1 *(Upper bound). Assume RH. For large real numbers t, we have*

$$\log\big|\zeta(\alpha+it)\big| \leqslant \begin{cases} \log(1+(\log t)^{1-2\alpha})\frac{\log t}{2\log\log t}+O\left(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2}\right), & \text{if } (\alpha-1/2)\log\log t = O(1);\\ \log(\log\log t)+O(1), & \text{if } (1-\alpha)\log\log t = O(1);\\ \left(\frac{1}{2}+\frac{2\alpha-1}{\alpha(1-\alpha)}\right)\frac{(\log t)^{2-2\alpha}}{\log\log t}+\log(2\log\log t)+O\left(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\right), & \text{otherwise}. \end{cases}$$

Theorem 2 *(Lower bound). Assume RH. For large real numbers t, we have*

$$\log\big|\zeta(\alpha+it)\big| \geqslant \begin{cases} \log(1-(\log t)^{1-2\alpha})\frac{\log t}{2\log\log t} - O\big(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2(1-(\log t)^{1-2\alpha})}\big), \\ \quad \text{if } (\alpha-1/2)\log\log t = O(1); \\ -\log(\log\log t) - O(1), \quad \text{if } (1-\alpha)\log\log t = O(1); \\ -\big(\frac{1}{2}+\frac{2\alpha-1}{\alpha(1-\alpha)}\big)\frac{(\log t)^{2-2\alpha}}{\log\log t} - \log(2\log\log t) - O\big(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\big), \quad \text{otherwise}. \end{cases}$$

Observe that when $\alpha \to 1/2$ in Theorem 1 we recover the main term of the bound (1.3). Also it is worth mentioning that the order of magnitude in the general upper bound in Theorem 1 is a classical result in the theory of the Riemann zeta-function (see for instance [14, Theorem 14.5]), and the novelty here is in fact the method with which we arrive at this upper bound and the explicit computation of the implied constant.

With a refined calculation we can find the constant term when $\alpha = 1$ and obtain Littlewood's result [9,10] for bounds at $\mathrm{Re}(s) = 1$.

Corollary 3. *Assume RH. For large real numbers t, we have*

$$\big|\zeta(1+it)\big| \leqslant \big(2e^{\gamma} + o(1)\big)\log\log t,$$

and

$$\frac{1}{|\zeta(1+it)|} \leqslant \left(\frac{12e^{\gamma}}{\pi^2} + o(1)\right)\log\log t,$$

where γ is the Euler constant.

The paper is divided in three sections plus an Appendix A. In Section 2 we prove the upper bounds for $\zeta(s)$ contained in Theorem 1 and Corollary 3. In Section 3 we prove the corresponding lower bounds for $\zeta(s)$ in Theorem 2 and Corollary 3. In these two sections we will state the necessary facts concerning the extremal functions as supporting lemmas that will be ultimately proved in Section 4. Appendix A in the end details some of the asymptotic calculations carried along the proofs.

2. Upper bound for $\zeta(s)$

2.1. Proof of Theorem 1

Let

$$\xi(s) = s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be the Riemann's ξ-function. This function is an entire function of order 1 and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Hadamard's factorization formula gives us

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $\rho = \frac{1}{2} + i\gamma$ runs over the non-trivial zeros of ζ, and from the functional equation we can show that $B = -\sum_\rho \operatorname{Re}(1/\rho)$. On RH, γ is real.

By the functional equation and Hadamard's factorization formula, we obtain

$$\left|\frac{\xi(\alpha+it)}{\xi(5/2-it)}\right| = \left|\frac{\xi(\alpha+it)}{\xi(-3/2+it)}\right| = \prod_{\rho=1/2+i\gamma}\left(\frac{(\alpha-1/2)^2+(t-\gamma)^2}{4+(t-\gamma)^2}\right)^{1/2}.$$

Recall Stirling's formula for the Gamma function [5, Chapter 10]

$$\log\Gamma(z) = \frac{1}{2}\log 2\pi - z + \left(z-\frac{1}{2}\right)\log z + O\left(|z|^{-1}\right),$$

for large $|z|$. Using Stirling's formula and the fact that $|\zeta(5/2-it)| \asymp 1$, we obtain

$$\log\left|\zeta(\alpha+it)\right| = \left(\frac{5}{4}-\frac{\alpha}{2}\right)\log\frac{t}{2} - \frac{1}{2}\sum_\gamma f_\alpha(t-\gamma) + O(1). \tag{2.1}$$

The sum of $f_\alpha(t-\gamma)$ over the non-trivial zeros is hard to evaluate, so the key idea here is to replace f_α by its appropriate minorant (with a compactly supported Fourier transform) and then apply the following explicit formula which connects the zeros of the zeta-function and prime powers. The proof of the following lemma can be found in [7, Theorem 5.12].

Lemma 4 *(Explicit formula). Let $h(s)$ be analytic in the strip $|\operatorname{Im}(s)| \leqslant 1/2+\epsilon$ for some $\epsilon > 0$, and such that $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\operatorname{Re}(s)| \to \infty$. Let $h(w)$ be real-valued for real w, and set $\hat{h}(x) = \int_{-\infty}^{\infty} h(w)e^{-2\pi ixw}\,\mathrm{d}w$. Then*

$$\sum_\rho h(\gamma) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\hat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(\frac{-\log n}{2\pi}\right)\right).$$

The properties of the minorant function that we are interested in are described in the next lemma, that shall be proved in Section 4.

Lemma 5 *(Extremal minorant). Let Δ denote a positive real number. There is a unique entire function g_Δ which satisfies the following properties*:

(i) *For all real x we have*

$$-\frac{C}{1+x^2} \leqslant g_\Delta(x) \leqslant f_\alpha(x), \tag{2.2}$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|g_\Delta(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|}e^{2\pi\Delta|y|}. \tag{2.3}$$

(ii) *The Fourier transform of g_Δ, namely*

$$\hat{g}_\Delta(\xi) = \int_{-\infty}^{\infty} g_\Delta(x) e^{-2\pi i x\xi}\, dx,$$

is a continuous real-valued function supported on the interval $[-\Delta, \Delta]$. For $0 \leqslant |\xi| \leqslant \Delta$ it is given by

$$\hat{g}_\Delta(\xi) = \sum_{k=0}^{\infty} (-1)^k \bigg(\frac{k+1}{|\xi| + k\Delta} \big(e^{-2\pi(|\xi|+k\Delta)(\alpha-1/2)} - e^{-4\pi(|\xi|+k\Delta)} \big) - \frac{k+1}{\Delta(k+2) - |\xi|} \big(e^{2\pi(|\xi|-(k+2)\Delta)(\alpha-1/2)} - e^{4\pi(|\xi|-(k+2)\Delta)} \big) \bigg).$$

In particular, if $\xi = 0$, we have

$$\hat{g}_\Delta(0) = 2\pi \left(\frac{5}{2} - \alpha \right) - \frac{2}{\Delta} \log \left(\frac{1 + e^{-(2\alpha-1)\pi\Delta}}{1 + e^{-4\pi\Delta}} \right).$$

(iii) *The L^1-distance between g_Δ and f_α equals to*

$$\int_{-\infty}^{\infty} \big\{ f_\alpha(x) - g_\Delta(x) \big\}\, dx = \frac{2}{\Delta} \big(\log\big(1 + e^{-(2\alpha-1)\pi\Delta}\big) - \log\big(1 + e^{-4\pi\Delta}\big) \big).$$

From (2.1) and (i) of Lemma 5 we obtain, for any $\Delta > 0$,

$$\log\big|\zeta(\alpha + it)\big| \leqslant \left(\frac{5}{4} - \frac{\alpha}{2} \right) \log \frac{t}{2} - \frac{1}{2} \sum_{\gamma} g_\Delta(t - \gamma) + O(1). \tag{2.4}$$

To bound the sum of $g_\Delta(t-\gamma)$ we let $h(z) = g_\Delta(t - z)$ and apply Lemma 4 to get (observe that the growth condition $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ can be derived from (2.2), or alternatively, directly from (4.5) and (4.6))

$$\begin{aligned} \sum_{\rho} g_\Delta(t - \gamma) = {} & \left\{ g_\Delta\left(t - \frac{1}{2i} \right) + g_\Delta\left(t + \frac{1}{2i} \right) \right\} - \frac{1}{2\pi} \hat{g}_\Delta(0) \log \pi \\ & + \frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta(x)\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i(t-x)}{2} \right) dx \\ & - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \hat{g}_\Delta\left(\frac{\log n}{2\pi} \right) \big(e^{-it\log n} + e^{it\log n} \big). \end{aligned} \tag{2.5}$$

We now proceed to the asymptotic analysis of each of the elements on the right-hand side of the expression (2.5).

2.1.1. First term

From (i) of Lemma 5 we get

$$\left|g_\Delta\left(t-\frac{1}{2i}\right)+g_\Delta\left(t+\frac{1}{2i}\right)\right| \ll \Delta^2\frac{e^{\pi\Delta}}{1+\Delta t}. \tag{2.6}$$

2.1.2. Second term

From (ii) of Lemma 5 we get

$$\frac{1}{2\pi}\hat{g}_\Delta(0)\log\pi = \left(\frac{5}{2}-\alpha\right)\log\pi - \frac{\log\pi}{\pi\Delta}\log\left(\frac{1+e^{-(2\alpha-1)\pi\Delta}}{1+e^{-4\pi\Delta}}\right). \tag{2.7}$$

2.1.3. Third term

We will now show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta(x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-x)}{2}\right)\mathrm{d}x = \left(\frac{5}{2}-\alpha\right)\log\frac{t}{2} - \frac{1}{\pi\Delta}\log\left(\frac{1+e^{-(2\alpha-1)\pi\Delta}}{1+e^{-4\pi\Delta}}\right)\log\frac{t}{2} + O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}}\right). \tag{2.8}$$

From (i) of Lemma 5, for $x\neq 0$, we get

$$-\frac{C}{1+x^2} \leqslant g_\Delta(x) \leqslant f_\alpha(x) \leqslant \frac{4}{x^2},$$

and hence

$$\left|g_\Delta(x)\right| \ll \min\left\{\frac{1}{x^2}, \frac{\Delta^2}{1+\Delta|x|}\right\}. \tag{2.9}$$

Since $\mathrm{Re}\,\frac{\Gamma'}{\Gamma}(\frac{1}{4}+iu) \ll \log(|u|+2)$, we see that for sufficiently large t,

$$\int_{4\sqrt{t}}^{\infty} g_\Delta(x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-x)}{2}\right)\mathrm{d}x \ll \int_{4\sqrt{t}}^{\infty}\frac{\log(x+2)}{x^2}\,\mathrm{d}x \ll \frac{\log t}{\sqrt{t}}. \tag{2.10}$$

By similar arguments,

$$\int_{-\infty}^{-4\sqrt{t}} g_\Delta(x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-x)}{2}\right)\mathrm{d}x \ll \frac{\log t}{\sqrt{t}}. \tag{2.11}$$

Finally, we use that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(|s|^{-1}\right)$$

for large s, together with (2.9), part (iii) of Lemma 5, and the fact that $\int_{-\infty}^{\infty} f_\alpha(x)\,dx = 2\pi(\frac{5}{2}-\alpha)$, to get

$$\begin{aligned}\int_{-4\sqrt{t}}^{4\sqrt{t}} g_\Delta(x)\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-x)}{2}\right)dx &= \int_{-4\sqrt{t}}^{4\sqrt{t}} g_\Delta(x)\left(\log\frac{t}{2}+O\left(\frac{1}{\sqrt{t}}\right)\right)dx\\ &= \log\frac{t}{2}\int_{-\infty}^{\infty} g_\Delta(x)\,dx + O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}}\right)\\ &= 2\pi\left(\frac{5}{2}-\alpha\right)\log\frac{t}{2} - \frac{2}{\Delta}\log\left(\frac{1+e^{-(2\alpha-1)\pi\Delta}}{1+e^{-4\pi\Delta}}\right)\log\frac{t}{2}\\ &\quad + O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}}\right). \end{aligned} \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) we arrive at (2.8).

2.1.4. Fourth term (sum over prime powers)

This is the hardest term to analyze. We will have to make use of the explicit expression for Fourier transform of g_Δ described in (ii) of Lemma 5 to get

$$\begin{aligned}&\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\hat{g}_\Delta\left(\frac{\log n}{2\pi}\right)\left(e^{-it\log n}+e^{it\log n}\right)\\ &= \sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}\left(\frac{k+1}{\log n+2\pi k\Delta}\frac{e^{-(2\alpha-1)\pi k\Delta}}{n^{\alpha-1/2}} - \frac{k+1}{(2\pi\Delta(k+2)-\log n)}\frac{n^{\alpha-1/2}}{e^{(2\alpha-1)\pi(k+2)\Delta}}\right)\\ &\quad\times(-1)^k\left(e^{-it\log n}+e^{it\log n}\right) - 2\operatorname{Re}\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{5/2+it}\log n} + O\left(e^{-3\pi\Delta}\right)\\ &= \sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}\left(\frac{k+1}{\log n+2\pi k\Delta}\frac{e^{-(2\alpha-1)\pi k\Delta}}{n^{\alpha-1/2}} - \frac{k+1}{(2\pi\Delta(k+2)-\log n)}\frac{n^{\alpha-1/2}}{e^{(2\alpha-1)\pi(k+2)\Delta}}\right)\\ &\quad\times(-1)^k\left(e^{-it\log n}+e^{it\log n}\right) - 2\log\left|\zeta\left(\frac{5}{2}+it\right)\right| + O\left(e^{-3\pi\Delta}\right). \end{aligned} \tag{2.13}$$

From now on we let $x = e^{2\pi\Delta}$. Since $\frac{1}{\log y}\frac{1}{y^{\alpha-1/2}}$ is a non-increasing function for $y>1$, we deduce that for all integers $k\geqslant 0$ (recall that $n\leqslant x$),

$$\frac{1}{\log nx^k}\frac{1}{(nx^k)^{\alpha-1/2}} - \frac{1}{(\log\frac{x^{k+2}}{n})}\frac{1}{(\frac{x^{k+2}}{n})^{\alpha-1/2}} \geqslant 0. \tag{2.14}$$

The following two lemmas will be used to bound the sum over prime powers above.

Lemma 6. *For all $k \geqslant 0$ and $n \leqslant x$,*

$$(k+1)\left(\frac{1}{\log nx^k}\frac{1}{(nx^k)^{\alpha-1/2}} - \frac{1}{\log \frac{x^{k+2}}{n}}\frac{1}{(\frac{x^{k+2}}{n})^{\alpha-1/2}}\right)$$
$$\geqslant (k+2)\left(\frac{1}{\log nx^{k+1}}\frac{1}{(nx^{k+1})^{\alpha-1/2}} - \frac{1}{\log \frac{x^{k+3}}{n}}\frac{1}{(\frac{x^{k+3}}{n})^{\alpha-1/2}}\right).$$

Proof. The above inequality is equivalent to

$$\frac{k+1}{x^{(\alpha-1/2)k}}\left(\frac{1}{n^{\alpha-1/2}\log nx^k} - \frac{n^{\alpha-1/2}}{x^{2\alpha-1}\log\frac{x^{k+2}}{n}}\right)$$
$$\geqslant \frac{k+2}{x^{(\alpha-1/2)(k+1)}}\left(\frac{1}{n^{\alpha-1/2}\log nx^{k+1}} - \frac{n^{\alpha-1/2}}{x^{2\alpha-1}\log\frac{x^{k+3}}{n}}\right).$$

Since $\frac{1}{x^{\alpha-1/2}} \leqslant 1$, it suffices to show that

$$\frac{1}{n^{\alpha-1/2}}\left(\frac{k+1}{k\log x+\log n} - \frac{k+2}{(k+1)\log x+\log n}\right)$$
$$\geqslant \frac{n^{\alpha-1/2}}{x^{2\alpha-1}}\left(\frac{k+1}{(k+2)\log x-\log n} - \frac{k+2}{(k+3)\log x-\log n}\right).$$

The above is true since

$$\frac{k+1}{k\log x+\log n} - \frac{k+2}{(k+1)\log x+\log n}$$
$$= \frac{\log x-\log n}{(k\log x+\log n)((k+1)\log x+\log n)} \geqslant 0,$$

while

$$\frac{k+1}{(k+2)\log x-\log n} - \frac{k+2}{(k+3)\log x-\log n}$$
$$= \frac{\log n-\log x}{((k+2)\log x-\log n)((k+3)\log x-\log n)} \leqslant 0. \qquad \square$$

Lemma 7. *For all $k \geqslant 1$ and positive real numbers $2 \leqslant n \leqslant x$,*

$$\frac{1}{\log x} - \frac{1}{x^{\alpha-1/2}\log x} \leqslant \frac{k+1}{k\log x+\log n} - \frac{k+2}{x^{\alpha-1/2}((k+1)\log x+\log n)},$$

and

$$\frac{1}{\log x} - \frac{1}{x^{\alpha-1/2}\log x} \geqslant \frac{k+1}{(k+2)\log x-\log n} - \frac{k+2}{x^{\alpha-1/2}((k+3)\log x-\log n)}.$$

Proof. We will only show the proof for one inequality. The proof of the other is quite similar. Let us show that

$$\frac{1}{\log x}-\frac{1}{x^{\alpha-1/2}\log x}\leqslant\frac{k+1}{k\log x+\log n}-\frac{k+2}{x^{\alpha-1/2}((k+1)\log x+\log n)}.$$

This is equivalent to

$$\begin{aligned}&\frac{1}{x^{\alpha-1/2}}\left(\frac{k+2}{(k+1)\log x+\log n}-\frac{1}{\log x}\right)\leqslant\frac{k+1}{k\log x+\log n}-\frac{1}{\log x}\\ &\quad\Leftrightarrow\quad\frac{1}{x^{\alpha-1/2}}\left(\frac{\log x-\log n}{((k+1)\log x+\log n)\log x}\right)\leqslant\left(\frac{\log x-\log n}{(k\log x+\log n)\log x}\right).\end{aligned}$$

The above inequality follows from the fact that $\frac{1}{x^{\alpha-1/2}}\leqslant 1$. This proves the lemma. □

From (2.13), (2.14) and Lemma 6, we have

$$\begin{aligned}&\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\hat{g}_{\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{-it\log n}+e^{it\log n}\right)\\ &\leqslant 2\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\ &\quad-2\log\left|\zeta\left(\frac{5}{2}+it\right)\right|+O\left(e^{-3\pi\Delta}\right).\end{aligned}\tag{2.15}$$

Rearranging the terms and using Lemma 7, we obtain that the sum over k is bounded above by

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\ &\leqslant\frac{1}{n^{\alpha-1/2}\log n}-\frac{n^{\alpha-1/2}}{(2\log x-\log n)x^{2\alpha-1}}+\frac{1}{\log x}\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\ &=\frac{1}{n^{\alpha-1/2}\log n}-\frac{n^{\alpha-1/2}}{(2\log x-\log n)x^{2\alpha-1}}-\frac{1}{\log x(x^{\alpha-1/2}+1)}\left(\frac{1}{n^{\alpha-1/2}}-\frac{n^{\alpha-1/2}}{x^{2\alpha-1}}\right).\end{aligned}\tag{2.16}$$

Recall that the prime number theorem on the Riemann Hypothesis is

$$\sum_{n\leqslant x}\Lambda(n)=x+O\left(x^{1/2}\log^2 x\right).\tag{2.17}$$

Therefore using partial summation, (2.16), and (2.17), we obtain that

$$\begin{aligned}
&\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\
&\quad\leqslant\int_2^x\left(\frac{1}{t^\alpha\log t}-\frac{1}{\log x(x^{\alpha-1/2}+1)t^\alpha}\right)\mathrm{d}t\\
&\qquad-\frac{1}{x^{2\alpha-1}}\int_2^x\left(\frac{1}{t^{1-\alpha}(2\log x-\log t)}-\frac{1}{t^{1-\alpha}\log x(x^{\alpha-1/2}+1)}\right)\mathrm{d}t\\
&\qquad+O\left(\min\left\{\log^2 x,\frac{\log x}{x^{\alpha-1/2}(\alpha-1/2)}\right\}\right)\\
&\quad=A(x)-B(x)+O\left(\min\left\{\log^2 x,\frac{\log x}{x^{\alpha-1/2}(\alpha-1/2)}\right\}\right),
\end{aligned}$$

where the asymptotics of $A(x)$ and $B(x)$ (calculated in Appendix A) are given by

$$A(x)=\begin{cases}\log\log x+O(1), & \text{if } (1-\alpha)\log x=O(1);\\ \frac{x^{1-\alpha}}{(1-\alpha)\log x}(\frac{x^{\alpha-1/2}}{x^{\alpha-1/2}+1})+\log\log x+O(\frac{x^{1-\alpha}}{(1-\alpha)^2\log^2 x}), & \text{otherwise,}\end{cases}$$

and

$$B(x)=\frac{1}{\alpha}\frac{x^{1-\alpha}}{\log x}\left(\frac{x^{\alpha-1/2}}{x^{\alpha-1/2}+1}\right)+O\left(\frac{x^{1-\alpha}}{\log^2 x}\right).$$

Therefore the sum over prime powers is

$$\begin{aligned}
&\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\
&\quad\leqslant\begin{cases}\log\log x+O(1), & \text{if } (1-\alpha)\log x=O(1);\\ \frac{2\alpha-1}{\alpha(1-\alpha)}\frac{x^{1-\alpha}}{\log x}(\frac{x^{\alpha-1/2}}{x^{\alpha-1/2}+1})+\log\log x+O(\frac{x^{1-\alpha}}{(1-\alpha)^2\log^2 x}), & \text{otherwise.}\end{cases}
\end{aligned}\tag{2.18}$$

2.1.5. Final analysis

Combining all the results above (Eqs. (2.4)–(2.8) and (2.18)), and recalling that $x=e^{2\pi\Delta}$, we obtain

$$\begin{aligned}
\log|\zeta(\alpha+it)|\leqslant{}&\frac{1}{2\pi\Delta}\log\left(\frac{1+e^{-(2\alpha-1)\pi\Delta}}{1+e^{-4\pi\Delta}}\right)\log\frac{t}{2}+O\left(\Delta^2\frac{e^{\pi\Delta}}{1+\Delta t}\right)+O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}}\right)\\
&+\begin{cases}\log 2\pi\Delta+O(1), & \text{if } (1-\alpha)\Delta=O(1);\\ \frac{2\alpha-1}{\alpha(1-\alpha)}\frac{e^{(2-2\alpha)\pi\Delta}}{2\pi\Delta}(\frac{e^{(2\alpha-1)\pi\Delta}}{e^{(2\alpha-1)\pi\Delta}+1})+\log 2\pi\Delta+O(\frac{e^{(2-2\alpha)\pi\Delta}}{(1-\alpha)^2\pi^2\Delta^2}), & \text{otherwise.}\end{cases}
\end{aligned}\tag{2.19}$$

An optimal bound in (2.19) occurs when $\pi\Delta=\log\log t$. This upper bound depends on how far α is from 1/2 and 1, and we examine three cases:

Case 1. $\alpha - 1/2 = O(\frac{1}{\log\log t})$.

For this case, $\frac{2\alpha-1}{\alpha(1-\alpha)} = O(\frac{1}{\log\log t})$, and the upper bound (2.19) becomes

$$\log\big|\zeta(\alpha+it)\big| \leqslant \log\big(1+(\log t)^{-(2\alpha-1)}\big)\frac{\log t}{2\log\log t} + O\left(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2}\right),$$

which, as mentioned in the Introduction, recovers the main term in (1.3) when $\alpha \to 1/2$.

Case 2. $1 - \alpha = O(\frac{1}{\log\log t})$.

The upper bound is

$$\log\big|\zeta(\alpha+it)\big| \leqslant \log(2\log\log t) + O(1).$$

In Section 2.2, we will bound explicitly what the constant term is for $|\zeta(1+it)|$.

Case 3. Otherwise, we have

$$\log\big(1+e^{-(2\alpha-1)\pi\Delta}\big) \asymp \frac{1}{(\log t)^{2\alpha-1}},$$

and the upper bound (2.19) becomes

$$\log\big|\zeta(\alpha+it)\big| \leqslant \left(\frac{1}{2} + \frac{2\alpha-1}{\alpha(1-\alpha)}\right)\frac{(\log t)^{2-2\alpha}}{\log\log t} + \log(2\log\log t) + O\left(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\right).$$

This completes the proof of Theorem 1.

2.2. An upper bound for $|\zeta(1+it)|$

In this section we will bound $\zeta(1+it)$ and rederive Littlewood's result [9], which is

$$\big|\zeta(1+it)\big| \leqslant \big(2e^{\gamma} + o(1)\big)\log\log t,$$

where γ is the Euler constant.

The method used to bound $|\zeta(1+it)|$ is the same as the above except that we will bound $\sum_{n\leqslant x}\frac{\Lambda(n)}{n\log n}$ with an error term $o(1)$. From [9] and Merten's formula [11], we have

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n\log n} = \log\log x + \gamma + O\left(\frac{1}{\log x}\right).$$

Moreover by the prime number theorem (2.17),

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{x(2\log x - \log n)} \leqslant \frac{1}{x\log x}\sum_{n\leqslant x}\Lambda(n) = O\left(\frac{1}{\log x}\right).$$

To obtain a constant term of the upper bound for $|\zeta(1+it)|$, we will exploit a refined upper bound for (2.1). To be precise, we can show that

$$\log\left|\zeta(\alpha+it)\right| = \left(\frac{5}{4}-\frac{\alpha}{2}\right)\log\frac{t}{2} - \frac{1}{2}\sum_{\gamma} f_\alpha(t-\gamma) + \log\left|\zeta\left(\frac{5}{2}+it\right)\right| - \left(\frac{5}{4}-\frac{\alpha}{2}\right)\log\pi + O\left(\frac{1}{t}\right) \tag{2.20}$$

using Stirling's formula.

Therefore by (2.20) and the bounds for each terms in the explicit formula, we obtain that

$$\log\left|\zeta(1+it)\right| \leqslant \log 2e^{\gamma}\pi\Delta + \frac{1}{2\pi\Delta}\log\left(\frac{1+e^{-\pi\Delta}}{1+e^{-4\pi\Delta}}\right)\log\frac{t}{2} + O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}} + \frac{\Delta e^{\pi\Delta}}{t} + \frac{1}{\pi\Delta} + \frac{1}{t}\right).$$

The upper bound of $|\zeta(1+it)|$ in Corollary 3 follows from choosing $\pi\Delta = \log\log t$.

3. Lower bound for $\zeta(s)$

3.1. Proof of Theorem 2

The method of computing a lower bound for $\zeta(\alpha+it)$ is similar to the one for the upper bound in Section 2, with the only difference being the use of a majorant function instead. The majorant function that we are interested in satisfies the following properties (that shall be proved in the next section).

Lemma 8 *(Extremal majorant). Let Δ denote a positive real number. There is a unique entire function m_Δ which satisfies the following properties*:

(i) *For all real x we have*

$$f_\alpha(x) \leqslant m_\Delta(x) \leqslant C\frac{1}{1+x^2},$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|}e^{2\pi\Delta|y|}.$$

(ii) *The Fourier transform of m_Δ, namely*

$$\hat{m}_\Delta(\xi) = \int_{-\infty}^{\infty} m_\Delta(x)e^{-2\pi ix\xi}\,dx,$$

is a continuous real-valued function supported on the interval $[-\Delta,\Delta]$. For $0\leqslant|\xi|\leqslant\Delta$, it is given by

$$\hat{m}_\Delta(\xi) = \sum_{k=0}^{\infty}\left(\frac{k+1}{|\xi|+k\Delta}\left(e^{-2\pi(|\xi|+k\Delta)(\alpha-1/2)} - e^{-4\pi(|\xi|+k\Delta)}\right) - \frac{k+1}{\Delta(k+2)-|\xi|}\left(e^{2\pi(|\xi|-(k+2)\Delta)(\alpha-1/2)} - e^{4\pi(|\xi|-(k+2)\Delta)}\right)\right).$$

In particular, if $\xi = 0$, we have

$$\hat{m}_\Delta(0) = 2\pi\left(\frac{5}{2} - \alpha\right) - \frac{2}{\Delta}\log\left(\frac{1 - e^{-(2\alpha-1)\pi\Delta}}{1 - e^{-4\pi\Delta}}\right).$$

(iii) *The L^1-distance between m_Δ and f_α equals to*

$$\int_{-\infty}^{\infty} \{m_\Delta(x) - f_\alpha(x)\}\,dx = \frac{2}{\Delta}\big(\log\big(1 - e^{-4\pi\Delta}\big) - \log\big(1 - e^{-(2\alpha-1)\pi\Delta}\big)\big).$$

From (2.1) and (i) of Lemma 8 we obtain, for any $\Delta > 0$,

$$\log\big|\zeta(\alpha + it)\big| \geqslant \left(\frac{5}{4} - \frac{\alpha}{2}\right)\log\frac{t}{2} - \frac{1}{2}\sum_\gamma m_\Delta(t - \gamma) + O(1). \tag{3.1}$$

We then apply the explicit formula (Lemma 4) to the majorant function $m_\Delta(z)$ to get

$$\begin{aligned}\sum_\rho m_\Delta(t - \gamma) = {} & \left\{m_\Delta\left(t - \frac{1}{2i}\right) + m_\Delta\left(t + \frac{1}{2i}\right)\right\} - \frac{1}{2\pi}\hat{m}_\Delta(0)\log\pi \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i(t-x)}{2}\right)dx \\ & - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\hat{m}_\Delta\left(\frac{\log n}{2\pi}\right)\big(e^{-it\log n} + e^{it\log n}\big),\end{aligned} \tag{3.2}$$

and the asymptotic analysis follows just as before.

3.1.1. First term

From (i) of Lemma 8 we have

$$\left|m_\Delta\left(t - \frac{1}{2i}\right) + m_\Delta\left(t + \frac{1}{2i}\right)\right| \ll \Delta^2\frac{e^{\pi\Delta}}{1 + \Delta t}. \tag{3.3}$$

3.1.2. Second term

From (ii) of Lemma 8 we have

$$\frac{1}{2\pi}\hat{m}_\Delta(0)\log\pi = \left(\frac{5}{2} - \alpha\right)\log\pi - \frac{\log\pi}{\pi\Delta}\log\left(\frac{1 - e^{-(2\alpha-1)\pi\Delta}}{1 - e^{-4\pi\Delta}}\right). \tag{3.4}$$

3.1.3. Third term

Proceeding as in Section 2.1.3 we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i(t-x)}{2}\right)dx = {} & \left(\frac{5}{2} - \alpha\right)\log\frac{t}{2} - \frac{1}{\pi\Delta}\log\left(\frac{1 - e^{-(2\alpha-1)\pi\Delta}}{1 - e^{-4\pi\Delta}}\right)\log\frac{t}{2} \\ & + O\left(\frac{\Delta\log(1 + \sqrt{t}\Delta)}{\sqrt{t}}\right).\end{aligned} \tag{3.5}$$

3.1.4. Fourth term (sum over prime powers)

By the same arguments that led to (2.15), using inequality (2.14), the sum over prime is bounded below as follows (recall that $x = e^{2\pi\Delta}$)

$$\begin{aligned}
&\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\hat{m}_{\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{-it\log n}+e^{it\log n}\right)\\
&\geqslant -2\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\
&\quad -2\log\left|\zeta\left(\frac{5}{2}+it\right)\right|+O\left(e^{-3\pi\Delta}\right). \qquad (3.6)
\end{aligned}$$

The sum over k in (3.6) is bounded above by

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\
&\leqslant\sum_{k=0}^{\infty}\frac{1}{(x^k)^{\alpha-1/2}}\left(\frac{1}{n^{\alpha-1/2}\log n}-\frac{n^{\alpha-1/2}}{(2\log x-\log n)(x^{2\alpha-1})}\right)\\
&=\frac{x^{\alpha-1/2}}{x^{\alpha-1/2}-1}\left(\frac{1}{n^{\alpha-1/2}\log n}-\frac{n^{\alpha-1/2}}{(2\log x-\log n)(x^{2\alpha-1})}\right). \qquad (3.7)
\end{aligned}$$

Using partial summation, the prime number theorem (2.17), Eq. (3.7), and the integrals on Appendix A, we obtain that

$$\begin{aligned}
&\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}\left(\frac{k+1}{k\log x+\log n}\frac{1}{(nx^k)^{\alpha-1/2}}-\frac{k+1}{((k+2)\log x-\log n)}\frac{n^{\alpha-1/2}}{(x^{k+2})^{\alpha-1/2}}\right)\\
&\leqslant\frac{x^{\alpha-1/2}}{x^{\alpha-1/2}-1}\left\{\int_2^x\left(\frac{1}{t^{\alpha}\log t}-\frac{1}{x^{2\alpha-1}}\frac{1}{t^{1-\alpha}(2\log x-\log t)}\right)dt\right.\\
&\quad\left.+O\left(\min\left\{\log^2 x,\frac{\log x}{x^{\alpha-1/2}(\alpha-1/2)}\right\}\right)\right\}\\
&=\begin{cases}\log\log x+O(1), & \text{if } (1-\alpha)\log x=O(1);\\ \frac{x^{\alpha-1/2}}{x^{\alpha-1/2}-1}\{\frac{2\alpha-1}{\alpha(1-\alpha)}\frac{x^{1-\alpha}}{\log x}+\log\log x+O(\frac{x^{1-\alpha}}{(1-\alpha)^2\log^2 x})\}, & \text{otherwise.}\end{cases} \qquad (3.8)
\end{aligned}$$

3.1.5. Final analysis

Combining the bounds (3.1)–(3.8) above, and using the fact that $x = e^{2\pi\Delta}$, we derive

$$\begin{aligned}
\log\left|\zeta(\alpha+it)\right| &\geqslant \frac{1}{2\pi\Delta}\log\left(\frac{1-e^{-(2\alpha-1)\pi\Delta}}{1-e^{-4\pi\Delta}}\right)\log\frac{t}{2}+O\left(\Delta^2\frac{e^{\pi\Delta}}{1+\Delta t}\right)+O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}}\right)\\
&\quad-\begin{cases}\log 2\pi\Delta+O(1), & \text{if } (1-\alpha)\Delta=O(1);\\ (\frac{e^{(2\alpha-1)\pi\Delta}}{e^{(2\alpha-1)\pi\Delta}-1})\{\frac{2\alpha-1}{\alpha(1-\alpha)}\frac{e^{(2-2\alpha)\pi\Delta}}{2\pi\Delta}+\log 2\pi\Delta+O(\frac{e^{(2-2\alpha)\pi\Delta}}{(1-\alpha)^2\pi^2\Delta^2})\}, & \text{otherwise.}\end{cases} \qquad (3.9)
\end{aligned}$$

An optimal bound in (3.9) occurs when $\pi\Delta = \log\log t$. Similar to the upper bound, the lower bound depends on the location of α, and again we examine three cases:

Case 1. $\alpha - 1/2 = O(\frac{1}{\log\log t})$.

For this case, $\frac{2\alpha-1}{\alpha(1-\alpha)} = O(\frac{1}{\log\log t})$ and the lower bound (3.9) becomes

$$\log\big|\zeta(\alpha+it)\big| \geqslant \log\big(1-(\log t)^{1-2\alpha}\big)\frac{\log t}{2\log\log t} - O\left(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2(1-(\log t)^{1-2\alpha})}\right).$$

Observe that when $\alpha \to 1/2$, the bound goes to $-\infty$, which corresponds to the case when $\zeta(1/2+it)=0$.

Case 2. $1-\alpha = O(\frac{1}{\log\log t})$.

For this case, the lower bound (3.9) is

$$\log\big|\zeta(\alpha+it)\big| \geqslant -\log(2\log\log t) - O(1).$$

In Section 3.2, we will bound explicitly what the constant term is for $|\zeta(1+it)|$.

Case 3. Otherwise, $\log(1-e^{-(2\alpha-1)\pi\Delta}) \asymp -\frac{1}{(\log t)^{2\alpha-1}}$, and the lower bound (3.9) becomes

$$\log\big|\zeta(\alpha+it)\big| \geqslant -\left(\frac{1}{2}+\frac{2\alpha-1}{\alpha(1-\alpha)}\right)\frac{(\log t)^{2-2\alpha}}{\log\log t} - \log(2\log\log t) - O\left(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\right).$$

This completes the proof of Theorem 2.

3.2. A lower bound for $|\zeta(1+it)|$

In this section we will bound $1/\zeta(1+it)$ and rederive Littlewood's result [10],

$$\frac{1}{|\zeta(1+it)|} \leqslant \left(\frac{12e^{\gamma}}{\pi^2}+o(1)\right)\log\log t,$$

where γ is the Euler constant.

To obtain this bound, we will use (2.20). The method exploited to derive the bound for $1/|\zeta(1+it)|$ is the same as the one in Section 3.1 except that we will bound $\operatorname{Re}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1+it}\log n}$ with the error term $o(1)$. The following identity from [10] is useful in bounding the sum over prime powers:

$$\begin{aligned}\operatorname{Re}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1+it}\log n} &= -\operatorname{Re}\log\prod_{p\leqslant x}\left(1-\frac{1}{p^{1+it}}\right)+O\left(\frac{1}{\sqrt{x}}\right)\\ &\geqslant -\operatorname{Re}\log\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)+O\left(\frac{1}{\sqrt{x}}\right)\\ &= -\log\left(\frac{6e^{\gamma}}{\pi^2}\log x\right)+o(1).\end{aligned}$$

Moreover by the prime number theorem, we have

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{x(2\log x-\log n)} \leqslant \frac{1}{x\log x}\sum_{n\leqslant x}\Lambda(n) = O\left(\frac{1}{\log x}\right).$$

By (2.20) and the same arguments contained in Section 3.1 we obtain

$$\log\left|\zeta(1+it)\right| \geqslant -\log\left(\frac{12e^{\gamma}}{\pi^2}\pi\Delta\right) + \frac{1}{2\pi\Delta}\log\left(\frac{1-e^{-\pi\Delta}}{1-e^{-4\pi\Delta}}\right)\log\frac{t}{2} + O\left(\frac{\Delta\log(1+\sqrt{t}\Delta)}{\sqrt{t}} + \frac{\Delta e^{\pi\Delta}}{t} + \frac{1}{\pi\Delta}\right).$$

If we pick $\pi\Delta = \log\log t$, we obtain the bound in Corollary 3.

4. Extremal functions

In this section we will discuss the extremal functions used in this paper, proving Lemmas 5 and 8. This study relies substantially on the recent work of Carneiro, Littmann and Vaaler [1] that contains the solution of the Beurling–Selberg extremal problem for the Gaussian and a general integration technique on the free parameter, producing a variety of new examples. In particular, the logarithmic family $f_\alpha(x)$ considered in this paper falls in the range of the ideas in [1].

Throughout this section we let $a = (\alpha - 1/2)\Delta$, and $b = 2\Delta$. We have the following identity

$$\log\left(\frac{x^2+b^2}{x^2+a^2}\right) = \int_0^\infty e^{-\pi\lambda x^2}\left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right)d\lambda. \tag{4.1}$$

Define $F_\Delta(x)$ to be the expression on the left-hand side of (4.1). It is clear that

$$f_\alpha(x) = F_\Delta(\Delta x).$$

By Corollary 17 in [1], there is a unique extremal minorant $G_\Delta(x)$ and a unique extremal majorant $M_\Delta(x)$ of exponential type 2π for $F_\Delta(x)$. We will let

$$g_\Delta(x) = G_\Delta(\Delta x) \quad \text{and} \quad m_\Delta(x) = M_\Delta(\Delta x). \tag{4.2}$$

From [1], we also have

$$G_\Delta(z) = \left(\frac{\cos\pi z}{\pi}\right)^2 \sum_{n=-\infty}^{\infty}\left\{\frac{F_\Delta(n-\frac{1}{2})}{(z-n+\frac{1}{2})^2} + \frac{F'_\Delta(n-\frac{1}{2})}{(z-n+\frac{1}{2})}\right\}, \tag{4.3}$$

and

$$M_\Delta(z) = \left(\frac{\sin\pi z}{\pi}\right)^2 \sum_{n=-\infty}^{\infty}\left\{\frac{F_\Delta(n)}{(z-n)^2} + \frac{F'_\Delta(n)}{(z-n)}\right\}. \tag{4.4}$$

4.1. Proof of Lemma 5

Part (iii) of Lemma 5 is contained in [1, Corollary 17, Example 3], and thus we will focus here in proving parts (i) and (ii).

4.1.1. Part (i)

Observe first that

$$G_\Delta(z) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi (z-n+\frac{1}{2})}{\pi (z-n+\frac{1}{2})}\right)^2 \left\{ f_\alpha\left(\frac{n-\frac{1}{2}}{\Delta}\right) + \frac{(z-n+\frac{1}{2})}{\Delta} f'_\alpha\left(\frac{n-\frac{1}{2}}{\Delta}\right)\right\}. \tag{4.5}$$

For any complex number ξ we have $(\sin(\pi\xi)/(\pi\xi))^2 \ll e^{2\pi|\operatorname{Im}\xi|}/(1+|\xi|^2)$. Using the fact that

$$f_\alpha(x) \leqslant \frac{4}{x^2+(\alpha-\frac{1}{2})^2} \quad \text{and} \quad f'_\alpha(x) \leqslant \frac{8|x|}{(x^2+4)(x^2+(\alpha-\frac{1}{2})^2)}, \tag{4.6}$$

we can split the sum (4.5) in two parts, where $n \leqslant |z|/2$ and $n \geqslant |z|/2$, to conclude that

$$\left|G_\Delta(x+iy)\right| \ll \frac{\Delta^2}{1+|x+iy|} e^{2\pi|y|},$$

and from (4.2) we arrive at (2.3).

For x real, we have $f_\alpha(x) \geqslant 0$ and $f'_\alpha(-x) = -f'_\alpha(x)$, and we can pair the terms $n \geqslant 1$ and $1-n \leqslant 0$ in the sum (4.5) to obtain

$$\begin{aligned} G_\Delta(x) &\geqslant \left(\frac{\cos \pi x}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{\Delta} f'_\alpha\left(\frac{n-\frac{1}{2}}{\Delta}\right)\left\{\frac{1}{(x-n+\frac{1}{2})} - \frac{1}{(x+n-\frac{1}{2})}\right\} \\ &= \sum_{n=1}^{\infty} \frac{\sin^2 \pi (x-n+\frac{1}{2})}{\pi^2 (x^2-(n-\frac{1}{2})^2)} \frac{2(n-\frac{1}{2})}{\Delta} f'_\alpha\left(\frac{n-\frac{1}{2}}{\Delta}\right). \end{aligned} \tag{4.7}$$

Using (4.6) and (4.7) we can show that there is a constant C such that

$$-C\frac{\Delta^2}{\Delta^2+x^2} \leqslant G_\Delta(x),$$

and thus from (4.2) we arrive at (2.2), completing the proof of part (i).

4.1.2. Part (ii)

It is sufficient to consider the Fourier transform of $G_\Delta(x)$ since $\hat{g}_\Delta(y) = \frac{1}{\Delta}\hat{G}_\Delta(\frac{y}{\Delta})$.

For $|y| \geqslant 1$, $\hat{G}_\Delta(y) = 0$. Therefore, in what follows we will consider $\hat{G}_\Delta(y)$ when $|y| < 1$. From (4.1) and [1, Theorem 4], we know that

$$\begin{aligned} \hat{G}_\Delta(y) = \int_0^\infty \Bigg\{ &(1-|y|) \sum_{n=-\infty}^{\infty} e^{-\pi\lambda(n+1/2)^2} e^{2\pi i y(n+1/2)} \\ &- \frac{\lambda}{2\pi} \operatorname{sgn}(y) \sum_{n=-\infty}^{\infty} 2\pi i \left(n+\frac{1}{2}\right) e^{-\pi\lambda(n+1/2)^2} e^{2\pi i y(n+1/2)} \Bigg\} \\ &\cdot \left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right) \mathrm{d}\lambda. \end{aligned} \tag{4.8}$$

It is easy to see that $\hat{G}_\Delta(y)$ is an even function. Therefore it is sufficient to consider the case $0 \leqslant y < 1$. We will evaluate the integrals of the first and second sums separately.

Integration of the first sum. By calculus, we can show that

$$\int_0^\infty (1-|y|)\left\{\sum_{n=-\infty}^{\infty} e^{-\pi\lambda(n+1/2)^2}e^{2\pi iy(n+1/2)}\right\}\cdot\left(\frac{e^{-\pi\lambda a^2}-e^{-\pi\lambda b^2}}{\lambda}\right)d\lambda$$
$$=(1-|y|)e^{\pi iy}\sum_{n=-\infty}^{\infty}\log\left(\frac{(n+1/2)^2+b^2}{(n+1/2)^2+a^2}\right)e^{2\pi iyn}. \tag{4.9}$$

Let

$$\hat{k}(x)=\log\left(\frac{(x+1/2)^2+b^2}{(x+1/2)^2+a^2}\right).$$

To evaluate the sum over n, we will use Poisson summation formula,

$$\sum_{n\in\mathbb{Z}}\hat{k}(n)e^{2\pi iyn}=\sum_{n\in\mathbb{Z}}k(y+n).$$

Therefore we need to compute $k(w)$. For $w\neq 0$ we use integration by parts to get

$$k(w)=\int_{-\infty}^{\infty}\log\left(\frac{(x+1/2)^2+b^2}{(x+1/2)^2+a^2}\right)e^{2\pi iwx}\,dx$$
$$=\frac{e^{-\pi iw}}{2\pi iw}\int_{-\infty}^{\infty}\frac{2x(b^2-a^2)}{(x^2+b^2)(x^2+a^2)}e^{2\pi iwx}\,dx. \tag{4.10}$$

For $w=0$ we will have

$$k(0)=\int_{-\infty}^{\infty}\log\left(\frac{(x+1/2)^2+b^2}{(x+1/2)^2+a^2}\right)dx=2\pi(b-a).$$

The integrals above can be computed via contour integration.

Case 1. $w>0$. The chosen contour is a rectangle with vertices $-X, X, X+iY, -X+iY$, where $X, Y>0$, and $X, Y\to\infty$. Therefore

$$\int_{-\infty}^{\infty}\frac{2x(b^2-a^2)}{(x^2+b^2)(x^2+a^2)}e^{2\pi iwx}\,dx=2\pi i\left(e^{-2\pi wa}-e^{-2\pi wb}\right). \tag{4.11}$$

Case 2. $w<0$. The contour is a rectangle with vertices $X, -X, -X-iY, X-iY$, where $X, Y>0$, and $X, Y\to\infty$. Therefore

$$\int_{-\infty}^{\infty}\frac{2x(b^2-a^2)}{(x^2+b^2)(x^2+a^2)}e^{2\pi iwx}\,dx=-2\pi i\left(e^{2\pi wa}-e^{2\pi wb}\right). \tag{4.12}$$

Combining Eqs. (4.9)–(4.12) above, for $y \neq 0$, we obtain

$$\int_0^\infty (1-|y|)\left\{\sum_{n=-\infty}^{\infty} e^{-\pi\lambda(n+1/2)^2} e^{2\pi i y(n+1/2)}\right\}\cdot\left(\frac{e^{-\pi\lambda a^2}-e^{-\pi\lambda b^2}}{\lambda}\right)d\lambda$$
$$= (1-|y|)\left\{\sum_{n=0}^{\infty}(-1)^n \frac{e^{-2\pi(y+n)a}-e^{-2\pi(y+n)b}}{y+n} - \sum_{n=1}^{\infty}(-1)^n \frac{e^{2\pi(y-n)a}-e^{2\pi(y-n)b}}{y-n}\right\}.$$

For $y=0$, the integral is

$$2\pi(b-a) - 2\log\left(\frac{1+e^{-2\pi a}}{1+e^{-2\pi b}}\right).$$

Integration of the second sum. For $y=0$, the second integral is 0. So we will compute its value for $0<y<1$. By calculus, we have

$$-\int_0^\infty \frac{\lambda}{2\pi}\left\{\sum_{n=-\infty}^{\infty} 2\pi i\left(n+\frac{1}{2}\right)e^{-\pi\lambda(n+1/2)^2} e^{2\pi i y(n+1/2)}\right\}\left(\frac{e^{-\pi\lambda a^2}-e^{-\pi\lambda b^2}}{\lambda}\right)d\lambda$$
$$= -i\frac{e^{\pi i y}}{\pi}\sum_{n=-\infty}^{\infty}\left(\frac{(n+1/2)}{(n+1/2)^2+a^2} - \frac{(n+1/2)}{(n+1/2)^2+b^2}\right)e^{2\pi i y n}. \tag{4.13}$$

Let

$$\hat{h}(x) = \frac{(x+1/2)}{(x+1/2)^2+a^2} - \frac{(x+1/2)}{(x+1/2)^2+b^2}.$$

Again we will compute the sum above by Poisson summation formula,

$$\sum_{n\in\mathbb{Z}} \hat{h}(n)e^{2\pi i y n} = \sum_{n\in\mathbb{Z}} h(y+n). \tag{4.14}$$

Since $y+n\neq 0$, we will compute $h(w)$, where $w\neq 0$,

$$h(w) = \int_{-\infty}^{\infty}\left(\frac{(x+1/2)}{(x+1/2)^2+a^2} - \frac{(x+1/2)}{(x+1/2)^2+b^2}\right)e^{2\pi i w x}\,dx$$
$$= e^{-\pi i w}\int_{-\infty}^{\infty}\left(\frac{x}{x^2+a^2} - \frac{x}{x^2+b^2}\right)e^{2\pi i w x}\,dx. \tag{4.15}$$

We now use contour integration again.

Case 1. $w>0$. The contour is a rectangle with vertices $-X, X, X+iY, -X+iY$, where $X, Y>0$, and $X, Y\to\infty$. Therefore

$$h(w) = 2\pi i e^{-\pi i w}\left(\operatorname{res}_{x=ia}\frac{x}{x^2+a^2}\cdot e^{2\pi i w x} - \operatorname{res}_{x=ib}\frac{x}{x^2+b^2}\cdot e^{2\pi i w x}\right)$$
$$= \pi i e^{-\pi i w}\left(e^{-2\pi w a} - e^{-2\pi w b}\right). \tag{4.16}$$

Case 2. $w < 0$. The contour is a rectangle with vertices $X, -X, -X - iY, X - iY$, where $X, Y > 0$, and $X, Y \to \infty$. Therefore

$$h(w) = -2\pi i e^{-\pi i w}\left(\operatorname{res}_{x=-ia}\frac{x}{x^2+a^2}\cdot e^{2\pi i w x} - \operatorname{res}_{x=-ib}\frac{x}{x^2+b^2}\cdot e^{2\pi i w x}\right)$$
$$= -\pi i e^{-\pi i w}\left(e^{2\pi w a} - e^{2\pi w b}\right). \tag{4.17}$$

Finally, combining (4.13)–(4.17) we obtain

$$-\int_0^\infty \frac{\lambda}{2\pi}\left\{\sum_{n=-\infty}^{\infty} 2\pi i\left(n+\frac{1}{2}\right)e^{-\pi\lambda(n+1/2)^2}e^{2\pi i y(n+1/2)}\right\}\left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right)\mathrm{d}\lambda$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(e^{-2\pi(y+n)a} - e^{-2\pi(y+n)b}\right) - \sum_{n=1}^{\infty}(-1)^n\left(e^{2\pi(y-n)a} - e^{2\pi(y-n)b}\right),$$

and this ultimately leads to part (ii) of Lemma 5.

4.2. Proof of Lemma 8

The proof of part (i) of Lemma 8 is very similar to the analogous part (i) of Lemma 5, proved in Section 4.1. Part (iii) of Lemma 8 is contained in [1, Corollary 17, Example 3], and thus we will only focus here on part (ii).

4.2.1. Part (ii)

Since $M_\Delta(y)$ is an even function, it suffices to consider $\hat{M}_\Delta(y)$ for $0 \leqslant y < 1$. We know from (4.1) and [1, Theorem 4] that

$$\hat{M}_\Delta(y) = \int_0^\infty \left\{(1-|y|)\sum_{n=-\infty}^{\infty} e^{-\pi\lambda n^2}e^{2\pi i y n} - \frac{\lambda}{2\pi}\operatorname{sgn}(y)\sum_{n=-\infty}^{\infty} 2\pi i n e^{-\pi\lambda n^2}e^{2\pi i y n}\right\}$$
$$\cdot\left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right)\mathrm{d}\lambda. \tag{4.18}$$

Integration of the first sum. By the same arguments used for $G_\Delta(y)$, we have

$$\int_0^\infty (1-|y|)\left\{\sum_{n=-\infty}^{\infty} e^{-\pi\lambda n^2}e^{2\pi i y n}\right\}\left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right)\mathrm{d}\lambda$$
$$= (1-|y|)\sum_{n=-\infty}^{\infty}\log\left(\frac{n^2+b^2}{n^2+a^2}\right)e^{2\pi i y n}$$
$$= (1-|y|)\left\{\sum_{n=0}^{\infty}\frac{e^{-2\pi(y+n)a} - e^{-2\pi(y+n)b}}{y+n} - \sum_{n=1}^{\infty}\frac{e^{2\pi(y-n)a} - e^{2\pi(y-n)b}}{y-n}\right\}, \tag{4.19}$$

for $y \neq 0$. For $y = 0$ we have the value

$$2\pi(b-a) - 2\log\left(\frac{1-e^{-2\pi a}}{1-e^{-2\pi b}}\right).$$

Integration of the second sum. By the same arguments used for $G_{\Delta}(y)$, the second term is equal to

$$-\int_0^{\infty} \frac{\lambda}{2\pi}\left\{\sum_{n=-\infty}^{\infty} 2\pi i n e^{-\pi\lambda n^2} e^{2\pi i y n}\right\}\left(\frac{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}}{\lambda}\right) d\lambda$$

$$= \sum_{n=0}^{\infty}\left(e^{-2\pi(y+n)a} - e^{-2\pi(y+n)b}\right) - \sum_{n=1}^{\infty}\left(e^{2\pi(y-n)a} - e^{2\pi(y-n)b}\right). \tag{4.20}$$

Combining (4.18), (4.19) and (4.20) we complete the proof of part (ii).

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Appendix A

Here we show the following asymptotics:

$$\int_2^x \frac{1}{t^{\alpha}\log t}\,dt = \begin{cases} \log\log x + O(1), & \text{if} (1-\alpha)\log x = O(1); \\ \frac{x^{1-\alpha}}{(1-\alpha)\log x} + \log\log x + O\left(\frac{x^{1-\alpha}}{(1-\alpha)^2\log^2 x}\right), & \text{otherwise,} \end{cases} \tag{A.1}$$

$$\int_2^x \frac{1}{t^{1-\alpha}(2\log x - \log t)}\,dt = \frac{1}{\alpha}\frac{x^{\alpha}}{\log x} + O\left(\frac{x^{\alpha}}{\log^2 x}\right). \tag{A.2}$$

Proof of (A.1). The left-hand side of (A.1) can be written as

$$\int_2^x \frac{1}{t^{\alpha}\log t}\,dt = \log\log x - \log\log 2 + \int_{(1-\alpha)\log 2}^{(1-\alpha)\log x} \frac{e^y - 1}{y}\,dy. \tag{A.3}$$

If $(1-\alpha)\log x = O(1)$, then $\frac{e^y-1}{y} \leqslant e^y = O(1)$ for $(1-\alpha)\log 2 \leqslant y \leqslant (1-\alpha)\log x$. Therefore the integral on the right-hand side of (A.3) is $O(1)$.

Otherwise, the integral on the right-hand side of (A.3) is $O(1)$ plus

$$\frac{e^y - y - 1}{y}\Big|_0^{(1-\alpha)\log x} + \int_0^{(1-\alpha)\log x} \frac{e^y - y - 1}{y^2}\,dy$$

$$= \frac{x^{1-\alpha}}{(1-\alpha)\log x} + \int_0^{(1-\alpha)\log x} \frac{e^y - y - 1}{y^2}\,dy + O(1).$$

Since $\lim_{y\to 0}\frac{e^y-y-1}{y^2}=\frac{1}{2}$,

$$\int_0^{(1-\alpha)\log x}\frac{e^y-y-1}{y^2}\,dy=\int_2^{\frac{(1-\alpha)\log x}{2}}\frac{e^y-y-1}{y^2}\,dy+\int_{\frac{(1-\alpha)\log x}{2}}^{(1-\alpha)\log x}\frac{e^y-y-1}{y^2}\,dy+O(1)$$

$$\leqslant\frac{1}{4}\int_2^{\frac{(1-\alpha)\log x}{2}}\left(e^y-y-1\right)dy+\frac{4}{(1-\alpha)^2\log^2 x}\int_{\frac{(1-\alpha)\log x}{2}}^{(1-\alpha)\log x}\left(e^y-y-1\right)dy+O(1)$$

$$=O\left(\frac{x^{1-\alpha}}{(1-\alpha)^2\log^2 x}\right).\qquad\square$$

Proof of (A.2). Let $y=x^2/t$. The integral (A.2) becomes

$$x^{2\alpha}\int_x^{x^2/2}\frac{1}{y^{1+\alpha}\log y}\,dy=\frac{1}{\alpha}\frac{x^\alpha}{\log x}-\frac{x^{2\alpha}}{\alpha}\int_x^{x^2/2}\frac{1}{y^{1+\alpha}\log^2 y}\,dy+O\left(\frac{1}{\log x}\right).$$

Expression (A.2) follows from the fact that

$$\int_x^{x^2/2}\frac{1}{y^{1+\alpha}\log^2 y}\,dy\ll\frac{1}{\log^2 x}\int_x^{x^2/2}\frac{1}{y^{1+\alpha}}\,dy\ll\frac{1}{x^\alpha\log^2 x}.\qquad\square$$

References

[1] E. Carneiro, F. Littmann, J.D. Vaaler, Gaussian subordination for the Beurling–Selberg extremal problem, preprint.
[2] E. Carneiro, J.D. Vaaler, Some extremal functions in Fourier analysis, II, Trans. Amer. Math. Soc. 62 (2010) 5803–5843.
[3] V. Chandee, Explicit upper bounds for L-functions on the critical line, Proc. Amer. Math. Soc. 137 (12) (2009) 4049–4063.
[4] V. Chandee, K. Soundararajan, Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann Hypothesis, Bull. Lond. Math. Soc., in press.
[5] H. Davenport, Multiplicative Number Theory, third ed., Grad. Texts in Math., vol. 74, Springer-Verlag, New York, 2000.
[6] D.A. Goldston, S.M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, Bull. Lond. Math. Soc. 39 (3) (2007) 482–486.
[7] H. Iwaniec, E. Kowalski, Analytic Number Theory, Amer. Math. Soc. Colloq. Publ., vol. 53, 2004.
[8] J.E. Littlewood, On the zeros of the Riemann zeta-function, Proc. Cambridge Philos. Soc. 22 (1924) 295–318.
[9] J.E. Littlewood, On the Riemann zeta-function, Proc. London Math. Soc. 24 (2) (1925) 175–201.
[10] J.E. Littlewood, On the function $1/\zeta(1+it)$, Proc. London Math. Soc. 27 (1928) 349–357.
[11] H.L. Montgomery, R.C. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge University Press, Cambridge, 2007.
[12] K. Ramachandra, A. Sankaranarayanan, On some theorems of Littlewood and Selberg, I, J. Number Theory 44 (3) (1993) 281–291.
[13] K. Soundararajan, Moments of the Riemann zeta-function, Ann. of Math. (2) 170 (2) (2009) 981–993.
[14] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, second ed., The Clarendon Press/Oxford University Press, New York, 1986.