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General Section

A generalization of a theorem of Hecke for $SL_2(\mathbb{F}_p)$ to fundamental discriminants

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ABSTRACT

Let $p > 3$ be an odd prime, $p \equiv 3 \bmod 4$ and let π^+, π^- be the pair of cuspidal representations of $SL_2(\mathbb{F}_p)$. It is well known by Hecke that the difference $m_{\pi^+} - m_{\pi^-}$ in the multiplicities of these two irreducible representations occurring in the space of weight 2 cusps forms with respect to the principal congruence subgroup $\Gamma(p)$, equals the class number $h(-p)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. We extend this result to all fundamental discriminants $-D$ of imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ and prove that an alternating sum of multiplicities of certain irreducibles of $SL_2(\mathbb{Z}/D\mathbb{Z})$ is an explicit multiple, up to a sign and a power of 2, of either the class number $h(-D)$ or of the sums $h(-D) + h(-D/2)$, $h(-D) + 2h(-D/2)$; the last two possibilities occur in some of the cases when $D \equiv 0 \bmod 8$. The proof uses the holomorphic Lefschetz number.

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0. Introduction

For $p > 3$ odd prime, there is, up to twist equivalence, a unique irreducible cuspidal representation π of $GL_2(\mathbb{F}_p)$ of dimension $p-1$, which, when restricted to $SL_2(\mathbb{F}_p)$, splits into a pair of irreducible representations π^+, π^- of the same dimension. The group $SL_2(\mathbb{F}_p)$ acts naturally on the space $\mathcal{S}_2(\Gamma(p))$ of cusp forms of weight 2 with respect to the principal congruence subgroup $\Gamma(p)$. One might think that π^+, π^- occur with the same multiplicity in $\mathcal{S}_2(\Gamma(p))$. Indeed, this holds true when $p \equiv 1 \bmod 4$. However, as Hecke showed in [4], the two cuspidal irreducible representations π^+, π^- of $SL_2(\mathbb{F}_p)$ have different multiplicities when $p \equiv 3 \bmod 4$. One could say that this was a precursor to the modern theory of L-indistinguishability [6]. Furthermore, Hecke showed that in this case, the difference in multiplicities $m_{\pi^+} - m_{\pi^-}$, is exactly $h(-p)$, the class number of $\mathbb{Q}(\sqrt{-p})$. Note that there is exactly one more, up to twist equivalence, irreducible representation τ of $GL_2(\mathbb{F}_p)$ that also splits into two irreducible representations τ^+, τ^- of $SL_2(\mathbb{F}_p)$, upon restriction. In this case, τ is in the principal series and $m_{\tau^+} = m_{\tau^-}$ for all odd p.

The purpose of this work is to extend Hecke's result to all fundamental discriminants $-D$, $D > 3$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ and to provide an alternate geometric proof even for the case when D is a prime $p \equiv 3 \bmod 4$. We write the fundamental discriminant $-D$ of K as $-D_0 p_1 \cdots p_t$ with $t \geq 0$ (the product $p_1 \cdots p_t$ being 1 if $t = 0$), $D_0 \in \{p_0, 4, 8\}$ and $p_0, p_1, \cdots, p_t$ distinct odd primes such that the typical fundamental discriminant congruences are satisfied. We consider certain distinctive irreducible representations of $G = SL_2(\mathbb{Z}/D\mathbb{Z})$ described in Section 1; for the moment, it suffices to say that these representations are (up to isomorphism) partitioned along tuples of the shape $(\epsilon_0, \cdots, \epsilon_t | e)$, where $\epsilon_i \in \{\pm\}$ and $e \in \{\pm 1\}$. Note that these distinctive representations agree with Hecke's representations in the case $D = p > 3$. Let $\mathcal{S}_2(\Gamma(D))$ be the space of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$. The natural action of G on $\mathcal{S}_2(\Gamma(D))$ gives a G-representation, which we denote by $(\rho, \mathcal{S}_2(\Gamma(D)))$. Let m_π be the multiplicity of a distinctive representation π in ρ and consider the alternating sum of multiplicities over distinctive G-representations of type $(\epsilon_0, \cdots, \epsilon_t | e)$

$$\Delta M_{t,e} = \sum_{(\epsilon_0, \cdots, \epsilon_t | e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi \in (\epsilon_0, \cdots, \epsilon_t | e)} m_\pi. \tag{1}$$

Note that by $\prod_{i=0}^{t} \epsilon_i$, we clearly mean the product of ± 1 when ϵ_i takes values in $\{\pm\}$. The main result we prove is as follows:

Theorem. *For $D > 3$, let $G = SL_2(\mathbb{Z}/D\mathbb{Z})$, where $-D$ is a fundamental discriminant associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. We may write D as $D_0 p_1 \cdots p_t$, with $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$. Consider the expression $\Delta M_{t,e}$ introduced in (1) above and* *let* $\Delta M_t = \Delta M_{t,1}$. *Then the following identity relating* ΔM_t *and the class number* $h(-D)$ *of* $\mathbb{Q}(\sqrt{-D})$ *holds*

$$\Delta M_t = \begin{cases} 0, & \text{if } D_0 = 4, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + h(-D/2)\big], & \text{if } D_0 = 8, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t h(-D), & \text{if } D_0 = p_0; D_0 = 4, t \geq 1; \\ & D_0 = 8, t \geq 1, p_1 \cdots p_t \equiv 3 \bmod 4 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + 2h(-D/2)\big], & \text{if } D_0 = 8, t \geq 1, p_1 \cdots p_t \equiv 1 \bmod 4, \end{cases}$$

where $\operatorname{sgn}_{D_0,t} \in \{\pm 1\}$ *is given by*

$$\operatorname{sgn}_{D_0,t} = \begin{cases} 1, & \text{if } D_0 = p_0, t = 0; D_0 = 4, t = 1; \\ & D_0 = 8, t \in \{0,1\} \\ \prod\limits_{0 \leq i < j \leq t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } D_0 = p_0, t \geq 1 \\ \prod\limits_{1 \leq i < j \leq t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } D_0 \in \{4, 8\}, t > 1. \end{cases}$$

Note that in the case when $D > 3$ is an odd prime $p \equiv 3 \bmod 4$, we have $\Delta M_0 = h(-p)$, so our result matches Hecke's original theorem. Towards the end of our work, we came to learn that this extension of Hecke's result has already been proved *up to a sign* for the case of *odd* discriminants D in a paper of McQuillan [7], though by a different method. We hope our result is still of some interest for two reasons. First, the even case is more subtle. Second, our method also makes explicit the sign $\operatorname{sgn}_{D_0,t}$, which was previously unknown even in the odd case.

We prove the main theorem by a geometric argument using the holomorphic Lefschetz number. The structure of the paper is as follows. The first section sets the necessary notation in introducing the desired distinctive irreducible representations of $G = SL_2(\mathbb{Z}/D\mathbb{Z})$ we are considering for the alternating sum ΔM_t of multiplicities of these representations into the space $\mathcal{S}_2(\Gamma(D))$. The general idea of the proof consists of computing the characters $\Delta\chi_{t,e}$, $\chi_{\mathcal{S}_2(\Gamma(D))}$ and then comparing the resulting expression for ΔM_t with an analytic formula for the class number. In Section 2, we find the values of the virtual character $\Delta\chi_{t,e}$. Since G acts on the modular curve $X(D) = \Gamma(D)\backslash\mathcal{H}^*$, we view $g : X(D) \to X(D)$ as a map on a one-dimensional compact complex manifold for all $g \in G$. We compute the fixed points of g on $X(D)$ in Section 3, which allows us to compute the holomorphic Lefschetz number of the map g in Section 4. As the Lefschetz numbers give us the characters $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$, we have all the ingredients to compute the alternating sum ΔM_t, which is done in the final two sections of the paper for both D_0 odd and even.

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We note that recently we have come to learn of various analogues of Hecke's result in different contexts; for instance, there is an extension for Maass cusp forms in an article by J. Stopple [10].

1. Ingredients of the main theorem

Let $G = SL_2(\mathbb{Z}/D\mathbb{Z})$, where $-D$ is a fundamental discriminant associated to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. We have the following possibilities for D:

$$D = \begin{cases} p_0 \prod_{i=1}^{t} p_i, & \text{with } p_0, p_i \text{ distinct odd primes s.t. } p_0 \prod_{i=1}^{t} p_i \equiv 3 \bmod 4, t \geq 0 \\ 4 \prod_{i=1}^{t} p_i, & \text{with } p_i \text{ distinct odd primes s.t. } \prod_{i=1}^{t} p_i \equiv 1 \bmod 4, t \geq 0 \\ 8 \prod_{i=1}^{t} p_i, & \text{with } p_i \text{ distinct odd primes, } t \geq 0. \end{cases}$$

Thus, we can let $D = D_0 \prod_{i=1}^{t} p_i$, where $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$ and the primes satisfy the above congruences. The object of interest of the paper is an expression for ΔM_t in terms of the class number $h(-D)$ of K. In the following, we first introduce the distinctive G-representations of type $(\epsilon_0, \cdots, \epsilon_t | e)$ that appear in the alternating sum $\Delta M_{t,e}$, as seen in (1).

Since $G \cong SL_2(\mathbb{Z}/D_0\mathbb{Z}) \times SL_2(\mathbb{F}_{p_1}) \times \cdots \times SL_2(\mathbb{F}_{p_t})$, all complex irreducible representations of G arise from the irreducible representations of $SL_2(\mathbb{Z}/D_0\mathbb{Z})$ and $SL_2(\mathbb{F}_{p_i})$, $i \in [1, t]$ an integer. Thus an irreducible representation of G can be written as $\pi = \otimes_i \pi_i = (\pi_0, \pi_1, \cdots, \pi_t)$, where π_0 is an irreducible of $SL_2(\mathbb{Z}/D_0\mathbb{Z})$ and π_i is an irreducible of $SL_2(\mathbb{F}_{p_i})$ for $i \in [1, t]$. If we denote by χ_π the character of π, we have $\chi_\pi = \chi_0 \prod_{i=1}^{t} \chi_i$, where χ_i is the character of π_i.

In order to describe a distinctive G-representation $\pi = (\pi_0, \pi_1, \cdots, \pi_t)$, we need to introduce the types of representations π_i that compose it. We are interested in irreducible representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $GL_2(\mathbb{F}_p)$ for p odd prime, that split into two irreducibles when restricted to $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $SL_2(\mathbb{F}_p)$. Let π_0, respectively π_i, be one of the two irreducible representations of $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $SL_2(\mathbb{F}_{p_i})$ that appear as constituents of this restriction from GL_2 to SL_2. We then call such a representation $\pi = (\pi_0, \pi_1, \cdots \pi_t)$ a *distinctive representation* of G. As we will see later, there are $4 \cdot 2^{2t}$ such distinctive representations if $D_0 \in \{p_0, 4\}$ and $20 \cdot 2^{2t}$ of them in the case $D_0 = 8$.

In view of the product representation of $SL_2(\mathbb{Z}/D\mathbb{Z})$, it suffices to describe what representations π_i can occur in a distinctive G-representation for the two basic cases, namely $SL_2(F_p)$, when p is an odd prime, and $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, when $D_0 \in \{4, 8\}$; we accomplish this in the following subsections.

1.1. *p odd prime case*

The case of p odd is well known, see [8], Chapters 1, 2, p. 1–48 or [2], Chapter 5, Section 5.2, p. 67–73, for example. There are two types of representations that appear, the ones *induced* from the Borel subgroup of upper triangular matrices $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, c \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}$ and the *cuspidal* ones. For the first type, if θ, ϕ are two characters of $\mathbb{F}_p^\times$, then we can define a character of B by $\tau \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \theta(a)\phi(c)$. The induced representation to $GL_2(\mathbb{F}_p)$ $\tau_{\theta,\phi} = \mathrm{Ind}_B^{GL_2(\mathbb{F}_p)} \tau$ will be irreducible of dimension $p+1$ iff $\theta \neq \phi$; we have $\tau_{\theta,\phi} \cong \tau_{\phi,\theta}$. Thus there are $\frac{1}{2}(p-1)(p-2)$ such representations. For α a character of $\mathbb{F}_p^\times$, we consider the characters of $GL_2(\mathbb{F}_p)$ given by the determinant function, $\chi_\alpha(g) = \alpha(\det g)$, which are trivial when restricted to $SL_2(\mathbb{F}_p)$. Since $\tau_{\theta,\phi} \otimes \chi_\alpha \cong \tau_{\theta\alpha,\phi\alpha}$, the induced representations above can be considered up to a twist equivalence. There is, up to twist equivalence, a unique irreducible induced representation $\tau_{\theta,1}$ that when restricted to $SL_2(\mathbb{F}_p)$ splits into two irreducibles τ^+, τ^- of the same dimension; the representation $\tau_{\theta,1}$ is given by the unique nontrivial *quadratic* character θ of $\mathbb{F}_p^\times$. We refer to the pair of representations τ^+, τ^- as irreducibles of $SL_2(\mathbb{F}_p)$ induced from the Borel subgroup.

On the other hand, the cuspidal representations of $GL_2(\mathbb{F}_p)$ are those that do not appear in a representation induced from the Borel subgroup. They are associated to characters λ of the cyclic group $\mathbb{F}_{p^2}^\times$ that do not come from characters of $\mathbb{F}_p^\times$, that is characters λ for which there exists no character μ of $\mathbb{F}_p^\times$ such that $\lambda(x) = \mu(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x))$ for all $x \in \mathbb{F}_{p^2}^\times$. For each such character λ, there is a corresponding irreducible cuspidal representation π_λ such that $\pi_\lambda \cong \pi_{\lambda'}$ iff $\lambda' = \lambda$ or $\lambda' = \lambda^p$. There are $\frac{1}{2}p(p-1)$ such irreducibles. Any character α of $\mathbb{F}_p^\times$ can be extended to a character λ_α of $\mathbb{F}_{p^2}^\times$ by $\lambda_\alpha(x) = \alpha(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x))$. We then have $\pi_\lambda \otimes \lambda_\alpha \cong \pi_{\lambda\lambda_\alpha}$, so we may partition the cuspidal representations π_λ according to twist equivalence. The restriction of π_λ to $SL_2(\mathbb{F}_p)$ depends only on the restriction of λ to the cyclic subgroup of order $p+1$ containing elements in $\mathbb{F}_{p^2}^\times$ of norm 1. There is, up to twist equivalence, a unique irreducible cuspidal representation π_λ that when restricted to $SL_2(\mathbb{F}_p)$ splits into a pair of two irreducibles π^+, π^- of the same dimension; the representation π_λ is given by the unique nontrivial *quadratic* character λ of order $p+1$. We refer to the pair of representations π^+, π^- as cuspidal irreducibles of $SL_2(\mathbb{F}_p)$.

We denote the above irreducible representations that can appear as components of a distinctive G-representation $\pi = (\pi_0, \pi_1, \cdots \pi_t)$ by π_δ^ϵ, where $\delta = +1$ if the representation is induced, $\delta = -1$ if it is cuspidal, $\epsilon \in \{\pm\}$. The characters of these representations take

Table 1
Characters of distinctive $SL_2(\mathbb{F}_p)$-representations.

$SL_2(\mathbb{F}_p)$	$\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\pm\begin{pmatrix}1&\eta_p\\0&1\end{pmatrix}$	a^l	b^m
χ_{-1}^{+}	$\lambda_p(\pm1)(p-1)/2$	$\lambda_p(\pm1)(-1+G_p)/2$	$\lambda_p(\pm1)(-1-G_p)/2$	0	$-\lambda_p(b)^m$
χ_{-1}^{-}	$\lambda_p(\pm1)(p-1)/2$	$\lambda_p(\pm1)(-1-G_p)/2$	$\lambda_p(\pm1)(-1+G_p)/2$	0	$-\lambda_p(b)^m$
χ_{+1}^{+}	$\theta_p(\pm1)(p+1)/2$	$\theta_p(\pm1)(1+G_p)/2$	$\theta_p(\pm1)(1-G_p)/2$	$\theta_p(\nu)^l$	0
χ_{+1}^{-}	$\theta_p(\pm1)(p+1)/2$	$\theta_p(\pm1)(1-G_p)/2$	$\theta_p(\pm1)(1+G_p)/2$	$\theta_p(\nu)^l$	0

Table 2
Characters of distinctive $SL_2(\mathbb{Z}/4\mathbb{Z})$-representations.

$SL_2(\mathbb{Z}/4\mathbb{Z})$	1 $\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$	6 $\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}$	3 $\pm\begin{pmatrix}1&2\\0&1\end{pmatrix}$	6 $\pm\begin{pmatrix}0&3\\1&0\end{pmatrix}$	8 $\pm\begin{pmatrix}1&1\\3&0\end{pmatrix}$
χ_1^{+}	±1	$\pm i$	$\pm(-1)$	$\pm i$	$\pm(-1)$
χ_1^{-}	±1	$\pm(-i)$	$\pm(-1)$	$\pm(-i)$	$\pm(-1)$
χ_3^{+}	±3	$\pm i$	±1	$\pm(-i)$	0
χ_3^{-}	±3	$\pm(-i)$	±1	$\pm i$	0

the value $(\delta+\epsilon G_p)/2$ on u_1, where $u_x=\begin{pmatrix}1&x\\0&1\end{pmatrix}$, as seen for example in [5], Section 7, p. 30. Clearly, $\epsilon G_p=\pm G_p$ depending on whether ϵ is $+$ or $-$ (Table 1).

Here η_p is a non-square mod p, $a=\begin{pmatrix}\nu&0\\0&\nu^{-1}\end{pmatrix}$, where ν generates the multiplicative group of $\mathbb{F}_p$, b is an element of order $p+1$ which is not diagonalizable over $\mathbb{F}_p$, $l\in[1,(p-3)/2]$, $m\in[1,(p-1)/2]$ integers. Also λ_p,θ_p are the unique nontrivial quadratic characters of cyclic groups of order $p+1$, respectively $p-1$, and G_p is the Gauss sum given by $\sum_{x\in\mathbb{F}_p^\times}\theta_p(x)\xi^x$, where $\xi=\exp(2\pi i/p)$. It is well known that $G_p=\sqrt{\theta_p(-1)p}$. In terms of notation, if $D_0=p_0$, π_0 will be of the type $\pi_{0,\delta_0}^{\epsilon_0}$ above, while π_i will be of type $\pi_{i,\delta_i}^{\epsilon_i}$, where $\epsilon_0,\epsilon_i\in\{\pm\}$, $\delta_0,\delta_i\in\{\pm1\}$, $i\in[1,t]$ an integer.

1.2. $D_0=4$ case

Recall that for the case D_0 even, we look at representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$ that split into two irreducibles of the same dimension when restricted to $SL_2(\mathbb{Z}/D_0\mathbb{Z})$. In the case $D_0=4$, there are 2 such representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$. Therefore, there are two pairs of representations that appear in a distinctive G-representation: a pair of one-dimensional representations and another pair of representations of dimension 3. We denote them by $\pi_{0,\delta_0}^{\epsilon_0}$, with $\epsilon_0\in\{\pm\},\delta_0\in\{1,3\}$, their characters appearing in Table 2.

1.3. $D_0=8$ case

The instance of $D_0=8$ is presented in Table 3, where $\alpha=\xi_8+\xi_8^3$, $\beta=\xi_8-\xi_8^3$ with $\xi_8=\exp(2\pi i/8)$.

Table 3
Characters of distinctive $SL_2(\mathbb{Z}/8\mathbb{Z})$-representations on the conjugacy classes of interest.

$SL_2(\mathbb{Z}/8\mathbb{Z})$	1 $\pm u_0$	12 $\pm u_1$	12 $\pm u_3$	12 $\pm u_5$	12 $\pm u_7$	24 $\pm a_0$	24 $\pm a_4$	6 $\pm u_2$	6 $\pm u_6$
$\chi^+_{1,1}$	± 1	$\pm i$	$\pm(-i)$	$\pm i$	$\pm(-i)$	$\pm(-i)$	$\pm(-i)$	$\pm(-1)$	$\pm(-1)$
$\chi^-_{1,1}$	± 1	$\pm(-i)$	$\pm i$	$\pm(-i)$	$\pm i$	$\pm i$	$\pm i$	$\pm(-1)$	$\pm(-1)$
$\chi^+_{2,1}$	2	α	α	$-\alpha$	$-\alpha$	0	0	0	0
$\chi^-_{2,1}$	2	$-\alpha$	$-\alpha$	α	α	0	0	0	0
$\chi^+_{2,2}$	± 2	$\pm\beta$	$\pm(-\beta)$	$\pm(-\beta)$	$\pm\beta$	0	0	0	0
$\chi^-_{2,2}$	± 2	$\pm(-\beta)$	$\pm\beta$	$\pm\beta$	$\pm(-\beta)$	0	0	0	0
$\chi^+_{3,1}$	± 3	$\pm i$	$\pm(-i)$	$\pm i$	$\pm(-i)$	$\pm i$	$\pm i$	± 1	± 1
$\chi^-_{3,1}$	± 3	$\pm(-i)$	$\pm i$	$\pm(-i)$	$\pm i$	$\pm(-i)$	$\pm(-i)$	± 1	± 1
$\chi^+_{3,2}$	3	i	$-i$	i	$-i$	-1	1	$-1+2i$	$-1-2i$
$\chi^-_{3,2}$	3	$-i$	i	$-i$	i	-1	1	$-1-2i$	$-1+2i$
$\chi^+_{3,3}$	3	i	$-i$	i	$-i$	1	-1	$-1-2i$	$-1+2i$
$\chi^-_{3,3}$	3	$-i$	i	$-i$	i	1	-1	$-1+2i$	$-1-2i$
$\chi^+_{3,4}$	± 3	$\pm(-1)$	$\pm(-1)$	$\pm(-1)$	$\pm(-1)$	$\pm(-i)$	$\pm i$	$\pm(1+2i)$	$\pm(1-2i)$
$\chi^-_{3,4}$	± 3	$\pm(-1)$	$\pm(-1)$	$\pm(-1)$	$\pm(-1)$	$\pm i$	$\pm(-i)$	$\pm(1-2i)$	$\pm(1+2i)$
$\chi^+_{3,5}$	± 3	± 1	± 1	± 1	± 1	$\pm(-i)$	$\pm i$	$\pm(1-2i)$	$\pm(1+2i)$
$\chi^-_{3,5}$	± 3	± 1	± 1	± 1	± 1	$\pm i$	$\pm(-i)$	$\pm(1+2i)$	$\pm(1-2i)$
$\chi^+_{6,1}$	6	β	$-\beta$	$-\beta$	β	0	0	0	0
$\chi^-_{6,1}$	6	$-\beta$	β	β	$-\beta$	0	0	0	0
$\chi^+_{6,2}$	± 6	$\pm\alpha$	$\pm\alpha$	$\pm(-\alpha)$	$\pm(-\alpha)$	0	0	0	0
$\chi^-_{6,2}$	± 6	$\pm(-\alpha)$	$\pm(-\alpha)$	$\pm\alpha$	$\pm\alpha$	0	0	0	0

In this case, there are 10 pairs of representations that appear in a distinctive G-representation. There is a pair of dimension 1, 2 pairs of dimension 2, 5 of dimension 3 and 2 of dimension 6. We denote them by $\pi^{\epsilon_0}_{0,\delta_0}$ with $\epsilon_0 \in \{\pm\}$, $\delta_0 \in \{(1,1),(2,1),(2,2),(3,1),(3,2),\cdots,(3,5),(6,1),(6,2)\}$. As we shall see in Lemma 3 below, we are only interested in the values of these characters on the conjugacy classes that take different values on π^+_{0,δ_0} and π^-_{0,δ_0}. As a result, the conjugacy classes that are of interest are represented by $\pm u_x$ with $x \in \{0,1,3,5,7,2,6\}$, where $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and by $\pm\begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}$ and $\pm\begin{pmatrix} 0 & 1 \\ 7 & 4 \end{pmatrix}$; we denoted the last 4 representatives by $\pm a_0$, respectively $\pm a_4$. The characters of the representations appearing in a distinctive G-representation on these conjugacy classes are as in Table 3.

Going back to the general setting, let π be a distinctive G-representation given by $(\pi^{\epsilon_0}_{0,\delta_0},\cdots,\pi^{\epsilon_t}_{t,\delta_t})$; note that $\pi^{\epsilon_i}_{i,\delta_i}$ can be either cuspidal or induced for $i\in[1,t]$ an integer, while the possible candidates for $\pi^{\epsilon_0}_{0,\delta_0}$ are those whose characters are given in the above tables. We call such a representation $\pi = (\pi^{\epsilon_0}_{0,\delta_0},\cdots,\pi^{\epsilon_t}_{t,\delta_t})$ of type $(\epsilon_0,\cdots,\epsilon_t)$. Since the action of $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathrm{Id}$ depends on $\chi^\epsilon_\delta(-\mathrm{Id}) = (-1)^{(p-\delta)/2}(p+\delta)/2$, for π_δ^ϵ either cuspidal or induced irreducible of $SL_2(\mathbb{F}_p)$, we get that $-\mathrm{Id}$ acts as $\mathrm{sgn}(\chi_0(-\mathrm{Id}))\prod_{i=1}^{t}(-1)^{(p_i-\delta_i)/2}\mathrm{Id}$. We say a distinctive representation of G is of type $(\epsilon_0,\cdots,\epsilon_t|e)$ if it is of type $(\epsilon_0,\cdots,\epsilon_t)$ and $\mathrm{sgn}(\chi_0(-\mathrm{Id}))\prod_{i=1}^{t}(-1)^{(p_i-\delta_i)/2}=e$, where $e\in\{\pm 1\}$.

As we saw in (1), we consider the following alternating sum over distinctive G-representations of type $(\epsilon_0,\cdots,\epsilon_t|e)$:

$$\Delta M_{t,e}=\sum_{(\epsilon_0,\cdots,\epsilon_t|e)}\prod_{i=0}^{t}\epsilon_i\sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)}m_\pi,$$

where m_π is the multiplicity of the representation π of type $(\epsilon_0,\cdots,\epsilon_t|e)$ in the G-representation ρ on the space $\mathcal{S}_2(\Gamma(D))$ of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$. If we let $\chi_{\mathcal{S}_2(\Gamma(D))}$ to be the character of $(\rho,\mathcal{S}_2(\Gamma(D)))$, we can rewrite $\Delta M_{t,e}$ as

$$\begin{aligned}\Delta M_{t,e}&=\sum_{(\epsilon_0,\cdots,\epsilon_t|e)}\prod_{i=0}^{t}\epsilon_i\sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)}\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)}\\&=\frac{1}{|G|}\sum_{g\in G}\Delta\chi_{t,e}(g)\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)},\end{aligned}\tag{2}$$

where the alternating sum of characters $\sum_{(\epsilon_0,\cdots,\epsilon_t|e)}\prod_{i=0}^{t}\epsilon_i\sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)}\chi_\pi$ was denoted by $\Delta\chi_{t,e}$.

After computing the values of $\Delta\chi_{t,e}$ and $\chi_{\mathcal{S}_2(\Gamma(D))}$ on the conjugacy classes of G, which will be done in Sections 2, 4 respectively, we get an analytic expression for ΔM_t. The goal is to rewrite this expression as a multiple involving $h(-D)$, which will be done by using the following modified version of the Dirichlet class number formula:

Lemma 1. *Let $D>4$ such that $-D$ is the fundamental discriminant associated to the imaginary quadratic field $K=\mathbb{Q}(\sqrt{-D})$. Then the class number of K is given by*

$$h(-D)=\begin{cases}-\frac{1}{D}\displaystyle\sum_{n=1}^{D-1}n\left(\frac{n}{D}\right), & \text{if } D_0 \text{ is odd}\\ -\frac{2}{D}\displaystyle\sum_{\substack{n=1,\\ n\equiv p_1\cdots p_t \bmod 4}}^{D-1}n\left(\frac{n}{D}\right), & \text{if } D_0 \text{ is even}\end{cases}$$

where D is written as $D_0p_1\cdots p_t$, with $D_0\in\{p_0,4,8\}$, $t\geq 0$ and p_0,p_i are distinct odd primes, $i\in[1,t]$ an integer.

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Proof. By the Dirichlet class number formula (cf. [1], Chapter 6, p. 49–50)

$$h(-D) = \frac{w}{2\pi}\sqrt{D}L(\chi, 1),$$

where w is the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$, and χ is the quadratic character of $\mathbb{Q}(\sqrt{-D})$, $\chi : \mathbb{Z}^+ \to \mathbb{C}^\times$, $\chi(m) = \left(\frac{-D}{m}\right)$. Since $-D < -4, w = 2$.

For a nonzero integer m, let m' denote its odd part, that is $m = 2^s m'$ with $(m', 2^s) = 1$. By the quadratic reciprocity of the Kronecker symbol we then have $\left(\frac{-D}{m}\right) = (-1)^{\frac{(m'-1)(-D'-1)}{4}} \left(\frac{m}{D}\right)$, where D' is the odd part of D. Moreover,

$$L(\chi, 1) = -\frac{\pi}{D\sqrt{D}} \sum_{n=1}^{D-1} n\chi(n),$$

so if D_0 is even we have

$$\begin{aligned} h(-D) &= -\frac{1}{D} \sum_{n=1}^{D-1} (-1)^{\frac{(n'-1)(-D'-1)}{4}} n \left(\frac{n}{D}\right) \\ &= -\frac{1}{D} \sum_{n=1}^{D-1} (-1)^{\frac{(n-1)(-p_1\cdots p_t-1)}{4}} n \left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right) \\ &= -\frac{1}{D} \sum_{n\equiv p_1\cdots p_t} n \left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right) \\ &\quad + \left(\frac{-1}{p_1\cdots p_t}\right) \frac{1}{D} \sum_{n\equiv 3p_1\cdots p_t} n \left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right) \\ &= -\frac{2}{D} \sum_{n\equiv p_1\cdots p_t} n \left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right), \end{aligned}$$

where the congruences are taken $\bmod 4$ and the summation is over integers $n \in [1, D-1]$.

On the other hand, for odd D_0 we get

$$\begin{aligned} h(-D) &= -\frac{1}{D} \sum_{n=1}^{D-1} (-1)^{\frac{(n'-1)(-D-1)}{4}} n \left(\frac{n}{D}\right) \\ &= -\frac{1}{D} \sum_{n=1}^{D-1} n \left(\frac{n}{D}\right). \quad \square \end{aligned}$$

As a side remark, note that we are only interested in finding an expression for $\Delta M_t = \Delta M_{t,1}$ as $\Delta M_{t,-1}$ always vanishes. As we shall see in the following section, this happens because the weight of cusp forms is even and thus forces the action of $-g$ on $\mathcal{S}_2(\Gamma(D))$ to be the same as that of g.

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2. A key virtual character

Consider the alternating sum over irreducibles of G of type $(\epsilon_0, \cdots, \epsilon_t|e)$ as introduced above:

$$\Delta\chi_{t,e} = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)} \chi_\pi.$$

As seen in (2), the values $\Delta\chi_{t,e}$ takes on the conjugacy classes of G appear in the expression ΔM_t. We obtain two results, see Lemmas 2, 3 below.

Lemma 2. *Let $g = (g_0, \cdots, g_t)$ represent a conjugacy class of G, where $g_i \in SL_2(\mathbb{F}_{p_i})$, for all $i \in [1, t]$ an integer and $g_0 \in SL_2(\mathbb{Z}/D_0\mathbb{Z})$. Then*

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t-1}\Big[\Delta\chi_{0,e}(g_0) + \Delta\chi_{0,-e}(g_0)\Big] \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (g_0, u_{x_1}, \cdots, u_{x_t}) \\ 2^{t-1}\Big[\Delta\chi_{0,e}(g_0) - \Delta\chi_{0,-e}(g_0)\Big] \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (g_0, -u_{x_1}, \cdots, -u_{x_t}) \\ 0, & \text{otherwise} \end{cases}$$

for $t \geq 1$, where $x_i \in \{1, \eta_{p_i}\}$ with η_{p_i} a non-square $\bmod p_i$ *for all $i \in [1, t]$ an integer.*

Proof. We have

$$\begin{aligned}
&\Delta\chi_{t,e}(g_0, \cdots, g_t) \\
&\quad = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)} \chi_\pi(g_0, \cdots, g_t) \\
&\quad = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t-1} \epsilon_i\epsilon_t \sum_{\delta_t} \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2})} \chi_\pi(g_0, \cdots, g_{t-1})\chi^{\epsilon_t}_{t,\delta_t}(g_t) \\
&\quad = \sum_{\delta_t} \sum_{(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2})} \prod_{i=0}^{t-1} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2}))} \chi_\pi(g_0, \cdots, g_{t-1})\Big[\chi^{+}_{t,\delta_t}(g_t) - \chi^{-}_{t,\delta_t}(g_t)\Big] \\
&\quad = \sum_{\delta_t} \Delta\chi_{t-1,e(-1)^{(p_t-\delta_t)/2}}(g_0, \cdots, g_{t-1})\Big[\chi^{+}_{t,\delta_t}(g_t) - \chi^{-}_{t,\delta_t}(g_t)\Big],
\end{aligned}$$

where $\chi^{\epsilon_t}_{t,\delta_t}$ denotes the character of the component $\pi^{\epsilon_t}_{t,\delta_t}$ in the G-representation $\pi = (\pi^{\epsilon_0}_{0,\delta_0}, \cdots, \pi^{\epsilon_t}_{t,\delta_t})$ of type $(\epsilon_0, \cdots, \epsilon_t|e)$.

For odd p, we get

$$\chi_\delta^+(g) - \chi_\delta^-(g) = \begin{cases} \left(\frac{x}{p}\right) G_p, & \text{if } g = u_x \text{ for } x \in \{1, \eta\} \\ (-1)^{(p-\delta)/2} \left(\frac{x}{p}\right) G_p, & \text{if } g = -u_x \text{ for } x \in \{1, \eta\} \\ 0, & \text{otherwise} \end{cases}$$

where η is a non-square $\bmod p$. Thus, we must have $\Delta\chi_{t,e}(g_0, \cdots, g_t) = 0$ for all $g_i \neq \pm u_{x_i}$, $i \in [1, t]$ an integer. On the other hand,

$$\begin{aligned} &\Delta\chi_{t,e}(g_0, \cdots, g_{t-1}, \pm u_{x_t}) \\ &\quad = \sum_{\delta_t} \Delta\chi_{t-1,e(-1)^{(p_t-\delta_t)/2}}(g_0, \cdots, g_{t-1})[\chi_{t,\delta_t}^+(\pm u_{x_t}) - \chi_{t,\delta_t}^-(\pm u_{x_t})] \\ &\quad = \Big[\Delta\chi_{t-1,e}(g_0, \cdots, g_{t-1}) \pm \Delta\chi_{t-1,-e}(g_0, \cdots, g_{t-1})\Big] \left(\frac{x_t}{p_t}\right) G_{p_t}, \end{aligned} \tag{3}$$

so

$$\Delta\chi_{t,e}(g_0, \cdots, g_{t-1}, \pm u_{x_t}) = \pm\Delta\chi_{t,-e}(g_0, \cdots, g_{t-1}, \pm u_{x_t}). \tag{4}$$

We claim that if there exists $i, j \in \{1, \cdots, t\}$ such that $g_i = u_{x_i}$ and $g_j = -u_{x_j}$ then we must have $\Delta\chi_{t,e}(g_0, \cdots, g_t) = 0$ for $t \geq 2$. Clearly, if $\Delta\chi_{s,e}(g_0, \cdots, g_s) = 0$ for some $s \leq t$, then by (3) and (4), we get $\Delta\chi_{t,e}(g_0, \cdots, g_s, \cdots, g_t) = 0$. Thus we can assume WLOG that $i = t, j = t-1$, the case $i = t-1, j = t$ being exactly the same. Since $\Delta\chi_{t-1,e}(g_0, \cdots, g_{t-2}, -u_{x_{t-1}}) = -\Delta\chi_{t-1,-e}(g_0, \cdots, g_{t-2}, -u_{x_{t-1}})$, we get $\Delta\chi_{t,e}(g_0, \cdots, -u_{x_{t-1}}, u_{x_t}) = 0$. As a result $\Delta\chi_{t,e}(g_0, \cdots, g_t)$ is zero outside the conjugacy classes of type $(g_0, u_{x_1}, \cdots, u_{x_t})$ or $(g_0, -u_{x_1}, \cdots, -u_{x_t})$.

Then

$$\begin{aligned} \Delta\chi_{t,e}(g_0, u_{x_1}, \cdots, u_{x_t}) &= 2\Delta\chi_{t-1,e}(g_0, u_{x_1}, \cdots, u_{x_{t-1}}) \left(\frac{x_t}{p_t}\right) G_{p_t} \\ &= 2^{t-1}\Big[\Delta\chi_{0,e}(g_0) + \Delta\chi_{0,-e}(g_0)\Big] \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, \end{aligned}$$

and similarly,

$$\Delta\chi_{t,e}(g_0, -u_{x_1}, \cdots, -u_{x_t}) = 2^{t-1}[\Delta\chi_{0,e}(g_0) - \Delta\chi_{0,-e}(g_0)] \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}. \quad \square$$

On the other hand, since

$$\Delta\chi_{0,e}(g_0) = \sum_{(\epsilon_0|e)} \epsilon_0 \sum_{\pi \in (\epsilon_0|e)} \chi_\pi(g_0),$$

we can compute $\Delta\chi_{0,1}(g_0) \pm \Delta\chi_{0,-1}(g_0)$ for the different values of D_0 as follows:

D_0	$\Delta\chi_{0,1}(g_0) + \Delta\chi_{0,-1}(g_0)$	$\Delta\chi_{0,1}(g_0) - \Delta\chi_{0,-1}(g_0)$
p_0	$\begin{cases} 2\left(\frac{x_0}{p_0}\right)G_{p_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{1, \eta_{p_0}\} \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 2\left(\frac{x_0}{p_0}\right)G_{p_0}, & \text{if } g_0 = -u_{x_0} \text{ for } x_0 \in \{1, \eta_{p_0}\} \\ 0, & \text{otherwise} \end{cases}$
4	$\begin{cases} 4\xi_4^{x_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{\pm 1\} \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 4\xi_4^{x_0}, & \text{if } g_0 = -u_{x_0} \text{ for } x_0 \in \{\pm 1\} \\ 0, & \text{otherwise} \end{cases}$
8	$\begin{cases} 8\xi_8^{x_0} + 8\xi_4^{x_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{\pm 1, \pm 3\} \\ \pm(-4\xi_4), & \text{if } g_0 = \pm a_0 \\ \pm 4\xi_4, & \text{if } g_0 = \pm a_4 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 8\xi_8^{x_0} + 8\xi_4^{x_0}, & \text{if } g_0 = -u_{x_0} \text{ for } x_0 \in \{\pm 1, \pm 3\} \\ \pm 4\xi_4, & \text{if } g_0 = \pm a_0 \\ \pm(-4\xi_4), & \text{if } g_0 = \pm a_4 \\ 0, & \text{otherwise.} \end{cases}$

Note that we used the fact that $-u_1$ and u_{-1} are in the same conjugacy class of $SL_2(\mathbb{Z}/4\mathbb{Z})$. Also, $\xi_{D_0} = \exp(2\pi i/D_0)$ for $D_0 \in \{4, 8\}$.

Thus, we can state the following result:

Lemma 3. *For all possible D_0, $\Delta\chi_{t,e}$ takes the following values on conjugacy classes $g = (g_0, \cdots, g_t)$:*

- *If $D_0 = p_0$,*

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^t \left(\frac{x_0}{p_0}\right) G_{p_0} \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}) \\ e2^t \left(\frac{x_0}{p_0}\right) G_{p_0} \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}) \\ 0, & \text{otherwise} \end{cases}$$

  for all $t \geq 0$.
- *If $D_0 = 4$,*

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t+1}\xi_4^{x_0} \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}), x_0 \in \{\pm 1\} \\ e2^{t+1}\xi_4^{x_0} \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}), x_0 \in \{\pm 1\} \\ 0, & \text{otherwise} \end{cases}$$

  for all $t \geq 1$. Also, $\Delta\chi_{0,1}(g) = 0$ and $\Delta\chi_{0,-1}(g) = \begin{cases} 2^2\xi_4^{x_0}, & \text{if } g = u_{x_0}, x_0 \in \{\pm 1\} \\ 0, & \text{otherwise.} \end{cases}$

- *If* $D_0 = 8$,

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t+2}(\xi_8^{x_0} + \xi_4^{x_0}) \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}), x_0 \in \{\pm 1, \pm 3\} \\ \mp 2^{t+1}\xi_4 \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (\pm a_0, \cdots, u_{x_t}) \\ \pm 2^{t+1}\xi_4 \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (\pm a_4, \cdots, u_{x_t}) \\ e2^{t+2}(\xi_8^{x_0} + \xi_4^{x_0}) \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}), \\ & x_0 \in \{\pm 1, \pm 3\} \\ \mp e2^{t+1}\xi_4 \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (\mp a_0, \cdots, -u_{x_t}) \\ \pm e2^{t+1}\xi_4 \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i}, & \text{if } g = (\mp a_4, \cdots, -u_{x_t}) \\ 0, & \text{otherwise} \end{cases}$$

for all $t \geq 1$. *Also,* $\Delta\chi_{0,1}(g) = \begin{cases} 2^2(\xi_8^{x_0} + \xi_4^{x_0}), & \text{if } g = \pm u_{x_0}, x_0 \in \{\pm 1, \pm 3\} \\ 0, & \text{otherwise} \end{cases}$

and $\Delta\chi_{0,-1}(g) = \begin{cases} \pm 2^2(\xi_8^{x_0} + \xi_4^{x_0}), & \text{if } g = \pm u_{x_0}, x_0 \in \{\pm 1, \pm 3\} \\ \mp 2^2\xi_4, & \text{if } g = \pm a_0 \\ \pm 2^2\xi_4, & \text{if } g = \pm a_4 \\ 0, & \text{otherwise.} \end{cases}$

When not specified, x_i *above takes values in* $\{1, \eta_{p_i}\}$, *where* η_{p_i} *is a non-square* $\bmod p_i$ *for* $i \in [0, t]$ *an integer. Also,* $\xi_{D_0} = \exp(2\pi i/D_0)$ *for* $D_0 \in \{4, 8\}$.

Using the above result in (2), the alternating sum $\Delta M_{t,e}$ can be rewritten as follows:

$$\begin{aligned} \Delta M_{t,e} &= \frac{1}{|G|} \sum_{g \in G} \Delta\chi_{t,e}(g) \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} \\ &= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{(p_i^2 - 1)p_i} \\ &\quad \times \Bigg[\sum_{(g_0, \cdots, u_{x_t})} c_0 \prod_{i=1}^{t} \frac{p_i^2 - 1}{2} \Delta\chi_{t,e}(g_0, \cdots, u_{x_t}) \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0, \cdots, u_{x_t})} \\ &\quad + \sum_{(-g_0, \cdots, -u_{x_t})} c_0 \prod_{i=1}^{t} \frac{p_i^2 - 1}{2} \Delta\chi_{t,e}(-g_0, \cdots, -u_{x_t}) \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(-g_0, \cdots, -u_{x_t})} \Bigg] \end{aligned}$$

$$= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{1}{2p_i}$$
$$\times \sum_{(g_0,\cdots,u_{x_t})} c_0(1+e)\Delta\chi_{t,e}(g_0,\cdots,u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})},$$

so

$$\Delta M_{t,e} = \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{1}{2p_i}$$
$$\times \sum_{(g_0,\cdots,u_{x_t})} c_0(1+e)\Delta\chi_{t,e}(g_0,\cdots,u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})}, \tag{5}$$

where c_0 is the size of the conjugacy class of g_0. The last equality follows since $\Delta\chi_{t,e}(-g) = e\Delta\chi_{t,e}(g)$ by the result of Lemma 3. Clearly $\Delta M_{t,-1} = 0$, as was previously mentioned.

3. Fixed points on the modular curve

Let M be the modular curve $X(D) = \Gamma(D)\backslash\mathcal{H}^*$. M is a one dimensional compact complex manifold. G acts on M and $g : M \to M$ is a holomorphic endomorphism. If we consider $\tilde{g}$ to be a lift of g to $SL_2(\mathbb{Z})$, then the map $g : M \to M$ is given by $g\pi(z) = \pi(\tilde{g}z)$, where π is the natural projection $\mathcal{H}^* \to M$. We are in the situation where we look at maps $g : M \to M$ whose fixed points, if they exist, are isolated and non-degenerate. Using the holomorphic Lefschetz number, one can compute $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$ by knowing the fixed points of g on M, as we shall see in the next section. In the following, we find the fixed points of maps of the form $g = (g_0,\cdots,u_{x_t})$ for which $\Delta\chi_{t,1}(g) \neq 0$, where g_0 depends on D_0 as seen in Lemma 3.

Lemma 4. *For $D > 3$, the map $g = (g_0,\cdots,u_{x_t})$ has no fixed points on $Y(D) = \Gamma(D)\backslash\mathcal{H}$; all the possible fixed points happen at the cusps of $\Gamma(D)$.*

Proof. If $\pi(z)$ is a fixed point on $\Gamma(D)\backslash\mathcal{H}^*$, $z \in \mathcal{H}^*$, then there exists $\eta \in \Gamma(D)$ such that $\tilde{g}z = \eta z$, so we need to look at the fixed points of $\eta^{-1}\tilde{g}$ on $\mathcal{H}^*$. Since $\operatorname{Tr}(\eta^{-1}\tilde{g}) \equiv \operatorname{Tr}\tilde{g} \bmod D$, we get $\operatorname{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod p_i$ for $i \in [1,t]$ an integer, so if $t \geq 1$ and $p_i > 3$ for some i, we must have $|\operatorname{Tr}(\eta^{-1}\tilde{g})| \geq 2$. As $D > 3$, we also have $|\operatorname{Tr}(\eta^{-1}\tilde{g})| \geq 2$ for the case $D_0 = p_0$. If $D_0 = 4$, we get $\operatorname{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod 4$, which gives us $|\operatorname{Tr}(\eta^{-1}\tilde{g})| \geq 2$ as well. If $D = 8$, $\operatorname{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod 8$. The last case to consider is $D = 8 \times 3$. If $g_0 = u_{x_0}$, $x_0 \in \{\pm 1, \pm 3\}$ or $g_0 = \pm a_4$ then $\operatorname{Tr}(\eta^{-1}\tilde{g})$ is either 2 or 4 mod 8. If $(g_0, u_{x_1}) = (a_0, u_1)$, a choice for $\tilde{g}$ is the matrix $\begin{pmatrix} -8 & 1 \\ 63 & -8 \end{pmatrix}$, while if $(g_0, u_{x_1}) = (a_0, u_{-1})$, we can choose a lift $\tilde{g} = \begin{pmatrix} -8 & -7 \\ -9 & -8 \end{pmatrix}$. Thus $\operatorname{Tr}(\eta^{-1}\tilde{g}) \equiv -16 \bmod 24$; similarly, one gets the same result if $g_0 = -a_0$. Therefore, $|\operatorname{Tr}(\eta^{-1}\tilde{g})| \geq 2$ holds for all possible values of $D > 3$, so $\eta^{-1}\tilde{g}$ is either parabolic or hyperbolic and thus it has either one or two fixed points on $\mathbb{R} \cup \{\infty\}$. $\square$

For the following, assume $D > 4$. Recall that two cusps $\frac{a}{b}$ and $\frac{c}{d}$ of $\Gamma(D)$ with integers a, b, c, d such that $(a, b) = 1, (c, d) = 1$ are $\Gamma(D)$-equivalent iff $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \pm \begin{pmatrix} c \\ d \end{pmatrix} \bmod D$ ([9], Chapter 1, Section 1.6, Lemma 1.42, p. 23). Now, if the cusp $\frac{a}{b}$ with $a, b \in \mathbb{Z}, (a, b) = 1$ is a fixed point of g, then $\frac{a}{b}$ and $\tilde{g}\frac{a}{b}$ are $\Gamma(D)$-equivalent. Depending on the values of D_0, we get the following cases:

- If $D_0 = p_0$, then $g_0 = u_{x_0}$ with $x_0 \in \{1, \eta_{p_0}\}$, where η_{p_0} is a non-square mod p_0. We have

$$a + bx_i \equiv \pm a \bmod p_i,$$
$$b \equiv \pm b \bmod p_i,$$

  for all $i \in [0, t]$ an integer. As $(a, b) = 1$, we must be in the case

$$a + bx_i \equiv a \bmod p_i,$$
$$b \equiv b \bmod p_i,$$

  for all $i \in [0, t]$ an integer, so $b \equiv 0 \bmod D$.
- If $D_0 = 4$, then $g_0 = u_{x_0}$, with $x_0 \in \{\pm 1\}$. Since $D > 4$, $t \geq 1$, so by the same reasoning as above we must have $b \equiv 0 \bmod \prod_{i=1}^{t} p_i$ for all $i \in [1, t]$ an integer. Moreover, we must be in the case

$$a + bx_0 \equiv a \bmod 4,$$
$$b \equiv b \bmod 4,$$

  so $b \equiv 0 \bmod 4$ and thus $b \equiv 0 \bmod D$.
- If $D_0 = 8$, first consider the case when $g_0 = u_{x_0}$, with $x_0 \in \{\pm 1, \pm 3\}$. If

$$a + bx_0 \equiv -a \bmod 8,$$
$$b \equiv -b \bmod 8,$$

  then $b \equiv 0 \bmod 4$ and $2a + bx_0 \equiv 0 \bmod 8$, so a must be even, contradiction. Thus, we must be in the case

$$a + bx_0 \equiv a \bmod 8,$$
$$b \equiv b \bmod 8,$$

so $b \equiv 0 \bmod 8$. If $t \geq 1$, by the same reasoning as above, we must have $b \equiv 0 \bmod \prod_{i=1}^{t} p_i$ and thus $b \equiv 0 \bmod D$.

If $g_0 = a_0$, then we have

$$b \equiv \pm a \bmod 8,$$
$$-a \equiv \pm b \bmod 8,$$

which forces both a, b to be even, so there are no fixed points in this case. Similarly, there are no fixed points for the cases $g_0 = -a_0$ and $g_0 = \pm a_4$.

Notice that $\Delta\chi_{t,1}(g) = 0$ when $D = 4$. Therefore, we can state the following result:

Lemma 5. *Let $D > 3$ and $g = (g_0, \cdots, u_{x_t})$ an element of G such that $\Delta\chi_{t,1}(g) \neq 0$ and g_0 depending on D_0 as seen in Lemma 3. Then the fixed points of g on M are as follows:*

- *If $D_0 \in \{p_0, 4\}$, $g_0 = u_{x_0}$ and g has fixed points $\frac{l}{D}$ with $(l, D) = 1$, $l \in [1, D/2]$ an integer.*
- *If $D_0 = 8$ and $g_0 = u_{x_0}$, then g has fixed points $\frac{l}{D}$ with $(l, D) = 1$, $l \in [1, D/2]$ an integer and there are no fixed points when $g_0 \in \{\pm a_0, \pm a_4\}$.*

In the above, we have $x_i \in \{1, \eta_{p_i}\}$, with η_{p_i} a non-square $\bmod p_i$ for all $i \in [1, t]$ an integer and

$$x_0 \in \begin{cases} \{1, \eta_{p_0}\}, \textit{with } \eta_{p_0} \textit{ a non-square } \bmod p_0, & \textit{if } D_0 = p_0 \\ (\mathbb{Z}/D_0\mathbb{Z})^\times, & \textit{if } D_0 \in \{4, 8\}. \end{cases}$$

4. The holomorphic Lefschetz number

For G acting on the one-dimensional compact complex manifold M, we identify any $g \in G$ with a map $g : M \to M$. Suppose the fixed points of g are isolated and non-degenerate. The holomorphic Lefschetz number of the map g relative to the holomorphic line bundle defined by the structure sheaf $\mathcal{O}$ is given by (cf. [3], Chapter 3, Section 4, p. 422–426)

$$L(g, \mathcal{O}) = \sum_q (-1)^q \operatorname{Tr}(g^* | H_{\bar\partial}^{0,q}(M)).$$

Let $dg_\kappa : T_\kappa(M) \to T_\kappa(M)$ be the differential induced by the map g on the holomorphic tangent space at the fixed point κ. By the holomorphic Lefschetz fixed-point formula we have

$$L(g, \mathcal{O}) = \sum_{g(\kappa)=\kappa} \frac{1}{\det(I - dg_\kappa)},$$

where, by abuse of notation, by dg_κ we mean the above differential evaluated at the fixed point. The goal of this section is to compute the characters $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$ which appear in the expression of ΔM_t in (2). We compute the Lefschetz numbers by using the fixed points in Lemma 5, which in turn give us the characters $\chi_{\mathcal{S}_2(\Gamma(D))}$.

We have $H^{0,q}_{\overline{\partial}}(M) \cong H^q(M, \mathcal{O})$. It is well-known that $H^q(M, \mathcal{O})$ vanishes for $q > 1$ and $H^0(M, \mathcal{O}) \cong \mathbb{C}$. Let Ω^i define the sheaf of holomorphic differentials of degree i on M; we have $\Omega^0 = \mathcal{O}$. By Hodge theory $H^1(M, \mathcal{O}) \cong \overline{H^0(M, \Omega^1)}$, where the space $H^0(M, \Omega^1)$ is exactly the space $\mathcal{S}_2(\Gamma(D))$ of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$.

As a result,

$$L(g, \mathcal{O}) = \mathrm{Tr}(g^*|\mathbb{C}) - \mathrm{Tr}(g^*|\overline{\mathcal{S}_2(\Gamma(D))}).$$

But $\mathrm{Tr}(g^*|\mathbb{C}) = 1$, since the action of g^* on $H^0(M, \mathcal{O})$ is trivial and $\mathrm{Tr}(g^*|\overline{\mathcal{S}_2(\Gamma(D))}) = \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)}$. Thus

$$\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} = 1 - L(g, \mathcal{O}). \tag{6}$$

Moreover, if g has no fixed points, the Lefschetz number is zero and we get $\chi_{\mathcal{S}_2(\Gamma(D))}(g) = 1$.

Next step is to compute the differentials dg_κ. As seen in Lemma 5, we are interested in the cases when $g = (g_0, \cdots, u_{x_t})$, with g_0 of the form u_{x_0}. The fixed points of g are given by $\frac{l}{D}$, with $(l, D) = 1$, $l \in [1, D/2]$ an integer. Note that the cusp $\frac{1}{D}$ is equivalent to infinity.

Lemma 6. *Let $D > 3$ and $g = (g_0, \cdots, u_{x_t})$ an element of G having fixed points on M such that $\Delta\chi_{t,1}(g) \neq 0$. We must have $g_0 = u_{x_0}$ with the values of x_0 depending on D_0 as seen in Lemma 5. The differential $dg_{\frac{l}{D}}$ at the cusp $\frac{l}{D}$ with $(l, D) = 1, l \in [1, D/2]$ an integer, is given by*

$$dg_{\frac{l}{D}} = \xi^{\lambda l^{-2}},$$

where $\xi = \exp(2\pi i/D)$, $\lambda \in \mathbb{Z}$ such that $\lambda \equiv x_0 \bmod D_0$, $\lambda \equiv x_i \bmod p_i$, for all $i \in [1, t]$ an integer.

Proof. The idea is to translate the cusp $\frac{l}{D}$ to ∞ and compute the differential there. Say the fixed points of g are at the cusps κ, so the complex structure on M is given locally by homeomorphisms into open sets of $\mathbb{C}$ through the map

$$\pi(z) \mapsto \exp(2\pi i \rho(z)/D),$$

where $\pi : \mathcal{H}^* \to M$ is the natural projection, $\rho \in SL_2(\mathbb{R})$ such that $\rho(\kappa) = \infty$.

For the cusp $\frac{l}{D}$, let $\gamma_l \in SL_2(\mathbb{Z})$ such that $\gamma_l(\frac{l}{D}) = \infty$. There exists an induced differential $d\gamma_l : T_{\frac{l}{D}}(M) \to T_\infty(M)$ such that the map $dg_{\frac{l}{D}}$ translated to ∞ is given by

$$d\gamma_l g \gamma_l^{-1} : T_\infty(M) \to T_\infty(M),$$

where the map $\gamma_l g \gamma_l^{-1}$ on M is given by $\pi(z) \mapsto \pi(\gamma_l \tilde{g} \gamma_l^{-1} z)$, with $\tilde{g}$ a lift of g to $SL_2(\mathbb{Z})$. For the cusp $\frac{l}{D}$ with $(l, D) = 1$, γ_l will be given by the matrix $\begin{pmatrix} b & a \\ -D & l \end{pmatrix}$, where $a, b \in \mathbb{Z}$ such that $aD + bl = 1$.

As

$$\begin{pmatrix} b & a \\ -D & l \end{pmatrix} \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l & -a \\ D & b \end{pmatrix} \equiv \begin{pmatrix} 1 + Dbx_0 & b^2 x_0 \\ -D^2 x_0 & 1 - Dbx_0 \end{pmatrix} \bmod D_0,$$

we get that $\gamma_l \tilde{g} \gamma_l^{-1} \equiv \begin{pmatrix} 1 & b^2 x_0 \\ 0 & 1 \end{pmatrix} \bmod D_0$. Similarly, $\gamma_l \tilde{g} \gamma_l^{-1} \equiv \begin{pmatrix} 1 & b^2 x_i \\ 0 & 1 \end{pmatrix} \bmod p_i$ for all $i \in [1, t]$ an integer. If $\lambda = \lambda_{x_0, \cdots, x_t} \in \mathbb{Z}$ such that $\lambda \equiv x_0 \bmod D_0$ and $\lambda \equiv x_i \bmod p_i$ for all $i \in [1, t]$ an integer, the action of $\gamma_l g \gamma_l^{-1}$ on M will be a translation by $b^2 \lambda$. Thus, if $\exp(2\pi i z/D)$ is the local coordinate for ∞ on M, then $\exp(2\pi i (z + b^2 \lambda)/D)$ will be the local coordinate for $\gamma_l g \gamma_l^{-1}(\infty)$. So

$$d\gamma_l g \gamma_l^{-1} = \frac{d \exp(2\pi i (z + b^2 \lambda)/D)}{d \exp(2\pi i z/D)},$$

and thus $dg_{\frac{l}{D}} = \xi^{\lambda b^2}$. Since $aD + bl = 1$, we have $b^2 \equiv l^{-2} \bmod D$. $\square$

Under the setting of Lemma 6, we get

$$L(g, \mathcal{O}) = \sum_{\substack{l=1, \\ (l,D)=1}}^{\lfloor D/2 \rfloor} \frac{1}{1 - \xi^{\lambda l^{-2}}} = \frac{1}{2} \sum_{\substack{l=1, \\ (l,D)=1}}^{D-1} \frac{1}{1 - \xi^{\lambda l^2}}$$

$$= 2^t n(D_0) \sum_{\substack{l \in [(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = 1}} \frac{1}{1 - \xi^{\lambda l}} = 2^t n(D_0) \sum_{\substack{l \in \lambda[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = \left(\frac{\lambda}{p_i}\right)}} \frac{1}{1 - \xi^l}$$

$$= 2^t n(D_0) \sum_{\substack{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = \left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^l},$$

where $n(D_0) = \begin{cases} 1, & \text{if } D_0 \in \{p_0, 4\} \\ 2, & \text{if } D_0 = 8 \end{cases}$ and the summation is over $l \in [1, D-1]$ an integer, $(l, D) = 1$.

Under the same conditions, from (6) we get

$$\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} = 1 - 2^t n(D_0) \sum_{\substack{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = \left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^l}.$$

Thus the expression in (5) gives us

$$\begin{aligned}
\Delta M_t &= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})} \\
&= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) \\
&\quad - \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) L((g_0,\cdots u_{x_t}),\mathcal{O}) \\
&= -\frac{2c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(u_{x_0},\cdots,u_{x_t})} \Delta(D_0, x_0) \\
&\quad \times \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i} 2^t n(D_0) \sum_{\substack{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = \left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^l} \\
&= -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \sum_{(u_{x_0},\cdots,u_{x_t})} \Delta(D_0, x_0) \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) \sum_{\substack{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2, \\ \left(\frac{l}{p_i}\right) = \left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^l},
\end{aligned}$$

so

$$\Delta M_t = -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \sum_{x_0} \Delta(D_0, x_0) \sum_{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2} \left(\frac{l}{p_1\cdots p_t}\right) \frac{1}{1-\xi^l}, \tag{7}$$

with $c_0 = \begin{cases} \frac{p_0^2-1}{2}, & \text{if } D_0 = p_0 \\ 6, & \text{if } D_0 = 4 \\ 12, & \text{if } D_0 = 8, \end{cases}$ $\qquad n(D_0) = \begin{cases} 1, & \text{if } D_0 \in \{p_0, 4\} \\ 2, & \text{if } D_0 = 8 \end{cases}$

and $\Delta(D_0, x_0) = \begin{cases} 2^t \left(\frac{x_0}{p_0}\right) G_{p_0}, & \text{if } D_0 = p_0, x_0 \in \{1, \eta_{p_0}\} \\ 2^{t+1}\xi_4^{x_0}, & \text{if } D_0 = 4, x_0 \in (\mathbb{Z}/4\mathbb{Z})^\times \\ 2^{t+2}(\xi_8^{x_0} + \xi_4^{x_0}), & \text{if } D_0 = 8, x_0 \in (\mathbb{Z}/8\mathbb{Z})^\times, \end{cases}$

where η_{p_0} is a non-square in $\mathbb{F}_{p_0}^\times$ and the summation is over $l \in [1, D-1]$ an integer. Note that the above result works for $D > 4$ and if $D = 4$ we have $\Delta M_0 = 0$. So for the rest of the paper we work with $D > 4$, unless mentioned otherwise.

5. Some useful lemmas

The following results provide key steps in bringing the expression for ΔM_t in (7) in the form of the analytical formula for $h(-D)$ appearing in Lemma 1.

Lemma 7. *If $D \in \mathbb{Z}_{>1}$, $\xi = \exp(2\pi i/D)$, then*

$$\frac{1}{1-\xi^l} = \frac{1}{D}\sum_{n=0}^{D-1} n\xi^{-l(n+1)},$$

for all $l \in [1, D-1]$ an integer.

Proof. Let θ_D be the polynomial $\theta_D(x) = \sum_{n=0}^{D-1} x^n = \prod_{n=1}^{D-1}(x - \xi^n)$. Then

$$\theta_D(x)' = \sum_{n=0}^{D-1} nx^{n-1} = \sum_{n=1}^{D-1}\prod_{j\neq n}(x - \xi^j),$$

so evaluating at ξ^l we get $\sum_{n=0}^{D-1} n\xi^{l(n-1)} = \prod_{j\neq l}(\xi^l - \xi^j) = \xi^{l(D-2)}\prod_{j\neq l}(1 - \xi^{j-l})$. Thus $\sum_{n=0}^{D-1} n\xi^{l(n-1)} = \xi^{-2l}\prod_{n\neq D-l}(1 - \xi^n)$, and as a result

$$\frac{1}{1-\xi^l} = \frac{1}{D}\prod_{n\neq l}(1 - \xi^n) = \frac{1}{D}\sum_{n=0}^{D-1} n\xi^{-l(n+1)}. \quad \square$$

Lemma 8. *Let $D \in \mathbb{Z}_{>1}$, $D = D_* p_1 \cdots p_t$, with p_i distinct odd primes, $(D_*, p_i) = 1$ for all $i \in [1, t]$ an integer, $x \in \mathbb{Z}$. Then*

$$\sum_{l\equiv x}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}=\frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv -nx}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l+\eta_t,$$

where $\xi=\exp(2\pi i/D)$, $\eta_t=\begin{cases}\frac{D-1}{D}, & \textit{if } t=0\\ 0, & \textit{otherwise,}\end{cases}$ *the summation is over* $l\in[1,D-1]$ *an integer and the congruences are* $\bmod D_*$.

Proof. If we denote $\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}$ by E, then from Lemma 7, we have

$$\begin{aligned}E&=\frac{1}{D}\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\sum_{n=0}^{D-1}n\xi^{-l(n+1)}\\ &=\frac{1}{D}\sum_{n=0}^{D-1}n\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\xi^{-l(n+1)}\\ &=\frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=0}^{D-2}n\left(\frac{n+1}{p_1\cdots p_t}\right)\sum_{l\equiv x \bmod D_*}\left(\frac{-l(n+1)}{p_1\cdots p_t}\right)\xi^{-l(n+1)}\\ &\quad+\frac{1}{D}(D-1)\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\\ &=\frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv x \bmod D_*}\left(\frac{-ln}{p_1\cdots p_t}\right)\xi^{-ln}+\eta_t\\ &=\frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv -nx \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\xi^{l}+\eta_t.\quad\square\end{aligned}$$

Lemma 9. *Let* p_i *be distinct odd primes,* $i\in[1,t]$ *an integer,* $t\geq 1$. *Then*

$$\sum_{l=1}^{p_1\cdots p_t-1}\left(\frac{l}{p_1\cdots p_t}\right)\xi_{p_1\cdots p_t}^l=e_t\prod_{i=0}^{t}G_{p_i},$$

where $\xi_{p_1\cdots p_t}=\exp(2\pi i/p_1\cdots p_t)$, $e_t=\begin{cases}\prod_{1\leq i<j\leq t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \textit{if } t>1\\ 1, & \textit{if } t=1.\end{cases}$

Proof. For an odd prime p we know $\sum_{l=1}^{p-1}\left(\frac{l}{p}\right)\xi_p^l=G_p$, where $\xi_p=\exp(2\pi i/p)$. If p,q distinct odd primes,

$$\sum_{i=1}^{p-1}\left(\frac{i}{p}\right)\xi_p^i\sum_{j=1}^{q-1}\left(\frac{j}{q}\right)\xi_q^j=\sum_{i,j}\left(\frac{i}{p}\right)\left(\frac{j}{q}\right)\xi_{pq}^{iq+jp}$$
$$=\sum_{i,j}\left(\frac{iq+jp}{pq}\right)\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)\xi_{pq}^{iq+jp}$$
$$=(-1)^{\frac{(p-1)(q-1)}{4}}\sum_{l=1}^{pq-1}\left(\frac{l}{pq}\right)\xi_{pq}^l,$$

and the result follows by induction. □

Lemma 10. *Let $D=D_0p_1\cdots p_t$, with $D_0\in\mathbb{Z}_{>1}$ and p_i distinct odd primes such that $(D_0,p_i)=1$ for all integers $i\in[1,t]$. Let $S\subset\mathbb{Z}$ finite set, $n,c\in\mathbb{Z}$. Then*

$$\sum_{x\in S}\xi_{D_0}^{cx}\sum_{\substack{l=1,\\ l\equiv -nx(p_1\cdots p_t)^2 \bmod D_0}}^{D-1}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l=\left(\frac{D_0}{p_1\cdots p_t}\right)e_t\prod_{i=1}^{t}G_{p_i}\sum_{x\in S}\xi_{D_0}^{x(c-np_1\cdots p_t)},$$

where $\xi=\exp(2\pi i/D)$, $\xi_{D_0}=\exp(2\pi i/D_0)$,

$$e_t=\begin{cases}\prod\limits_{1\le i<j\le t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } t>1\\ 1, & \text{if } t\in\{0,1\}.\end{cases}$$

Proof. There is nothing to prove if $t=0$. For $t\ge 1$, by Lemma 9, we have

$$\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{j}{p_1\cdots p_t}\right)\xi_{p_1\cdots p_t}^j=e_t\prod_{i=1}^{t}G_{p_i}.$$

Then

$$\left(\sum_{x\in S}\xi_{D_0}^{cx}\xi_{D_0}^{-nxp_1\cdots p_t}\right)\left[\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{j}{p_1\cdots p_t}\right)\xi_{p_1\cdots p_t}^j\right]$$
$$=\sum_{x\in S}\xi_{D_0}^{cx}\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{j}{p_1\cdots p_t}\right)\xi^{jD_0-nx(p_1\cdots p_t)^2}$$
$$=\left(\frac{D_0}{p_1\cdots p_t}\right)\sum_{x\in S}\xi_{D_0}^{cx}\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{jD_0}{p_1\cdots p_t}\right)\xi^{jD_0-nx(p_1\cdots p_t)^2}$$
$$=\left(\frac{D_0}{p_1\cdots p_t}\right)\sum_{x\in S}\xi_{D_0}^{cx}\sum_{\substack{l=1,\\ l\equiv -nx(p_1\ldots p_t)^2 \bmod D_0}}^{D-1}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l.\quad\square$$

6. Proof of the main theorem

We prove the main result in both the odd and even cases, by using the key lemmas from the previous section in the expression (7) for ΔM_t.

6.1. The odd case $D_0 = p_0$

From (7) we have

$$\Delta M_t = -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{G_{p_i}}{p_i}\sum_{x\in\{1,\eta_{p_0}\}}\Delta(D_0,x_0)\sum_{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}$$

$$= -2^t\frac{G_{p_0}}{p_0}\prod_{i=1}^{t}\frac{G_{p_i}}{p_i}\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l}.$$

We have $D = p_0\prod_{i=1}^{t}p_i$, with $-D\equiv 1 \bmod 4$ and let's first assume $t\geq 1$. We need to compute $\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l}$, which we denote by Δ_{p_0}. From Lemma 8 for $D_* = 1$ we get

$$\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l} = \frac{1}{D}\left(\frac{-1}{D}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{D}\right)\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\xi^l,$$

and using the results of Lemmas 9 and 1 we have

$$\Delta_{p_0} = \prod_{0\leq i<j\leq t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}\frac{1}{D}\left(\frac{-1}{D}\right)G_{p_0}\prod_{i=1}^{t}G_{p_i}\left[\sum_{n=1}^{D-1}n\left(\frac{n}{D}\right)-\sum_{n=1}^{D-1}\left(\frac{n}{D}\right)\right]$$

$$= \prod_{0\leq i<j\leq t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}\frac{1}{D}\left(\frac{-1}{D}\right)G_{p_0}\prod_{i=1}^{t}G_{p_i}\sum_{n=1}^{D-1}n\left(\frac{n}{D}\right)$$

$$= -\prod_{0\leq i<j\leq t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}\left(\frac{-1}{D}\right)G_{p_0}\prod_{i=1}^{t}G_{p_i}h(-D).$$

Thus for $t\geq 1$ we have

$$\Delta M_t = -2^t\prod_{i=0}^{t}\frac{G_{p_i}}{p_i}\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l}$$

$$= \prod_{0\leq i<j\leq t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}2^t\prod_{i=0}^{t}\frac{G_{p_i}^2}{p_i}\left(\frac{-1}{D}\right)h(-D)$$

$$= \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} 2^t h(-D),$$

since $G_p = \sqrt{(-1)^{\frac{p-1}{2}} p} = \sqrt{\left(\frac{-1}{p}\right) p}$.

The case $t = 0$ works similarly and we get $\Delta M_0 = h(-p)$, when $p \equiv 3 \bmod 4$, which is Hecke's initial result.

6.2. The even case $D_0 \in \{4, 8\}$

From (7) we get

$$\begin{aligned}\Delta M_t &= -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \\ &\quad \times \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \Delta(D_0, x_0) \sum_{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2} \left(\frac{l}{p_1\cdots p_t}\right) \frac{1}{1-\xi^l} \\ &= -2^{t-1} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \Delta_{D_0},\end{aligned}$$

where

$$\Delta_{D_0} = \begin{cases} \displaystyle\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{x_0} \sum_{\substack{l=1,\\ l\equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1\cdots p_t}\right) \frac{1}{1-\xi^l}, & \text{if } D_0 = 4 \\ \displaystyle\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} (\xi_{D_0}^{x_0} + \xi_{D_0}^{2x_0}) \sum_{\substack{l=1,\\ l\equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1\cdots p_t}\right) \frac{1}{1-\xi^l}, & \text{if } D_0 = 8. \end{cases}$$

Note that for $D_0 \in \{4, 8\}$, we have $(p_1 \cdots p_t)^2 \equiv 1 \bmod D_0$. Thus, using Lemma 8 for $D_* = D_0$ and Lemma 10 for $S = (\mathbb{Z}/D_0\mathbb{Z})^\times$, the expression

$$E := \sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \sum_{\substack{l=1,\\ l\equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1\cdots p_t}\right) \frac{1}{1-\xi^l},$$

where $c = 1$ when $D_0 = 4$ and $c \in \{1, 2\}$ when $D_0 = 8$, can be rewritten as

$$\begin{aligned}E = &\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \left[\frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right) \sum_{n=1}^{D-1} (n-1) \left(\frac{n}{p_1\cdots p_t}\right)\right. \\ &\left.\times \sum_{l\equiv -nx_0 \bmod D_0} \left(\frac{l}{p_1\cdots p_t}\right) \xi^l + \eta_t\right]\end{aligned}$$

$$= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times}\xi_{D_0}^{cx_0}\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv -nx_0 \bmod D_0}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l$$

$$= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times}\xi_{D_0}^{cx_0}\sum_{l\equiv -nx_0 \bmod D_0}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l$$

$$= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\left(\frac{D_0}{p_1\cdots p_t}\right)e_t\prod_{i=1}^{t}G_{p_i}\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)$$

$$\times \sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times}\xi_{D_0}^{x_0(c-np_1\cdots p_t)},$$

where $e_t = \begin{cases}\prod_{1\le i<j\le t}(-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } t>1\\ 1, & \text{if } t\in\{0,1\}.\end{cases}$

An easy computation gives us the following:

$$\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times}\xi_{D_0}^{x_0(1-np_1\cdots p_t)}$$

$$= \begin{cases}\pm\frac{D_0}{2}, & \text{if } n\equiv p_1\cdots p_t, -p_1\cdots p_t \bmod D_0,\ \text{when } D_0=4\\ \pm\frac{D_0}{2}, & \text{if } n\equiv p_1\cdots p_t, -3p_1\cdots p_t \bmod D_0,\ \text{when } D_0=8\\ 0, & \text{otherwise}\end{cases}$$

and

$$\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times}\xi_{D_0}^{x_0(2-np_1\cdots p_t)}$$

$$= \begin{cases}\pm\frac{D_0}{2}, & \text{if } n\equiv 2p_1\cdots p_t, -2p_1\cdots p_t \bmod D_0,\ \text{when } D_0=8\\ 0, & \text{otherwise.}\end{cases}$$

As a result, we get

$$\Delta_{D_0} = \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\left(\frac{D_0}{p_1\cdots p_t}\right)e_t\prod_{i=0}^{t}G_{p_i}\frac{D_0}{2}\Delta^*_{D_0},$$

where

$$\Delta^*_{D_0} = \begin{cases}\displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{n\equiv -p_1\cdots p_t \bmod 4} n\left(\frac{n}{p_1\cdots p_t}\right), & \text{if } D_0=4\\ \displaystyle\sum_{\substack{n\equiv p_1\cdots p_t \bmod 8,\\ n\equiv 2p_1\cdots p_t \bmod 8}} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{\substack{n\equiv -3p_1\cdots p_t \bmod 8,\\ n\equiv -2p_1\cdots p_t \bmod 8}} n\left(\frac{n}{p_1\cdots p_t}\right), & \text{if } D_0=8.\end{cases}$$

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For $D_0 = 8$, we have

$$\sum_{\substack{n=1,\\ n\equiv 2p_1\cdots p_t \bmod 8}}^{D-1} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{\substack{n=1,\\ n\equiv -2p_1\cdots p_t \bmod 8}}^{D-1} n\left(\frac{n}{p_1\cdots p_t}\right)$$
$$= \begin{cases} 2\left(\frac{2}{p_1\cdots p_t}\right)\Delta^*_{\frac{D_0}{2}}, & \text{if } \left(\frac{-1}{p_1\cdots p_t}\right)=1 \\ 0, & \text{otherwise;} \end{cases}$$

we denote this difference by δ^*_8.

A trivial check gives us

$$\Delta^*_{D_0} = \begin{cases} 2\left(\frac{D_0}{p_1\cdots p_t}\right) \displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right), & \text{if } D_0=4, t\geq 1 \\ \left(\frac{D_0}{p_1\cdots p_t}\right) \displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right) + \delta^*_8, & \text{if } D_0 = 8 \end{cases}$$

where the summations are over $n \in [1, D-1]$ an integer.

Using the result of Lemma 1, we have

$$\Delta_{D_0} = \begin{cases} -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2\displaystyle\prod_{i=1}^{t} G_{p_i} h(-D), & \text{if } D_0 = 4, t\geq 1 \\ -e_t 2\Big[h(-D)+h(-D/2)\Big], & \text{if } D_0=8, t=0 \\ -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2\displaystyle\prod_{i=1}^{t} G_{p_i}\Big[h(-D)+2h(-D/2)\Big], & \text{if } D_0=8, t\geq 1, \\ & p_1\cdots p_t \equiv 1 \bmod 4 \\ -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2\displaystyle\prod_{i=1}^{t} G_{p_i} h(-D), & \text{if } D_0=8, t\geq 1, \\ & p_1\cdots p_t \equiv 3 \bmod 4, \end{cases}$$

so we get

$$\Delta M_t = \begin{cases} e_t 2^t h(-D), & \text{if } D_0 = 4, t\geq 1 \\ e_t 2^t \left[h(-D)+h(-D/2)\right], & \text{if } D_0=8, t=0 \\ e_t 2^t \left[h(-D)+2h(-D/2)\right], & \text{if } D_0=8, t\geq 1, p_1\cdots p_t\equiv 1 \bmod 4 \\ e_t 2^t h(-D), & \text{if } D_0=8, t\geq 1, p_1\cdots p_t \equiv 3 \bmod 4, \end{cases}$$

which is what we want for the case $D > 4$ even. This concludes the proof of the main theorem for all cases $D_0 \in \{p_0, 4, 8\}$.

Remark. Note that when $D_0 = 8$, there is a multiple of $h(-D/2)$ appearing in the expression for ΔM_t. Morally, this term comes from the distinctive G-representations whose $SL_2(\mathbb{Z}/8\mathbb{Z})$ part can be factored through $SL_2(\mathbb{Z}/4\mathbb{Z})$. There are two such pairs of irreducibles of $SL_2(\mathbb{Z}/8\mathbb{Z})$ that can appear in a distinctive G-representation, that is $\pi^+_{1,1}, \pi^-_{1,1}$, respectively $\pi^+_{3,1}, \pi^-_{3,1}$. Interchanging π^+ and π^- for some of the irreducibles appearing in the $SL_2(\mathbb{Z}/8\mathbb{Z})$ part and discarding those above that factor through $SL_2(\mathbb{Z}/4\mathbb{Z})$ will give us $\Delta M_t = \mathrm{sgn}_{D_0,t} 2^t h(-D)$ for all cases $D_0 = 8, t \geq 1$; here $\mathrm{sgn}_{D_0,t}$ is as given in the statement of the main theorem. For example, in order to get such a result, one can interchange π^+ and π^- for $\pi_{3,3}$ and $\pi_{3,4}$ in the $SL_2(\mathbb{Z}/8\mathbb{Z})$ part of a distinctive G-representation.

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