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Congruences modulo powers of 5 for k-colored partitions

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ABSTRACT

Let $p_{-k}(n)$ enumerate the number of k-colored partitions of n. In this paper, we establish some infinite families of congruences modulo 25 for k-colored partitions. Furthermore, we prove some infinite families of Ramanujan-type congruences modulo powers of 5 for $p_{-k}(n)$ with $k = 2, 6$, and 7. For example, for all integers $n \geq 0$ and $\alpha \geq 1$, we prove that

$$p_{-2}\left(5^{2\alpha-1}n + \frac{7 \times 5^{2\alpha-1}+1}{12}\right) \equiv 0 \pmod{5^{\alpha}}$$

and

$$p_{-2}\left(5^{2\alpha}n + \frac{11 \times 5^{2\alpha}+1}{12}\right) \equiv 0 \pmod{5^{\alpha+1}}.$$

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1. Introduction

A *partition* [1] of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i's are called the *parts* of the partition. Let $p(n)$ denote the number of partitions of n, then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout the paper, we adopt the following customary notation on partitions and q-series:

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

A partition is called a k-colored partition if each part can appear as k colors. Let $p_{-k}(n)$ count the number of k-colored partitions of n. The generating function of $p_{-k}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n = \frac{1}{(q;q)_\infty^k}.$$

For convention, we denote $p_{-1}(n) = p(n)$.

Many congruences modulo 5 and 25 enjoyed by $p_{-k}(n)$ have been found. For example, Ramanujan's so-called "most beautiful identity" for partition function $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

which readily implies one of his three classical partition congruences, namely,

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.2}$$

Further, we have, modulo 25,

$$\begin{aligned} \sum_{n=0}^{\infty} p(5n+4)q^n &= 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \equiv 5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty} \\ &= 5(q^5;q^5)_\infty^4 \sum_{n=0}^{\infty} p(n)q^n, \end{aligned}$$

from which it follows easily from (1.2) that

$$p(25n+24) \equiv 0 \pmod{25}.$$

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For $k = 2$, Hammond and Lewis [11] as well as Baruah and Sarmah [4, Eq. (5.4)] proved that

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5. \tag{1.3}$$

Later on, Chen et al. [5, Eq. (1.17)] proved the following congruence modulo 25 for $p_{-2}(n)$ via modular forms:

$$p_{-2}(25n+23) \equiv 0 \pmod{25}.$$

Quite recently, with the help of modular forms, Lazarev et al. [15] provided a criterion which can be used for searching for congruences of k-colored partitions. As applications, they obtained that

$$p_{-(25r+1)}(25n+24) \equiv p_{-(25r+6)}(25n+19) \equiv p_{-(25r+11)}(25n+14) \equiv 0 \pmod{25}. \tag{1.4}$$

Following the work of Chen et al. as well as Lazarev et al., and relating to our recent work on k-colored partitions [7,8], we further consider arithmetic properties for k-colored partitions.

In this paper, we give an elementary proof of (1.4) and also find the following new congruences for $p_{-k}(n)$ modulo 25 for $k \equiv 2, 7, 17 \pmod{25}$.

$$p_{-(25r+2)}(25n+23) \equiv p_{-(25r+7)}(25n+18) \equiv p_{-(25r+17)}(25n+8) \equiv 0 \pmod{25}. \tag{1.5}$$

Interestingly, all six infinite families of congruences (1.4)–(1.5) can be written as a combined result.

Theorem 1.1. *If $r \geq 0$ and $k \in \{1, 2, 6, 7, 11, 17\}$, then for any non-negative integer n, we have*

$$p_{-(25r+k)}(25n+25-k) \equiv 0 \pmod{25}. \tag{1.6}$$

In 1919, Ramanujan [17] conjectured that for any integer $\alpha \geq 1$,

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha},$$

where δ_α is the reciprocal modulo 5^α of 24, and this was first proved by Watson [18].

Following the strategy of Hirschhorn [12,13] as well as Garvan [9], we will obtain many infinite families of congruences modulo any powers of 5 for k-colored partition functions $p_{-k}(n)$ with $k = 2, 6$, and 7.

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Theorem 1.2. *For all integers $n \geq 0$ and $\alpha \geq 1$, we have*

$$p_{-2}\left(5^{2\alpha-1}n + \frac{7 \times 5^{2\alpha-1}+1}{12}\right) \equiv 0 \pmod{5^{\alpha}}, \tag{1.7}$$

$$p_{-2}\left(5^{2\alpha}n + \frac{11 \times 5^{2\alpha}+1}{12}\right) \equiv 0 \pmod{5^{\alpha+1}}. \tag{1.8}$$

Theorem 1.3. *For all integers $n \geq 0$ and $\alpha \geq 1$, we have*

$$p_{-6}\left(5^{\alpha}n + \frac{3 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod{5^{\alpha}}, \tag{1.9}$$

$$p_{-6}\left(5^{\alpha+1}n + \frac{11 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod{5^{\alpha+1}}, \tag{1.10}$$

$$p_{-6}\left(5^{\alpha+1}n + \frac{19 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod{5^{\alpha+1}}. \tag{1.11}$$

Theorem 1.4. *For all integers $n \geq 0$ and $\alpha \geq 1$, we have*

$$p_{-7}\left(5^{2\alpha-1}n + \frac{13 \times 5^{2\alpha-1}+7}{24}\right) \equiv 0 \pmod{5^{\alpha}}, \tag{1.12}$$

$$p_{-7}\left(5^{2\alpha}n + \frac{17 \times 5^{2\alpha}+7}{24}\right) \equiv 0 \pmod{5^{\alpha+1}}, \tag{1.13}$$

$$p_{-7}\left(5^{2\alpha}n + \frac{61 \times 5^{2\alpha-1}+7}{24}\right) \equiv 0 \pmod{5^{\alpha+1}}, \tag{1.14}$$

$$p_{-7}\left(5^{2\alpha}n + \frac{109 \times 5^{2\alpha-1}+7}{24}\right) \equiv 0 \pmod{5^{\alpha+1}}. \tag{1.15}$$

The rest of the paper is organized as follows. In Section 2, we present the background material on the H operator and some useful lemmas. In Section 3, we will provide an elementary proof for Theorem 1.1, and in Section 4 we will prove Theorems 1.2–1.4.

2. Preliminary results

To prove the main results of this paper, we introduce some useful notations and terminology on q-series.

For convenience, we denote that

$$E_j := (q^j; q^j)_\infty.$$

Denote

$$R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

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From [12, Eqs. (8.4.1), (8.4.2) and (8.4.4)], we see that

$$\frac{E_1}{E_{25}} = \frac{1}{R(q^5)} - q - q^2 R(q^5), \tag{2.1}$$

$$\frac{E_5^6}{E_{25}^6} = \frac{1}{R(q^5)^5} - 11q^5 - q^{10} R(q^5)^5, \tag{2.2}$$

$$\begin{aligned} \frac{1}{E_1} = \frac{E_{25}^5}{E_5^6} \Bigg(& \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \\ & - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \Bigg). \end{aligned} \tag{2.3}$$

Define

$$\zeta = \frac{E_1}{qE_{25}}, \quad T = \frac{E_5^6}{q^5 E_{25}^6}.$$

Following Hirschhorn [12,13], we introduce a "huffing" operator H, which operates on a series by picking out those terms of the form q^{5n}, and huffing the rest away. That is,

$$H\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_{5n} q^{5n}.$$

As in Hirschhorn [12], we define an infinite matrix $\{m(i,j)\}_{i,j\geq 1}$ by

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2\times 5 & 5^3 & 0 & 0 & 0 & 0 & \cdots \\ 9 & 3\times 5^3 & 5^5 & 0 & 0 & 0 & \cdots \\ 4 & 22\times 5^2 & 4\times 5^5 & 5^7 & 0 & 0 & \cdots \\ 1 & 4\times 5^3 & 8\times 5^5 & 5^8 & 5^9 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and for $i \geq 6$, $m(i,1) = 0$, and for $j \geq 2$,

$$\begin{aligned} m(i,j) = {} & 25m(i-1,j-1) + 25m(i-2,j-1) + 15m(i-3,j-1) \\ & + 5m(i-4,j-1) + m(i-5,j-1). \end{aligned} \tag{2.4}$$

By induction, it follows immediately that

Lemma 2.1 *([12]). We have*

- $m(i,j) = 0$ *for* $j > i$.
- $m(i,j) = 0$ *for* $i > 5j$.

The following lemma is important for our proof.

Lemma 2.2 *(Eq. (6.4.9), [12]). For $j \geq 1$, we have*

$$H\left(\frac{1}{\zeta^i}\right) = \sum_{j=1}^{\infty} \frac{m(i,j)}{T^j} = \sum_{j=1}^{i} \frac{m(i,j)}{T^j}.$$

Employing the binomial theorem, we can easily establish the following congruences, which will be frequently used without explicit mention.

Lemma 2.3. *If p is a prime, α is a positive integer, then*

$$(q^\alpha; q^\alpha)_\infty^p \equiv (q^{p\alpha}; q^{p\alpha})_\infty \pmod{p},$$
$$(q;q)_\infty^{p^\alpha} \equiv (q^p; q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

3. Congruences for k-colored partitions modulo 25

Proof of Theorem 1.1. We will first prove Theorem 1.1 for $r = 0$, then we will explain its connection with the remaining cases.

Case 1. For $k = 2$, according to (2.3), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-2}(n)q^n &= \frac{1}{E_1^2} \\ &= \frac{E_{25}^{10}}{E_5^{12}}\Bigg(\frac{1}{R(q^5)^8} + \frac{2q}{R(q^5)^7} + \frac{5q^2}{R(q^5)^6} + \frac{10q^3}{R(q^5)^5} + \frac{20q^4}{R(q^5)^4} + \frac{16q^5}{R(q^5)^3} \\ &\quad + \frac{27q^6}{R(q^5)^2} + \frac{20q^7}{R(q^5)} + 15q^8 - 20q^9R(q^5) + 27q^{10}R(q^5)^2 - 16q^{11}R(q^5)^3 \\ &\quad + 20q^{12}R(q^5)^4 - 10q^{13}R(q^5)^5 + 5q^{14}R(q^5)^6 - 2q^{15}R(q^5)^7 + q^{16}R(q^5)^8\Bigg).\end{aligned}$$

Invoking (2.2), we find that

$$\sum_{n=0}^{\infty} p_{-2}(5n+3)q^n = 10\frac{E_5^4}{E_1^6} + 125q\frac{E_5^{10}}{E_1^{12}}. \tag{3.1}$$

Furthermore, we get, (all the following congruences are modulo 25 in this section unless otherwise specified)

$$\sum_{n=0}^{\infty} p_{-2}(5n+3)q^n \equiv 10\frac{E_5^3}{E_1} = 10E_5^3\sum_{n=0}^{\infty} p(n)q^n.$$

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It follows from (1.2) that

$$p_{-2}(25n+23) \equiv 0.$$

Case 2. For $k = 17$, by (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{-17}(n)q^n &= \frac{1}{E_1^{17}} = \frac{E_1^8}{E_1^{25}} \equiv \frac{E_1^8}{E_5^5} \\
&= \frac{E_{25}^8}{E_5^5}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)^8 \\
&= \frac{E_{25}^8}{E_5^5}\left(\frac{1}{R(q^5)^8} - \frac{8q}{R(q^5)^7} + \frac{20q^2}{R(q^5)^6} - \frac{70q^4}{R(q^5)^4} + \frac{56q^5}{R(q^5)^3} + \frac{112q^6}{R(q^5)^2}\right. \\
&\quad - \frac{120q^7}{R(q^5)} - 125q^8 + 120q^9R(q^5) + 112q^{10}R(q^5)^2 - 56q^{11}R(q^5)^3 \\
&\quad \left. - 70q^{12}R(q^5)^4 + 20q^{14}R(q^5)^6 + 8q^{15}R(q^5)^7 + q^{16}R(q^5)^8\right).
\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} p_{-17}(5n+3)q^n \equiv 0.$$

So

$$p_{-17}(5n+3) \equiv 0 \tag{3.2}$$

and, in particular,

$$p_{-17}(25n+8) \equiv 0,$$

which is what we wanted to prove. But note that we achieved a stronger result.

Case 3. Similarly, for $k = 11$, we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{-11}(n)q^n &= \frac{1}{E_1^{11}} = \frac{E_1^{14}}{E_1^{25}} \equiv \frac{E_1^{14}}{E_5^5} \\
&= \frac{E_{25}^{14}}{E_5^5}\left(\frac{1}{R(q^5)^{14}} - \frac{14q}{R(q^5)^{13}} + \frac{77q^2}{R(q^5)^{12}} - \frac{182q^3}{R(q^5)^{11}} + \frac{910q^5}{R(q^5)^9} - \frac{1365q^6}{R(q^5)^8}\right. \\
&\quad - \frac{1430q^7}{R(q^5)^7} + \frac{5005q^8}{R(q^5)^6} - \frac{10010q^{10}}{R(q^5)^4} + \frac{3640q^{11}}{R(q^5)^3} + \frac{14105q^{12}}{R(q^5)^2} - \frac{6930q^{13}}{R(q^5)} \\
&\quad - 15625q^{14} + 6930q^{15}R(q^5) + 14105q^{16}R(q^5)^2 - 3640q^{17}R(q^5)^3 \\
&\quad - 10010q^{18}R(q^5)^4 + 5005q^{20}R(q^5)^6 + 1430q^{21}R(q^5)^7 - 1365q^{22}R(q^5)^8
\end{aligned}$$

$$- 910q^{23}R(q^5)^9 + 182q^{25}R(q^5)^{11} + 77q^{26}R(q^5)^{12} + 14q^{27}R(q^5)^{13} + q^{28}R(q^5)^{14}\Big).$$

It follows that

$$p_{-11}(5n+4) \equiv 0. \tag{3.3}$$

In particular,

$$p_{-11}(25n+14) \equiv 0.$$

Case 4. For $k = 6$ and 7, we need the following lemma.

Lemma 3.1. *We have*

$$H\left(q^{i-5}\frac{E_5^{6i-1}}{E_1^{6i}}\right) = \sum_{j=1}^{\infty} m(6i, i+j)q^{5j-5}\frac{E_{25}^{6j}}{E_5^{6j+1}}, \tag{3.4}$$

$$H\left(q^{i-4}\frac{E_5^{6i}}{E_1^{6i+1}}\right) = \sum_{j=1}^{\infty} m(6i+1, i+j)q^{5j-5}\frac{E_{25}^{6j-1}}{E_5^{6j}}. \tag{3.5}$$

Proof. It follows immediately from Lemma 2.2 and induction on i. $\square$

Taking $i = 1$ in (3.4), according to (2.4) and the definition of H, then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p_{-6}(5n+4)q^n = 315\frac{E_5^6}{E_1^{12}} + 52 \times 5^4 q\frac{E_5^{12}}{E_1^{18}} + 63 \times 5^6 q^2\frac{E_5^{18}}{E_1^{24}} + 6 \times 5^9 q^3\frac{E_5^{24}}{E_1^{30}} + 5^{11}q^4\frac{E_5^{30}}{E_1^{36}} \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} p_{-6}(5n+4)q^n \equiv 315\frac{E_5^4}{E_1^2} = 315E_5^4\sum_{n=0}^{\infty} p_{-2}(n)q^n.$$

By (1.3), we get

$$p_{-6}(25n+14) \equiv p_{-6}(25n+19) \equiv p_{-6}(25n+24) \equiv 0.$$

Putting $i = 1$ in (3.5) and by (2.4), we have

$$\sum_{n=0}^{\infty} p_{-7}(5n+3)q^n = 140\frac{E_5^5}{E_1^{12}} + 49\times 5^4 q\frac{E_5^{11}}{E_1^{18}} + 21\times 5^7 q^2\frac{E_5^{17}}{E_1^{24}} + 91\times 5^8 q^3\frac{E_5^{23}}{E_1^{30}} + 7\times 5^{11} q^4\frac{E_5^{29}}{E_1^{36}} + 5^{13} q^5\frac{E_5^{35}}{E_1^{42}} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} p_{-7}(5n+3)q^n \equiv 140\frac{E_5^3}{E_1^2} = 140E_5^3\sum_{n=0}^{\infty} p_{-2}(n)q^n.$$

Similarly, we obtain

$$p_{-7}(25n+13) \equiv p_{-7}(25n+18) \equiv p_{-7}(25n+23) \equiv 0.$$

Case 5. For $r \geq 1$ and $k \in \{1, 2, 6, 7, 11, 17\}$, assume that $k = 5s + t$ $(1 \leq t \leq 4)$. We consider the following two cases:

1) $k \in \{11, 17\}$.

$$\sum_{n=0}^{\infty} p_{-(25r+k)}(n)q^n = \frac{1}{E_1^{25r+k}} \equiv \frac{1}{E_1^k E_5^{5r}} = \frac{1}{E_5^{5r}}\sum_{n=0}^{\infty} p_{-k}(n)q^n.$$

Hence

$$\sum_{n=0}^{\infty} p_{-(25r+k)}(5n+5-t)q^n \equiv \frac{1}{E_5^{5r}}\sum_{n=0}^{\infty} p_{-k}(5n+5-t)q^n.$$

The Eqs. (3.2) and (3.3) imply

$$p_{-k}\left(5n+5-t\right) \equiv 0.$$

It follows immediately that

$$p_{-(25r+k)}\left(5n+5-t\right) \equiv 0,$$

since all terms in the factor $1/E_5^{5r}$ are of the form q^{5i}.

2) $k \in \{1, 2, 6, 7\}$.

$$\sum_{n=0}^{\infty} p_{-(25r+k)}(n)q^n = \frac{1}{E_1^{25r+5s+t}} = \zeta^{-25r-5s-t}\frac{1}{q^{25r+5s+t}E_{25}^{25r+5s+t}}.$$

Picking out those terms of the form q^{5n+5-t} and applying Lemma 2.2, we find that

$$\sum_{n=0}^{\infty} p_{-(25r+k)}(5n+5-t)q^n = \sum_{5r+s+1}^{25r+5s+t} m(25r+5s+t, h)\frac{q^{h-5r-s-1}}{E_1^{6h}E_5^{25r+5s+t-6h}}.$$

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According to Lemma 4.2, we know that $\pi_5(m(25r+5s+t,h)) \geq 2$ if $h \geq 5r+s+2$, and $\pi_5(m\,(25r+5s+t,5r+s+1)) \geq 1$ if $t=1$ or 2. Therefore

$$\sum_{n=0}^{\infty} p_{-(25r+k)}(5n+5-t)q^n \equiv m\,(25r+5s+t,5r+s+1)\,\frac{E_5^{5r+s-t+6}}{E_1^{30r+6s+6}}$$
$$\equiv m\,(25r+5s+t,5r+s+1)\,\frac{E_5^{5-r-t}}{E_1^{s+1}}.$$

- If $k \in \{1,2\}$, then $s=0$. By (1.2), we obtain

$$p_{-(25r+k)}(25n+25-t) \equiv 0.$$

- If $k \in \{6,7\}$, then $s=1$. Upon (1.3), we get

$$p_{-(25r+k)}(25n+15-t) \equiv p_{-(25r+k)}(25n+20-t) \equiv p_{-(25r+k)}(25n+25-t) \equiv 0.$$

This proves (1.6). $\square$

As an immediate consequence, we obtain the following corollary.

Corollary 3.2. *For all non-negative integers r and n, we have*

$$p_{25r+16}(25n+19) \equiv p_{25r+16}(25n+24) \equiv 0,$$
$$p_{25r+21}(25n+9) \equiv p_{25r+21}(25n+14) \equiv p_{25r+21}(25n+19) \equiv p_{25r+21}(25n+24) \equiv 0,$$
$$p_{25r+22}(25n+8) \equiv p_{25r+22}(25n+13) \equiv p_{25r+22}(25n+18) \equiv p_{25r+22}(25n+23) \equiv 0.$$

4. Congruences for k-colored partitions modulo powers of 5

4.1. Congruences for $p_{-2}(n)$ modulo powers of 5

In this subsection, we will prove (1.7) and (1.8). To present the generating functions for the sequence in (1.7) and (1.8), we need to define another infinite matrix of natural numbers $\{a(j,k)\}_{j,k\geq 1}$ by

1) $a(1,1)=10$, $a(1,2)=125$, and $a(1,k)=0$ for $k \geq 3$.
2)

$$a(j+1,k) = \begin{cases} \sum_{i=1}^{\infty} a(j,i)m(6i,i+k) & \text{if } j \text{ is odd,} \\ \sum_{i=1}^{\infty} a(j,i)m(6i+2,i+k) & \text{if } j \text{ is even.} \end{cases}$$

According to Lemma 2.1, the summation in 2) is indeed finite.

To prove (1.7)–(1.8), we need the following key theorem and lemmas.

Theorem 4.1. *For any positive integer j, we have*

$$\sum_{n=0}^{\infty} p_{-2}\left(5^{2j-1}n+\frac{7\times 5^{2j-1}+1}{12}\right)q^n=\sum_{l=1}^{\infty} a(2j-1,l)q^{l-1}\frac{E_5^{6l-2}}{E_1^{6l}}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} p_{-2}\left(5^{2j}n+\frac{11\times 5^{2j}+1}{12}\right)q^n=\sum_{l=1}^{\infty} a(2j,l)q^{l-1}\frac{E_5^{6l}}{E_1^{6l+2}}. \tag{4.2}$$

Proof. We proceed by induction on j. By (3.1), we know that (4.1) holds for $j=1$. Assume that (4.1) is true for some positive integer $j\geq 1$. We restate it as

$$\sum_{n=0}^{\infty} p_{-2}\left(5^{2j-1}n+\frac{7\times 5^{2j-1}+1}{12}\right)q^n=\frac{1}{qE_5^2}\sum_{l=1}^{\infty} a(2j-1,l)T^l\zeta^{-6l}.$$

Picking out those terms of the form q^{5n+4} and applying Lemma 2.2, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-2}\left(5^{2j-1}(5n+4)+\frac{7\times 5^{2j-1}+1}{12}\right)q^{5n+4}\\ =&\frac{1}{qE_5^2}\sum_{l=1}^{\infty} a(2j-1,l)T^l\left(\sum_{k=1}^{\infty} m(6l,k)T^{-k}\right).\end{aligned} \tag{4.3}$$

According to Lemma 2.1, we know that $m(6l,k)\neq 0$ implies $k\geq l+1$. Now (4.3) implies

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-2}\left(5^{2j}n+\frac{11\times 5^{2j}+1}{12}\right)q^n\\ =&\frac{1}{qE_1^2}\sum_{l=1}^{\infty}\sum_{k=l+1}^{\infty} a(2j-1,l)m(6l,k)\left(q\frac{E_5^6}{E_1^6}\right)^{k-l} \qquad (\text{replace } k \text{ by } k+l)\\ =&\frac{1}{qE_1^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a(2j-1,l)m(6l,k+l)\left(q\frac{E_5^6}{E_1^6}\right)^{k}\\ =&\sum_{k=1}^{\infty} a(2j,k)q^{k-1}\frac{E_5^{6k}}{E_1^{6k+2}}.\end{aligned}$$

This implies that (4.2) holds for j. Similarly, we rewrite (4.2) as

$$\sum_{n=0}^{\infty} p_{-2}\left(5^{2j}n+\frac{11\times 5^{2j}+1}{12}\right)q^n=\frac{1}{q^3E_{25}^2}\sum_{l=1}^{\infty} a(2j,l)T^l\zeta^{-(6l+2)}.$$

Taking out those terms of the form q^{5n+2} and applying Lemma 2.2, we find

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$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-2}\left(5^{2j}(5n+2)+\frac{11\times 5^{2j}+1}{12}\right)q^{5n+2}\\ =&\frac{1}{q^3E_{25}^2}\sum_{l=1}^{\infty} a(2j,l)T^l\left(\sum_{k=1}^{\infty} m(6l+2,k)T^{-k}\right).\end{aligned} \tag{4.4}$$

By Lemma 2.1, we know that $m(6l+2,k)\neq 0$ implies $k\geq l+1$. Now (4.4) implies

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-2}\left(5^{2j+1}n+\frac{7\times 5^{2j+1}+1}{12}\right)q^{n}\\ =&\frac{1}{qE_5^2}\sum_{l=1}^{\infty}\sum_{k=l+1}^{\infty} a(2j,l)m(6l+2,k)\left(q\frac{E_5^6}{E_1^6}\right)^{k-l} \quad (\text{replace } k \text{ by } k+l)\\ =&\frac{1}{qE_5^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a(2j,l)m(6l+2,k+l)\left(q\frac{E_5^6}{E_1^6}\right)^{k}\\ =&\sum_{k=1}^{\infty} a(2j+1,k)q^{k-1}\frac{E_5^{6k-2}}{E_1^{6k}}.\end{aligned}$$

This implies that (4.1) holds for $j+1$. This finishes the proof by induction. □

For any positive integer n, let $\pi_5(n)$ enumerate the highest power of 5 that divides n. For convention, we define $\pi_5(0)=+\infty$. To prove (1.7)–(1.8), we need the following lemma to estimate $\pi_5(a(j,k))$.

Lemma 4.2 *(Lemma 4.1, [13]). For any positive integers $j\geq 1$, we have*

$$\pi_5(m(i,j))\geq\left\lfloor\frac{5j-i-1}{2}\right\rfloor.$$

Lemma 4.3. *For any positive integers $j\geq 1$ and $k\geq 1$, we have*

$$\pi_5(a(2j-1,k))\geq j+\left\lfloor\frac{5k-5}{2}\right\rfloor, \tag{4.5}$$

$$\pi_5(a(2j,k))\geq j+\left\lfloor\frac{5k-3}{2}\right\rfloor. \tag{4.6}$$

Proof. It is easy to see that (4.5) holds for $j=1$. Assume (4.5) is true for $j\geq 1$. By definition of π_5 and Lemma 4.2, we get

$$\begin{aligned}\pi_5(a(2j,k))&=\pi_5\left(\sum_{i=1}^{\infty} a(2j-1,i)m(6i,k+i)\right)\\ &\geq\min_{i\geq 1}\left(\pi_5(a(2j-1,i))+\pi_5(m(6i,k+i))\right)\\ &\geq\min_{i\geq 1}\left(j+\left\lfloor\frac{5i-5}{2}\right\rfloor+\left\lfloor\frac{5k-i-1}{2}\right\rfloor\right).\end{aligned} \tag{4.7}$$

Let

$$g(i,k) = \left\lfloor \frac{5i-5}{2} \right\rfloor + \left\lfloor \frac{5k-i-1}{2} \right\rfloor .$$

Notice that for fixed k, if we increase i by 1, $\left\lfloor \frac{5i-5}{2} \right\rfloor$ increases by at least 2, but $\left\lfloor \frac{5k-i-1}{2} \right\rfloor$ decreases by at most 1. Hence $g(i+1,k) \geq g(i,k)+1$. Thus we obtain

$$g(i,k) \geq g(1,k) = \left\lfloor \frac{5k-2}{2} \right\rfloor \geq \left\lfloor \frac{5k-3}{2} \right\rfloor .$$

Thus, we derive form (4.7) that

$$\pi_5(a(2j,k)) \geq j + \left\lfloor \frac{5k-3}{2} \right\rfloor .$$

This proves that (4.6) holds for j.

Similarly, we find

$$\begin{aligned} \pi_5(a(2j+1,k)) &= \pi_5\left(\sum_{i=1}^{\infty} a(2j,i)m(6i+2,k+i)\right) \\ &\geq \min_{i\geq 1}\left(j + \left\lfloor \frac{5i-3}{2} \right\rfloor + \left\lfloor \frac{5k-i-3}{2} \right\rfloor\right) \\ &\geq j+1+\left\lfloor \frac{5k-5}{2} \right\rfloor . \end{aligned} \tag{4.8}$$

Here the last equality in (4.8) because the minimal value occurs at $i = 1$. This shows that (4.5) holds for $j+1$. The proof is completed by induction. □

The congruence (1.7) follows from (4.1) together with (4.5), and the congruence (1.8) follows from (4.2) together with (4.6).

4.2. Congruences for $p_{-6}(n)$ modulo powers of 5

Now, we apply the same method to investigate the arithmetic properties of $p_{-6}(n)$. Define

1) $b(1,1) = 315$, $a(1,2) = 52 \times 5^4$, $b(1,3) = 63 \times 5^6$, $b(1,4) = 6 \times 5^9$, $b(1,5) = 5^{11}$ and $b(1,k) = 0$ for $k \geq 6$.
2)

$$b(j+1,k) = \sum_{i=1}^{\infty} b(j,i)m(6i+6,k+i+1), \quad j \geq 1, \quad k \geq 1.$$

Theorem 4.4. *For any positive integer j, we have*

$$\sum_{n=0}^{\infty} p_{-6}\left(5^j n + \frac{3\times 5^j+1}{4}\right) q^n = \sum_{l=1}^{\infty} b(j,l) q^{l-1} \frac{E_5^{6l}}{E_1^{6l+6}}. \tag{4.9}$$

Proof. We proceed by induction on j. According to (3.6), we know that (4.9) is true for $j=1$. Assume that (4.9) holds for some natural number $j \geq 1$. We rewrite it as

$$\sum_{n=0}^{\infty} p_{-6}\left(5^j n + \frac{3\times 5^j+1}{4}\right) q^n = \frac{1}{q^7 E_{25}^6} \sum_{l=1}^{\infty} b(j,l) T^l \zeta^{-(6l+6)}.$$

Picking out those terms of the form q^{5n+3} and applying Lemma 2.2, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-6}\left(5^j(5n+3) + \frac{3\times 5^j+1}{4}\right) q^{5n+3} &= \frac{1}{q^7 E_{25}^6} \sum_{l=1}^{\infty} b(j,l) T^l H\left(\zeta^{-(6l+6)}\right) \\ &= \frac{1}{q^7 E_{25}^6} \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} b(j,l) m(6l+6,k) T^{l-k}.\end{aligned}$$

By Lemma 2.1, we know that $m(6l+6,k) \neq 0$ implies $k \geq l+2$. Dividing both sides by q^3 and replacing q^5 by q, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-6}\left(5^{j+1} n + \frac{3\times 5^{j+1}+1}{4}\right) q^n \\ =&\frac{1}{q^2 E_5^6} \sum_{l=1}^{\infty}\sum_{k=l+2}^{\infty} b(j,l) m(6l+6,k) \left(q\frac{E_5^6}{E_1^6}\right)^{k-l} \qquad \text{(replace } k \text{ by } k+l+1) \\ =&\frac{1}{q^2 E_5^6} \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} b(j,l) m(6l+6,k+l+1) \left(q\frac{E_5^6}{E_1^6}\right)^{k+1} \\ =&\sum_{k=1}^{\infty} b(j+1,k) q^{k-1} \frac{E_5^{6l}}{E_1^{6l+6}}.\end{aligned}$$

This implies that (4.9) holds for $j+1$. Thus we complete the proof by induction. □

Lemma 4.5. *For any positive integers $j \geq 1$ and $k \geq 1$, we have*

$$\pi_5\left(b(j,k)\right) \geq j + \left\lfloor \frac{5k-5}{2} \right\rfloor. \tag{4.10}$$

Proof. It is obvious that (4.10) holds for $j=1$. Assume (4.10) holds for some $j \geq 1$, then by Lemma 4.2 we get

$$\begin{aligned}\pi\left(b(j+1,k)\right) &= \pi\left(\sum_{i=1}^{\infty} b(j,i)m(6i+6,k+i+1)\right)\\ &\geq \min_{i\geq 1}\Big(\pi\big(b(j,i)\big)+\pi\big(m(6i+6,k+i+1)\big)\Big)\\ &\geq j+\left\lfloor\frac{5i-5}{2}\right\rfloor+\left\lfloor\frac{5k-i-2}{2}\right\rfloor\\ &\geq j+1+\left\lfloor\frac{5k-5}{2}\right\rfloor.\end{aligned}$$

Hence, (4.10) holds for $j+1$ and therefore for all integers $j \geq 1$ by induction. □

It follows easily from (4.10) that

$$\pi_5\left(b(j,k)\right) \geq j+\left\lfloor\frac{5k-5}{2}\right\rfloor \geq j+2$$

for $k \geq 2$.

By (4.9), we get, modulo 5^{j+1},

$$\sum_{n=0}^{\infty} p_{-6}\left(5^j n+\frac{3\times 5^j+1}{4}\right)q^n \equiv b(j,1)\frac{E_5^6}{E_1^{12}} \equiv b(j,1)E_5^4\sum_{n=0}^{\infty} p_{-2}(n)q^n. \tag{4.11}$$

Since $\pi_5(b(j,1)) \geq j$, the congruences (1.10) and (1.11) follow from (4.11) together with (1.3).

4.3. Congruences for $p_{-7}(n)$ modulo powers of 5

This case is similar to the case $k=2$, we present here the main results and omit their proofs. Let

1) $c(1,1)=140$, $a(1,2)=49\times 5^4$, $c(1,3)=21\times 5^7$, $c(1,4)=91\times 5^8$, $c(1,5)=7\times 5^{11}$, $c(1,6)=5^{13}$ and $c(1,k)=0$ for $k\geq 7$.
2)

$$c(j+1,k)=\begin{cases}\sum_{i=1}^{\infty} c(j,i)m(6i+6,i+k+1) & \text{if } j \text{ is odd},\\ \sum_{i=1}^{\infty} c(j,i)m(6i+7,i+k+1) & \text{if } j \text{ is even}.\end{cases}$$

Theorem 4.6. *For any positive integer j, we have*

$$\sum_{n=0}^{\infty} p_{-7}\left(5^{2j-1}+\frac{13\times 5^{2j-1}+7}{24}\right)q^n = \sum_{l=1}^{\infty} c(2j-1,l)q^{l-1}\frac{E_5^{6l-1}}{E_1^{6l+6}}, \tag{4.12}$$

$$\sum_{n=0}^{\infty} p_{-7}\left(5^{2j}+\frac{17\times 5^{2j}+7}{24}\right)q^n = \sum_{l=1}^{\infty} c(2j,l)q^{l-1}\frac{E_5^{6l}}{E_1^{6l+7}}.$$

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Lemma 4.7. *For any positive integers $j \geq 1$ and $k \geq 1$, we have*

$$\pi_5(c(2j-1,k)) \geq j + \left\lfloor \frac{5k-5}{2} \right\rfloor, \tag{4.13}$$

$$\pi_5(c(2j,k)) \geq j + \left\lfloor \frac{5k-3}{2} \right\rfloor.$$

It follows immediately from (4.13) that

$$\pi_5(c(2j-1,k)) \geq j + \left\lfloor \frac{5k-5}{2} \right\rfloor \geq j+2$$

for $k \geq 2$.

According to (4.12), we have, modulo 5^{j+1},

$$\sum_{n=0}^{\infty} p_{-7}\left(5^{2j-1} + \frac{13 \times 5^{2j-1}+7}{24}\right) q^n \equiv c(2j-1,1)\frac{E_5^5}{E_1^{12}} \equiv c(2j-1,1)E_5^3 \sum_{n=0}^{\infty} p_{-2}(n)q^n. \tag{4.14}$$

Eqs. (1.12)–(1.15) are immediate consequence of (4.13), (4.14) and (1.3).

5. Final remarks

A number of congruences satisfied by k-colored partitions have been found (see [2,3, 6,10,14,16], to name a few). For example, Atkin [3] proved the following infinite families of congruences modulo powers of prime.

Theorem 5.1 *(Theorem 1.1, [3]). Suppose $k > 0$ and $q = 2, 3, 5, 7$ or 13. If $24n \equiv k \pmod{q^r}$, then $p_{-k}(n) \equiv 0 \pmod{q^{\frac{1}{2}\alpha r+\epsilon}}$, where $\epsilon = \epsilon(q,k) = O(\log k)$, and where α depends on q and the residue of k modulo 24 according to a certain table.*

Applying the operator H, we also obtain some infinite families of congruences modulo powers of 5 for $k = 11$ by following the same line of proving Theorems 1.2 and 1.3. However, for $k = 17$, it seems that there do not exist congruences modulo powers of 5 similar to the types of (1.7)–(1.11). Interestingly, Atkin's results also assert that $\alpha = 0$ when $k \equiv 17 \pmod{24}$ and $q = 5$.

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References

[1] G.E. Andrews, in: G.-C. Rota (Ed.), The Theory of Partitions, in: Encyclopedia of Mathematics and Its Applications, vol. 2, Addison–Wesley, Reading, 1976 (Reprinted: Cambridge Univ. Press, London and New York, 1984).
[2] G.E. Andrews, A survey of multipartitions: congruences and identities, in: Surveys in Number Theory, Springer, New York, 2008, pp. 1–19.
[3] A.O.L. Atkin, Ramanujan congruences for $p_{-k}(n)$, Canad. J. Math. 20 (1968) 67–78.
[4] N.D. Baruah, B.K. Sarmah, Identities and congruences for general partition and Ramanujan's tau functions, Indian J. Pure Appl. Math. 44 (2013) 643–671.
[5] W.Y.C. Chen, D.K. Du, Q.H. Hou, L.H. Sun, Congruences of multipartition functions modulo powers of primes, Ramanujan J. 35 (2014) 1–19.
[6] A. Folsom, Z. Kent, K. Ono, ℓ-adic properties of the partition function, Adv. Math. 229 (3) (2012) 1586–1609.
[7] S. Fu, D. Tang, Multiranks and classical theta functions, Int. J. Number Theory (2017), https://doi.org/10.1142/S1793042118500343, in press.
[8] S. Fu, D. Tang, On a generalized crank for k-colored partitions, J. Number Theory 184C (2018) 485–497.
[9] F.G. Garvan, A simple proof of Waston's partition congruences for powers of 7, J. Aust. Math. Soc. 36 (1984) 316–334.
[10] B. Gordon, Ramanujan congruences for $p_{-k} \pmod{11^r}$, Glas. J. Math. 24 (2) (1981) 107–123.
[11] P. Hammond, R. Lewis, Congruences in ordered pairs of partitions, Int. J. Math. Math. Sci. (2004) 45–48, 2509–2512.
[12] M.D. Hirschhorn, The Power of q, Developments in Mathematics, vol. 49, Springer, 2017.
[13] M.D. Hirschhorn, D.C. Hunt, A simple proof of the Ramanujan conjecture for powers of 5, J. Reine Angew. Math. 326 (1981) 1–17.
[14] I. Kiming, J.B. Olsson, Congruences like Ramanujan's for powers of the partition function, Arch. Math. 59 (4) (1992) 348–360.
[15] O. Lazarev, M.S. Mizuhara, B. Reid, H. Swisher, Extension of a proof of the Ramanujan congruences for multipartitions, Ramanujan J. (2017), https://doi.org/10.1007/s11139-016-9817-x, in press.
[16] M. Locus, I. Wagner, Congruences for powers of the partition function, Ann. Comb. 21 (2017) 83–93.
[17] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, in: Paper 25 of Collected Papers of S. Ramanujan, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962, reprinted by the American Mathematical Society, Providence, RI, 2000.
[18] G.N. Watson, Ramanujans Vermutung über Zerfallungsanzahlen, J. Reine Angew. Math. 179 (1938) 97–128.

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