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On the characteristic p valued measure associated to Drinfeld discriminant

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ABSTRACT

In this paper, after reviewing known results on functions over Bruhat–Tits trees and the theory of characteristic p valued modular forms, we present some structure of the tempered distributions on the projective space $\mathbb{P}^1(\mathbf{k}_\infty)$ over a complete function field $\mathbf{k}_\infty$ of characteristic p, and calculate the characteristic p valued measure associated to the Drinfeld discriminant and the characteristic p valued measure associated to the Poincaré series.

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Throughout this paper, q is a fixed power of a prime p, $\mathbb{F}_q$ is the finite field of q elements. Let $\mathbf{A} = \mathbb{F}_q[T]$ be the polynomial ring with the discrete valuation at ∞, and $\mathbf{k}$ be the fraction field of $\mathbf{A}$. For an element α in $\mathbf{A}$, we denote by $\deg_T(\alpha)$ the degree of α as a polynomial in T, the subscript T is omitted if it does not cause any confusion. By convention, $\deg(0) = -\infty$. On the completion $\mathbf{k}_\infty = \mathbb{F}_q((\frac{1}{T}))$ of $\mathbf{k}$ at ∞, we take $\pi = \frac{1}{T}$ as the uniformizer, and set $\mathbf{A}_\infty = \mathbb{F}_q[\![\pi]\!] = \mathbb{F}_q[\![\frac{1}{T}]\!]$. The valuation v and absolute value $|\cdot|$ of $\mathbf{k}_\infty$ are normalized such that $v(\pi) = 1$, and $|\pi| = \frac{1}{q}$. Let $\mathbf{C}_\infty$ be the completion of an algebraic closure of $\mathbf{k}_\infty$ with the absolute value also denoted by $|\cdot|$. We set $\Gamma = \mathrm{GL}_2(\mathbf{A})$.

1. Introduction

Let $\mathbf{A}^+ = \{\text{monic polynomials in } \mathbb{F}_q[T]\}$. Over the function field $\mathbf{k}$, the analogues of the Riemann-zeta values [Ca] is given as

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$$\mathrm{z}(n) = \sum_{a\in\mathbf{A}^+} \frac{1}{a^n}$$

for $n \in \mathbb{Z}^+$. The above series clearly converges and $\mathrm{z}(n) \in \mathbf{k}_\infty$. The analogous zeta function is given by Goss [Go6, Chapter 8] as an analytic function $\zeta(s)$ over the "complex plane" $S_\infty = \mathbf{C}_\infty^* \times \mathbb{Z}_p$ with $s = (x, y) \in S_\infty$:

$$\zeta(s) = \sum_{a\in\mathbf{A}^+} \frac{1}{a^s} = \sum_{j=0}^{\infty} x^{-j} \Bigg(\sum_{\substack{a\in\mathbf{A}^+ \\ \deg(a)=j}} \langle a\rangle^{-y} \Bigg), \tag{1.1}$$

where $\langle a\rangle = T^{-\deg(a)} \cdot a$ is the 1-unit part of $a \in \mathbf{k}_\infty^\times$, and $a^s = x^{\deg(a)} \cdot (\langle a\rangle)^y$. Under this construction, we get the power sum $\mathrm{z}(n) = \zeta(T^n, n)$ for an integer $n \geqslant 1$. Goss [Go6] also constructs the general L-series over the "complex plane" S_∞.

One aspect of the analytic theory over function fields is the study of the Drinfeld's upper half plane $\Omega = \mathbf{C}_\infty - \mathbf{k}_\infty$, which comes as the rank 2 case of the analytic structures related to the moduli spaces of Drinfeld modules, see [Ge1] for an exposition. There are rich structures of rigid analytic spaces on Ω, so that we can talk about the concepts like modular forms and Hecke operators over function fields, see [Go1,Go2,Go3,Ge1,Ge2,GR], etc. But the Hecke operators thus defined so far can't distinguish the modular forms, and we don't know if there is any analogue of Mellin transforms relating the modular forms to the L-series over S_∞ (in Goss' sense), even for the L-series coming from geometry as [Bo] points out. Since there are no Haar measures of characteristic p, a general idea of dealing with this problem is to study the measures (more precisely, distributions) which define the L-series through integrations over power functions, and the measures associated to modular forms.

On the measures associated to the Goss zeta function $\zeta(s)$ in Eq. (1.1), we have the computations by Thakur [Th] at finite places and by Yang [Ya2] at the place ∞. The special values $\zeta(x, -j)$ are determined by the formula [Ya2]

$$\zeta(x, -j) = \int_{\mathbf{A}_\infty} t^j \, d\mu_x^{(\infty)}(t) = \int_{\mathcal{U}_1} t^j \, d\big(x\nu_x^{(\infty)}\big)(t) \tag{1.2}$$

for integers $j \geqslant 0$, where $\mathcal{U}_1$ is the 1-units of $\mathbf{A}_\infty$, and $\mu_x^{(\infty)}$ and $\nu_x^{(\infty)}$ are measures on $\mathbf{A}_\infty$ depending on the variable x.

On the measures associated to the modular forms, Teitelbaum [Te1] established an isomorphism between the vector space of the cusp forms of weight n of an arithmetic subgroup G of $\mathrm{GL}_2(\mathbf{k})$ and the vector space of the G-invariant harmonic co-cycles on the Bruhat–Tits tree $\mathcal{T}$ associated to $\mathbf{k}_\infty$. A harmonic co-cycle on $\mathcal{T}$ can also be viewed as a measure on $\mathbb{P}^1(\mathbf{k}_\infty)$, therefore Teitelbaum's theorem gives a way to compare the relevant measures we have discussed about. Goss [Go5] has pointed out some implications from Teitelbaum's construction of measures. But the construction of the harmonic co-cycles in Teitelbaum's theorem is quite abstract, it is not easy to understand what is behind these measures. We will carry out an explicit computation of the measure associated to the Drinfeld discriminant in Teitelbaum's theorem.

We summarize some well-known facts about Bruhat–Tits trees in Section 2, and give a simple introduction to characteristic p valued modular forms in Section 3.

In Section 4, we talk about the space of C^h functions on an open compact subset S of $\mathbb{P}^1(\mathbf{k}_\infty)$ and study its dual space, which is proved to be the space of h-admissible measures on S.

In Section 5, we introduce some known results of the theory of functions on Bruhat–Tits trees, and Teitelbaum's theorems on the correspondence between cusp forms of an arithmetic subgroup G of $\mathrm{GL}_2(\mathbf{k})$ and G-equivariant harmonic cocycles on Bruhat–Tits trees.

In Section 6, we explicitly determine the characteristic p valued measure associated to the Drinfeld discriminant $\Delta(z)$.

In Section 7, we calculate some special values of the L-function associated to the Drinfeld discriminant, and also give similar results on the characteristic p valued measure associated to the Poincaré series. Some comments about characteristic p valued modular forms and harmonic functions on Bruhat–Tits trees are made, and some possible application to algebraic ergodic theory is also mentioned in this section.

2. Bruhat–Tits trees

Let $V = \mathbf{k}_\infty \oplus \mathbf{k}_\infty$ be a two-dimensional vector space over $\mathbf{k}_\infty$. A lattice L of V is a finitely generated free $\mathbf{A}_\infty$-module of rank 2. Two lattices L and L' are said to be equivalent if there exists a $\lambda \in \mathbf{k}_\infty^*$ such that $L' = \lambda L$. Let $V_{\mathcal{T}}$ denote the set of all equivalence classes $[L]$ of lattices L of V. For any elements $\Lambda, \Lambda' \in V_{\mathcal{T}}$, we can find lattices $L \in \Lambda$ and $L' \in \Lambda'$ such that $L' \subset L$ and $L/L' \cong \mathbf{A}_\infty/\pi^n\mathbf{A}_\infty \cong \mathbf{A}/\pi^n\mathbf{A}$ for some non-negative integer n, which we denote by $d(\Lambda, \Lambda') = n$. We define

$$E_{\mathcal{T}} = \{e_{\Lambda\Lambda'} \colon d(\Lambda, \Lambda') = 1\},$$

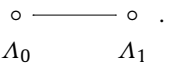
where $e_{\Lambda\Lambda'}$ denotes the edge connecting Λ and Λ'. Then the Bruhat–Tits tree $\mathcal{T}$ consists of the set $V_{\mathcal{T}}$ of vertices and the set $E_{\mathcal{T}}$ of (oriented, with orientation to be determined later in this section) edges. The action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on V induces an action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $\mathcal{T}$ (in fact, an action of $\mathrm{PGL}_2(\mathbf{k}_\infty)$ on $\mathcal{T}$).

Suppose that G is a group acting on a graph X. Then we have the quotient graph $G\backslash X$. A fundamental domain of $G\backslash X$ is a subgraph Z of X such that the natural map $Z \to G\backslash X$ is an isomorphism of graphs. A fundamental domain of a graph by a group action doesn't necessarily exist, but we have results in some special cases. Let

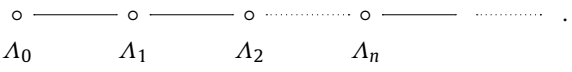
$$\mathrm{GL}_2(\mathbf{k}_\infty)^+ = \{\alpha \in \mathrm{GL}_2(\mathbf{k}_\infty) \colon v(\det\alpha) \text{ is an even integer}\}.$$

Let Λ_n denote the class of the lattice $\mathbf{A}_\infty \oplus \pi^n\mathbf{A}_\infty \sim T^n\mathbf{A}_\infty \oplus \mathbf{A}_\infty$ for $n \in \mathbb{Z}$. We know the following properties of the actions of subgroups of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $\mathcal{T}$, the detail can be found in Serre's book [Se].

- Let G be a subgroup of $\mathrm{GL}_2(\mathbf{k}_\infty)^+$. If the closure of G in $\mathrm{GL}_2(\mathbf{k}_\infty)$ contains $\mathrm{SL}_2(\mathbf{k}_\infty)$, then the fundamental domain of the action of G on $\mathcal{T}$ is



- The fundamental domain of the action of $\Gamma = \mathrm{GL}_2(\mathbf{A})$ on $\mathcal{T}$ is



(2.1)

The action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $V_{\mathcal{T}}$ is transitive, thus we have the bijections

$$V_{\mathcal{T}} \overset{\cong}{\longleftrightarrow} \mathrm{GL}_2(\mathbf{k}_\infty)/\mathrm{Stab}(\Lambda_0) = \mathrm{GL}_2(\mathbf{k}_\infty)/\big(\mathbf{k}_\infty^* \cdot \mathrm{GL}_2(\mathbf{A}_\infty)\big), \tag{2.2}$$

where $\mathrm{Stab}(\Lambda_0)$ denotes the stabilizer of Λ_0 in $\mathrm{GL}_2(\mathbf{k}_\infty)$. Therefore we can use matrices to express the vertices of $\mathcal{T}$ in terms of the coset representatives in the above relation:

$$V_{\mathcal{T}} = \left\{ \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \colon k \in \mathbb{Z},\ u \in \mathbf{k}_\infty,\ u \bmod \pi^k \mathbf{A}_\infty \right\}. \tag{2.3}$$

Let $\mathbf{v}_1 = (1,0)^T$, $\mathbf{v}_2 = (0,1)^T$ be the standard basis of $V = \mathbf{k}_\infty \oplus \mathbf{k}_\infty$. Under the one-to-one correspondence (2.2) and the representations of the coset representatives (2.3), we have

$$\left[\pi^k \mathbf{v}_1, u\mathbf{v}_1 + \mathbf{v}_2\right] \longleftrightarrow \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

where k and u are as in (2.3), and $[\pi^k \mathbf{v}_1, u\mathbf{v}_1 + \mathbf{v}_2]$ denotes the equivalence class of the $\mathbf{A}_\infty$-lattice generated by $\pi^k \mathbf{v}_1$ and $u\mathbf{v}_1 + \mathbf{v}_2$.

An end of the tree $\mathcal{T}$ is an infinite path starting at some vertex and without back-tracking, for example, the half line in (2.1) from the vertex Λ_0 to Λ_1, Λ_2, and so on. Two ends are equivalent if and only if they differ by a finite number of vertices and edges. The set of ends of $\mathcal{T}$ is in bijection with $\mathbb{P}^1(\mathbf{k}_\infty)$. This bijection is set up in this paper as follows. We choose the vertex Λ_0 as the starting point of any end (in some equivalence class), then $\infty \in \mathbb{P}^1(\mathbf{k}_\infty)$ corresponds to the end (2.1). For $x = \sum_{n=n_0}^{\infty} c_n \pi^n \in \mathbf{k}_\infty$, where $c_n \in \mathbb{F}_q$, the corresponding end (which is in some equivalence class; but the end of the following graph may not start with the vertex Λ_0) is

$$\underset{[\mathbf{v}_1,\, x\cdot\mathbf{v}_1+\mathbf{v}_2]}{\circ} \text{———} \underset{[\pi\mathbf{v}_1,\, x\cdot\mathbf{v}_1+\mathbf{v}_2]}{\circ} \text{———} \circ \cdots\cdots \underset{[\pi^n\mathbf{v}_1,\, x\cdot\mathbf{v}_1+\mathbf{v}_2]}{\circ} \text{———} \cdots\cdots \tag{2.5}$$

where we notice that $[\pi^n \mathbf{v}_1, x\cdot\mathbf{v}_1 + \mathbf{v}_2] = [\pi^n \mathbf{v}_1, x_{[n-1]}\cdot\mathbf{v}_1 + \mathbf{v}_2]$, with $x_{[n]} = \sum_{k=n_0}^{n} c_k \pi^k$.

The edges of $\mathcal{T}$ are oriented, with the set $E_{\mathcal{T}}^+$ of edges of positive orientation given in terms of the ∞ end of $\mathcal{T}$:

$$\underset{\Lambda_0}{\circ} \longrightarrow \underset{\Lambda_1}{\circ} \longrightarrow \underset{\Lambda_2}{\circ} \cdots\cdots\!\!> \underset{\Lambda_n}{\circ} \longrightarrow \cdots\cdots, \tag{2.6}$$

any edge with a consistent orientation with the above is in $E_{\mathcal{T}}^+$ (since $\mathcal{T}$ is connected).

Let $\Gamma_0 = \mathrm{GL}_2(\mathbb{F}_q)$, a subgroup of $\Gamma = \mathrm{GL}_2(\mathbf{A})$, and

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \colon a, d \in \mathbb{F}_q^*,\ b \in \mathbb{F}_q[T],\ \deg_T(b) \leqslant n \right\},$$

for $n \geqslant 1$. Then we have

Proposition 2.1. *(See Serre [Se].)*

(1) *Γ_n is the stabilizer of Λ_n.*
(2) *Γ_0 acts transitively on the set of edges with origin Λ_0.*
(3) *For $n \geqslant 1$, Γ_n fixes the edge $\Lambda_n\Lambda_{n+1}$ and acts transitively on the set of edges with origin Λ_n but distinct from the edge $\Lambda_n\Lambda_{n+1}$.*

3. Modular forms

The group $\mathrm{GL}_2(\mathbf{k}_\infty)$ acts on the Drinfeld's upper half plane $\Omega = \mathbf{C}_\infty - \mathbf{k}_\infty = \mathbb{P}^1(\mathbf{C}_\infty) - \mathbb{P}^1(\mathbf{k}_\infty)$ by

$$\gamma \cdot z = \frac{az+b}{cz+d}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{k}_\infty)$ and $z \in \Omega$. This action is also written as $\gamma(z)$ or γz. The space Ω has a good covering $\{D_i\}_{i\in I}$, where $I = \{(n,x)\colon\ n \in \mathbb{Z},\ x \in \mathbf{k}_\infty/\pi^{n+1}\mathbf{A}_\infty\}$, $D_{(n,x)} = x + D_n$, and

$$D_n = \left\{ z \in \mathbf{C}_\infty\colon \begin{array}{l} |\pi^{n+1}| \leqslant |z| \leqslant |\pi^n|, \\ |z - c\pi^n| \geqslant |\pi^n|,\ |z - c\pi^{n+1}| \geqslant |\pi^{n+1}| \text{ for all } c \in \mathbb{F}_q^* \end{array} \right\}.$$

These subsets D_i's of Ω are affinoid spaces and Ω has a rigid analytic structure, see [GR] and [GP] for the details. Hence we can talk about the rigid analytic functions on Ω and study their properties.

A finitely generated $\mathbf{A}$-submodule $L \subset \mathbf{C}_\infty$ is said to be an $\mathbf{A}$-lattice of $\mathbf{C}_\infty$ if L is discrete in the topology of $\mathbf{C}_\infty$. Hence an $\mathbf{A}$-lattice L of $\mathbf{C}_\infty$ is a projective $\mathbf{A}$-module of finite rank d, and we have an isomorphism of $\mathbf{A}$-modules

$$L \cong \mathbf{A}^d$$

since $\mathbf{A} = \mathbb{F}_q[T]$ is a principal ideal domain.

Example 3.1. Let $L \subset \mathbf{C}_\infty$ be an $\mathbf{A}$-lattice of rank 1, in particular, a fractional ideal of $\mathbf{A}$. Then

$$e_L(z) = z \prod_{0 \neq a \in L} \left(1 - \frac{z}{a}\right)$$

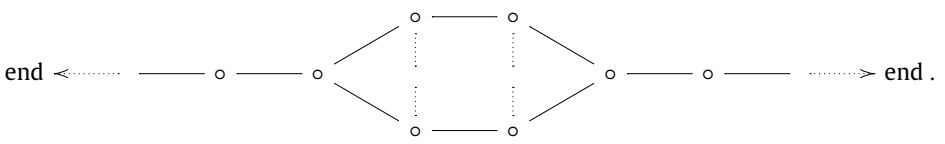
and

$$t_L(z) = e_L^{-1}(z) = \sum_{a \in L} \frac{1}{z-a} \tag{3.1}$$

are $\mathbb{F}_q$-linear rigid analytic functions on Ω. These two functions are invariant under L-translations. See [Go6], or [Ge2].

Generally, a subgroup G of $\mathrm{GL}_2(\mathbf{k})$ is called arithmetic if G is contained in $\mathrm{GL}(Y)$ and contains the kernel $G(Y, \mathfrak{a})$ of the reduction map $\mathrm{GL}(Y) \to \mathrm{GL}(Y/\mathfrak{a}Y)$ for some rank 2 projective $\mathbf{A}$ submodule Y of $\mathbf{k} \oplus \mathbf{k}$ and some ideal $\mathfrak{a}$ of $\mathbf{A}$. In this paper, we assume that $\mathbf{A}$ is a polynomial ring over $\mathbb{F}_q$, hence the rank 2 projective $\mathbf{A}$-module Y is free: $Y \cong \mathbf{A} \oplus \mathbf{A}$, thus we only consider the arithmetic subgroups of $\mathrm{GL}_2(\mathbf{A})$.

As a subgroup of $\mathrm{GL}_2(\mathbf{k}_\infty)$, an arithmetic subgroup G acts on the Bruhat–Tits tree $\mathcal{T}$. The quotient $G\backslash\mathcal{T}$ is a finite graph joined by a finite number of ends. For example, in the following graph, there are two ends on the left and on the right, and the middle is finite.



These ends are called the cusps of the arithmetic subgroup G. We have the bijection

$$\{\text{cusps of } G\} \xrightarrow{\cong} G\backslash\mathbb{P}^1(\mathbf{k}).$$

We refer §2, Chapter II of [Se] for the details. Thus we see from (2.1) that the group Γ has exactly one cusp ∞.

Definition 3.1. Let G be an arithmetic subgroup of $\mathrm{GL}_2(\mathbf{k})$. A rigid analytic function $f : \Omega \to \mathbf{C}_\infty$ is called a modular form with respect to the arithmetic subgroup G of weight k and type m for a non-negative integer k and a class m in $\mathbb{Z}/(q-1)$ if the following two conditions are satisfied:

(1) for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, the following equation holds

$$f|[\gamma]_{k,m}(z) = f(z),$$

where $f|[\gamma]_{k,m}(z) := (\det(\gamma))^m j(\gamma, z)^{-k} f(\gamma z)$, and

$$j(\gamma, z) = cz + d; \tag{3.2}$$

(2) f is holomorphic at every cusp of G.

In the above definition, the second condition is interpreted as follows. For a cusp p of G, we take an element $\rho \in \mathrm{GL}_2(\mathbf{k})$ such that $\rho(\infty) = p$. Then the stabilizer of ∞ in $\rho^{-1}G\rho$ contains a maximal subgroup of G consisting of elements of the form $\gamma_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, where $x \in L$ for some fractional ideal L of $\mathbf{A}$. Such an element γ_x acts on Ω as a translation by x, therefore the first condition gives $f(z+x) = f(\gamma_x z) = f(z)$. Compared with the classical situation, the rigid analytic function $t_L(z)$ of (3.1) serves as a parameter at the infinity point ∞. So f is holomorphic at p if and only if f can be expanded as a series in terms of the parameter $t_L(z)$:

$$f|[\rho]_{k,m}(z) = \sum_{i \geqslant 0} c_i t_L(z)^i, \quad \text{where } c_i \in \mathbf{C}_\infty. \tag{3.3}$$

If the expansion coefficient $c_0 = 0$ in (3.3) for all cusps of G, then f is called a cusp form of G with weight k and type m.

Remark 3.1. In this paper, we assume $\mathbf{A} = \mathbb{F}_q[T]$, and $\mathbf{k} = \mathbb{F}_q(T)$, but the same definition as above can be stated for the general case when $\mathbf{k}$ is the field of functions on a complete, geometrically irreducible curve over the field $\mathbb{F}_q$ and $\mathbf{A}$ is the ring of regular functions away from the point ∞.

Example 3.2. The Eisenstein series $E_k(z)$ is defined as

$$E_k(z) = \sum_{(0,0)\neq(c,d)\in\mathbf{A}^2} \frac{1}{(cz+d)^k} \tag{3.4}$$

for an integer $k \geqslant 0$. If $k \not\equiv 0 \bmod (q-1)$, then $E_k(z)$ is identically equal to 0. For $k \equiv 0 \bmod (q-1)$, the Eisenstein series $E_k(z)$ is a non-cuspidal modular form of the arithmetic subgroup $\Gamma = \mathrm{GL}_2(\mathbf{A})$ with weight k and type 0. See [Go2] for the detail.

Example 3.3. This is an example given by Gekeler [Ge2]. Let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \colon a \in \mathbb{F}_q{}^*,\ b \in \mathbf{A} \right\}$$

be a subgroup of Γ, and $t(z)$ be the rigid analytic function defined as (3.1) for the rank 1 lattice $\mathbf{A}\bar{\pi}$, where the constant $\bar{\pi} \in \mathbf{C}_\infty^*$ is some "period". Then the Poincaré series

$$P_{k,m}(z) = \sum_{\gamma \in H \backslash \Gamma} \big(\det(\gamma)\big)^m j(\gamma, z)^{-k} t^m(\gamma z) \tag{3.5}$$

is a cusp form of Γ with weight k and type m mod $(q-1)$, where $j(\gamma, z)$ is defined as in (3.2).

Example 3.4. Let $L \subset \mathbf{C}_\infty$ be an $\mathbf{A}$-lattice of $\mathbf{C}_\infty$ of rank 2. Such a lattice L is associated with an $\mathbb{F}_q$-linear function

$$e_L(z) = z \prod_{0 \neq \lambda \in L} \left(1 - \frac{z}{\lambda}\right)$$

which is rigid analytic on Ω. Then there exists a ring homomorphism

$$\begin{aligned} \phi^L : \mathbf{A} &\to \mathbf{C}_\infty\{\tau\} = \left\{ \sum_{i \geqslant 0} a_i \tau^i \colon a_i \in \mathbf{C}_\infty \text{ for each integer } i \geqslant 0 \right\}, \\ a &\mapsto \phi_a^L \end{aligned}$$

such that

$$\phi_a^L \cdot e_L = e_L \cdot a \tag{3.6}$$

for any $a \in \mathbf{A}$, where $\tau(x) = x^q$ for any $x \in \mathbf{C}_\infty$ is the Frobenius map. In the above Eq. (3.6), the dot notion "$\cdot$" denotes the composition map, a is regarded as the multiplication map from $\mathbf{C}_\infty$ to itself, and $e_L : \mathbf{C}_\infty \to \mathbf{C}_\infty$ is the additive map given by the function $e_L(z)$.

The ring homomorphism ϕ^L is a Drinfeld module of rank 2 over $\mathbf{C}_\infty$ associated to the $\mathbf{A}$-lattice L of $\mathbf{C}_\infty$. For $0 \neq a \in \mathbf{A}$, we can see that $\phi_a^L(z)$ is a polynomial in z and is given as

$$\phi_a^L(z) = az \prod_{0 \neq \lambda \in a^{-1}L/L} \left(1 - \frac{z}{e_L(\lambda)}\right)$$

by checking the zeroes of $e_L(az)$ and $\phi_a^L(e_L(z))$ and the coefficients of the first terms of the expansions of these two rigid analytic functions in terms of z. The homomorphism ϕ is determined by

$$\phi_T^L = T\tau^0 + g_L\tau + \Delta_L\tau^2. \tag{3.7}$$

We notice that λL is also an $\mathbf{A}$-lattice of $\mathbf{C}_\infty$ for $\lambda \in \mathbf{C}_\infty^*$ and $e_{\lambda L}(\lambda z) = \lambda e_L(z)$, therefore

$$\begin{aligned}\phi_T^{\lambda L}\big(\lambda e_L(z)\big) &= \phi_T^{\lambda L}\big(e_{\lambda L}(\lambda z)\big)\\ &= e_{\lambda L}(T\lambda z)\\ &= \lambda e_L(Tz)\\ &= \lambda\phi_T^{L}\big(e_L(z)\big).\end{aligned}$$

Putting the above equation into Eq. (3.7) for ϕ_T^L and $\phi_T^{\lambda L}$, we get

$$T\lambda e_L(z) + g_{\lambda L}\lambda^q e_L(z)^q + \Delta_{\lambda L}\lambda^{q^2} e_L(z)^{q^2} = \lambda T e_L(z) + \lambda g_L e_L(z)^q + \lambda\Delta_L e_L(z)^{q^2}.$$

Thus we have

$$g_{\lambda L} = \lambda^{1-q} g_L, \qquad \Delta_{\lambda L} = \lambda^{1-q^2}\Delta_L. \tag{3.8}$$

An $\mathbf{A}$-lattice of $\mathbf{C}_\infty$ of rank 2 can be written as $L = \mathbf{A}\omega_1 \oplus \mathbf{A}\omega_2 = \omega_2(\mathbf{A}z \oplus \mathbf{A})$ with $\omega_1, \omega_2 \in \mathbf{C}_\infty^*$ and $z = \omega_1/\omega_2 \in \Omega$ (because L is discrete in the topology of $\mathbf{C}_\infty$). We define for the lattice $L_z = \mathbf{A}z \oplus \mathbf{A} \subset \mathbf{C}_\infty$ with $z \in \Omega$

$$g(z) = g_{L_z}, \qquad \Delta(z) = \Delta_{L_z}. \tag{3.9}$$

From the above Eq. (3.9), it is not difficult to see that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$

$$g(\gamma z) = (cz+d)^{q-1} g(z), \qquad \Delta(\gamma z) = (cz+d)^{q^2-1}\Delta(z).$$

The two functions $g(z)$ and $\Delta(z)$ on Ω can be expressed in terms of Eisenstein series $E_k(z)$ (see [Go1] or Section 2, Chapter II of [Ge1]), so they are rigid analytic functions on Ω and are holomorphic at the cusp ∞ of Γ. Therefore $g(z)$ is a modular form of Γ of weight $q-1$ and type 0 and $\Delta(z)$ is a modular form of Γ of weight q^2-1 and type 0.

The function $\Delta(z)$ on the Drinfeld's upper half plane Ω is called the Drinfeld discriminant. We refer Chapter 4 of [Go6] for more details and related topics of Drinfeld modules, see also [Ge1].

Theorem 3.1. *We have the following results with respect to the modular forms of Γ:*

(1) (Goss [Go1].) *The graded ring of modular forms of type* 0 *associated with the group Γ has dimension* 2 *and is equal to the polynomial ring $\mathbf{C}_\infty[g, \Delta]$.*
(2) (Gekeler [Ge2].) *The graded ring of modular forms associated with the group Γ has dimension* 2 *and is equal to the polynomial ring $\mathbf{C}_\infty[g, P_{q+1,1}]$, where $P_{q+1,1}$ is given in (3.5). Moreover, $\Delta = -(P_{q+1,1})^{q-1}$.*
(3) (Goss [Go1]; also see [Ge2].) *The following two equalities hold*

$$\begin{aligned} g(z) &= \big(T^q - T\big)E_{q-1}(z),\\ \Delta(z) &= \big(T^{q^2} - T\big)E_{q^2-1}(z) + \big(T^q - T\big)^q E_{q-1}(z)^{q+1}.\end{aligned} \tag{3.10}$$

4. Characteristic p valued measures on $\mathbb{P}^1(\mathbf{k}_\infty)$

For $a \in \mathbf{k}_\infty$ and $r > 0$ a real number, we denote the closed ball of radius r and centered at a by $B_a(r) = \{x \in \mathbf{k}_\infty\colon\ |x-a| \leqslant r\}$. As the set of absolute values of $\mathbf{k}_\infty$ is $\{|\pi|^l\colon\ l \in \mathbb{Z}\} = \{q^l\colon\ l \in \mathbb{Z}\}$, we need only consider the closed balls $B_a(|\pi|^l) = B_a(q^{-l})$ of radius q^{-l}, for $l \in \mathbb{Z}$. We also denote by $B_\infty(|\pi|^l) = \{x \in \mathbf{k}_\infty\colon\ |x| \geqslant |\pi|^{-l}\} \cup \{\infty\}$, which is viewed as a closed ball of radius q^{-l} centered at ∞. A function $f : B_a(|\pi|^l) \to \mathbf{C}_\infty$ on the ball $B_a(|\pi|^l)$ is said to be analytic if f can be expanded as a Taylor series

$$f(x) = \sum_{n=0}^{\infty} c_n \left(\frac{x-a}{\pi^l}\right)^n \quad (c_n \in \mathbf{C}_\infty \text{ for } n \geqslant 0) \tag{4.1}$$

which is convergent for any $x \in B_a(|\pi|^l)$, i.e., $c_n \to 0$ as $n \to \infty$. And a function $f : B_\infty(|\pi|^l) \to \mathbf{C}_\infty$ is said to be analytic on $B_\infty(|\pi|^l)$ if $g(x) := f(\frac{1}{x})$ is analytic on the ball $B_0(|\pi|^l)$.

Let $S \subset \mathbb{P}^1(\mathbf{k}_\infty)$ be an open compact nonempty subset. Hence S is a finite disjoint union of closed balls of positive radii.

A function $f : S \to \mathbf{C}_\infty$ is said to be locally analytic (of order l, respectively) at the point $a \in S$ if f is analytic on some closed ball B_a which is centered at a and of positive radius (at least $|\pi|^l$, respectively). And f is said to be locally analytic (of order l, respectively) on S if f is locally analytic (of order l, respectively) at every point of S. The space of all locally analytic functions (of order l, respectively) (taking values in $\mathbf{C}_\infty$) on S is denoted by $LA(S)$ ($LA_l(S)$, respectively).

Remark 4.1. The above definition of local analyticity of order l is a little ambiguous, since the balls $B_a(|\pi|^l)$ may not be contained in S for a given positive integer l. But we have assumed that S is open compact, therefore $B_a(|\pi|^l) \subset S$ as long as the integer l is sufficiently large (say $|\pi|^l$ is less than or equal to the smallest value among all radii in a decomposition of S into finite disjoint union of closed balls of positive radii). So the definition makes sense, and we always assume that such an integer l in the above is sufficiently large.

The space $LA_l(S)$ is equipped with a norm $\|\cdot\|$ as follows. We have a decomposition of S into a finite disjoint union

$$S = \bigsqcup_i B_i, \tag{4.2}$$

where each B_i is a closed ball of radius $|\pi|^l$. We notice that if one of these closed balls, say B_{i_0}, is centered at ∞, then $B_{i_0} = B_\infty(|\pi|^l) = \{x \in \mathbf{k}_\infty\colon\ |x| \geqslant |\pi|^{-l}\} \cup \{\infty\}$. Suppose $f \in LA_l(S)$. As every element of a closed ball is a center under a non-Archimedean absolute value, f has an expansion as Eq. (4.1) on each closed ball B_i in the decomposition (4.2):

$$f|_{B_i}(x) = \sum_{n \geqslant 0} c_{in} \left(\frac{x-a_i}{\pi^l}\right)^n, \tag{4.3}$$

for any $a_i \in B_i$, and the expansion should be replaced by the following if B_{i_0} is the closed ball centered at ∞:

$$g_{i_0}(x) = \sum_{n \geqslant 0} c_{i_0,n} \left(\frac{x}{\pi^l}\right)^n, \qquad g_{i_0}(x) := f|_{B_{i_0}}\left(\frac{1}{x}\right).$$

Then we define

$$\|f\|_{B_i} = \max_{n\geqslant 0}\{|c_{in}|\}, \quad \text{and} \quad \|f\| = \max_i\{\|f\|_{B_i}\}. \tag{4.4}$$

To see (4.4) is well defined, we need to show

Lemma 4.1. *$\|f\|_{B_i}$ is independent of the choices of the center a_i.*

Proof. Suppose $b_i \in B_i$. Then b_i is also a center of B_i. And $f|_{B_i}(x)$ has expansion (4.3) at a_i, and has the following expansion at b_i as well:

$$f|_{B_i}(x) = \sum_{n\geqslant 0} c'_{in}\left(\frac{x-b_i}{\pi^l}\right)^n, \quad \text{where } c'_{in} = \sum_{j\geqslant n} c_{ij}\binom{j}{n}\left(\frac{b_i-a_i}{\pi^l}\right)^{j-n}.$$

As $|b_i-a_i| \leqslant |\pi^l|$, $|\binom{j}{n}| \leqslant 1$, and $|c_{ij}| \leqslant \|f\|_{B_i}$, we see that $\|f\|'_{B_i} := \max_{n\geqslant 0}\{|c'_{in}|\} \leqslant \|f\|_{B_i}$. In the same way, we have $\|f\|_{B_i} \leqslant \|f\|'_{B_i}$, hence we get the conclusion. □

It is easy to see that $LA_l(S)$ is a Banach space with the norm defined above. And we have the following sequence of closed inclusions of Banach spaces

$$\cdots \hookrightarrow LA_l(S) \hookrightarrow LA_{l+1}(S) \hookrightarrow LA_{l+2}(S) \hookrightarrow \cdots$$

and $LA(S) = \varinjlim LA_l(S) = \bigcup_l LA_l(S)$. We equip $LA(S)$ with the topology of direct limit. In our case, a subset $W \subset LA(S)$ is closed if and only if $W \cap LA_l(S)$ is closed in $LA_l(S)$ for each sufficiently large integer l. This is equivalent to saying that for any topological space Z, a map $f : LA(S) \to Z$ is continuous if and only if $f|_{LA_l(S)} : LA_l(S) \to Z$ is continuous for each sufficiently large l.

We denote by $M^* = \mathrm{Hom}_{\mathbf{C}_\infty}(M, \mathbf{C}_\infty)$ the space of all $\mathbf{C}_\infty$-linear continuous maps from M to $\mathbf{C}_\infty$ for a given topological vector space M over $\mathbf{C}_\infty$.

Remark 4.2. Suppose $(M, \|\cdot\|)$ is a separable $\mathbf{C}_\infty$-Banach space. For any map $f : M \to \mathbf{C}_\infty$ we define $\|f\| = \sup_{0\neq x\in M}\{|f(x)|/\|x\|\}$. Then

$$\begin{aligned} M^* &= \mathrm{Hom}_{\mathbf{C}_\infty}(M, \mathbf{C}_\infty) \\ &= \{f\colon\ f \text{ is } \mathbf{C}_\infty\text{-linear and continuous from } M \text{ to } \mathbf{C}_\infty\} \\ &= \big\{f\colon\ f \text{ is } \mathbf{C}_\infty\text{-linear and } \|f\| < \infty\big\}. \end{aligned}$$

Take now $M = LA(S)$ (with the topology given above). The elements of $LA(S)^*$ are called tempered distributions on S (with values taken in $\mathbf{C}_\infty$), and we write $(f, \mu) := \mu(f)$ for $\mu \in LA(S)^*$ and $f \in LA(S)$.

Remark 4.3. Let $LC(S)$ denote the $\mathbf{C}_\infty$-vector space of locally constant functions on

$$P(S) = \{U\colon\ U \subset S \text{ is open and compact}\}.$$

A distribution on S is a linear functional $\mu \in (LC(S))^*$. We put

$$\mu(U) := \big(\xi_U(x), \mu\big),$$

where $\xi_U(x)$ is the characteristic function of the subset U. Then μ can be viewed as a function from $P(S)$ to $\mathbf{C}_\infty$ which satisfies the finite additivity property: if $U, V \in P(S)$ and $U \cap V = \emptyset$, then $\mu(U \cup V) = \mu(U) + \mu(V)$. If $\{\mu(U)\colon\ U \in P(S)\}$ is bounded, then μ is called a measure on S. Since locally constant functions are locally analytic, we see that any element $\mu \in LA(S)^*$ is a distribution on S, thus it is natural to write

$$\int_S f(x)\,d\mu(x) := (f, \mu)$$

for $f \in LA(S)$. We also write for an open compact subset $U \subset S$

$$\int_U f(x)\,d\mu(x) = \int_S f(x)\xi_U(x)\,d\mu(x)$$

provided that $f(x)$ is locally analytic over U, and we extend f by 0 outside of U in the second integral.

Definition 4.1.

(1) For integers $l, j \geqslant 0$, and $a \in S$, we define the following functions on S:

$$\chi(a, l; j; x) = \begin{cases} (x-a)^j, & \text{if } x \in B_a(|\pi|^l), \\ 0, & \text{if } x \notin B_a(|\pi|^l), \end{cases}$$

for $B_a(|\pi|^l) \subset S$, provided that $a \neq \infty$, and

$$\chi(\infty, l; j; x) = \begin{cases} (\frac{1}{x})^j, & \text{if } x \in B_\infty(|\pi|^l), \\ 0, & \text{if } x \notin B_\infty(|\pi|^l), \end{cases}$$

for $B_\infty(|\pi|^l) \subset S$, where $B_\infty(|\pi|^l) = \{x \in \mathbf{k}_\infty\colon\ |x| \geqslant |\pi^{-1}|^l\}$. In what follows, we understand the term $(x-a)^j$ on $B_a(|\pi|^l)$ needs to be replaced by $(1/x)^j$ if $a = \infty$.

(2) Let l, n be non-negative integers. The $\mathbf{C}_\infty$-vector spaces $P_l^{(n)}$ are defined as

$$P_l^{(n)} = \sum_{\substack{a \in S \\ 0 \leqslant j \leqslant n}} \mathbf{C}_\infty \chi(a, l; j; x),$$

this is the vector space generated by all the functions $\chi(a, l; j; x)$, with $0 \leqslant j \leqslant n, a \in S$, over $\mathbf{C}_\infty$. We put

$$P^{(n)} = \varinjlim P_l^{(n)},$$

where the direct limit is taken with respect to the injective maps $P_l^{(n)} \to P_{l+1}^{(n)}$.

(3) Denote

$$P^{(\infty)} = \varinjlim P^{(n)}, \quad \text{and} \quad P_l^{(\infty)} = \varinjlim P_l^{(n)},$$

where the direct limits are taken over the injective maps $P^{(n)} \to P^{(n+1)}$ and $P_l^{(n)} \to P_l^{(n+1)}$ respectively. $P^{(\infty)}$ is in fact the space of all locally polynomial functions over S, and $P_l^{(\infty)}$ those defined for balls of radius $|\pi|^l$ (which are contained in S).

We notice that for $\mu \in LA(S)^*$, the set $\{\mu(U) = (\xi_U(x), \mu)\colon\ U \subset S \text{ open compact}\}$ does not determine μ, instead we have

Proposition 4.1.

(1) $\mu \in (P_l^{(\infty)})^* = \mathrm{Hom}_{\mathbf{C}_\infty}(P_l^{(\infty)}, \mathbf{C}_\infty)$ *can be extended to an element of* $(LA_l(S))^*$ *if and only if*

$$\left| \int\limits_{B_a(|\pi|^l)} \chi(a,l;j;x)\,d\mu(x) \right| = \left| \int\limits_{S} \chi(a,l;j;x)\,d\mu(x) \right| \leqslant C \cdot |\pi|^{lj},$$

for any $j \geqslant 0$ *and any* $B_a(|\pi|^l) \subset S$, *where* C *is a constant depending only on* μ.

(2) $\mu \in (P^{(\infty)})^* = \mathrm{Hom}_{\mathbf{C}_\infty}(P^{(\infty)}, \mathbf{C}_\infty)$ *can be extended to an element of* $(LA(S))^*$ *if and only if*

$$\left| \int\limits_{B_a(|\pi|^l)} \chi(a,l;j;x)\,d\mu(x) \right| = \left| \int\limits_{S} \chi(a,l;j;x)\,d\mu(x) \right| \leqslant C(l) \cdot |\pi|^{lj},$$

for any sufficiently large integer l, *any* $j \geqslant 0$, *and any* $B_a(|\pi|^l) \subset S$, *where* $C(l)$ *is a constant depending only on* l *and* μ.

Proof. Since the topology on $LA(S)$ is induced from those of all $LA_l(S)$'s, we need only prove (1). Suppose $\mu \in (P_l^{(\infty)})^*$ can be extended to an element of $LA_l(S)$, which is still denoted by μ. By Remark 4.2,

$$\begin{aligned}\left| \int\limits_{B_a(|\pi|^l)} \chi(a,l;j;x)\,d\mu(x) \right| &= \left| \int\limits_{B_a(|\pi|^l)} (x-a)^j\,d\mu(x) \right| \\ &\leqslant \|\mu\| \cdot |\pi|^{lj} \cdot \left\| \left(\frac{x-a}{\pi^l} \right)^j \right\| \\ &\leqslant \|\mu\| \cdot |\pi|^{lj},\end{aligned}$$

if $\infty \notin B_a(|\pi|^l)$. And similarly if $\infty \in B_a(|\pi|^l)$ (therefore $a = \infty$ by the assumption on the notation $B_a(|\pi|^l)$).

Conversely, suppose the condition holds, and $f \in LA_l(S)$. Then we decompose $S = \bigsqcup_i B_i$ as a finite disjoint union of closed balls of radius $|\pi|^l$. On each B_i, the function f can be expanded as

$$f|_{B_i}(x) = \sum_{n \geqslant 0} c_{in} \left(\frac{x - a_i}{\pi^l} \right)^n, \quad \text{with } a_i \in B_i, \text{ and } c_{in} \to 0 \text{ as } n \to \infty$$

(if $\infty \in B_i$, the function $f(x) = g(1/x)$ for some analytic function g on $B_0(|\pi|^l)$, and we get the expansion of f in terms of the parameter $1/x$ by the expansion of g, then we proceed similarly). Therefore we extend $\mu \in (P_l^{(\infty)})^*$ by defining

$$(f, \mu) = \sum_i \sum_{n \geqslant 0} \pi^{-ln} c_{in} \int\limits_{B_i} (x - a_i)^n\,d\mu(x),$$

which is convergent under the given assumption. The assumption on $\mu \in (P_l^{(\infty)})^*$ also implies that the extension of μ to $LA_l(S)$ is continuous. □

Remark 4.4. In Proposition 4.1, the condition for $\mu \in (P^{(\infty)})^*$ to be extendable to an element of $(LA(S))^*$ is based on the topology of the space of locally analytic functions. In the following definition, we are going to specify the constant $C(l)$ in (2) of Proposition 4.1 and define the "h-admissible" measures, which will be proved dual to the C^h-functions (see Definition 4.3).

Definition 4.2. For a non-negative integer h, a linear functional $\mu \in (P^{(h)})^*$ is called an h-admissible measure on S, if it satisfies the following growth condition:

$$\left| \int_{B_a(|\pi|^l)} (x-a)^j \, d\mu(x) \right| \leqslant C \cdot |\pi|^{l(j-h)} \tag{4.5}$$

for $0 \leqslant j \leqslant h$, any l sufficiently large, and any $B_a(|\pi|^l) \subset S$, where C is a constant depending only on μ. If $a = \infty$, then (4.5) should be understood as

$$\left| \int_{B_a(|\pi|^l)} \left(\frac{1}{x}\right)^j d\mu(x) \right| \leqslant C \cdot |\pi|^{l(j-h)}.$$

Remark 4.5. This definition is a little bit different from [Vi1] and [Vi2] by Vishik. The right side of (4.5) is $o(1) \cdot |\pi|^{l(j-h)}$ in Vishik's papers. This change is made in order to relate the h-admissible measures with C^h-functions on S instead of functions satisfying the "h-th order Lipschitz" condition (as in Vishik's paper).

It is clear from the above definition that the 0-admissible measures on S are exactly the measures on S. In general, the h-admissible measures on S are dual to the space of C^h functions on S. Since $\mathbf{k}_\infty$ is of characteristic $p > 0$, the definition of differentiability of a function is a little different to the usual one in the case of characteristic 0. At first, we consider the functions on closed balls of $\mathbb{P}^1(\mathbf{k}_\infty)$ and follow Schikhof's definition of C^n-functions [Sc] that we present now.

Definition 4.3. Let $B \subset \mathbf{k}_\infty$ be a closed ball of positive radius. For an integer $n > 0$, set

$$\Delta^n B = \left\{(x_1, x_2, \ldots, x_n) \in B^n \colon x_i \neq x_j \text{ if } i \neq j\right\}.$$

The n-th order difference quotient $\Phi_n f : \Delta^{n+1} B \to \mathbf{C}_\infty$ of a function $f : B \to \mathbf{C}_\infty$ is inductively defined by

$$\Phi_0 f = f,$$
$$(\Phi_n f)(x_1, x_2, \ldots, x_{n+1}) = \frac{(\Phi_{n-1} f)(x_1, x_3, \ldots, x_{n+1}) - (\Phi_{n-1} f)(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}$$

for $n \geqslant 1$ and $(x_1, x_2, \ldots, x_{n+1}) \in \Delta^{n+1} B$. The function f is called a C^n-function on B if $\Phi_n f$ can be extended to a continuous function $\overline{\Phi_n f} : B^{n+1} \to \mathbf{C}_\infty$. For $B = B_\infty(|\pi|^l)$, we change the parameter of B by the transform $\phi : x \to 1/x$ and make a similar definition. Since ϕ is one-to-one and continuous on B, the definition of differentiability of functions on $B' \subset B_\infty(|\pi|^l)$ for B' a closed ball not containing ∞ agrees with that on $B_\infty(|\pi|^l)$. For convenience, it is understood throughout this paper that such a transform is automatically applied to change the parameter whenever the closed ball B is centered at ∞.

The space of C^n-functions from B to $\mathbf{C}_\infty$ is denoted by $C^n(B)$. For $f \in C^n(B)$ and $x \in B$, set

$$D_n f(x) = \overline{\Phi_n f}(x, x, \ldots, x) \quad \text{for } n > 0, \quad \text{and} \quad D_0 f(x) = f(x).$$

$D_n f$ is the n-th order hyper-derivative of the function f. We have the following characterization of C^n-functions on B, in terms of Taylor expansions:

Theorem 4.1. *(See Section 29 and Section 83 [Sc].) Let B be a closed ball of positive radius. If $f \in C^n(B)$, then for all $x, y \in B$,*

$$\begin{aligned} f(x) &= f(y) + \sum_{j=1}^{n-1} (x-y)^j D_j f(y) + (x-y)^n \overline{(\Phi_n f)}(x, y, \ldots, y) \\ &= f(y) + \sum_{j=1}^{n} (x-y)^j D_j f(y) + (x-y)^n \big(\overline{(\Phi_n f)}(x, y, \ldots, y) - D_n f(y) \big). \end{aligned}$$

Conversely, let f be a function on B. If there exist continuous functions $\lambda_1, \ldots, \lambda_{n-1} : B \to \mathbf{C}_\infty$ and $\Lambda_n : B \times B \to \mathbf{C}_\infty$ such that

$$f(x) = f(y) + \sum_{j=1}^{n-1} (x-y)^j \lambda_j(y) + (x-y)^n \Lambda_n(x, y) \quad (x, y \in \mathbf{A}_\infty),$$

then $f \in C^n(B)$.

$C^n(B)$ is a Banach space over $\mathbf{C}_\infty$ with the norm $\|\cdot\|_n$ defined by

$$\|f\|_n = \max\Big\{ \big\| \overline{(\Phi_0 f)} \big\|_\infty, \big\| \overline{(\Phi_1 f)} \big\|_\infty, \ldots, \big\| \overline{(\Phi_n f)} \big\|_\infty \Big\} \quad \text{for } f \in C^n(\mathbf{A}_\infty) \tag{4.6}$$

where the functions $\overline{(\Phi_j f)}$ for $0 \leqslant j \leqslant n$ are as in Definition 4.3, and $\|\cdot\|_\infty$ is the sup norm of the functions on the respective topological spaces. As the spaces $B^j = B \times B \times \cdots \times B$ (j times, $1 \leqslant j \leqslant n+1$) are compact, the right side of (4.6) is finite. On the notation of the norm, we use $\|\cdot\|_{C^n(B)}$ for $\|\cdot\|_n$ if any confusion may occur.

In the case $B = \mathbf{A}_\infty$. The functions in $C^n(\mathbf{A}_\infty)$ (as well as $LA_n(\mathbf{A}_\infty)$ and $LA(\mathbf{A}_\infty)$) can be described by the Carlitz basis $\{G_n(x)\}_{n \geqslant 0}$, where $G_n(x)$ is a polynomial of degree n for each integer $n \geqslant 0$, they are defined in the following. Set

- $[i] = \pi^{q^i} - \pi = (1/T)^{q^i} - (1/T)$ for i positive integer;
- $L_i = 1$ if $i = 0$; and $L_i = [i] \cdot [i-1] \cdots [1]$ if i is a positive integer;
- $D_i = 1$ if $i = 0$; and $D_i = [i] \cdot [i-1]^q \cdots [1]^{q^{i-1}}$ if i is a positive integer;
- $e_i(x) = x$, if $i = 0$; $e_i(x) = \prod_{\alpha \in \mathbb{F}_q[\pi],\, \deg_\pi(\alpha) < i} (x - \alpha)$, if i is a positive integer;
- $E_i(x) = e_i(x)/D_i$, for each non-negative integer i.

Thus we see that $e_i(x)$ and $E_i(x)$ are $\mathbb{F}_q$-linear polynomials of degree q^i. For any non-negative integer n, write n in q-digit expansion: $n = n_0 + n_1 q + \cdots + n_s q^s$ with $0 \leqslant n_i < q$, the Carlitz polynomial $G_n(x)$ is defined by

$$G_n(x) = \prod_{i=0}^{s} \big(E_i(x) \big)^{n_i}.$$

Theorem 4.2. *(See Wagner [Wa], also see [Go4] or [Ya3].) The Carlitz polynomials $G_n(x)$ $(n \geqslant 0)$ constitute an orthonormal basis of the Banach space $C(\mathbf{A}_\infty)$ of continuous functions from $\mathbf{A}_\infty$ to $\mathbf{C}_\infty$ with the sup norm, that is, any $f \in C(\mathbf{A}_\infty)$ can be expressed as*

$$f(x) = \sum_{n=0}^{\infty} a_n G_n(x), \quad \text{with } a_n \in \mathbf{C}_\infty, \text{ and } a_n \to 0 \text{ as } n \to \infty.$$

And the norm $\|f\|$ (which is also denoted by $\|f\|_0$ in (4.6)) is given by

$$\|f\| := \sup_{x \in \mathbf{A}_\infty} \{|f(x)|\} = \max_{n \geqslant 0} \{|a_n|\}.$$

Theorem 4.3. *(See [Ya1,Ya3].) For integers $j, l \geqslant 0$, let $\mu_{j,l} = \sum_{i=l+1}^{\infty} [j/q^i]$.*

(1) *The polynomials $\pi^{\mu_{j,l}} G_j(x)$ with $j \geqslant 0$ constitute an orthonormal basis of the Banach space $LA_l(\mathbf{A}_\infty)$, that is, any $f \in LA_l(\mathbf{A}_\infty)$ can be expanded as $f(x) = \sum_{j=0}^{\infty} a_j \pi^{\mu_{j,l}} G_j(x)$ with $a_j \to 0$ as $j \to \infty$, and $\|f\|_{LA_l(\mathbf{A}_\infty)} = \max_j \{|a_j|\}$.*
(2) *Let $f(x) = \sum_{j=0}^{\infty} a_j G_j(x)$ be a continuous function from $\mathbf{A}_\infty$ to $\mathbf{C}_\infty$. Then $f \in C^n(\mathbf{A}_\infty)$ if and only if $\lim_{j \to \infty} |a_j| j^n = 0$.*

From (2) of the above theorem, we can define a norm $\|\cdot\|'_n$ on $C^n(\mathbf{A}_\infty)$ by

$$\|f\|'_n := \max_j \{|a_j| \cdot |\pi|^{-n[\log_q j]}\} \quad \text{for } f(x) = \sum_{j \geqslant 0} a_j G_j(x) \in C^n(\mathbf{A}_\infty)$$

so that $(C^n(\mathbf{A}_\infty), \|\cdot\|'_n)$ is a Banach space over $\mathbf{C}_\infty$.

Remark 4.6. The norm $\|\cdot\|'_n$ and the norm $\|\cdot\|_n$ defined in (4.6) (take $B = \mathbf{A}_\infty$) on $C^n(\mathbf{A}_\infty)$ are equivalent, that is, there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda_1 \|\cdot\|_n \leqslant \|\cdot\|'_n \leqslant \lambda_2 \|\cdot\|_n$. This can be seen through the computation of the difference quotients of the Carlitz polynomials $G_n(x)$, which is carried out in the proof of Theorem 5.1 in [Ya3].

Due to the above remark, we will use the same notation $\|\cdot\|_n$ for the two norms $\|\cdot\|_n$ and $\|\cdot\|'_n$ on $C^n(\mathbf{A}_\infty)$.

Corollary 4.1. *For $n \geqslant 0$ an integer, $(C^n(\mathbf{A}_\infty), \|\cdot\|_n)$ is a $\mathbf{C}_\infty$-Banach space with an orthonormal basis $\{\pi^{n[\log_q j]} G_j(x)\}_{j \geqslant 0}$.*

On a closed ball $B = B_a(|\pi|^{l_0})$ for $a \in \mathbf{k}_\infty$ and l_0 a non-negative integer, a function f can be expressed as $f(x) = g((x-a)/\pi^{l_0})$ for some function g on $\mathbf{A}_\infty$, and $f \in C^n(B)$ if and only if $g \in C^n(\mathbf{A}_\infty)$. For $a = \infty$ and l_0 a non-negative integer, on $B = B_\infty(|\pi|^{l_0}) = \{x \in \mathbf{k}_\infty : |x| \geqslant |\pi|^{-l_0}\}$, a function f can be expressed as $f(x) = g(1/(\pi^{l_0} x))$ for some function g on $\mathbf{A}_\infty$, and $f \in C^n(B)$ if and only if $g \in C^n(\mathbf{A}_\infty)$. Hence we get the following

Corollary 4.2. *Let $l_0, n \in \mathbb{Z}$ with $n \geqslant 0$, and $B = B_a(|\pi|^{l_0})$.*

(1) *If $a \in \mathbf{k}_\infty$, then $(C^n(B), \|\cdot\|_n)$ is a $\mathbf{C}_\infty$-Banach space with an orthonormal basis $\{\pi^{n[\log_q j]} G_j((x-a)/\pi^{l_0})\}_{j \geqslant 0}$.*
(2) *If $a = \infty$, then $(C^n(B), \|\cdot\|_n)$ is a $\mathbf{C}_\infty$-Banach space with an orthonormal basis $\{\pi^{n[\log_q j]} G_j(1/(\pi^{l_0} x))\}_{j \geqslant 0}$.*

Corollary 4.3. *Let $l_0, n \in \mathbb{Z}$ with $n \geqslant 0$, and $B = B_a(|\pi|^{l_0})$. For integers $j, l \geqslant 0$, let $\mu_{j,l} = \sum_{i=l+1}^{\infty} [j/q^i]$ be as in Theorem 4.3.*

(1) *If $a \in \mathbf{k}_\infty$, then the polynomials $\pi^{\mu_{j,l}} G_j((x-a)/\pi^{l_0})$ with $j \geqslant 0$ constitute an orthonormal basis of the Banach space $LA_l(B)$.*
(2) *If $a = \infty$, then the polynomials $\pi^{\mu_{j,l}} G_j(1/(\pi^{l_0} x))$ with $j \geqslant 0$ constitute an orthonormal basis of the Banach space $LA_l(B)$.*

To study the C^n-functions on the open compact subset $S \subset \mathbb{P}1(\mathbf{k}_\infty)$, we fix a decomposition (4.2): $S = \bigsqcup_i B_i$, where each B_i is a closed ball of radius $|\pi|^{l_0}$. A function f on S is called of C^n if and only if $f|_{B_i}$ is of C^n, and the space of all C^n-functions on S is denoted by $C^n(S)$. We define a norm $\|\cdot\|_n$ on $C^n(S)$ by

$$\|f\|_n = \max_i \big\{ \|f\|_{C^n(B_i)} \big\}.$$

Then $(C^n(S), \|\cdot\|_n)$ is a Banach space over $\mathbf{C}_\infty$. We have

Theorem 4.4. *Let h be a non-negative integer. The space of h-admissible measures on S is dual to the $\mathbf{C}_\infty$-Banach space $(C^h(S), \|\cdot\|_h)$.*

Proof. The conclusion follows from subsequent Proposition 4.2 and Proposition 4.3, and the proof will occupy the rest of this section. □

Lemma 4.2. *Let $\mu_{n,l} = \sum_{i \geqslant l+1} [n/q^i]$. Then*

$$\mu_{n,l} \geqslant \frac{n}{(q-1)q^l} + l - \log_q n - 1.$$

Proof. Express n in q-digit expansion: $n = n_w q^w + \cdots + n_1 q + n_0$ with $n_w \neq 0$ and $0 \leqslant n_i \leqslant q-1$ for $0 \leqslant i \leqslant w$. Let $s(n,l) = n_w + \cdots + n_{l+2} + n_{l+1}$, and $t(n,l) = n_l q^l + \cdots + n_1 q + n_0$. We have

$$\begin{aligned}
\big[n/q^{l+1}\big] &= n_w q^{w-l-1} + \cdots + n_{l+2} q + n_{l+1}, \\
\big[n/q^{l+2}\big] &= n_w q^{w-l-2} + \cdots + n_{l+2}, \\
&\cdots \\
\big[n/q^{l+w}\big] &= n_w.
\end{aligned}$$

Add up the above equations, column by column on the right sides, then

$$\begin{aligned}
\mu_{n,l} &= n_w \frac{q^{w-l}-1}{q-1} + n_{w-1} \frac{q^{w-l-1}-1}{q-1} + \cdots + n_{l+1} \frac{q-1}{q-1} \\
&= \frac{q^{-l}(n - t(n,l)) - s(n,l)}{q-1} \\
&= \frac{n - t(n,l) - q^l s(n,l)}{(q-1)q^l}.
\end{aligned}$$

It's easily seen that $s(n,l) \leqslant (q-1)(w-l) \leqslant (q-1)(\log_q n - l)$ and $t(n,l) \leqslant (q-1)q^l$, therefore $\mu_{n,l} \geqslant \frac{n}{(q-1)q^l} + l - \log_q n - 1$. □

Proposition 4.2. *Let h be a non-negative integer. If $\mu \in (C^h(S))^*$, then μ is an h-admissible measure on S.*

Proof. We need to prove the inequality (4.5). As $S \subset \mathbb{P}^1(\mathbf{k}_\infty)$ is open compact, we have the decomposition $S = \bigsqcup_i B_i$ for a fixed sufficiently large integer l_0, where $B_i = B_{a_i}(|\pi|^{l_0})$. Then for any integer $l \geqslant l_0$, any closed ball $B_a(|\pi|^l) \subset S$ is contained in B_i for some i. Hence $((x-a)/(\pi^l))^j \cdot \xi_{B_a(|\pi|^l)}(x)$ is locally analytic of order l and has norm 1 in the space $LA_l(B_i)$ or $LA_l(S)$. Corollary 4.3 implies

$$\left(\frac{x-a}{\pi^l}\right)^j \xi_{B_a(|\pi|^l)}(x) = \sum_{n=0}^{\infty} a_n \pi^{\mu_{n,l}} G_n\big((x-a_i)/\pi^{l_0}\big), \tag{4.7}$$

where $a_n \to 0$ as $n \to \infty$, $|a_n| \leqslant 1$ for all $n \geqslant 0$. And $\mu_{n,l} \geqslant \frac{n}{(q-1)q^l} + l - 1 - \log_q n$, by Lemma 4.2. Also Corollary 4.2 implies

$$\left| \int_{B_i} \pi^{h[\log_q n]} G_n\big((x-a_i)/\pi^{l_0}\big)\, d\mu(x) \right| \leqslant \|\mu\|_{(C^h(B_i))^*} \leqslant \|\mu\|_{(C^h(S))^*}$$

since $\mu \in (C^h(S))^*$ and $C^h(B_i) \subset C^h(S)$. Therefore

$$v\left(a_n \pi^{\mu_{n,l}} \int_{B_i} G_n\big((x-a_i)/\pi^{l_0}\big)\, d\mu(x)\right) \geqslant \frac{n}{(q-1)q^l} - 1 - \log_q n + l - C_1 - (\log_q n)h \tag{4.8}$$

where C_1 does not depend on l. A little calculation on the minimum value of the function $m(x) = \frac{x}{(q-1)q^l} - (h+1)(\log_q x)$ gives $\frac{n}{(q-1)q^l} - (h+1)(\log_q n) \geqslant C_2 - l(h+1)$, where the constant C_2 does not depend on l or n. Therefore the right side of (4.8) is greater than or equal to $C_3 - lh$ with C_3 a constant not depending on l or n. Integrating Eq. (4.7), we get the required estimation. □

Now we are going to show that the converse of Proposition 4.2 is also true. Suppose μ is an h-admissible measure on S, we are going to show that $\mu \in (C^h(S))^*$.

For each fixed integer l which is sufficiently large, we have the finite disjoint decomposition

$$S = \bigsqcup_{0 \leqslant i \leqslant r(l)} B_i\big(|\pi|^l\big)$$

where $r(l)$ is a positive integer depending on l. Let us fix a point $a_{l,i}$ inside each closed ball $B_i(|\pi|^l)$.

Let $f \in C^h(S)$. For any $a \in S$, there exists a $B_i(|\pi|^l)$ containing this element a. From Theorem 4.1, the function f has a Taylor expansion on $B_i(|\pi|^l)$:

$$f(x) = f(a) + Df(a)(x-a) + \cdots + D_h f(a)(x-a)^h + \Lambda_h(f; x, a)(x-a)^h \tag{4.9}$$

where $\Lambda_h(f; x, a)$ is continuous on $B_i(|\pi|^l) \times B_i(|\pi|^l)$ and $\lim_{x \to a} \Lambda_h(f; x, a) = 0$. The Riemann sum of f with respect to the h-admissible measure μ is defined by

$$R_l\big(f, \mu, \{a_{l,i}\}\big) = \sum_{0 \leqslant i \leqslant r(l)} \sum_{j=0}^{h} D_j f(a_{l,i})\big(\chi(l, j, a_{l,i}; x), \mu\big). \tag{4.10}$$

Lemma 4.3. *(See Schikhof, Theorem 78.2 of [Sc].) Let h be any non-negative integer and $f \in C^h(S)$. Then $D_j f \in C^{h-j}(S)$ and $D_i D_j f = \binom{i+j}{i} D_{i+j} f$ for $0 \leqslant i$, $0 \leqslant j$, and $i + j \leqslant h$.*

Lemma 4.4. *The limit* $\lim_{l\to\infty} R_l(f,\mu,\{a_{l,i}\})$ *exists, and does not depend on the choices of* $a_{l,i}\in B_i(|\pi|^l)$ *for any* l *sufficiently large and* $0\leqslant i\leqslant r(l)$. *And we define*

$$\int_S f(x)\,d\mu(x)=\lim_{l\to\infty} R_l\big(f,\mu,\{a_{l,i}\}\big). \tag{4.11}$$

Proof. The proof here follows Vishik's paper [Vi1], where the integral is with respect to the h-th order Lipschitz functions instead of C^h-function. At first we prove the limit does not depend on the choices of $\{a_{l,i}\}_{0\leqslant i\leqslant r(l)}$ if it exists. Let $b_{l,i}\in B_i(|\pi|^l)$ be another choice for each l and i. Then

$$\begin{aligned}
R_l\big(f,\mu,\{b_{l,i}\}\big)&=\sum_{0\leqslant i\leqslant r(l)}\sum_{0\leqslant j\leqslant h} D_j f(b_{l,i})\big(\chi(b_{l,i},l;j;x),\mu(x)\big)\\
&=\sum_{0\leqslant i\leqslant r(l)}\sum_{0\leqslant j\leqslant h}\sum_{0\leqslant k\leqslant j}\binom{j}{k} D_j f(b_{l,i})(a_{l,i}-b_{l,i})^{j-k}\big(\chi(a_{l,i},l;k;x),\mu(x)\big)\\
&=\sum_{0\leqslant i\leqslant r(l)}\sum_{k=0}^{h}\sum_{j=0}^{h-k}\binom{j+k}{k} D_{j+k} f(b_{l,i})(a_{l,i}-b_{l,i})^{j}\big(\chi(a_{l,i},l;k;x),\mu(x)\big).
\end{aligned}$$

But

$$\sum_{j=0}^{h-k}\binom{j+k}{k} D_{j+k} f(b_{l,i})(a_{l,i}-b_{l,i})^{j}=D_k f(a_{l,i})-\Lambda_h(D_k f;a_{l,i},b_{l,i})(a_{l,i}-b_{l,i})^{h-k}$$

by making an appropriate substitution in (4.9) and applying Lemma 4.3, therefore

$$R_l\big(f,\mu,\{b_{l,i}\}\big)=R_l\big(f,\mu,\{a_{l,i}\}\big)-\sum_{i=0}^{r(l)}\sum_{k=0}^{h}\Lambda_h(D_k f;a_{l,i},b_{l,i})(a_{l,i}-b_{l,i})^{h-k}\big(\chi(a_{l,i},l;k;x),\mu(x)\big).$$

This proves $\lim_{l\to\infty}(R_l(f,\mu,\{b_{l,i}\})-R_l(f,\mu,\{a_{l,i}\}))=0$, by the condition (4.5) on μ and $\lim_{l\to\infty}\Lambda_h(D_k f;a_{l,i},b_{l,i})=0$ for each k with $0\leqslant k\leqslant h$. Therefore the limit $\lim_{l\to\infty} R_l(f,\mu,\{a_{l,i}\})$, if it exists, does not depend on the choices of $a_{l,i}\in B_i(|\pi|^l)$ for each i and l.

Now let $m>l$. Then $B_i(|\pi|^l)$ can be decomposed disjointly into q^{m-l} smaller balls for each i:

$$B_i\big(|\pi|^l\big)=\bigsqcup_{s(i)\in I(l,m,i)} B_{s(i)}\big(|\pi|^m\big).$$

Therefore

$$\begin{aligned}
R_l\big(f,\mu,\{a_{l,i}\}\big)&=\sum_{i=0}^{r(l)}\sum_{j=0}^{h} D_j f(a_{l,i})\int_{B_i(|\pi|^l)}(x-a_{l,i})^j\,d\mu(x)\\
&=\sum_{i=0}^{r(l)}\sum_{j=0}^{h} D_j f(a_{l,i})\sum_{s(i)\in I(l,m,i)}\sum_{k=0}^{j}\int_{B_{s(i)}(|\pi|^m)}\binom{j}{k}(a_{m,s(i)}-a_{l,i})^{j-k}(x-a_{m,s(i)})^k\,d\mu(x)
\end{aligned}$$

$$= \sum_{i=0}^{r(l)} \sum_{s(i)\in I(l,m,i)} \sum_{k=0}^{h} \sum_{j=k}^{h} \binom{j}{k} D_j f(a_{l,i})(a_{m,s(i)} - a_{l,i})^{j-k} \int_{B_{s(i)}(|\pi|^m)} (x - a_{m,s(i)})^k \, d\mu(x)$$

$$= \sum_{i=0}^{r(l)} \sum_{s(i)\in I(l,m,i)} \sum_{k=0}^{h} \big(D_k f(a_{m,s(i)}) - \Lambda_h(D_k f; a_{m,s(i)}, a_{l,i})\big) \times \int_{B_{s(i)}(|\pi|^m)} (x - a_{m,s(i)})^k \, d\mu(x).$$

Applying the definition of Riemann sum for h-admissible measures, we get

$$R_l\big(f, \mu, \{a_{l,i}\}\big) = R_m\big(f, \mu, \{a_{m,i}\}\big) - \sum_{i=0}^{r(l)} \sum_{s(i)\in I(l,m,i)} \sum_{k=0}^{h} \Lambda_h(D_k f; a_{m,s(i)}, a_{l,i})(a_{m,s(i)} - a_{l,i})^{h-k} \times \int_{B_{s(i)}(|\pi|^m)} (x - a_{m,s(i)})^k \, d\mu(x). \tag{4.12}$$

Let $m = l + 1$. We see from (4.12) that

$$\big|R_l\big(f, \mu, \{a_{l,i}\}\big) - R_{l+1}\big(f, \mu, \{a_{l+1,i}\}\big)\big| \leqslant C \cdot \sup_{k,i,s(i)} \big\{\big|\Lambda_h(D_k f; a_{m,s(i)}, a_{l,i})\big|\big\},$$

where C is a constant. So for any $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\big|R_l\big(f, \mu, \{a_{l,i}\}\big) - R_{l+1}\big(f, \mu, \{a_{l+1,i}\}\big)\big| \leqslant \epsilon$$

for any $l \geqslant N$. Therefore $\{R_l(f, \mu, \{a_{l,i}\})\}_l$ is a Cauchy sequence in $\mathbf{C}_\infty$. Thus the limit $\lim_{l\to\infty} R_l(f, \mu, \{a_{l,i}\})$ exists. □

From this lemma, an h-admissible measure μ on S extends to a linear functional on $C^h(S)$. Its extension on $C^h(S)$ is still denoted by μ.

Corollary 4.4. *An h-admissible measure μ satisfies*

$$\left| \int_{B(|\pi|^l)} (x - a)^j \, d\mu(x) \right| \leqslant C \cdot |\pi|^{l(j-h)}$$

for all $j \geqslant 0$, and $a \in B(|\pi|^l) \subset S$ (where C is a constant not depending on $B(|\pi|^l)$).

Proof. Apply the definition of Riemann sum for an h-admissible measure. □

Proposition 4.3. *The extended functional μ on $C^h(S)$ is continuous, so $\mu \in (C^h(S))^*$.*

Proof. We need to prove μ is bounded. Let $f \in C^h(S)$. In the proof of Lemma 4.4, take $\epsilon = \|f\|_h$, then there exists a sufficiently large integer N, such that

$$\big|R_N\big(f, \mu, \{a_{N,i}\}\big) - R_l\big(f, \mu, \{a_{l,i}\}\big)\big| \leqslant \|f\|_h \tag{4.13}$$

for any $l \geqslant N$. Meanwhile,

$$R_N\big(f,\mu,\{a_{N,i}\}\big) = \sum_{i=0}^{r(N)} \sum_{j=0}^{h} D_j f(a_{N,i}) \int\limits_{B_i(|\pi|^N)} (x - a_{N,i})^j \, d\mu(x).$$

From $\max_{j,i}\{|D_j f(a_{N,i})|\} \leqslant \max_j\{\|\Phi_j f\|_\infty\} \leqslant C_1 \cdot \|f\|_h$ and the inequality (4.5) in the definition of h-admissible measures, it is clear that $|R_N(f,\mu,\{a_{N,i}\})| \leqslant C_2 \cdot \|f\|_h$. Therefore $|R_l(f,\mu,\{a_{l,i}\})| \leqslant C_3 \cdot \|f\|_h$ from (4.13) for any $l \geqslant N$. This implies

$$\left| \int\limits_S f(x)\, d\mu(x) \right| \leqslant C_3 \cdot \|f\|_h,$$

thus μ is bounded as a functional on $C^h(S)$. □

5. Functions on Bruhat–Tits trees

In this section, we'll recall Teitelbaum's results [Te1] on the correspondence between the space of cusp forms and the space of harmonic functions on the edges of the Bruhat–Tits tree $\mathcal{T}$. Recall that $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ is the Bruhat–Tits tree associated to the field $\mathbf{k}_\infty$ with an orientation chosen in Section 2. And $E_{\mathcal{T}} = E_{\mathcal{T}}^+ \sqcup E_{\mathcal{T}}^-$ with $E_{\mathcal{T}}^+$ the set of edges of positive orientation and $E_{\mathcal{T}}^-$ the set of edges of negative orientation.

Let M be an abelian group. An M-valued function $\mathfrak{c}$ on the set $E_{\mathcal{T}}$ of edges of $\mathcal{T}$ is called harmonic if the following two conditions are satisfied:

(1) $\sum_{e \mapsto v} \mathfrak{c}(e) = 0$ for any vertex $v \in V_{\mathcal{T}}$, where the sum is taken over all the oriented edges with final vertex v;
(2) for each $e \in E_{\mathcal{T}}$, we have $\mathfrak{c}(e) + \mathfrak{c}(\bar{e}) = 0$, where $\bar{e}$ denotes the edge e with opposite direction.

For a positive integer n, let

$$V(n) = \big\{F(X,Y) \in \mathbf{C}_\infty[X,Y]\colon\ F \text{ is homogeneous of degree } n-1\big\}.$$

There is a natural action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $V(n)$ given by

$$(\gamma \cdot F)(X,Y) = F\big((dX - bY)/(ad-bc), (-cX + aY)/(ad-bc)\big),$$

or equivalently,

$$\big(\gamma^{-1} \cdot F\big)(X,Y) = F(aX + bY, cX + dY)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{k}_\infty)$ and $F \in V(n)$. We assume $\mathrm{GL}_2(\mathbf{k}_\infty)$ acts trivially on the field $\mathbf{C}_\infty$, and acts on $V(-n) = \mathrm{Hom}_{\mathbf{C}_\infty}(V(n), \mathbf{C}_\infty)$ by diagonal, that is,

$$(\gamma \cdot \lambda)(F) = \lambda\big(\gamma^{-1} \cdot F\big) \quad \text{for } \gamma \in \mathrm{GL}_2(\mathbf{k}_\infty),\ \lambda \in V(-n),\ \text{and } F \in V(n).$$

Let G be an arithmetic subgroup of $\mathrm{GL}_2(\mathbf{k})$ acting on the Bruhat–Tits tree $\mathcal{T}$, with the action denoted by "$\star$". For an integer $n > 1$, we take $M = V(1-n)$ as an abelian group, and let G act on M

by $V(1-n)\otimes\det$ diagonally, where det denotes the determinantal representation of G, that is, for $\lambda\in V(1-n)$ regarded as an element of M, $\gamma\in G$, and $F\in V(n-1)$, the action given by

$$(\gamma\cdot\lambda)(F)=\det(\gamma)\cdot\lambda\big(\gamma^{-1}\cdot F\big).$$

Therefore we can define the diagonal action of G on the M-valued functions on $E_{\mathcal{T}}$ by

$$\big((\gamma\cdot\mathfrak{c})(e)\big)(F)=\big(\gamma\cdot\mathfrak{c}\big(\gamma^{-1}\star e\big)\big)(F)=\det(\gamma)\cdot\mathfrak{c}\big(\gamma^{-1}\star e\big)\big(\gamma^{-1}\cdot F\big),$$

where $\mathfrak{c}$ is an M-valued function on $E_{\mathcal{T}}$, $e\in E_{\mathcal{T}}$, γ and F are as above. The space $C_{\mathrm{har}}(G,n)$ of harmonic cocycles of weight n for G is defined to be the space of G-invariant $V(1-n)\otimes\det$ valued harmonic functions on $E_{\mathcal{T}}$, that is, $C_{\mathrm{har}}(G,n)$ consists of harmonic functions $\mathfrak{c}$ on $E_{\mathcal{T}}$ which satisfy the equation $\gamma\star\mathfrak{c}=\mathfrak{c}$ for any $\gamma\in G$. Therefore $\mathfrak{c}\in C_{\mathrm{har}}(G,n)$ if and only if

$$\mathfrak{c}(\gamma\star e)\big(F(X,Y)\big)=\det(\gamma)\cdot\mathfrak{c}(e)\big(F(aX+bY,cX+dY)\big)\tag{5.1}$$

for $e\in E_{\mathcal{T}}$, $F\in V(n-1)$, γ as above, and this condition on $\mathfrak{c}$ is also equivalent to

$$\mathfrak{c}(\gamma\star e)\big(X^iY^{n-2-i}\big)=\det(\gamma)\cdot\mathfrak{c}(e)\big((aX+bY)^i(cX+dY)^{(n-2-i)}\big).\tag{5.2}$$

Remark 5.1. As G is a subgroup of $\mathrm{GL}_2(\mathbf{k})\subset\mathrm{GL}_2(\mathbf{k}_\infty)$, it has an induced action on the Bruhat–Tits tree $\mathcal{T}$ from the natural action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $\mathbf{k}_\infty\oplus\mathbf{k}_\infty$. In the subsequent content, we are also going to use an action "$*$" defined by

$$\text{for }\gamma\in\mathrm{GL}_2(\mathbf{k}_\infty)\text{ and }v\in\mathbf{k}_\infty\oplus\mathbf{k}_\infty,\quad \gamma*v:=\big(\gamma^{-1}\big)^T\cdot v$$

where "$\cdot$" denotes the natural action.

To any oriented edge $e\in E_{\mathcal{T}}$, we associate with the set $U(e)$ of ends of $\mathcal{T}$ which pass through e. By using the bijection between the set of ends of $\mathcal{T}$ and $\mathbb{P}^1(\mathbf{k}_\infty)$ chosen in Section 2, we also denote the corresponding subset of $\mathbb{P}^1(\mathbf{k}_\infty)$ by $U(e)$. Then $U(e)$ is an open compact subset of $\mathbb{P}^1(\mathbf{k}_\infty)$.

Example 5.1. For the edges $e=\Lambda_0\Lambda_1$, $\Lambda_1\Lambda_0$, $\Lambda_1\Lambda_2$ on the half line (2.6), we have

$$\begin{aligned}
U(\Lambda_0\Lambda_1)&=\big\{x\in\mathbf{k}_\infty\colon\ |x|\geqslant|\pi|^{-1}\big\}\cup\{\infty\}=B_\infty\big(|\pi|^1\big),\\
U(\Lambda_1\Lambda_0)&=\big\{x\in\mathbf{k}_\infty\colon\ |x|\leqslant|\pi|^0\big\}=B_0\big(|\pi|^0\big)=\mathbf{A}_\infty,\\
U(\Lambda_1\Lambda_2)&=\big\{x\in\mathbf{k}_\infty\colon\ |x|\geqslant|\pi|^{-2}\big\}\cup\{\infty\}=B_\infty\big(|\pi|^2\big).
\end{aligned}$$

And more generally, for an integer $m\geqslant 0$

$$\begin{aligned}
U(\Lambda_m\Lambda_{m+1})&=\big\{x\in\mathbf{k}_\infty\colon\ |x|\geqslant|\pi|^{-(m+1)}\big\}\cup\{\infty\}=B_\infty\big(|\pi|^{m+1}\big),\\
U(\Lambda_{m+1}\Lambda_m)&=\big\{x\in\mathbf{k}_\infty\colon\ |x|\leqslant|\pi|^{-m}\big\}=B_0\big(|\pi|^{-m}\big).
\end{aligned}$$

Example 5.2. Consider the edges on the half line:

$$\underset{\Lambda_0}{\circ}\ \text{———}\ \underset{\Lambda_{-1}}{\circ}\ \text{———}\ \underset{\Lambda_{-2}}{\circ}\ \cdots\cdots\ \underset{\Lambda_{-m}}{\circ}\ \text{———}\ \cdots\cdots\tag{5.3}$$

where Λ_{-m} is the equivalence class of the lattice $=\pi^m\mathbf{A}_\infty\oplus\mathbf{A}_\infty$ for each integer $m\geqslant 0$. Then

$$U(\Lambda_{-m}\Lambda_{-(m+1)})=\{x\in\mathbf{k}_\infty\colon |x|\leqslant|\pi|^{m+1}\}=B_0(|\pi|^{m+1}),$$
$$U(\Lambda_{-(m+1)}\Lambda_{-m})=\{x\in\mathbf{k}_\infty\colon |x|\geqslant|\pi|^{m}\}\cup\{\infty\}=B_\infty(|\pi|^{-m}).$$

Example 5.3. In general, let $M_j=[\pi^j\mathbf{v}_1, x\mathbf{v}_1+\mathbf{v}_2]$ denote the equivalence class of lattice generated by $\pi^j\mathbf{v}_1$ and $x\mathbf{v}_1+\mathbf{v}_2$ (the two vectors $\mathbf{v}_1=(1,0)^T$, $\mathbf{v}_2=(0,1)^T$ are the standard basis of $V=\mathbf{k}_\infty\oplus\mathbf{k}_\infty$) on the half line (2.5), where $x=\sum_{j=j_0}^{\infty}c_j\pi^j\in\mathbf{k}_\infty$ and $c_j\in\mathbb{F}_q$, then a little computation shows that

$$\begin{aligned}U(M_jM_{j+1})&=\{y\in\mathbf{k}_\infty\colon |y-x|\leqslant|\pi|^{j+1}\}=B_x(|\pi|^{j+1}),\\ U(M_{j+1}M_j)&=\{y\in\mathbf{k}_\infty\colon |y-x|=|\pi|^{j}\}\cup\{y\in\mathbf{k}_\infty\colon |y-x|=|\pi|^{j-1}\}\\ &\quad\cup\cdots\cup\{y\in\mathbf{k}_\infty\colon |y-x|=|\pi|^{j_0}\}\cup\{y\in\mathbf{k}_\infty\colon |y|\geqslant|\pi|^{j_0-1}\}\cup\{\infty\}\\ &=\{y\in\mathbf{k}_\infty\colon |y-x|\geqslant|\pi|^{j}\}.\end{aligned}$$

Notice that the set of ends of $\mathcal{T}$ is in bijection with $\mathbb{P}1(\mathbf{k}_\infty)$, as explained in Section 2, we conclude from the above examples that

Proposition 5.1.

(1) *For any $j\in\mathbb{Z}$ and any $x\in\mathbf{k}_\infty$, we have*

$$B_x(|\pi|^j)=U(M_{j-1}M_j),\qquad B_\infty(|\pi|^j)=U(\Lambda_{j-1}\Lambda_j),$$

where the notations are as in Examples 5.1–5.3.

(2) *For any $e\in E_{\mathcal{T}}$, we have $\mathbb{P}^1(\mathbf{k}_\infty)=U(e)\sqcup U(\bar{e})$. And $\infty\in U(e)$ if and only if $e\in E^+_{\mathcal{T}}$.*

For an integer $n\geqslant 2$, a harmonic cocycle $\mathfrak{c}\in C_{\mathrm{har}}(G,n)$ can be associated with a $\mu_{\mathfrak{c}}\in(P^{(n-2)})^*=\mathrm{Hom}_{\mathbf{C}_\infty}(P^{(n-2)},\mathbf{C}_\infty)$ (the space $P^{(n)}$ is defined in Definition 4.1) as follows. Given an $e\in E_{\mathcal{T}}$ and an integer i with $0\leqslant i\leqslant n-2$, we define

$$\int_{U(e)}x^i\,d\mu_{\mathfrak{c}}(x)=\mathfrak{c}(e)(X^iY^{n-2-i}),\tag{5.4}$$

and extend to $P^{(n-2)}$ by linearity. Proposition 5.1 assures $\mu_{\mathfrak{c}}\in(P^{(n-2)})^*$.

Lemma 5.1. *(See [Te1].) For $\mathfrak{c}\in C_{\mathrm{har}}(G,n)$, $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G$, $e\in E_{\mathcal{T}}$, and f a polynomial of degree at most $n-2$, the following hold.*

(1) *We have*

$$\int_{U(\gamma*e)}f(x)\,d\mu_{\mathfrak{c}}(x)=\int_{U(e)}\det(\gamma)\cdot f(\gamma x)(cx+d)^{n-2}\,d\mu_{\mathfrak{c}}(x),\tag{5.5}$$

where the action of $G\subset\mathrm{GL}_2(\mathbf{k}_\infty)$ on $\mathbb{P}^1(\mathbf{k}_\infty)$ is given by $\gamma x=(ax+b)/(cx+d)$.

(2)
$$\int_{\mathbb{P}^1(\mathbf{k}_\infty)} f(x)\,d\mu_{\mathfrak{c}}(x) = 0. \tag{5.6}$$

(3) *There exists a constant $C > 0$ such that for all $0 \leqslant i \leqslant n-2$,*

$$\left| \int_{U(e)} (x-a)^i\,d\mu_{\mathfrak{c}}(x) \right| \leqslant C\rho(e)^{i-(n-2)/2}, \tag{5.7}$$

where $e \in E_{\mathcal{T}}^-$, $a \in U(e)$, and $\rho(e) = \sup_{x,y\in U(e)} |x-y|$ is the diameter of $U(e)$.

Proof. See [Te1]. □

Let h be the smallest integer greater than or equal to $(n-2)/2$.

By Proposition 5.1, every closed ball is of the form $U(e)$ for some edge e of the tree $\mathcal{T}$, therefore the inequality (5.7) implies that the measure $\mu_{\mathfrak{c}}$ is h-admissible on any open compact subset S of $\mathbf{k}_\infty$. Hence the functions of $C^h(S)$ are integrable against $\mu_{\mathfrak{c}}$ by Theorem 4.4. For $B_\infty(|\pi|^m) = U(\Lambda_{m-1}\Lambda_m) = \{x \in \mathbf{k}_\infty\colon\ |x| \geqslant |\pi|^{-m}\} \cup \{\infty\}$ where m is an integer, we can decompose it as

$$\begin{aligned} B_\infty\big(|\pi|^m\big) &= \Bigg(\bigsqcup_{m\leqslant s\leqslant l} \big\{x \in \mathbf{k}_\infty\colon\ |x| = |\pi|^{-s}\big\} \Bigg) \sqcup \Big(\big\{x \in \mathbf{k}_\infty\colon\ |x| \geqslant |\pi|^{-(l+1)}\big\} \cup \{\infty\} \Big) \\ &= \Bigg(\bigsqcup_{c\in\mathbb{F}_q^*} \bigsqcup_{m\leqslant s\leqslant l} B_{c\pi^{-s}}\big(|\pi|^{-s+1}\big) \Bigg) \sqcup B_\infty\big(|\pi|^{l+1}\big) \end{aligned}$$

for $l \geqslant m$ a large integer, and define for an integer $i > 0$

$$\int_{B_\infty(|\pi|^m)} x^{-i}\,d\mu_{\mathfrak{c}}(x) = \lim_{l\to\infty} \sum_{c\in\mathbb{F}_q^*} \sum_{m\leqslant s\leqslant l} \int_{B_{c\pi^{-s}}(|\pi|^{-s+1})} x^{-i}\,d\mu_{\mathfrak{c}}(x). \tag{5.8}$$

The limit of (5.8) exists because

$$\begin{aligned} \left| \int_{B_{c\pi^{-l}}(|\pi|^{-l+1})} x^{-i}\,d\mu_{\mathfrak{c}}(x) \right| &\leqslant \left| \int_{B_{c\pi^{-l}}(|\pi|^{-l+1})} \left(\frac{1}{c\pi^{-l}}\right)^i \left(\frac{1}{1+(x-c\pi^{-l})/(c\pi^{-l})}\right)^i d\mu_{\mathfrak{c}}(x) \right| \\ &\leqslant C\cdot|\pi|^{l(i+(n-2)/2)} \end{aligned}$$

where C is a constant independent of the closed balls contained in $\mathbb{P}^1(\mathbf{k}_\infty)$, and we have the estimate

$$\left| \int_{B_\infty(|\pi|^m)} x^{-i}\,d\mu_{\mathfrak{c}}(x) \right| \leqslant C\cdot\big(|\pi|^m\big)^{(i+(n-2)/2)} \quad \text{for } i \geqslant 0 \tag{5.9}$$

where C is a constant independent of the closed balls contained in $\mathbb{P}^1(\mathbf{k}_\infty)$.

In Lemma 5.1, we take $e = \Lambda_{m-1}\Lambda_m$. Then $\bar{e} = \Lambda_m\Lambda_{m-1}$ is the edge with opposite orientation of that of e, and $U(e) = B_\infty(|\pi|^m)$, $U(\bar{e}) = B_0(|\pi|^{-(m-1)})$. Therefore $\mathfrak{c}(e) + \mathfrak{c}(\bar{e}) = 0$ implies

$$\left| \int_{B_\infty(|\pi|^m)} x^i\,d\mu_{\mathfrak{c}}(x) \right| = \left| \int_{B_0(|\pi|^{-(m-1)})} x^i\,d\mu_{\mathfrak{c}}(x) \right| \leqslant C\cdot\big(|\pi|^m\big)^{-i+(n-2)/2}. \tag{5.10}$$

Combine (5.9) and (5.10), we have

Proposition 5.2. *(See [Te1].) There exists a constant C such that for $m \in \mathbb{Z}$ and $-\infty \leqslant i \leqslant n-2$*

$$\left| \int\limits_{B_\infty(|\pi|^m)} x^i \, d\mu_{\mathfrak{c}}(x) \right| \leqslant C \cdot \left(|\pi|^m\right)^{-i+(n-2)/2}. \tag{5.11}$$

Therefore by recalling Theorem 4.4 of Section 4, we have

Corollary 5.1. *The map*

$$C_{\mathrm{har}}(G,n) \to \left(C^h\left(\mathbb{P}^1(\mathbf{k}_\infty)\right)\right)^*,$$
$$\mathfrak{c} \mapsto \mu_{\mathfrak{c}}$$

injects $C_{\mathrm{har}}(G,n)$ into $(C^h(\mathbb{P}^1(\mathbf{k}_\infty)))^*$ as the subspace of h-admissible measures satisfying Eqs. (5.5) and (5.6) for a polynomial f of degree at most $n-2$.

Proof. As every closed ball of $\mathbb{P}^1(\mathbf{k}_\infty)$ is of the form $U(e)$ for some $e \in E_{\mathcal{T}}$ by Proposition 5.1, this is direct from the estimates (5.7) and (5.11). □

Remark 5.2. Due to the inequality (5.10), not only we can integrate C^h functions on $\mathbb{P}^1(\mathbf{k}_\infty)$ against $\mu_{\mathfrak{c}}$, but also we can integrate those functions with a pole at ∞ of order at most $n-2$. More generally, if the integral $\int_{U(e)} f(x)\, d\mu_{\mathfrak{c}}(x)$ is defined for a C^h function f and an edge e of $\mathcal{T}$, then we can also formally define

$$\int\limits_{U(\bar{e})} f(x)\, d\mu_{\mathfrak{c}}(x) := -\int\limits_{U(e)} f(x)\, d\mu_{\mathfrak{c}}(x). \tag{5.12}$$

We'll always apply this convention in the subsequent discussion. Finally, we notice that although $\mu_{\mathfrak{c}}$ is constructed as an element of $(P^{(n-2)})^*$ by definition, but we see that $\mu_{\mathfrak{c}}$ is determined by its values on $P^{(h)}$.

Corollary 5.2. *Eq. (5.5) induces an action of G on $C_{\mathrm{har}}(G,n)$, when viewed as a subspace of $(C^h(\mathbb{P}^1(\mathbf{k}_\infty)))^*$:*

$$\gamma \cdot d\mu_{\mathfrak{c}}(x) = \det(\gamma)(cx+d)^{n-2}\, d\mu_{\mathfrak{c}}(x) \quad \text{for } \mathfrak{c} \in C_{\mathrm{har}}(G,n) \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.13}$$

There is a natural relation between the Bruhat–Tits tree $\mathcal{T}$ and the Drinfeld's upper half plane Ω, which we will describe below, more exposition about it can be found in [GR]. The Bruhat–Tits tree $\mathcal{T}$ is in fact a pair of sets $V_{\mathcal{T}}$ and $E_{\mathcal{T}}$ with two maps $o, t : E_{\mathcal{T}} \to V_{\mathcal{T}}$ designating the origin $o(e)$ and the terminal $t(e)$ of an element e of $E_{\mathcal{T}}$, and an orientation given in Section 2. Let $\mathcal{T}(\mathbb{R})$ be the realization of $\mathcal{T}$ consisting of a unit interval for every edge (without considering orientation) of $\mathcal{T}$ which is glued at the extremities according to the incidence relations of $\mathcal{T}$. The set of vertices of $\mathcal{T}(\mathbb{R})$ is $\mathcal{T}(\mathbb{Z})$, which corresponds to $V_{\mathcal{T}}$.

An edge e of $\mathcal{T}(\mathbb{R})$ corresponds to some edge $[L][L']$, where L and L' are $\mathbf{A}_\infty$-lattices of $V = \mathbf{k}_\infty \oplus \mathbf{k}_\infty$ satisfying $\pi L' \subset L \subset L'$ (which is the same as saying that $d([L],[L']) = 1$ in Section 2). Then every point P of e can be formally expressed as

$$P = (1-t)[L] + t\left[L'\right] \quad \text{for } 0 \leqslant t \leqslant 1. \tag{5.14}$$

We have defined an action $\gamma\cdot[L]$ for $\gamma\in \mathrm{GL}_2(\mathbf{k}_\infty)$ and L an $\mathbf{A}_\infty$-lattice in Section 2, which is induced from the ordinary action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $V=\mathbf{k}_\infty\oplus\mathbf{k}_\infty$. There is another action

$$\begin{aligned}&\mathrm{GL}_2(\mathbf{k}_\infty)\times V_{\mathcal{T}}\to V_{\mathcal{T}},\\ &\qquad\big(\gamma,[L]\big)\mapsto \gamma*[L]:=\big(\gamma^T\big)^{-1}\cdot[L]=\big[\big(\gamma^T\big)^{-1}\cdot L\big]\end{aligned}\tag{5.15}$$

which induces an action "$*$" of $\mathrm{GL}_2(\mathbf{k}_\infty)$ on $\mathcal{T}(\mathbb{R})$.

To any lattice L of V, we associate a norm ν_L over $\mathbf{k}_\infty$ by

$$\nu_L(v)=\inf\big\{|x|\colon\ x\in\mathbf{k}_\infty,\ v\in xL\big\}\quad\text{for any } v\in V.\tag{5.16}$$

This is equivalent to saying that L is the unit ball with respect to ν_L. For a point $P\in\mathcal{T}(\mathbb{R})$ in (5.14), we define a norm ν_P by

$$\nu_P(v)=\max\big\{\nu_L(v),q^t\nu_{L'}(v)\big\}\quad\text{for } v\in V.\tag{5.17}$$

Two norms ν_1 and ν_2 on V are said to be similar if there exists a real number $C>0$ such that $\nu_2=C\nu_1$. Similarity of norms on V is an equivalence relation. The equivalence class of ν is denoted by $[\nu]$. It is easy to see that equivalent lattices correspond to similar norms, and we have the well-defined notions $[\nu_{[L]}]=[\nu_L]$ and $[\nu_P]$ according to (5.14)–(5.17).

Example 5.4. If we denote by $L=\mathbf{A}_\infty\oplus\mathbf{A}_\infty$, and $L'=\pi\mathbf{A}_\infty\oplus\mathbf{A}_\infty$, then for $v=(a,b)^T\in V=\mathbf{k}_\infty\oplus\mathbf{k}_\infty$,

$$\nu_L(v)=\max\big\{|a|,|b|\big\},\qquad \nu_{L'}(v)=\max\big\{q\cdot|a|,|b|\big\}.$$

The group $\mathrm{GL}_2(\mathbf{k}_\infty)$ acts on the space of similarity classes of norms on V by

$$(\gamma\cdot\nu)(v)=\nu\big(\gamma^T\cdot v\big)\tag{5.18}$$

for $\gamma\in\mathrm{GL}_2(\mathbf{k}_\infty)$, $v\in V$, and ν a norm on V (this induces an action $\gamma\cdot[\nu]$).

Theorem 5.1 *(Goldman–Iwahori). (See [GR].) The association of a point P in $\mathcal{T}(\mathbb{R})$ with $[\nu_P]$ establishes a canonical $\mathrm{GL}_2(\mathbf{k}_\infty)$-equivariant bijection between $\mathcal{T}(\mathbb{R})$ (with respect to the "$*$" action defined by Eq. (5.15)) and the space of similarity classes of norms on V.*

To any $z\in\Omega$ we also associate $[\nu_z]$ of the norm ν_z on V by

$$\nu_z((u,v)):=|uz+v|\quad\text{for } u,v\in\mathbf{k}_\infty.\tag{5.19}$$

Due to Theorem 5.1, we can identify $\mathcal{T}(\mathbb{R})$ with the space of similarity classes of norms on V. Therefore (5.19) gives the "building map"

$$\begin{aligned}&\lambda:\Omega\to\mathcal{T}(\mathbb{R}),\\ &\qquad z\mapsto[\nu_z].\end{aligned}$$

The image of λ is $\mathcal{T}(\mathbb{Q})$.

Proposition 5.3. *(See [GR].) The building map λ is $\mathrm{GL}_2(\mathbf{k}_\infty)$-equivariant, that is, it satisfies*

$$\lambda(\gamma z) = [\nu_{\gamma z}] = \gamma \cdot [\nu_z]$$

for any $z \in \Omega$ and any $\gamma \in \mathrm{GL}_2(\mathbf{k}_\infty)$.

Proof. This is a direct verification. Notice that the notion of the action of $\mathrm{GL}_2(\mathbf{k}_\infty)$ given by (5.18) is a little different from [GR], the group $\mathrm{GL}_2(\mathbf{k}_\infty)$ always acts on the left in this paper. □

Example 5.5. Let Λ_n be the equivalence class of the $\mathbf{A}_\infty$-lattice $L_n = \pi^{-n}\mathbf{A}_\infty \oplus \mathbf{A}_\infty$ as in Section 2.

(1) $\lambda^{-1}(\Lambda_n) = \{z \in \Omega\colon\ |z - c\pi^n| = |\pi^n| \text{ for all } c \in \mathbb{F}_q\}$.
(2) Let $\tilde{e}_n = \Lambda_n\Lambda_{n+1} \in \mathcal{T}(\mathbb{R})$ be the edge with the two end points Λ_n and Λ_{n+1}, and e_n the edge with the two end points removed. Then

$$\lambda^{-1}(e_n) = \left\{z \in \Omega\colon\ \left|\pi^{n+1}\right| < |z| < \left|\pi^n\right|\right\},$$
$$\lambda^{-1}(\tilde{e}_n) = \left\{z \in \Omega:\ \begin{array}{l} |\pi^{n+1}| \leqslant |z| \leqslant |\pi^n|, \\ |z - c\pi^n| \geqslant |\pi^n|,\ |z - c\pi^{n+1}| \geqslant |\pi^{n+1}| \text{ for all } c \in \mathbb{F}_q^* \end{array}\right\}.$$

These relations can be verified directly from the definition of the building map λ. A natural way to understand them is by considering some analytic reduction of Ω to a locally finite scheme over $\mathbb{F}_q$, which gives rise to the Bruhat–Tits tree $\mathcal{T}$, as in [GR].

Now we identify $\mathcal{T}$ with its realization $\mathcal{T}(\mathbb{R})$ and let e_n be as in Example 5.5. Let $e \in E_\mathcal{T}$ with the two endpoints removed and $D(e) = \lambda^{-1}(e)$. Then

$$D := D(e_0) = \left\{z \in \Omega\colon\ |\pi| < |z| < 1\right\}$$

is a fundamental region of Ω. Since $\tilde{e}_0 = \Lambda_0\Lambda_1$ is a fundamental domain of the action of $\mathrm{GL}_2(\mathbf{k}_\infty)^+$ on $\mathcal{T}$ as explained in Section 2, we have

$$e = \gamma * e_0 = \left(\gamma^{-1}\right)^T \cdot e_0 \tag{5.20}$$

and there is an isomorphism

$$\begin{aligned} &D \to D(e), \\ &v \mapsto z = \gamma v \end{aligned} \tag{5.21}$$

given by an element $\gamma \in \mathrm{GL}_2(\mathbf{k}_\infty)^+$, from Theorem 5.1 and Proposition 5.3. Then any rigid differential form $f\,dz$ on $D(e)$ can be expressed by

$$\sum_{i\in\mathbb{Z}} a_i v^i\,dv. \tag{5.22}$$

We have the following definitions (see [Te1]).

- Define $\mathrm{Res}_e f\,dz := a_{-1}$, where a_{-1} is the coefficient of $v^{-1}\,dv$ in Eq. (5.22). Here $e \in E_\mathcal{T}$ is an oriented edge which is related to e_0 by (5.20).

- Suppose f is a modular form of weight n for an arithmetic subgroup $G \subset \mathrm{GL}_2(\mathbf{k})$. Then define a harmonic cocycle $\mathrm{Res}(f)$ of weight n by

$$\mathrm{Res}(f)(e)\big(X^i Y^{n-2-i}\big) := \mathrm{Res}_e\, z^i f(z)\, dz. \tag{5.23}$$

For convenience, in the discussions throughout the rest of this paper, we will use the same notation for an edge with the end points or without the end points if no confusion occurs in the context. It can be verified that $\mathrm{Res}(f)$ is harmonic and G-equivariant (note that we need to use the action "$*$" of G on $\mathcal{T}$ defined in (5.15) for $\mathrm{Res}(f)$ to be G-equivariant).

Theorem 5.2. *(See Teitelbaum [Te1].) Let $\mathfrak{c} \in C_{\mathrm{har}}(G, n)$ be a harmonic cocycle of weight $n \geqslant 2$ for the arithmetic group G, and let $\mu_{\mathfrak{c}}$ be the associated measure. Define f by the integral*

$$f(z) = \int_{\mathbb{P}^1(\mathbf{k}_\infty)} \frac{1}{z-x}\, d\mu_{\mathfrak{c}}(x).$$

Then f is a rigid analytic cusp form for G of weight n and $\mathrm{Res}(f) = \mathfrak{c}$. Moreover, let $e \in E_{\mathcal{T}}^-$ (thus $\infty \notin U(e)$) and a be a center of $U(e)$. Then for all $m \in \mathbb{Z}$, we have

$$\mathrm{Res}_e (z-a)^m f(z)\, dz = \int_{U(e)} (x-a)^m\, d\mu_{\mathfrak{c}}(x). \tag{5.24}$$

Remark 5.3. In Eq. (5.24) of Theorem 5.2, the integral $\int_{U(e)} (x-a)^m\, d\mu_{\mathfrak{c}}(x)$ is certainly well defined for all $m \geqslant 0$, because $\infty \notin U(e)$ and so $(x-a)^m$ is an analytic function on $U(e)$ when $m \geqslant 0$. For $m < 0$, the function $(x-a)^m$ has a pole at the point $a \in U(e)$, therefore the integral cannot be defined directly. But the function $(x-a)^m$ is analytic on $U(\bar{e})$, thus we set

$$\int_{U(e)} (x-a)^m\, d\mu_{\mathfrak{c}}(x) := -\int_{U(\bar{e})} (x-a)^m\, d\mu_{\mathfrak{c}}(x)$$

as Remark 5.2 explains.

Theorem 5.3. *(See Teitelbaum [Te1].) Let G be an arithmetic subgroup of $\mathrm{GL}_2(\mathbf{k})$, and let $S_n(G)$ denote the space of rigid cusp forms of weight $n \geqslant 2$ for G. Then the maps*

$$\mathrm{Res} : S_n(G) \to C_{\mathrm{har}}(G, n)$$

and

$$\begin{aligned} \iota : C_{\mathrm{har}}(G, n) &\to S_n(G), \\ \mathfrak{c} &\mapsto \int_{\mathbb{P}^1(\mathbf{k}_\infty)} \frac{1}{z-x}\, d\mu_{\mathfrak{c}}(x) \end{aligned}$$

are mutually inverse isomorphisms.

The set $\mathcal{U}_1 = \{x \in \mathbf{k}_\infty :\ |x-1| < 1\}$ of 1-units of $\mathbf{k}_\infty$ can be given as

$$\mathcal{U}_1 = U(M_0M_1)$$

where M_0M_1 is the edge with initial vertex $M_0 = [\mathbf{v}_1, \mathbf{v}_2]$ and terminal vertex $M_1 = [\pi\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2]$, as explained in Example 5.3. For $G = \Gamma = \mathrm{GL}_2(\mathbf{A})$ and $f \in S_n(\Gamma)$, an analogue of L-functions of classical cusp forms is $L_f(s) : \mathbb{Z}_p \to \mathbf{C}_\infty$ given by (see [Go5])

$$L_f(s) = \int_{\mathcal{U}_1} x^s \frac{d\mu_f(x)}{x} = \int_{U(M_0M_1)} x^{s-1}\, d\mu_f(x) \tag{5.25}$$

where μ_f is the measure associated to the cocycle $\iota^{-1}(f)$ under the correspondence of Theorem 5.3. In general, let G be an arithmetic subgroup of $\mathrm{GL}_2(\mathbf{k})$ and $f \in S_n(G)$ with $n \geqslant 2$. Then for an edge e of $\mathcal{T}$, we define

$$L(f; e; j) = \int_{U(e)} x^{j-1}\, d\mu_f(x) \quad \text{for } j \in \mathbb{Z}. \tag{5.26}$$

In the case when the edge $e = M_0M_1$, the function $L(f; e; j) : \mathbb{Z} \to \mathbf{C}_\infty$ in the variable j can be extended to the function $L(f; e; s)$ for $s \in \mathbb{Z}_p$ by continuity, which is the L-function $L_f(s)$ defined in (5.25).

6. The measure associated to the Drinfeld discriminant

We have defined the Drinfeld discriminant Δ in Example 3.4. In this section, we will compute $C_{\mathrm{har}}(\Gamma, q^2-1)$. Since this space is one-dimensional, generated by the image of Δ under the isomorphism of Theorem 5.3, we will get the measure associated to the Drinfeld discriminant Δ.

Theorem 6.1. *(See Goss [Go1].) The dimension of the space of cusp forms of weight $m = m_0(q-1)$ and type 0 of the arithmetic subgroup Γ is $[m_0/(q+1)]$. The dimension of the space of all Drinfeld modular forms of weight $m = m_0(q-1)$ and type 0 of the arithmetic subgroup Γ is $[m_0/(q+1)]+1$.*

By the above theorem and Theorem 5.3, we see that $C_{\mathrm{har}}(\Gamma, q^2-1)$, as well as $S_{q^2-1}(\Gamma)$, is a one-dimensional vector space over $\mathbf{C}_\infty$. The space $S_{q^2-1}(\Gamma)$ of cusp forms is generated by the Drinfeld discriminant Δ. Let $\mathfrak{c}_\Delta = \mathrm{Res}(\Delta)$ be the harmonic cocycle in Theorem 5.3. Then $\mathfrak{c}_\Delta$ is a basis of $C_{\mathrm{har}}(\Gamma, q^2-1)$. We can calculate $\mathfrak{c}_\Delta$ by

$$\mathfrak{c}_\Delta(e)\big(X^iY^{q^2-3-i}\big) = \mathrm{Res}_e\, z^i\Delta(z)\,dz$$

for $0 \leqslant i \leqslant q^2-3$ from the definition (5.23) of harmonic cocycles coming from modular forms. Since the harmonic cocycle associated to Δ is Γ-invariant and the half line (2.1) consisting of Λ_m ($m \geqslant 0$) is the fundamental domain of $\Gamma\backslash\mathcal{T}$, the values $\mathrm{Res}_{e_m} z^i\Delta(z)\,dz$ (for $0 \leqslant i \leqslant q^2-3$) determine $\mathfrak{c}_\Delta$, where e_m is the oriented edge $\Lambda_m\Lambda_{m+1}$.

At first, we are going to calculate $\mathrm{Res}_{e_0} z^i\Delta(z)\,dz$, the residue of $z^i\Delta(z)\,dz$ over the region $D = \lambda^{-1}(e_0) = \{z \in \Omega :\ |\pi| < |z| < 1\}$. As $\Delta(z) = (T^{q^2}-T)E_{q^2-1}(z) + (T^q-T)^qE_{q-1}(z)^{q+1}$, we will expand $(E_{q-1}(z))^{q+1}$ and $E_{q^2-1}(z)$ on the region D in the following.

$$E_{q-1}(z) = \sum_{(0,0)\neq(c,d)\in\mathbf{A}^2} \frac{1}{(cz+d)^{q-1}} = \sum_1 \frac{1}{(cz+d)^{q-1}} + \sum_2 \frac{1}{(cz+d)^{q-1}} \tag{6.1}$$

where $\sum_1$ is taken over all $(c,d)\in\mathbf{A}^2$ such that $d\neq 0$ and $\deg(d)\geqslant\deg(c)$, and $\sum_2$ is taken over all $(c,d)\in\mathbf{A}^2$ such that $c\neq 0$ and $\deg(c)\geqslant\deg(d)+1$. Here we set $\deg(0)=-\infty$ by convention. We'll compute $\sum_1$ and $\sum_2$ separately.

$$\begin{aligned} S_1 &:= \sum_1 \frac{1}{(cz+d)^{q-1}} \\ &= \sum_1 \left(\frac{1}{1+\frac{c}{d}z}\right)^{q-1} \cdot \frac{1}{d^{q-1}} \\ &= \sum_1 \frac{1}{d^{q-1}} \left(\sum_{i\geqslant 0} \left(\frac{c}{d}\right)^i z^i\right)^{q-1} \\ &= \sum_1 \frac{1}{d^{q-1}} \left(1 - \frac{c}{d}z + \left(\frac{c}{d}\right)^q z^q - \left(\frac{c}{d}\right)^{q+1} z^{q+1} \right. \\ &\quad \left. + \left(\frac{c}{d}\right)^{2q} z^{2q} - \left(\frac{c}{d}\right)^{2q+1} z^{2q+1} + \cdots\right) \end{aligned} \tag{6.2}$$

where we have applied the identity in the following lemma.

Lemma 6.1. *For $B\in\mathbf{C}_\infty$, we have*

$$\left(\sum_{i\geqslant 0} B^i z^i\right)^{q-1} = 1 - Bz + B^q z^q - B^{q+1} z^{q+1} + B^{2q} z^{2q} - B^{2q+1} z^{2q+1} + \cdots.$$

Proof. We directly verify that

$$\left(\sum_{i\geqslant 0} B^i z^i\right)^{q} = \left(\sum_{i\geqslant 0} B^i z^i\right)\left(1 - Bz + B^q z^q - B^{q+1} z^{q+1} + B^{2q} z^{2q} - B^{2q+1} z^{2q+1} + \cdots\right)$$

and get the identity in the lemma. □

For the sum $\sum_2$ of (6.1),

$$\begin{aligned} S_2 &:= \sum_2 \frac{1}{(cz+d)^{q-1}} \\ &= \sum_2 \frac{1}{(cz)^{q-1}} \left(\frac{1}{1-\frac{d}{cz}}\right)^{q-1} \end{aligned}$$

$$= \sum_2 \frac{1}{(cz)^{q-1}} \left(\sum_{i \geqslant 0} \left(\frac{d}{c}\right)^i \frac{1}{z^i} \right)^{q-1}$$

$$= \sum_2 \frac{1}{(cz)^{q-1}} \left(1 - \frac{d}{c}\frac{1}{z} + \left(\frac{d}{c}\right)^q \frac{1}{z^q} - \left(\frac{d}{c}\right)^{q+1} \frac{1}{z^{q+1}} + \left(\frac{d}{c}\right)^{2q} \frac{1}{z^{2q}} - \left(\frac{d}{c}\right)^{2q+1} \frac{1}{z^{2q+1}} + \cdots \right). \tag{6.3}$$

In the summation (6.2), we notice that

- the condition $\deg(d) \geqslant \deg(c)$ for $\sum_1$ ensures that $d \neq 0$, since otherwise $\deg(c) \leqslant \deg(0) = -\infty$ implies $c = 0$ and the couple $(0, 0)$ is excluded in the summation;
- the summation can be taken term by term inside the parenthesis, and we have

$$\sum_1 \frac{1}{d^{q-1}} \left(\frac{c}{d}\right)^n z^n = \sum_{\substack{(c,d) \in \mathbf{A}^2 \\ \deg(c) \leqslant \deg(d)}} \frac{1}{d^{q-1}} \left(\frac{c}{d}\right)^n z^n = 0 \quad \text{if } (q-1) \nmid n,$$

since we can pull out the leading coefficients of $d \in \mathbf{A} = \mathbb{F}_q[T]$ in the terms of a fixed degree of d and sum these terms up.

Therefore the summation S_1 in (6.2) equals

$$\sum_{\substack{(c,d) \in \mathbf{A}^2 \\ \deg(d) \geqslant \deg(c)}} \frac{1}{d^{q-1}} \left[1 + \left(\frac{c}{d}\right)^{(q-1)q} z^{(q-1)q} + \left(\frac{c}{d}\right)^{2(q-1)q} z^{2(q-1)q} + \cdots - \left(\frac{c}{d}\right)^{(q-1)^2} z^{(q-1)^2} - \left(\frac{c}{d}\right)^{(2q-1)(q-1)} z^{(2q-1)(q-1)} - \left(\frac{c}{d}\right)^{(3q-1)(q-1)} z^{(3q-1)(q-1)} - \cdots \right]. \tag{6.4}$$

In the same way, we notice that in summation (6.3)

- c cannot be equal to 0, otherwise $\deg(d) \leqslant \deg(c) - 1 = -\infty$ implies $d = 0$,
- we can take the summation term by term inside the parenthesis, and

$$\sum_2 \frac{1}{c^{q-1}} \left(\frac{d}{c}\right)^n \frac{1}{z^{n+q-1}} = \sum_{\substack{(c,d) \in \mathbf{A}^2 \\ \deg(c) \geqslant \deg(d)+1}} \frac{1}{c^{q-1}} \left(\frac{d}{c}\right)^n \frac{1}{z^{n+q-1}} = 0 \quad \text{if } (q-1) \nmid n,$$

for example, the first term is

$$\sum_{\substack{(c,d) \in \mathbf{A}^2 \\ \deg(c) \geqslant \deg(d)+1}} \frac{1}{c^{q-1} z^{q-1}} = \sum_{\substack{(c,d) \in \mathbf{A}^2 \\ \deg(c) = 0,\, d = 0}} \frac{1}{c^{q-1} z^{q-1}} = -\frac{1}{z^{q-1}}.$$

Therefore the summation S_2 in (6.3) equals

$$\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{q-1}z^{q-1}}\left(1+\left(\frac{d}{c}\right)^{(q-1)q}\frac{1}{z^{(q-1)q}}+\left(\frac{d}{c}\right)^{2(q-1)q}\frac{1}{z^{2(q-1)q}}+\cdots-\left(\frac{d}{c}\right)^{(q-1)^2}\frac{1}{z^{(q-1)^2}}\right.$$
$$\left.-\left(\frac{d}{c}\right)^{(2q-1)(q-1)}\frac{1}{z^{(2q-1)(q-1)}}-\left(\frac{d}{c}\right)^{(3q-1)(q-1)}\frac{1}{z^{(3q-1)(q-1)}}-\cdots\right). \tag{6.5}$$

Hence we get the Laurent series expansion of $E_{q-1}(z)$:

$$E_{q-1}(z)=S_1+S_2$$

which is the summation of (6.4) and (6.5). In Eq. (3.10) for $\Delta(z)$, the part with $E_{q^2-1}(z)$ doesn't contribute to the residue of $z^i\Delta(z)\,dz$ on the region $D=\{z\in\Omega\colon\ |\pi|<|z|<1\}$ for $0\leqslant i\leqslant q^2-3$, since

$$E_{q^2-1}(z)=\sum_{(0,0)\neq(c,d)\in\mathbf{A}^2}\frac{1}{(cz+d)^{q^2-1}},$$

and the residue of $z^i\,dz/(cz+d)^{q^2-1}$ is 0 as long as $0\leqslant i\leqslant q^2-3$. Therefore we need only compute the coefficient of dz/z in $(T^q-T)^qE_{q-1}(z)^{q+1}z^i\,dz$. So consider

$$E_{q-1}(z)^{q+1}=(S_1+S_2)^{q+1}=S_1^{q+1}+S_1^qS_2+S_2^qS_1+S_2^{q+1}. \tag{6.6}$$

It is clear that the terms S_1^{q+1} and S_2^{q+1} in (6.6) don't contribute to the wanted residues. Hence we need only compute the expansions of $S_1^qS_2$ and $S_2^qS_1$. As

$$S_1^qS_2=\left(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}}\frac{1}{d^{(q-1)q}}\left(\sum_{i=0}^{\infty}\left(\frac{c}{d}\right)^{i(q-1)q^2}z^{i(q-1)q^2}-\sum_{j=1}^{\infty}\left(\frac{c}{d}\right)^{(jq-1)(q-1)q}z^{(jq-1)(q-1)q}\right)\right)$$
$$\cdot\left(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}}\frac{1}{c^{q-1}z^{q-1}}\left(\sum_{s=0}^{\infty}\left(\frac{d}{c}\right)^{s(q-1)q}\frac{1}{z^{s(q-1)q}}-\sum_{t=1}^{\infty}\left(\frac{d}{c}\right)^{(tq-1)(q-1)}\frac{1}{z^{(tq-1)(q-1)}}\right)\right), \tag{6.7}$$

we consider all possible powers $(1/z)^m$ for $m\in\mathbb{Z}$ in the above expansion:

(1) for $1\leqslant m=-i(q-1)q^2+s(q-1)q+q-1\leqslant q^2-2$ with $i\geqslant 0$ and $s\geqslant 0$, then the only solution for m is $m=q-1$ when $s=iq$ for $i\geqslant 0$; in general, $m=(q-1)+l(q-1)q$ for $l\in\mathbb{Z}$;
(2) for $1\leqslant m=-i(q-1)q^2+(tq-1)(q-1)+q-1\leqslant q^2-2$ with $i\geqslant 0$ and $t\geqslant 1$, then the only solution for m is $m=(q-1)q$ when $t=iq+1$ for $i\geqslant 0$; in general, $m=l(q-1)q$ for $l\in\mathbb{Z}$;
(3) for $1\leqslant m=-(jq-1)(q-1)q+s(q-1)q+q-1\leqslant q^2-2$ with $j\geqslant 1$ and $s\geqslant 0$, then the only solution for m is $m=q-1$ when $s=jq-1$ for $j\geqslant 1$; in general, $m=(q-1)+l(q-1)q$ for $l\in\mathbb{Z}$;
(4) for $1\leqslant m=-(jq-1)(q-1)q+(tq-1)(q-1)+q-1\leqslant q^2-2$ with $j\geqslant 1$ and $t\geqslant 1$, then the only solution for m is $m=(q-1)q$ when $t=jq$ for $j\geqslant 1$; in general, $m=l(q-1)q$ for $l\in\mathbb{Z}$.

Therefore we get from (6.7)

$$S_1^q S_2 = \frac{\Upsilon_{10}}{z^{q-1}} + \frac{\Xi_{11}}{z^{(q-1)q}} + \sum_{0\neq i\in\mathbb{Z}} \frac{\Upsilon_{1i}}{z^{q-1+i(q-1)q}} + \sum_{1\neq j\in\mathbb{Z}} \frac{\Xi_{1j}}{z^{j(q-1)q}}, \tag{6.8}$$

where

$$\begin{aligned}
\Upsilon_{10} &= \sum_{i=0}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{(q-1)q}}\left(\frac{c}{d}\right)^{i(q-1)q^2}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{q-1}}\left(\frac{d}{c}\right)^{i(q-1)q^2}\Bigg)\\
&\quad - \sum_{j=1}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{(q-1)q}}\left(\frac{c}{d}\right)^{(jq-1)(q-1)q}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{q-1}}\left(\frac{d}{c}\right)^{(jq-1)(q-1)q}\Bigg),\\
\Xi_{11} &= -\sum_{i=0}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{(q-1)q}}\left(\frac{c}{d}\right)^{i(q-1)q^2}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{q-1}}\left(\frac{d}{c}\right)^{((iq+1)q-1)(q-1)}\Bigg)\\
&\quad + \sum_{j=1}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{(q-1)q}}\left(\frac{c}{d}\right)^{(jq-1)(q-1)q}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{q-1}}\left(\frac{d}{c}\right)^{(jq^2-1)(q-1)}\Bigg).
\end{aligned}$$

We compute the expansion of $S_2^q S_1$ in the same way. In the following expansion of

$$\begin{aligned}
S_2^q S_1 = &\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{(q-1)q} z^{(q-1)q}}\left(\sum_{s=0}^{\infty}\left(\frac{d}{c}\right)^{s(q-1)q^2}\frac{1}{z^{s(q-1)q^2}}\right.\\
&\left. - \sum_{t=1}^{\infty}\left(\frac{d}{c}\right)^{(tq-1)(q-1)q}\frac{1}{z^{(tq-1)(q-1)q}}\right)\\
&\cdot \sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{(q-1)}}\left(\sum_{i=0}^{\infty}\left(\frac{c}{d}\right)^{i(q-1)q} z^{i(q-1)q}\right.\\
&\left. - \sum_{j=1}^{\infty}\left(\frac{c}{d}\right)^{(jq-1)(q-1)} z^{(jq-1)(q-1)}\right),
\end{aligned} \tag{6.9}$$

we find all possible powers $(1/z)^m$ for $m\in\mathbb{Z}$:

(1) for $1\leqslant m = s(q-1)q^2 + (q-1)q - i(q-1)q \leqslant q^2-2$ with $i\geqslant 0$ and $s\geqslant 0$, the only solution is $m=(q-1)q$ when $i=sq$ for $s\geqslant 0$; in general, $m=l(q-1)q$ for $l\in\mathbb{Z}$;
(2) for $1\leqslant m = s(q-1)q^2 + (q-1)q - (jq-1)(q-1) \leqslant q^2-2$ with $j\geqslant 1$ and $s\geqslant 0$, the only solution is $m=q-1$ when $j=sq-1$ for $s\geqslant 1$; in general, $m=(q-1)+l(q-1)q$ for $l\in\mathbb{Z}$;
(3) for $1\leqslant m = (tq-1)(q-1)q + (q-1)q - i(q-1)q \leqslant q^2-2$ with $i\geqslant 0$ and $t\geqslant 1$, the only solution is $m=(q-1)q$ when $i=tq-1$ for $t\geqslant 1$; in general, $m=l(q-1)q$ for $l\in\mathbb{Z}$;

(4) for $1 \leqslant m = (tq-1)(q-1)q + (q-1)q - (jq-1)(q-1) \leqslant q^2 - 2$ with $j \geqslant 1$ and $t \geqslant 1$, the only solution is $m = q-1$ when $j = tq$ for $t \geqslant 1$; in general, $m = (q-1) + l(q-1)q$ for $l \in \mathbb{Z}$.

Therefore we get from (6.9)

$$S_2^q S_1 = \frac{\Upsilon_{20}}{z^{q-1}} + \frac{\Xi_{21}}{z^{(q-1)q}} + \sum_{0 \neq i \in \mathbb{Z}} \frac{\Upsilon_{2i}}{z^{q-1+i(q-1)q}} + \sum_{1 \neq j \in \mathbb{Z}} \frac{\Xi_{2j}}{z^{j(q-1)q}}, \tag{6.10}$$

where

$$\begin{aligned}
\Upsilon_{20} = &-\sum_{s=1}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{(q-1)q}}\left(\frac{d}{c}\right)^{s(q-1)q^2}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{q-1}}\left(\frac{c}{d}\right)^{((sq-1)q-1)(q-1)}\Bigg)\\
&+\sum_{t=1}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{(q-1)q}}\left(\frac{d}{c}\right)^{(tq-1)(q-1)q}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{q-1}}\left(\frac{c}{d}\right)^{(tq^2-1)(q-1)}\Bigg),\\
\Xi_{21} = &\sum_{s=0}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{(q-1)q}}\left(\frac{d}{c}\right)^{s(q-1)q^2}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{q-1}}\left(\frac{c}{d}\right)^{s(q-1)q^2}\Bigg)\\
&-\sum_{t=1}^{\infty}\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(c)\geqslant\deg(d)+1}} \frac{1}{c^{(q-1)q}}\left(\frac{d}{c}\right)^{(tq-1)(q-1)q}\Bigg)\cdot\Bigg(\sum_{\substack{(c,d)\in\mathbf{A}^2\\ \deg(d)\geqslant\deg(c)}} \frac{1}{d^{q-1}}\left(\frac{c}{d}\right)^{(tq-1)(q-1)q}\Bigg).
\end{aligned}$$

Hence we have the expansion on $D = \{z \in \Omega\colon\ |\pi| < |z| < 1\}$ from (6.8) and (6.10)

$$\Delta(z) = \frac{\Upsilon_0}{z^{q-1}} + \frac{\Xi_1}{z^{(q-1)q}} + \sum_{m \geqslant 0} * \cdot z^m + \sum_{m \leqslant -(q^2-2)} * \cdot z^m, \tag{6.11}$$

$$\Upsilon_0 = \left(T^q - T\right)^q (\Upsilon_{10} + \Upsilon_{20}), \tag{6.12}$$

$$\Xi_1 = \left(T^q - T\right)^q (\Xi_{11} + \Xi_{21}). \tag{6.13}$$

Let μ_Δ be the measure associated to the Drinfeld discriminant $\Delta(z)$, which is by definition, the measure associated to the harmonic cocycle $\mathfrak{c}_\Delta$ given in Eq. (5.4). Then for $0 \leqslant i \leqslant q^2 - 3$,

$$\begin{aligned}
\int_{U(e_0)} x^i \, d\mu_\Delta(x) &= \mathfrak{c}_\Delta(e)\left(X^i Y^{q^2-3-i}\right) = \operatorname{Res}_{e_0} z^i \Delta(z)\, dz\\
&= \operatorname{Res}_{e_0}\left(\Upsilon_0 \frac{z^i\, dz}{z^{q-1}} + \Xi_1 \frac{z^i\, dz}{z^{(q-1)q}}\right)
\end{aligned}$$

where $U(e_0) = U(\Lambda_0\Lambda_1) = \{x \in \mathbf{k}_\infty : |x| \geqslant |\pi|^{-1}\} \cup \{\infty\} = B_\infty(|\pi|^1)$ is given in Example 5.1. From the above discussion, we get

Lemma 6.2. *For $0 \leqslant i \leqslant q^2-3$, we have the following formulas*:

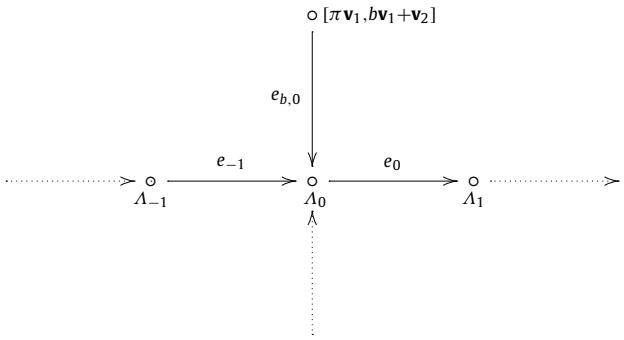
$$\int_{U(e_0)} x^i \, d\mu_\Delta(x) = 0 \quad \text{if } i \neq q-2, q^2-q-1; \tag{1}$$

$$\int_{U(e_0)} x^{q-2} \, d\mu_\Delta(x) = \Upsilon_0; \tag{2}$$

$$\int_{U(e_0)} x^{q^2-q-1} \, d\mu_\Delta(x) = \Xi_1. \tag{3}$$

Lemma 6.3. $\Xi_1 = 0$.

Proof. We consider the edges around the vertex Λ_0 of the Bruhat–Tits tree $\mathcal{T}$:



(6.14)

where the notations are as in Section 2, and $b \in \mathbb{F}_q{}^*$. Under the correspondence (2.4), the vertex $[\pi \mathbf{v}_1, b\mathbf{v}_1 + \mathbf{v}_2]$ corresponds to the matrix $\left(\begin{smallmatrix} \pi & b \\ 0 & 1 \end{smallmatrix}\right)$ and Λ_{-1} corresponds to the matrix $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\gamma_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for $b \in \mathbb{F}_q{}^*$, and $\delta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $(\gamma_b^{-1})^T = \left(\begin{smallmatrix} 1 & 0 \\ -b & 1 \end{smallmatrix}\right)$. The matrices γ_b and δ fix the vertex Λ_0 under the "$*$" action. By performing column operations, we see that

$$\begin{aligned} \delta * \Lambda_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \pi^{-1} & 1 \end{pmatrix} \sim \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} = \Lambda_{-1}, \\ \gamma_b * \Lambda_1 &= \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi^{-1} & 0 \\ -b\pi^{-1} & 1 \end{pmatrix} \sim \begin{pmatrix} \pi & -b^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \left[\pi \mathbf{v}_1, -b^{-1}\mathbf{v}_1 + \mathbf{v}_2\right], \end{aligned}$$

where the symbol $\sim$ means that two matrices are in the same class in the quotient space $\mathrm{GL}_2(\mathbf{k}_\infty)/(\mathbf{k}_\infty^* \cdot \mathrm{GL}_2(\mathbf{A}_\infty))$. Hence $\overline{e_{b,0}} = \gamma_{-b^{-1}} * e_0$ and $\overline{e_{-1}} = \delta * e_0$, where $\bar{e}$ denotes the edge with opposite orientation of e.

By the definitions in Section 5, we have for an edge $e = \gamma * e_0$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{k}_\infty)^+$ and $0 \leqslant i \leqslant q^2-3$,

$$\begin{aligned}\int_{U(e)} x^i\,d\mu_\Delta(x) &= \mathfrak{c}_\Delta(e)\big(X^iY^{q^2-3-i}\big)\\ &= \operatorname{Res}_e z^i\Delta(z)\,dz\\ &= \operatorname{Res}_{e_0}(\gamma z)^i\Delta(\gamma z)\,d\gamma z\\ &= \int_{U(e_0)} \det(\gamma)(ax+b)^i(cx+d)^{q^2-3-i}\,d\mu_\Delta(x).\end{aligned}$$

Therefore for $0\leqslant j\leqslant q^2-3$,

$$\int_{U(\gamma_b*e_0)} x^j\,d\mu_\Delta(x) = \int_{U(e_0)} (x+b)^j\,d\mu_\Delta(x),$$

$$\int_{U(e_{-1})} x^j\,d\mu_\Delta(x) = -\int_{U(\delta*e_0)} x^j\,d\mu_\Delta(x) = \int_{U(e_0)} x^{q^2-3-j}\,d\mu_\Delta(x).$$

Hence from

$$\begin{aligned}&\sum_{b\in\mathbb{F}_q^*}\int_{U(\gamma_b*e_0)} x^j\,d\mu_\Delta(x) + \int_{U(e_0)} x^j\,d\mu_\Delta(x) - \int_{U(e_{-1})} x^j\,d\mu_\Delta(x)\\ &= \sum_{b\in\mathbb{F}_q^*}\int_{U(\gamma_{-b^{-1}}*e_0)} x^j\,d\mu_\Delta(x) + \int_{U(e_0)} x^j\,d\mu_\Delta(x) - \int_{U(e_{-1})} x^j\,d\mu_\Delta(x)\\ &= -\Bigg(\sum_{b\in\mathbb{F}_q^*}\int_{U(e_{b,0})} x^j\,d\mu_\Delta(x) + \int_{U(\overline{e_0})} x^j\,d\mu_\Delta(x) + \int_{U(e_{-1})} x^j\,d\mu_\Delta(x)\Bigg)\\ &= 0,\end{aligned}$$

we get

$$\int_{U(e_0)} x^{q^2-3-j}\,d\mu_\Delta(x) = \sum_{b\in\mathbb{F}_q^*}\int_{U(e_0)} (x+b)^j\,d\mu_\Delta(x) + \int_{U(e_0)} x^j\,d\mu_\Delta(x).$$

We plug in $j=q-2$ in the above equation and apply (1) of Lemma 6.2, then get

$$\int_{U(e_0)} x^{q^2-q-1}\,d\mu_\Delta(x) = 0,$$

therefore $\Xi_1=0$. □

Recall that the Λ_n ($n\geqslant 0$) are the vertices of the half line (2.6), and $e_n=\Lambda_n\Lambda_{n+1}$ is the oriented edge with origin Λ_n and terminal Λ_{n+1}.

Theorem 6.2. *The measure μ_Δ on $\mathbb{P}^1(\mathbf{k}_\infty)$ associated to the Drinfeld discriminant $\Delta(z)$ is h-admissible with h being the smallest integer greater than or equal to $(q^2-3)/2$, and is completely determined by the following values*:

$$\int_{U(e_0)} x^j \, d\mu_\Delta(x) = \begin{cases} 0, & \text{if } 0 \leqslant j \leqslant q^2-3,\ j \neq q-2, \\ \Upsilon_0, & \text{if } j = q-2. \end{cases} \tag{1}$$

$$\int_{U(e_n)} x^j \, d\mu_\Delta(x) = 0, \quad \text{for } 0 \leqslant j \leqslant q^2-3 \text{ and any } n \geqslant 1. \tag{2}$$

Proof. That the measure μ_Δ is h-admissible follows from Corollary 5.1 directly.

Since the fundamental domain of $\Gamma\backslash\mathcal{T}$ is the half line (2.1) consisting of Λ_n ($n \geqslant 0$) (for both the ordinary action and the action "$*$") and the measure μ_Δ satisfies Eq. (5.5), μ_Δ is determined by the values $\int_{U(e_n)} x^j \, d\mu_\Delta(x)$ for $0 \leqslant j \leqslant q^2-3$ and all integers $n \geqslant 0$.

From Proposition 2.1, we know that under the ordinary action, the group

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \ a, d \in \mathbb{F}_q^*, \ b \in \mathbb{F}_q[T], \ \deg_T(b) \leqslant n \right\}, \quad n \geqslant 1$$

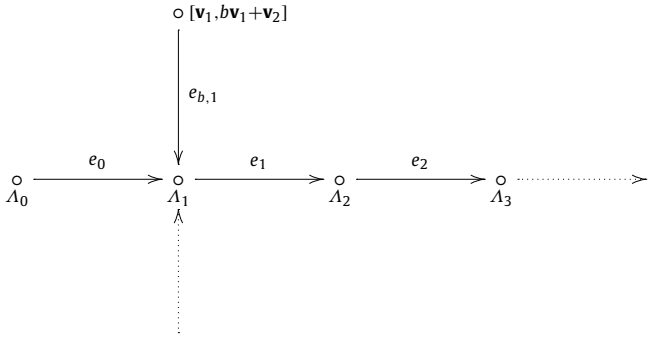
fixes the edge $\Lambda_n\Lambda_{n+1}$, and acts transitively on the set of edges with origin Λ_n but distinct from the edge $\Lambda_n\Lambda_{n+1}$. In fact, the element $\gamma_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for $b = \beta T^n$ with $\beta \in \mathbb{F}_q$ fixes Λ_n and Λ_{n+1} and takes Λ_{n-1} to the vertex $[\pi^{-(n-1)}\mathbf{v}_1, \beta\pi^{-n}\mathbf{v}_1 + \mathbf{v}_2]$, since Λ_{n-1} is represented by the lattice $L_{n-1} = \mathbf{A}_\infty \oplus \pi^{n-1}\mathbf{A}_\infty \sim \pi^{-(n-1)}\mathbf{A}_\infty \oplus \mathbf{A}_\infty$, which is represented by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi^{n-1} \end{smallmatrix}\right)$, and

$$\begin{pmatrix} 1 & \beta T^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & \beta T \\ 0 & \pi^{n-1} \end{pmatrix} \sim \begin{pmatrix} \pi^{-(n-1)} & \beta\pi^{-n} \\ 0 & 1 \end{pmatrix}$$

where the notion '$\sim$' means that the two matrices define the same element of $\mathrm{GL}_2(\mathbf{k}_\infty)/(\mathbf{k}_\infty^* \cdot \mathrm{GL}_2(\mathbf{A}_\infty))$ as before, thus they represent the same vertex of the Bruhat–Tits tree $\mathcal{T}$ in terms of the correspondence (2.2).

Identities (1) follow from Lemma 6.2 and Lemma 6.3 directly.

For identities (2), we consider the case $n = 1$ at first. The $q+1$ edges with terminal vertex Λ_1 are $e_{b,1}$ (for $b = \beta T$ and $\beta \in \mathbb{F}_q$) and $\overline{e_1}$, as illustrated in the following diagram:



where the origin vertex of $e_{b,1}$ is

$$\left(\gamma_b^{-1}\right)^T * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta T \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \beta\pi^{-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned}\int_{U(e_1)} x^j\,d\mu_\Delta(x) &= \sum_{b\in T\cdot\mathbb{F}_q{}^*}\int_{U(e_{b,1})} x^j\,d\mu_\Delta(x)+\int_{U(e_0)} x^j\,d\mu_\Delta(x)\\ &= \sum_{b\in T\cdot\mathbb{F}_q{}^*}\int_{U((\gamma_b^{-1})^T*e_0)} x^j\,d\mu_\Delta(x)+\int_{U(e_0)} x^j\,d\mu_\Delta(x)\\ &= \sum_{b\in T\cdot\mathbb{F}_q{}^*}\int_{U(e_0)} x^j(-bx+1)^{q^2-3-j}\,d\mu_\Delta(x)+\int_{U(e_0)} x^j\,d\mu_\Delta(x)\\ &= \sum_{i=0}^{q^2-3-j}\binom{q^2-3-j}{i}\sum_{b\in T\cdot\mathbb{F}_q{}^*}(-b)^i\int_{U(e_0)} x^{i+j}\,d\mu_\Delta(x)\\ &\quad+\int_{U(e_0)} x^j\,d\mu_\Delta(x).\end{aligned}\tag{6.15}$$

In the expression (6.15), the integral of the first term is equal to 0 unless $i+j=q-2$, thus $\int_{U(e_1)} x^j\,d\mu_\Delta(x)=0$ if $j>q-2$ by Lemmas 6.2 and 6.3. For $j\leqslant q-2$, we get from (6.15) that

$$\int_{U(e_1)} x^j\,d\mu_\Delta(x)=\binom{q^2-3-j}{q-2-j}\sum_{b\in T\cdot\mathbb{F}_q{}^*}(-b)^{q-2-j}\int_{U(e_0)} x^{q-2}\,d\mu_\Delta(x)+\int_{U(e_0)} x^j\,d\mu_\Delta(x).$$

The summation $\sum_{b\in T\cdot\mathbb{F}_q{}^*}(-b)^{q-2-j}=0$ unless $j=q-2$, so it is clear that the integral $\int_{U(e_1)} x^j\,d\mu_\Delta(x)$ is equal to 0 for $j<q-2$, because the integral $\int_{U(e_0)} x^j\,d\mu_\Delta(x)$ is also equal to 0 in this case. And for $j=q-2$, we have

$$\begin{aligned}\int_{U(e_1)} x^{q-2}\,d\mu_\Delta(x) &= \binom{q^2-q-1}{0}\sum_{b\in T\cdot\mathbb{F}_q{}^*}1\cdot\int_{U(e_0)} x^{q-2}\,d\mu_\Delta(x)+\int_{U(e_0)} x^{q-2}\,d\mu_\Delta(x)\\ &=0.\end{aligned}$$

Therefore we have showed that

$$\int_{U(e_1)} x^j\,d\mu_\Delta(x)=0$$

for $0\leqslant j\leqslant q^2-3$.

For $n>1$, we can show in a similar way that $\int_{U(e_n)} x^j\,d\mu_\Delta(x)=0$ for $0\leqslant j\leqslant q^2-3$ by induction on n. □

Corollary 6.1. $\Upsilon_0\neq 0$.

Proof. Suppose $\Upsilon_0=0$. Then $\int_{U(e_0)} x^j\,d\mu_\Delta(x)=0$ for all integers j between 0 and q^2-3, thus we can show in the same way as the proof of Theorem 6.2 that $\int_{U(e_n)} x^j\,d\mu_\Delta(x)=0$ for all integers $n\geqslant 0$

and $0 \leqslant j \leqslant q^2-3$, which implies that $\int_{U(e)} x^j\,d\mu_\Delta(x)=0$ for all edges e of the Bruhat–Tits tree $\mathcal{T}$. Therefore the harmonic cocycle $\mathfrak{c}_\Delta = 0$, and then $\Delta(z)=0$ by Theorem 5.3, which is absurd. □

Corollary 6.2. *For any edge e of the Bruhat–Tits tree $\mathcal{T}$ and $0 \leqslant j \leqslant q^2-3$, we have*

$$\int_{U(e)} x^j\,d\mu_\Delta(x) = r(e,j)\cdot \Upsilon_0$$

where $r(e,j)\in \mathbf{A}$.

Proof. The edge e can be obtained as $e=\gamma * e_n$ for some integer $n\geqslant 0$ and some $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$. Hence

$$\begin{aligned}\int_{U(e)} x^j\,d\mu_\Delta(x) &= \int_{U(\gamma * e_n)} x^j\,d\mu_\Delta(x)\\ &= \int_{U(e_n)} \det(\gamma)(ax+b)^j(cx+d)^{q^2-3-j}\,d\mu_\Delta(x). \end{aligned}\tag{6.16}$$

In the integral of (6.16), the integrand is a polynomial with coefficients in $\mathbf{A}$, therefore the integral $\int_{U(e)} x^j\,d\mu_\Delta(x)$ is 0 if $n\neq 0$. And if $n=0$, the integral $\int_{U(e)} x^j\,d\mu_\Delta(x)$ is equal to $r(e,j)\cdot\int_{U(e_0)} x^{q-2}\,d\mu_\Delta(x) = r(e,j)\cdot\Upsilon_0$ for some $r(e,j)\in\mathbf{A}$. □

7. Complements

In Theorem 6.2, we have determined the integrals $\int_{U(e_0)} x^j\,d\mu_\Delta(x)$ for $0\leqslant j\leqslant q^2-3$ and we can see what the values $\int_{U(e)} x^j\,d\mu_\Delta(x)$ look like in Corollary 6.2 for a general edge e of $\mathcal{T}$ and $0\leqslant j\leqslant q^2-3$. In general, we set

$$L(\Delta;e;j)=\int_{U(e)} x^{j-1}\,d\mu_\Delta(x)\quad \text{for } e\in E_{\mathcal{T}} \text{ and } j\in\mathbb{Z}\tag{7.1}$$

as given in Eq. (5.26). Since $\delta=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in\Gamma$, we have the following equation for these values:

$$L(\Delta;\delta * e;j) = -L\big(\Delta;e,q^2-1-j\big)\quad \text{for } e\in E_{\mathcal{T}} \text{ and } j\in\mathbb{Z}.\tag{7.2}$$

When the edge e is $\overline{e_{1,0}}$ in (6.14), the corresponding $U(\overline{e_{1,0}})$ is the set $\mathcal{U}_1$ of 1-units of $\mathbf{k}_\infty$, so $L(\Delta;\overline{e_{1,0}};j)$ can be extended to $L(\Delta;\overline{e_{1,0}};s)$ for $s\in\mathbb{Z}_p$, which is $L_\Delta(s)$ in terms of the notations in (5.25), and (7.2) becomes the functional equation

$$L_\Delta(s) = -L_\Delta\big(q^2-1-s\big)\tag{7.3}$$

as pointed out by D. Goss in [Go5]. About the measure associated to the Drinfeld discriminant Δ, it is also interesting to see what the values $L_\Delta(j)$ might be for $j\geqslant 1$.

As a special case of integrals of functions in $C^h(\mathbb{P}^1(\mathbf{k}_\infty))$ against the measure μ_Δ (see Corollary 5.1 and Theorem 5.3), the values $L(\Delta;e;j)$ for $j\geqslant q^2-1$ or $j\leqslant 0$ are obtained through a limit process by those for $1\leqslant j\leqslant q^2-2$, but they can also be calculated by applying Eq. (5.24) in terms of the residues. At first, we will consider $L(\Delta;e_0;j+1)=\int_{U(e_0)} x^j\,d\mu_\Delta(x)$ by expanding the Drinfeld discriminant

$\Delta(z)$ over the region $\lambda^{-1}(e_0) = \{z \in \Omega\colon\ |\pi| < |z| < 1\}$ better than Eq. (6.11), where the terms with z^m for $m \geqslant 0$ or $m \leqslant -(q^2-2)$ are omitted.

In Eq. (6.6), we've already calculated $S_1^q S_2$ in (6.8) and $S_2^q S_1$ in (6.10). And we do similar calculations to get

$$S_1^{q+1} = \sum_{l=0}^{\infty} * \cdot z^{l(q-1)q} + \sum_{l=1}^{\infty} * \cdot z^{-(q-1)+l(q-1)q}, \tag{7.4}$$

$$S_2^{q+1} = \sum_{l=1}^{\infty} \frac{*}{z^{q-1+l(q-1)q}} + \sum_{l=2}^{\infty} \frac{*}{z^{l(q-1)q}}, \tag{7.5}$$

where the symbol $*$ (which also appears below) denotes a coefficient which may not be equal to 0 and we don't want to calculate out explicitly. Therefore we get from (6.8), (6.10), (7.4), and (7.5) that

$$\left(T^q - T\right)^q E_{q-1}(z)^{q+1} = \frac{\Upsilon_0}{z^{q-1}} + \frac{\Xi_1}{z^{(q-1)q}} + \sum_{0 \neq l \in \mathbb{Z}} \frac{*}{z^{q-1+l(q-1)q}} + \sum_{1 \neq l \in \mathbb{Z}} \frac{*}{z^{l(q-1)q}}.$$

We also expand $E_{q^2-1}(z)$ over the region $\lambda^{-1}(e_0)$ as

$$E_{q^2-1}(z) = \sum_{l \in \mathbb{Z}} \frac{*}{z^{l(q-1)q^2}} + \sum_{l \in \mathbb{Z}} \frac{*}{z^{(q-1)+(q-1)q+l(q-1)q^2}}.$$

Therefore we get the expansion of $\Delta(z)$ over $\lambda^{-1}(e_0)$:

$$\begin{aligned} \Delta(z) &= \left(T^{q^2} - T\right) E_{q^2-1}(z) + \left(T^q - T\right)^q E_{q-1}(z)^{q+1} \\ &= \sum_{i \in \mathbb{Z}} \frac{\Upsilon_i}{z^{q-1+i(q-1)q}} + \sum_{i \in \mathbb{Z}} \frac{\Xi_i}{z^{i(q-1)q}} \end{aligned} \tag{7.6}$$

where Υ_0 and Ξ_1 are given in (6.12) and (6.13), respectively.

Proposition 7.1.

(1) *We have*

$$L(\Delta; e_0; j) = \begin{cases} \Upsilon_i, & \text{if } j = q-1+i(q-1)q \text{ with } i \in \mathbb{Z}, \\ \Xi_i, & \text{if } j = i(q-1)q \text{ with } i \in \mathbb{Z}, \\ 0, & \text{if } j \neq q-1+i(q-1)q, i(q-1)q \text{ with } i \in \mathbb{Z}. \end{cases}$$

(2) *For an integer $m \geqslant 0$, we have*

$$\begin{aligned} L_\Delta(m+1) = &\sum_{0 \leqslant q-2+i(q-1)q \leqslant m} (-1)^{m-(q-2+i(q-1)q)} \binom{m}{q-2+i(q-1)q} \Upsilon_i \\ &+ \sum_{0 \leqslant -1+i(q-1)q \leqslant m} (-1)^{m-(-1+i(q-1)q)} \binom{m}{-1+i(q-1)q} \Xi_i \end{aligned} \tag{7.7}$$

where the two summations are taken over the integers i.

Proof. Conclusion (1) directly follows from the residue formula (5.24) and Eq. (7.6) above.

To prove the conclusion (2), we use the action of $\gamma := \gamma_{-1} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ on the edge e_0 as in the proof of Lemma 6.3 to get $\overline{e_{1,0}} = \gamma * e_0$. Since $\mathcal{U}_1 = U(\overline{e_{1,0}})$, we have for an integer $m \geqslant 0$

$$\begin{aligned} L_\Delta(m+1) &= \int_{\mathcal{U}_1} x^m \, d\mu_\Delta(x) = \int_{U(\gamma * e_0)} x^m \, d\mu_\Delta(x) = \int_{U(e_0)} (x-1)^m \, d\mu_\Delta(x) \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \int_{U(e_0)} x^j \, d\mu_\Delta(x) \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} L_\Delta(j+1) \\ &= \sum_{0 \leqslant q-2+i(q-1)q \leqslant m} (-1)^{m-(q-2+i(q-1)q)} \binom{m}{q-2+i(q-1)q} \Upsilon_i \\ &\quad + \sum_{0 \leqslant -1+i(q-1)q \leqslant m} (-1)^{m-(-1+i(q-1)q)} \binom{m}{-1+i(q-1)q} \Xi_i. \qquad \square \end{aligned}$$

Remark 7.1. We have some remarks related to the values $L_\Delta(j)$ below.

(1) Other than $\Upsilon_0 \neq 0$ and $\Xi_1 = 0$ which we prove in Section 6, we don't know whether the other elements $\Upsilon_i, \Xi_i \in \mathbf{k}_\infty$ vanish or not.
(2) We don't know whether the elements Υ_i and Ξ_i are transcendental over $\mathbf{k}$ (if they are not equal to 0).
(3) The values $L_\Delta(j)$ for $1 \leqslant j \leqslant q^2 - 2$ are calculated in Example 7.1 in the following.

We'll state the Lucas' formula of binomial numbers in characteristic p, which is useful in the computation in our cases. Denote by

$$((i_1, i_2, \ldots, i_s)) = \frac{(i_1 + i_2 + \cdots + i_s)!}{i_1!\, i_2! \cdots i_s!}$$

for any integers $i_1, i_2, \ldots, i_s \geqslant 0$. We have the following assertion about the multinomial numbers by Lucas [Lu]:

Proposition 7.2 *(Lucas). For non-negative integers $n_0, n_1, \ldots, n_s$,*

$$((n_0, n_1, \ldots, n_s)) \equiv \prod_{j \geqslant 0} ((n_{0,j}, n_{1,j}, \ldots, n_{s,j})) \mod p \tag{7.8}$$

where $n_i = \sum_{j \geqslant 0} n_{i,j} q^j$ is the q-digit expansion for $i = 0, 1, \ldots, s$.

Remark 7.2. Proposition 7.2 is useful when $s = 1$. In this case formula (7.8) is expressed in the form: let $n = \sum_j n_j q^j$ and $k = \sum_j k_j q^j$ be q-digit expansion for non-negative integers n and k, then

$$\binom{n}{k} \equiv \prod_{j \geqslant 0} \binom{n_j}{k_j} \mod p. \tag{7.9}$$

Example 7.1. The values $L_\Delta(j)$ for $1 \leqslant j \leqslant q^2-2$ can be calculated quickly by putting in $m = j-1$ in the formula (7.7), where the index $i = 0$ in the first summation and $i = 1$ in the second summation on the right side of the formula. As $\Xi_1 = 0$, we get

$$L_\Delta(j) = (-1)^{j-q+1}\binom{j-1}{q-2}\Upsilon_0, \quad 1 \leqslant j \leqslant q^2-2.$$

After applying Lucas' formula (7.9), we see that $L_\Delta(j) \neq 0$ if and only if $j = q-1+lq, q+lq$ for $l = 0, 1, \ldots, q-2$, and

$$L_\Delta(q-1+lq) = (-1)^l\Upsilon_0, \qquad L_\Delta(q+lq) = (-1)^l\Upsilon_0.$$

Therefore the functional equation (7.3) for $L_\Delta(j)$ with $1 \leqslant j \leqslant q^2-2$ essentially becomes

$$\begin{aligned} L_\Delta(q-1+lq) &= (-1)^l\Upsilon_0 = (-1)\cdot(-1)^{q-l-2}\Upsilon_0 = -L_\Delta\big((q-l-1)q\big) \\ &= -L_\Delta\big(q^2-1-(q-1+lq)\big). \end{aligned}$$

Due to Gekeler's work ((2) of Theorem 3.1), we have $\Delta(z) = -(P_{q+1,1}(z))^{q-1}$, where the Poincaré series $P_{q+1,1}(z)$ is a cusp form of Γ of weight $q+1$ and type 1 mod $(q-1)$. For a cusp form f of Γ with weight n and type m, Eq. (5.5) becomes

$$\int\limits_{U(\gamma\star e)} g(x)\,d\mu_f(x) = \int\limits_{U(e)} \big(\det(\gamma)\big)^{1-m}(cx+d)^{n-2}g(\gamma x)\,d\mu_f(x) \tag{7.10}$$

where the action "$\star$" of Γ on $\mathcal{T}$ is actually "$*$", see [Go5] and [Te2] for detailed exposition on the above equation. Let $\mathcal{P}(z)$ denote by the Poincaré series $P_{q+1,1}(z)$ and $\mu_{\mathcal{P}}$ the associated measure on $\mathbb{P}^1(\mathbf{k}_\infty)$. And let

$$\mathcal{X}_0 = \int\limits_{U(e_0)} d\mu_{\mathcal{P}}.$$

Proposition 7.3. *The measure $\mu_{\mathcal{P}}$ on $\mathbb{P}^1(\mathbf{k}_\infty)$ is h-admissible with h being the smallest integer greater than or equal to $(q-1)/2$, and is completely determined by the following values*:

(1) $$\int\limits_{U(e_0)} x^j\,d\mu_{\mathcal{P}}(x) = \begin{cases} 0, & \text{if } 0 < j \leqslant q-1, \\ \mathcal{X}_0 \neq 0, & \text{if } j = 0. \end{cases}$$

(2) $$\int\limits_{U(e_n)} x^j\,d\mu_{\mathcal{P}}(x) = 0, \quad \text{for } 0 \leqslant j \leqslant q-1 \text{ and any } n \geqslant 1.$$

The notations e_n for $n \in \mathbb{Z}$ are the same as those in Section 6.

Proof. We'll only show that Eq. (1) of the theorem holds here, the rest of the proof is the same as those of Lemma 6.3, Theorem 6.2, and Corollary 6.1.

The action of $\delta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on e_0 gives $\overline{e_{-1}} = \delta * e_0$. Therefore

$$\int\limits_{U(e_{-1})} x^j\,d\mu_{\mathcal{P}}(x) = -\int\limits_{U(\delta * e_0)} x^j\,d\mu_{\mathcal{P}}(x) = -\int\limits_{U(e_0)} x^{q-1-j}\,d\mu_{\mathcal{P}}(x),$$

where in the second equality we need to apply Eq. (7.10) since the type of the Poincaré series $\mathcal{P}(z)$ is 1 mod $(q-1)$. Then from

$$\sum_{b\in\mathbb{F}_q^*}\int_{U(\gamma_b * e_0)} x^j\,d\mu_{\mathcal{P}}(x)+\int_{U(e_0)} x^j\,d\mu_{\mathcal{P}}(x)-\int_{U(e_{-1})} x^j\,d\mu_{\mathcal{P}}(x)=0$$

we get

$$\begin{aligned}-\int_{U(e_0)} x^{q-1-j}\,d\mu_{\mathcal{P}}(x)&=\int_{U(e_0)} x^j\,d\mu_{\mathcal{P}}(x)+\sum_{b\in\mathbb{F}_q^*}\int_{U(e_0)}(x+b)^j\,d\mu_{\mathcal{P}}(x)\\&=\int_{U(e_0)} x^j\,d\mu_{\mathcal{P}}(x)+\sum_{i=0}^{j}\binom{j}{i}\sum_{b\in\mathbb{F}_q^*} b^i\int_{U(e_0)} x^{j-i}\,d\mu_{\mathcal{P}}(x).\end{aligned}\tag{7.11}$$

As $0\leqslant i\leqslant q-1$, we have the equality

$$\sum_{b\in\mathbb{F}_q^*} b^i=\begin{cases}0, & \text{if } 0<i<q-1,\\ -1, & \text{if } i=0,\ q-1,\end{cases}$$

thus, Eq. (7.11) can be written as

$$-\int_{U(e_0)} x^{q-1-j}\,d\mu_{\mathcal{P}}(x)=\begin{cases}0, & \text{if } 0\leqslant j<q-1,\\ -\int_{U(e_0)} d\mu_{\mathcal{P}}(x), & \text{if } j=q-1.\end{cases}$$

Therefore we get (1) of the theorem except for $\mathcal{X}_0\neq 0$. But this conclusion can be proved in the same way as Corollary 6.1. □

We denote an element $\gamma\in \mathrm{GL}_2(\mathbf{A})$ by $\gamma=\left(\begin{smallmatrix}a_\gamma & b_\gamma\\ c_\gamma & d_\gamma\end{smallmatrix}\right)$, then get

Corollary 7.1. *For $0\leqslant j\leqslant q-1$,*

$$\int_{U(e)} x^j\,d\mu_{\mathcal{P}}(x)=\begin{cases}0, & \text{if } e=\gamma * e_n \text{ for some } n\geqslant 1 \text{ and some } \gamma\in\Gamma,\\ b_\gamma^j d_\gamma^{q-1-j}\mathcal{X}_0, & \text{if } e=\gamma * e_0 \text{ for some } \gamma\in\Gamma.\end{cases}$$

Remark 7.3. Although the Drinfeld discriminant $\Delta(z)$ and the Poincaré series $\mathcal{P}(z)$ are related by $\Delta(z)=-(\mathcal{P}(z))^{q-1}$, we don't know if there are direct relations in the space of h-admissible measures (h big enough) between their associated measures μ_Δ and $\mu_{\mathcal{P}}$, or even there are direct relations between the constants Υ_0 and $\mathcal{X}_0$ as elements of $\mathbf{C}_\infty$. In the general case, for any $f\in S_n(\Gamma)$, Teitelbaum [Te1] has proved that $\int_{U(e_i)} x^j\,d\mu_f(x)=0$ for $0\leqslant j\leqslant n-2$ and all $i\geqslant i_0$ for some integer i_0, but we don't know much information about the values $\int_{U(e_i)} x^j\,d\mu_f(x)$ for $0\leqslant i\leqslant i_0-1$.

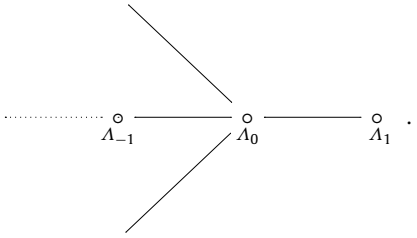
Remark 7.4. The zeta function $\zeta(s)$ for $s=(x,y)\in S_\infty=\mathbf{C}_\infty^*\times\mathbb{Z}_p$ in Section 1 has special values $\zeta(x,-j)$ for integers $j\geqslant 0$. And $\zeta(T^j,j)=\mathrm{z}(j)=\sum_{a\in\mathbf{A}^+}\frac{1}{a^j}$ as mentioned in the beginning of Section 1. The zeta measure $\mu_x^{(\infty)}=x\nu_x^{(\infty)}$ is a measure on $\mathbf{A}_\infty$ given by (see Section 3 of [Ya2])

$$\nu_x^{(\infty)} = \sum_{k=1}^{\infty} (-1)^k x^{-k} G^*_{q^k-1}$$

where the measures $\{G^*_{q^k-1}\}_{k\geqslant 1}$ are given as follows: let $\mathbf{A}_\infty = \bigsqcup_i B_{\alpha_i}(|\pi|^l)$ be a disjoint decomposition into closed balls, where l is chosen such that $l \geqslant k$, $\alpha_i \in \mathbb{F}_q[\pi]$, and $\deg_\pi(\alpha_i) < l$ for each i. Each measure $G^*_{q^k-1}$ is 0-admissible, and is given by (see the computation in Section 3 of [Ya2])

$$G^*_{q^k-1}\big(B_{\alpha_i}(|\pi|^l)\big) = \begin{cases} (-1)^k, & \text{if } \deg_\pi(\alpha_i) < k, \\ 0, & \text{if } k \leqslant \deg_\pi(\alpha_i) < l. \end{cases} \tag{7.12}$$

By using Example 5.3, we see that the measures $\{G^*_{q^k-1}\}_{k\geqslant 1}$ can be easily expressed as functions on the subtree $\mathcal{W}$ of $\mathcal{T}$, where $\mathcal{W}$ is obtained from $\mathcal{T}$ by cutting every vertex and every edge in the paths starting at the vertex Λ_1 except for those containing the edge $\Lambda_1\Lambda_0$:



There are many ways to extend such functions to harmonic functions on the (oriented) edges of $\mathcal{T}$. But unlike the measures associated to cusp forms, the author doesn't know the measures $G^*_{q^k-1}$, $\nu_x^{(\infty)}$, and $\mu_x^{(\infty)}$ have any symmetries under the action of Γ (or subgroups of Γ). Further study is necessary in order to understand them better.

In Section 4, we have discussed the characteristic p valued distributions by using C^h functions. Although C^h functions and the functions with Lipschitz conditions are not studied very often in rigid analytic geometry and related topics, the study of their dual spaces (the spaces of h-admissible measures and their variants) may have interesting applications in the theory of ergodic functions over $\mathbb{F}_q[\![\pi]\!]$. In the case $q = 2$, an ergodic function $f : \mathbb{F}_2[\![\pi]\!] \to \mathbb{F}_2[\![\pi]\!]$ is a continuous function

$$f(x) = \sum_{j=0}^{\infty} a_j G_j(x)$$

(where $G_j(x)$ for $j \geqslant 0$ are the Carlitz polynomials given in Section 4) which satisfies the following conditions on the coefficients a_j for $j \geqslant 0$ (see [LSY]):

(1) $a_0 \equiv 1 \bmod \pi$, $a_1 \equiv 1 + \pi \bmod \pi^2$, $a_3 \equiv \pi^2 \bmod \pi^3$;
(2) $|a_j| < |\pi|^{[\log_2(j)]} = 2^{-[\log_2(j)]}$, for $j \geqslant 2$;
(3) $a_{2^j-1} \equiv \pi^j \bmod \pi^{j+1}$ for $j \geqslant 2$.

We can see from the above descriptions and (2) of Theorem 4.3 that ergodic functions on $\mathbb{F}_2[\![\pi]\!]$ are not C^1 functions (therefore certainly are not locally analytic functions), but are continuous functions satisfying the so-called 1-Lipschitz condition (see Chapter 3 of [AK] or [LSY] for the concept "1-Lipschitz" condition). In the case $q = 2$, the measures μ_Δ and $\mu_{\mathcal{P}}$ are 1-admissible. After checking

Teitelbaum's estimation (5.7) and the construction of integrals in Lemma 4.4 more carefully, we can see that the ergodic functions on $\mathbb{F}_2[\![\pi]\!]$ are integrable against μ_Δ or $\mu_{\mathcal{P}}$. Due to the applications of the theory of ergodic functions in non-Archimedean analysis to cryptography, we hope that the duality between functions and measures in characteristic $p > 0$ will be helpful in further applications.

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