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General Section

On Drinfeld modular forms of higher rank IV:
Modular forms with level

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ARTICLE INFO

Article history:

Received 11 January 2019

Accepted 26 April 2019

Available online xxxx

Communicated by F. Pellarin

Dedicated to the memory of David
Goss*Keywords:*

Drinfeld modular forms

Eisenstein series

Compactification of moduli schemes

ABSTRACT

We construct and study a natural compactification $\overline{M}^r(N)$ of the moduli scheme $M^r(N)$ for rank- r Drinfeld $\mathbb{F}_q[T]$ -modules with a structure of level $N \in \mathbb{F}_q[T]$. Namely, $\overline{M}^r(N) = \text{Proj } \mathbf{Eis}(N)$, the projective variety associated with the graded ring $\mathbf{Eis}(N)$ generated by the Eisenstein series of rank r and level N . We use this to define the ring $\mathbf{Mod}(N)$ of all modular forms of rank r and level N . It equals the integral closure of $\mathbf{Eis}(N)$ in their common quotient field $\widehat{\mathcal{F}}_r(N)$. Modular forms are characterized as those holomorphic functions on the Drinfeld space Ω^r with the right transformation behavior under the congruence subgroup $\Gamma(N)$ of $\Gamma = \text{GL}(r, \mathbb{F}_q[T])$ (“weak modular forms”) which, along with all their conjugates under $\Gamma/\Gamma(N)$, are bounded on the natural fundamental domain \mathbf{F} for Γ on Ω^r .

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0. Introduction

(0.1). This is the fourth of a series of papers (see [18], [21], [19]) which aim to lay the foundations for a theory of Drinfeld modular forms of higher rank. These are modular forms for the modular group $\Gamma = \text{GL}(r, \mathbb{F}_q[T])$ or its congruence subgroups, where “higher

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<https://doi.org/10.1016/j.jnt.2019.04.019>

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rank” refers to r larger or equal to 2. The case of $r = 2$, remarkably similar in some aspects but rather different in others to the theory of classical elliptic modular forms for $\mathrm{SL}(2, \mathbb{Z})$ or its congruence subgroups, is meanwhile well-established and the subject of several hundred publications since about 1980.

We leave aside to deal with more general Drinfeld coefficient rings A than $A = \mathbb{F}_q[T]$, as the amount of technical and notational efforts required would obscure the overall picture. The interested reader may consult [15] to get an impression of the complications that - even for $r = 2$ - result from class numbers $h(A) > 1$ for general A .

(0.2). While we developed some of the theory of modular forms “without level” in [18] and [19] and focussed on the connection with the geometry of the Bruhat-Tits building in [18] and [21], the current part IV is devoted to forms “with level”, i.e., forms for congruence subgroups of Γ . Again we restrict to the most simple case of full congruence subgroups $\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{N}\}$ for $N \in A$. Finer arithmetic/geometric properties of modular forms (or varieties) for other congruence subgroups $\Gamma' \supset \Gamma(N)$ may be derived in the course of the further development of the theory from those for $\Gamma(N)$, by taking invariants (or quotients) of the finite group $\Gamma'/\Gamma(N)$.

(0.3). Let us introduce a bit of notation: $\mathbb{F} = \mathbb{F}_q$ is the finite field with q elements, $A = \mathbb{F}[T]$ the polynomial ring in an indeterminate T , with quotient field $K = \mathbb{F}(T)$, and its completion $K_\infty = \mathbb{F}((T^{-1}))$ at infinity, and C_∞ the completed algebraic closure of K_∞ . The Drinfeld symmetric space Ω^r (where $r \geq 2$) is the complement in $\mathbb{P}^{r-1}(C_\infty)$ of the K_∞ -rational hyperplanes. The modular group $\Gamma = \mathrm{GL}(r, A)$ acts in the usual fashion on Ω^r , and we let $M^r(N)$ be the quotient analytic space $\Gamma(N) \backslash \Omega^r$ (which is also the set of C_∞ -points of an affine variety labelled by the same symbol, and which is smooth if $N \in A$ is non-constant).

The modular forms dealt with will be holomorphic functions on Ω^r with certain additional properties; so the theory is “over C_∞ ”; we will only briefly touch on questions of rationality.

(0.4). Our approach is based on

- (i) the use of the natural fundamental domain \mathbf{F} for Γ on Ω^r introduced in [20]; it relies on the notion of successive minimum basis (SMB) of an A -lattice in C_∞ . On \mathbf{F} , one may perform explicit calculations;
- (ii) a natural compactification $\overline{M}^r(N)$ of $M^r(N)$, the *Eisenstein compactification*, whose construction is influenced by but different from Kapranov’s in [29].

The obvious examples of modular forms-to-be for $\Gamma(N)$ are the Eisenstein series of level N . They generate a graded C_∞ -algebra $\mathbf{Eis}(N)$ (generated in dimension 1 if N is non-constant), and $\overline{M}^r(N)$ will be the associated projective variety $\mathrm{Proj}(\mathbf{Eis}(N))$, see Theorem 5.9. It is a closed subvariety of a certain projective space \mathbb{P}^{c-1} , where c is the

number of cusps of $\Gamma(N)$ (Corollary 4.7, Theorem 5.9), and is therefore supplied with a natural very ample line bundle \mathfrak{M} . We define strong modular forms of weight k for $\Gamma(N)$ as sections of $\mathfrak{M}^{\otimes k}$, and thereby get the graded ring $\mathbf{Mod}^{\text{st}}(N)$ of strong modular forms, which encompasses $\mathbf{Eis}(N)$.

(0.5). The Eisenstein compactification is natural and explicit, and has good functorial properties (see Remark 5.10; it is, e.g., compatible with level change), but unfortunately we presently cannot assure that it is normal. Correspondingly, strong modular forms are integral over $\mathbf{Eis}(N)$ (and in fact over $\mathbf{Mod} = \mathbf{Mod}(1)$, the ring of modular forms of type 0 for $\Gamma(1) = \Gamma$), but we don't know whether $\mathbf{Mod}^{\text{st}}(N)$ is integrally closed. We define the Satake compactification $M^r(N)^{\text{Sat}}$ of $M^r(N)$ as the normalization of $\overline{M}^r(N)$ (as Kapranov does) and a modular form of weight k for $\Gamma(N)$ as a section of the pull-back of $\mathfrak{M}^{\otimes k}$ to $M^r(N)^{\text{Sat}}$. This yields the graded ring $\mathbf{Mod}(N)$ of all modular forms. Hence we have inclusions

$$\mathbf{Eis}(N) \subset \mathbf{Mod}^{\text{st}}(N) \subset \mathbf{Mod}(N) \quad (0.6)$$

of finitely generated graded integral C_∞ -algebras, where $\mathbf{Mod}(N)$ is the integral closure of $\mathbf{Eis}(N)$ in their common quotient field $\widetilde{\mathcal{F}}_r(N)$. Elements of $\mathbf{Mod}(N)$ have a nice characterization given by Theorem 7.9: A weak modular form f of weight k is modular if and only if f , together with all its conjugates $f_{[\gamma]_k}$ ($\gamma \in \Gamma/\Gamma(N)$), is bounded on the fundamental domain \mathbf{F} . Further $\mathbf{Eis}(N)$ has always finite codimension in $\mathbf{Mod}^{\text{st}}(N)$ (Corollary 7.11), while $\dim(\mathbf{Mod}(N)/\mathbf{Mod}^{\text{st}}(N))$ is either zero or infinite, according to whether $M^r(N)^{\text{Sat}}$ agrees with $\overline{M}^r(N)$ or not (Corollary 7.14). Except for some examples presented in Section 8, where the two compactifications and also the three rings in (0.6) agree (these examples depend crucially on work of Cornelissen [8] and Pink-Schieder [33]), we don't know what happens in general: more research is needed! At least $M^r(N)^{\text{Sat}}$ is not very far from $\overline{M}^r(N)$: the normalization map

$$\nu : M^r(N)^{\text{Sat}} \longrightarrow \overline{M}^r(N)$$

is bijective on C_∞ -points (Corollary 7.6, see also Proposition 1.18 in [29]), and is an isomorphism on the complement of a closed subvariety of codimension ≥ 2 (Corollary 6.10).

(0.7). We now describe the plan of the paper. In the first section, we introduce the space $\overline{\Omega}^r$ with its strong topology, which upon dividing out the action of $\Gamma(N)$ will yield the underlying topological space for the Eisenstein compactification $\overline{M}^r(N)$. Its points correspond to homothety classes of pairs (U, i) , where $U \neq 0$ is a K -subspace of K^r and i is a discrete embedding of $U \cap A^r$ into C_∞ . For technical purposes we also consider the \mathbb{G}_m -torsor $\overline{\Psi}^r$ over $\overline{\Omega}^r$ whose points correspond to pairs (not homothety classes) (U, i) as above. Further, the fundamental domains $\widetilde{\mathbf{F}}$ on Ψ^r and \mathbf{F} on Ω^r for Γ are introduced. Although $\overline{\Omega}^r$ and $\overline{\Psi}^r$ come with the same information, it will sometimes be

more convenient to work with $\overline{\Psi}^r$ and $\widetilde{\mathbf{F}}$ instead of $\overline{\Omega}^r$ and \mathbf{F} . We take particular care to give a consistent description of the group actions on $\overline{\Omega}^r$ and related objects.

In Section 2 the (well-known) relationship of Ω^r with the moduli of Drinfeld modules of rank r is presented. We further show the crucial technical result Theorem 2.3, which asserts that the bijection

$$j : \Gamma \setminus \overline{\Omega}^r \xrightarrow{\cong} \text{Proj}(\mathbf{Mod})$$

is a homeomorphism for the strong topologies on both sides. We further introduce and describe the function fields of the analytic spaces $M^r(N) = \Gamma(N) \setminus \Omega^r$ and $\widetilde{M}^r(N) = \Gamma(N) \setminus \Psi^r$.

In Sections 3 and 4, the boundary components and the (non-)vanishing of Eisenstein series on them are studied. We find in Corollary 4.7 that the space $\text{Eis}_k(N)$ of Eisenstein series of level N and weight k has dimension $c_r(N)$, the number of cuspidal divisors of $\Gamma(N) \setminus \overline{\Omega}^r$, independently of k . Further (Proposition 4.8), $\text{Eis}_1(N)$ separates points of $\Gamma(N) \setminus \overline{\Omega}^r$, which will give rise to its projective embedding. This latter is defined and investigated in Section 5; we thereby interpret $\Gamma(N) \setminus \overline{\Omega}^r$ as the Eisenstein compactification $\overline{M}^r(N)$ of $M^r(N)$.

Section 6 is of a more technical nature. There we construct tubular neighborhoods along the cuspidal divisors of $\overline{M}^r(N)$, see Theorem 6.9.

In Section 7, the rings $\mathbf{Mod}^{\text{st}}(N)$ and $\mathbf{Mod}(N)$ of modular forms are introduced and their relation with the Eisenstein ring $\mathbf{Eis}(N)$ and the compactifications $\overline{M}^r(N)$ and $M^r(N)^{\text{Sat}}$ is discussed.

We conclude in Section 8 with the two classes of examples where our knowledge is more satisfactory than in the general situation, namely the special cases where either the rank r equals 2 or where the conductor N has degree 1.

(0.8). The point of view (and the notation, see below) of this paper widely agrees with that of the preceding [18], [21], [19], to which we often refer. As in these, our basic references for rigid analytic geometry are the books [12] by Fresnel-van der Put and [7] of Bosch-Güntzer-Remmert. The canonical topology on the set $X(C_\infty)$ of C_∞ -points of an analytic space X ([7] Section 7.2) is labelled as the *strong topology*, so functions continuous with respect to it are strongly continuous, etc. In general, we don't distinguish in notation between X and $X(C_\infty)$; ditto, a C_∞ -variety and its analytification are usually described by the same symbol. It will (hopefully) always be clear from the context whether e.g. the “algebraic” or the “analytic” local ring is intended.

(0.9). After this paper was largely completed, I got access to the recent preprints [4], [5], [6] of Dirk Basson, Florian Breuer, and Richard Pink, which go about the same topic: providing a foundation for the theory of higher rank Drinfeld modular forms. As it turns out, the relative perspectives of Basson-Breuer-Pink's work and of the current paper are rather different. While BBP deal with the most general Drinfeld coefficient rings A and

arithmetic subgroups of $\mathrm{GL}(r, A)$, for which they establish basic but sophisticated facts like e.g. the existence of expansions around infinity of weak modular forms, we restricted to the coefficient ring $A = \mathbb{F}_q[T]$ and full congruence subgroups and focus on the role of Eisenstein series, their arithmetic properties, and their impact on compactifications of the moduli schemes. Apart from examples, there is little overlap between the two works; so the reader who wants to enter into the field might profit from studying the two of them.

Finally, I wish to point to the recent thesis [25] of Simon Häberli, whose purpose is similar. In contrast with the present article, Häberli gives a direct construction of the Satake compactification, which he uses for the description of modular forms.

Notation.

$\mathbb{F} = \mathbb{F}_q$ the finite field with q elements;

$A = \mathbb{F}[T]$ the polynomial ring in an indeterminate T , with quotient field $K = \mathbb{F}(T)$ and its completion $K_\infty = \mathbb{F}((T^{-1}))$ at infinity;

$C_\infty =$ completed algebraic closure of K_∞ , with absolute value $|\cdot|$ and valuation $v: C_\infty^* \rightarrow \mathbb{Q}$ normalized by $v(T) = -1$, $|T| = q$;

$\Psi^r = \{\omega = (\omega_1, \dots, \omega_r) \in C_\infty^r \mid \text{the } \omega_i \text{ are } K_\infty\text{-linearly independent}\}$;

$\Omega^r = \{\omega = (\omega_1 : \dots : \omega_r) \in \mathbb{P}^{r-1}(C_\infty) \mid \omega \text{ represented by } (\omega_1, \dots, \omega_r) \in \Psi^r\}$;

$\Gamma = \Gamma_r = \mathrm{GL}(r, A)$ with center $Z \cong \mathbb{F}^*$ of scalar matrices;

$\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{N}\}$, $N \in A$;

$\mathfrak{U} =$ set of K -subspaces $U \neq 0$ of $V = K^n$;

$\Psi_U \cong \Psi^s$, $\Omega_U \cong \Omega^s$ attached to $U \in \mathfrak{U}$, where $\dim U = s$;

$\overline{\Psi}^r = \bigcup_{U \in \mathfrak{U}} \Psi_U$, $\overline{\Omega}^r = \bigcup_{U \in \mathfrak{U}} \Omega_U$.

If the group G acts on the space X then G_x , Gx and $G \backslash X$ denote the stabilizer of $x \in X$, its orbit, and the space of all orbits, respectively. Also, for $Y \subset X$, $G \backslash Y$ is the image of Y in $G \backslash X$. The multiplicative group of the ring R is R^* ; the R -module generated by x_1, \dots, x_r is written either as $\sum R x_i$ or as $\langle x_1, \dots, x_r \rangle_R$. We use the convention $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

1. The spaces $\overline{\Psi}^r$ and $\overline{\Omega}^r$

(1.1). We let V be the K -vector space K^r , where $r \geq 2$, and \mathfrak{U} the set of K -subspaces $U \neq 0$ of V . An A -lattice in $U \in \mathfrak{U}$ is a free A -submodule L of U of full rank $\mathrm{rk}_A(L) = \dim_K(U)$, that is $K \otimes L = KL = U$. A subset of C_∞ is *discrete* if the intersection with each ball of finite radius in C_∞ is finite. A *discrete embedding* of $U \in \mathfrak{U}$ (“embedding” for short) is some K -linear injective map $i: U \rightarrow C_\infty$ such that $i(L)$ is discrete in C_∞ for one fixed (or equivalently, for each) A -lattice L in U . We put

$$\begin{aligned}\Psi_U &:= \text{set of discrete embeddings of } U, \text{ and} \\ \Omega_U &:= C_\infty^* \setminus \Psi_U, \text{ the quotient of } \Psi_U \text{ modulo the} \\ &\quad \text{action of the multiplicative group } C_\infty^*.\end{aligned}\tag{1.2}$$

Further, $\Psi^r := \Psi_V = \Psi_{K^r}$, $\Omega^r := \Omega_V$, and

$$\overline{\Psi}^r := \bigcup_{U \in \mathfrak{U}} \Psi_U, \quad \overline{\Omega}^r := \bigcup_{U \in \mathfrak{U}} \Omega_U.$$

If $U \subset U' \in \mathfrak{U}$, restriction to U defines canonical maps

$$\Psi_{U'} \longrightarrow \Psi_U \text{ and } \Omega_{U'} \longrightarrow \Omega_U.\tag{1.2.1}$$

(1.3). We let $L_V := A^r$ and $L_U := L_V \cap U$ be the standard lattices in V and U , respectively. As a K -linear map $i : V \rightarrow C_\infty$ is discrete if and only if the images $\omega_j := i(e_j)$ of the standard basis vectors e_j ($1 \leq j \leq r$) are K_∞ -linearly independent (l.i.), we see that

$$\Psi^r = \{\omega = (\omega_1, \dots, \omega_r) \in C_\infty^* \mid \omega_1, \dots, \omega_r \text{ l.i.}\}.$$

After choosing bases of the subspaces U , we get similar descriptions for Ψ_U and the quotients Ω^r and Ω_U . In particular, we find for $r = 2$ the familiar Drinfeld upper half-plane

$$\begin{aligned}\Omega^2 &= C_\infty^* \setminus \{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \text{ l.i.}\} \xrightarrow{\cong} C_\infty \setminus K_\infty. \\ &(\omega_1, \omega_2) \longmapsto \omega_1/\omega_2\end{aligned}$$

(1.4). The sets Ψ^r and Ω^r (and therefore also Ψ_U and Ω_U) are equipped with structures of C_∞ -analytic spaces (actually defined over K_∞), namely as admissible open subspaces of $\mathbb{A}^r(C_\infty) = C_\infty^r$ or of $\mathbb{P}^{r-1}(C_\infty)$, respectively, see [11], [10], or [34].

(1.5). The group $\mathrm{GL}(r, K)$ acts as a matrix group from the right on V , which induces left actions on $\overline{\Psi}^r$ and $\overline{\Omega}^r$, viz.: For $\gamma \in \mathrm{GL}(r, K)$, let $r_\gamma : V \rightarrow V$ be the map $x \mapsto x\gamma$. Then γ maps $(U, i : U \hookrightarrow C_\infty) \in \Psi_U$ to $\gamma(U, i) := (U\gamma^{-1}, i \circ r_\gamma)$. The reader may verify that this, together with the description of Ψ^r in (1.3), yields the standard left matrix action of γ on Ψ^r , the elements of Ψ^r being regarded as column vectors $(\omega_1, \dots, \omega_r)^t$.

(1.6). Since A is a principal ideal domain, the theory of finitely generated modules over such (e.g. [30] XV Sect. 2) shows that $\Gamma := \mathrm{GL}(r, A)$ acts transitively on the set \mathfrak{U}_s of $U \in \mathfrak{U}$ of fixed dimension s . We use as a standard representative for \mathfrak{U}_s the space

$$V_s := \{(0, \dots, 0, *, \dots, *) \in V\}\tag{1.6.1}$$

of vectors whose first $r - s$ entries vanish. The fixed group of V_s ($1 \leq s < r$) in $\mathrm{GL}(r, K)$ is the maximal parabolic subgroup

$$P_s := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad (1.6.2)$$

of matrices with an $(r - s, s)$ -block structure whose lower left block vanishes. The action of P_s on V_s is via the group

$$M_s := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \right\} \quad (1.6.3)$$

regarded as a factor group of P_s .

(1.7). As explained in (1.3), the choice of a K -basis of $U \in \mathfrak{U}$ yields an embedding of Ψ_U into $C_\infty^{\dim U}$. The Hausdorff topology induced on Ψ_U is independent of that choice, and is referred to as the *strong topology* on Ψ_U . Similarly, using embeddings into projective spaces, we define the strong topologies on the Ω_U .

(1.8). Our next aim is to define reasonable strong topologies on $\overline{\Psi}^r$ and $\overline{\Omega}^r$ extending the topologies on the strata. For this we recall the concept of successive minimum bases. An *A-lattice* in C_∞ is a discrete A -submodule Λ of finite rank. A *successive minimum basis* (SMB) of Λ is an ordered A -basis $\{\omega_1, \dots, \omega_r\}$ of Λ (note this differs from usual set-theoretic notation) subject to: For each $1 \leq j \leq r$, $|\omega_j|$ is minimal among

$$\{|\omega| \mid \omega \in \Lambda \setminus (A\omega_1 + \dots + A\omega_{j-1})\}.$$

(For $j = 1$ this means: ω_1 is a lattice vector of minimal non-zero length.) It is shown in [20] Proposition 3.1 that each A -lattice Λ in C_∞ possesses an SMB $\{\omega_1, \dots, \omega_r\}$, and it has the following additional properties:

(1.8.1) The ω_i are orthogonal, that is, given $a_1, \dots, a_r \in K_\infty$,

$$\left| \sum_{1 \leq i \leq r} a_i \omega_i \right| = \max_i |a_i| |\omega_i|;$$

(1.8.2) The series of positive real numbers $|\omega_1| \leq |\omega_2| \leq \dots \leq |\omega_r|$ is an invariant of Λ , that is, independent of the choice of the SMB.

(1.9). We define the strong topology on $\overline{\Psi}^r$ as the unique Hausdorff topology which satisfies for each $U' \in \mathfrak{A}$:

(1.9.1) Restricted to $\Psi_{U'}$, it agrees with the strong topology given there by (1.7);

(1.9.2) The topological closure $\overline{\Psi}_{U'}$ of $\Psi_{U'}$ equals $\bigcup_{U \subset U'}^{\bullet} \Psi_U$;

(1.9.3) Assume $U' \supset U \in \mathfrak{U}$, and let $i : U \rightarrow C_\infty$ and $i_k : U' \rightarrow C_\infty$ ($k \in \mathbb{N}$) be discrete embeddings. Then $(U, i) = \lim_{k \rightarrow \infty} (U', i_k)$ if and only if

- (a) for each $\lambda \in L_U$, $i(\lambda) = \lim_{k \rightarrow \infty} i_k(\lambda)$ and
- (b) for each $\lambda \in L_{U'} \setminus L_U$, $\lim_{k \rightarrow \infty} |i_k(\lambda)| = \infty$, uniformly in λ .

Note that it suffices to require (a) for the elements of a basis of L_U . In qualitative terms, $i_k : U' \hookrightarrow C_\infty$ is very close to $i : U \hookrightarrow C_\infty$ iff

- (a') $i_k(\lambda)$ is very close to $i(\lambda)$ for the elements λ of an A -basis of L_U , and
- (b') for each $\lambda \in L_{U'} \setminus L_U$, $|i_k(\lambda)|$ is very large compared to the $|\omega_j|$, where $\{\omega_j\}$ is an SMB of $i(L_U)$.

Furthermore, (b') may be replaced by

- (b'') for each $\lambda \in L_{U'} \setminus L_U$, $i_k(\lambda)$ has very large distance $d(i_k(\lambda), K_\infty i(U))$ to the K_∞ -space generated by $i(U)$.

The strong topology on $\overline{\Omega}^r = C_\infty^* \setminus \overline{\Psi}^r$ is the quotient topology; it has properties analogous to (1.9.1)–(1.9.3). Obviously, the action of $\mathrm{GL}(r, K)$ on both $\overline{\Psi}^r$ and $\overline{\Omega}^r$ is through homeomorphisms w.r.t. the so defined topologies.

(1.10). A continuous function $f : \overline{\Psi}^r \rightarrow C_\infty$ has *weight* $k \in \mathbb{Z}$ if

$$f(U, c \cdot i) = c^{-k} f(U, i)$$

holds for $c \in C_\infty^*$ and $(U, i) \in \overline{\Psi}^r$.

(1.11). The basic examples of functions with weight are the various types of *Eisenstein series* defined below. For $k \in \mathbb{N}$ put

$$E_k(U, i) := \sum'_{\lambda \in L_U} i(\lambda)^{-k}.$$

(The prime \sum' indicates that the sum is over the non-zero elements of the index set.) The following are obvious or easy to show:

- (i) The sum converges and defines a continuous (even analytic) function E_k on Ψ_U ($U \in \mathfrak{U}$), which is non-trivial if and only if $k \equiv 0 \pmod{q-1}$;

- (ii) E_k is continuous on the whole of $\overline{\Psi}^r$ with respect to the strong topology (due to the very definition of the latter);
- (ii) E_k has weight k ;
- (iv) E_k is invariant under $\mathrm{GL}(r, A)$.

(1.12). Now let N be a non-constant monic element of A and $\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{N}\}$ be the *full congruence subgroup* of level N . Fix some vector $\mathbf{u} = (u_1, \dots, u_r) \in V = K^r$ with $N\mathbf{u} \in L_V = A^r$, and put

$$E_{k,\mathbf{u}}(U, i) := \sum'_{\substack{\lambda \in U \\ \lambda \equiv \mathbf{u} \pmod{L_V}}} i(\lambda)^{-k}.$$

The following hold:

- (i') The sum converges and defines a continuous (even analytic) function $E_{k,\mathbf{u}}$ on Ψ_U ; it depends only on the residue class of \mathbf{u} modulo L_V , and is called the *partial Eisenstein series* with congruence condition \mathbf{u} ;
- (ii), (iii) (see (1.11)), and
- (iv') $E_{k,\mathbf{u}\gamma}(U, i) = E_{k,\mathbf{u}}(\gamma(U, i))$, $\gamma \in \Gamma$. In particular, $E_{k,\mathbf{u}}$ is invariant under $\Gamma(N)$.

Remark.

$$E_{k,\mathbf{u}}(U, i) = N^{-k} \sum'_{\substack{\lambda \in L_U \\ \lambda \equiv N\mathbf{u} \pmod{NL_V}}} i(\lambda)^{-k},$$

which up to the factor N^{-k} is a partial sum of $E_k(U, i)$. This explains the notation “partial Eisenstein series”.

(1.13). By definition, $\omega_r \neq 0$ for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Psi^r$. Therefore we can normalize projective coordinates on $\Omega^r \subset \mathbb{P}^{r-1}(C_\infty)$ so that

$$(1.13.1) \quad \omega_r = 1, \text{ i.e.,}$$

$$\Omega^r = \{(\omega_1, \dots, \omega_{r-1}) = (\omega_1 : \dots : \omega_{r-1} : 1) \mid \omega_1, \dots, \omega_{r-1}, \omega_r = 1 \text{ l.i.}\}.$$

Similarly we usually assume $\omega_r = 1$ for $\boldsymbol{\omega} = (\omega_1 : \dots : \omega_r) \in \Omega_U$ if U is one of the spaces V_s of (1.6). With that convention, the Eisenstein series E_k and $E_{k,\mathbf{u}}$ may be regarded as functions on $\bigcup_{1 \leq s \leq r} \Omega_{V_s}$. If $\boldsymbol{\omega} \in \Omega^r$ then (iii), (iv), (iv') imply

$$E_k(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^k E_k(\boldsymbol{\omega}) \quad (1.13.2)$$

and

$$E_{k,\mathbf{u}}(\gamma\boldsymbol{\omega}) = \text{aut}(\gamma, \boldsymbol{\omega})^k E_{k,\mathbf{u}\gamma}(\boldsymbol{\omega}). \quad (1.13.3)$$

Here $\gamma \in \Gamma$, $\boldsymbol{\omega} = (\omega_1 : \dots : \omega_r)$ with $\omega_r = 1$, and $\text{aut}(\gamma, \boldsymbol{\omega})$ is the factor of automorphy

$$\text{aut}(\gamma, \boldsymbol{\omega}) = \sum_{1 \leq i \leq r} \gamma_{r,i} \omega_i \neq 0. \quad (1.13.4)$$

We assign no value to $E_k(\boldsymbol{\omega})$ or $E_{k,\mathbf{u}}(\boldsymbol{\omega})$ if $\boldsymbol{\omega} = C_\infty^*(U, i) \in \overline{\Omega}^r$ does not belong to $\bigcup \Omega_{V_s}$, but are content with the distinction (always well-defined) of whether E_k (resp. $E_{k,\mathbf{u}}$) vanishes at $\boldsymbol{\omega}$ or not.

1.14 Remark (on notation). In order to avoid notational overflow, we use the same symbol E_k for both occurrences: as a Γ -invariant function on $\overline{\Psi}^r$ of weight k , or as a function on $\bigcup \Omega_{V_s}$ subject to (1.13.2). A similar remark applies to $E_{k,\mathbf{u}}$ and to other functions with weight.

(1.15). We finally define fundamental domains for the actions of Γ on Ψ^r and Ω^r . To wit, put

$$\begin{aligned} \tilde{\mathbf{F}} &:= \{\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Psi^r \mid \{\omega_r, \omega_{r-1}, \dots, \omega_1\} \text{ is an SMB} \\ &\quad \text{of its lattice } \Lambda_{\boldsymbol{\omega}} = \langle \omega_1, \dots, \omega_r \rangle_A\} \\ \mathbf{F} &:= C_\infty^* \setminus \tilde{\mathbf{F}}. \end{aligned}$$

(Note the reverse order of the ω_i !) They have the following properties.

(1.15.1) As the condition for $\boldsymbol{\omega} \in \tilde{\mathbf{F}}$ is stable under the multiplicative group, $\tilde{\mathbf{F}}$ is the full cone above \mathbf{F} .

(1.15.2) Each $\boldsymbol{\omega} \in \Psi^r$ (resp. Ω^r) is Γ -equivalent with at least one and at most a finite number of $\boldsymbol{\omega}' \in \tilde{\mathbf{F}}$ (resp. $\boldsymbol{\omega}' \in \mathbf{F}$).

Proof. It suffices to treat the case $\tilde{\mathbf{F}}$. As each A -lattice Λ in C_∞ has an SMB, the existence of a representative $\boldsymbol{\omega}' \in \tilde{\mathbf{F}}$ for $\boldsymbol{\omega} \in \Psi^r$ is obvious. Given $\boldsymbol{\omega} \in \tilde{\mathbf{F}}$, the condition $\gamma\boldsymbol{\omega} \in \tilde{\mathbf{F}}$ on $\gamma \in \Gamma$ together with (1.8.1) leads to bounds on the entries of γ , which can be satisfied for a finite number of γ 's only. \square

(1.15.3) $\tilde{\mathbf{F}}$ resp. \mathbf{F} is an admissible open subspace of Ψ^r resp. Ω^r .

The most intuitive way to see this comes from identifying \mathbf{F} as the inverse image under the building map $\lambda : \Omega^r \rightarrow \mathcal{BT}(\mathbb{Q})$ of a subcomplex W of the Bruhat-Tits building \mathcal{BT} of $\text{PGL}(r, K_\infty)$: see [18] Sect. 2. In fact, W is a fundamental domain for Γ on \mathcal{BT} .

In view of the above, we refer to $\tilde{\mathbf{F}}$ resp. \mathbf{F} as the *fundamental domain* for Γ on Ψ^r resp. Ω^r . As uniqueness of the representative in $\tilde{\mathbf{F}}$ resp. \mathbf{F} fails, this is weaker than the classical notion of fundamental domain, but is still useful. Property (1.8.1) turns out particularly valuable for explicit calculations with modular forms, as exemplified in [18]. Also useful is the following observation, which is immediate from definitions. We formulate it for \mathbf{F} only, but it holds true also for $\tilde{\mathbf{F}}$.

(1.15.4) Let \mathbf{F}_s be the fundamental domain for $\Gamma_s = \mathrm{GL}(s, A)$ in $\Omega_{V_s} \xrightarrow{\cong} \Omega^s$ ($1 \leq s \leq r$). Then the strong closure of \mathbf{F} in $\overline{\Omega}^r$ is $\overline{\mathbf{F}} = \bigcup_{1 \leq s \leq r} \mathbf{F}_s$. Each point of $\overline{\Omega}^r$ is Γ -equivalent with at least one and at most a finite number of points of $\overline{\mathbf{F}}$.

Therefore, we can regard $\overline{\mathbf{F}}$ as a fundamental domain for Γ on $\overline{\Omega}^r$.

2. Quotients by congruence subgroups and moduli schemes

(2.1). Given an A -lattice Λ in C_∞ of rank $r \in \mathbb{N}$, we dispose of

- the exponential function $e_\Lambda : C_\infty \longrightarrow C_\infty$

$$e_\Lambda(z) = z \prod'_{\lambda \in \Lambda} (1 - z/\lambda) = \sum_{i \geq 0} \alpha_i(\Lambda) z^{q^i}; \quad (2.1.1)$$

- the Drinfeld A -module ϕ^Λ of rank r , defined by the operator polynomial

$$\phi_T^\Lambda(X) = TX + g_1(\Lambda)X^q + \cdots + g_r(\Lambda)X^{q^r}, \text{ and} \quad (2.1.2)$$

- the Eisenstein series

$$E_k(\Lambda) = \sum'_{\lambda \in \Lambda} \lambda^{-k} \quad (k \in \mathbb{N}). \quad (2.1.3)$$

We further put $g_0(\Lambda) = T$, $E_0(\Lambda) = -1$. These are connected by

$$e_\Lambda(Tz) = \phi_T^\Lambda(e_\Lambda(z)); \quad (2.1.4)$$

$$\sum_{\substack{i, j \geq 0 \\ i+j=k}} \alpha_i E_{q^j-1}^{q^i} = \sum_{i+j=k} \alpha_i^{q^j} E_{q^j-1} = 1 \text{ if } k=0 \text{ and } 0 \text{ otherwise,} \quad (2.1.5)$$

which determines a number of further relations, see e.g. [16] Sect. 2.

If $\Lambda = \Lambda_\omega = \sum_{1 \leq i \leq r} A\omega_i$ with $\omega = (\omega_1, \dots, \omega_r) \in \Psi^r$, we use ω instead of Λ as the argument. Thus $\phi^\omega = \phi^{\Lambda_\omega}$, $e_\omega = e_{\Lambda_\omega}$, etc. As functions on Ψ^r , g_i , α_i are - like the Eisenstein series - holomorphic and Γ -invariant of weight $q^i - 1$, while considered as functions on Ω^r , g_i (and α_i) satisfies

$$g_i(\gamma\omega) = \text{aut}(\gamma, \omega)^{q^i-1} g_i(\omega)$$

(see Remark 1.14).

(2.1.6) The three systems of functions on Ψ^r : $\{g_1, \dots, g_r\}$, $\{\alpha_1, \dots, \alpha_r\}$, $\{E_{q^i-1} \mid 1 \leq i \leq r\}$ are each algebraically independent, and the relations between them are such that the ring

$$\mathbf{Mod} = \bigoplus_{k \geq 0} \mathbf{Mod}_k = C_\infty[g_1, \dots, g_r],$$

graded by the weight $\text{wt}(g_i) := q^i - 1$, may also be described as

$$C_\infty[\alpha_1, \dots, \alpha_r] = C_\infty[\alpha_i \mid i \in \mathbb{N}] = C_\infty[E_{q^i-1} \mid 1 \leq i \leq r] = C_\infty[E_{q^i-1} \mid i \in \mathbb{N}].$$

(Actually \mathbf{Mod} is the ring of modular forms of type 0 for Γ , see [18].)

(2.1.7) As a consequence, since the g_i and α_i may be expressed through Eisenstein series, they have strongly continuous extensions to $\overline{\Psi}^r$ and may therefore be evaluated on arbitrary points $\omega = (U, i) \in \overline{\Psi}^r$.

(2.2). The Drinfeld modules ϕ^ω and $\phi^{\omega'}$ ($\omega, \omega' \in \Omega^r$) are isomorphic if and only if $\omega' = \gamma\omega$ with some $\gamma \in \Gamma$.

Hence the map

$$\begin{aligned} j : \Gamma \backslash \Omega^r &\hookrightarrow \text{Proj } \mathbf{Mod} \\ \omega &\longmapsto (g_1(\omega) : \dots : g_r(\omega)) \end{aligned}$$

identifies the quotient analytic space of Ω^r modulo Γ with the complement of the vanishing locus of $\Delta := g_r$ in the weighted projective space $\overline{M}^r = \text{Proj } \mathbf{Mod}$. (We remind the reader that we do not distinguish in notation between a C_∞ -variety, its associated analytic space, and the set of its C_∞ -points.) Here the g_i are considered as formal variables of weight $q^i - 1$, that is $(x'_1 : \dots : x'_r) = (x_1 : \dots : x_r)$ in $\text{Proj } \mathbf{Mod}$ if and only if there exists $c \in C_\infty^*$ such that $x'_i = c^{q^i-1} x_i$ for all i . In other words, via j

$$\Gamma \backslash \Omega^r \xrightarrow{\cong} M^r := (\text{Proj } \mathbf{Mod})_{(g_r \neq 0)} \quad (2.2.1)$$

equals (the set of C_∞ -points of) the moduli scheme M^r for Drinfeld A -modules of rank r over C_∞ . The natural compactification of M^r is

$$\text{Proj } \mathbf{Mod} = \overline{M}^r = M^r \cup M^{r-1} \cup \dots \cup M^1, \quad (2.2.2)$$

where for $1 \leq s \leq r$,

$$(\Gamma \cap P_s) \backslash \Omega_{V_s} = \text{GL}(s, A) \backslash \Omega^s \xrightarrow{\cong} M^s \quad (\text{see (1.6)})$$

and $\Omega^1 = M^1 = \{\text{point}\}$. Hence the stratification of the variety \overline{M}^r corresponds to that of

$$\Gamma \setminus \overline{\Omega}^r = \Gamma \setminus \left(\bigcup_{\substack{1 \leq s \leq r \\ U \in \mathfrak{U}_s}} \Omega_U \right) = \bigcup_{1 \leq s \leq r} \text{GL}(s, A) \setminus \Omega^s \quad (2.2.3)$$

under the bijection

$$\begin{aligned} j : \Gamma \setminus \overline{\Omega}^r &\xrightarrow{\cong} \overline{M}^r \\ \omega &\mapsto (g_1(\omega) : \dots : g_r(\omega)), \end{aligned} \quad (2.2.4)$$

which is well-defined in view of (2.1.7).

In a similar way (although this looks a bit artificial), we may describe $\Gamma \setminus \Psi^r$ via

$$\begin{aligned} \tilde{j} : \Gamma \setminus \Psi^r &\hookrightarrow \mathbb{A}^r(C_\infty) \\ \omega &\mapsto (g_1(\omega), \dots, g_r(\omega)), \end{aligned} \quad (2.2.5)$$

as the complement \widetilde{M}^r of $(g_r = 0)$ in \mathbb{A}^r . It is the moduli scheme of rank- r Drinfeld A -modules over C_∞ with a “non-vanishing differential”, that is, with an identification of the underlying additive group with \mathbb{G}_a or, what is the same, with explicit coefficients g_i of its T -operator polynomial. The horizontal compactification $\Gamma \setminus \overline{\Psi}^r$ then becomes

$$\begin{aligned} \Gamma \setminus \overline{\Psi}^r &= \Gamma \setminus \left(\bigcup_{\substack{1 \leq s \leq r \\ U \in \mathfrak{U}_s}} \Psi_U \right) = \bigcup_{1 \leq s \leq r} \text{GL}(s, A) \setminus \Psi^s \\ &\xrightarrow[\tilde{j}]{\cong} \bigcup_{1 \leq s \leq r} \widetilde{M}^s =: \widetilde{M}^r = C_\infty^r \setminus \{0\}, \end{aligned} \quad (2.2.6)$$

in analogy with (2.2.2), (2.2.3), (2.2.4).

(2.2.7) In the sequel, whenever writing $\Gamma \setminus \Omega^r = M^r$ or $\Gamma \setminus \Psi^r = \widetilde{M}^r$, the identification is via j or \tilde{j} , respectively.

2.3 Theorem. *The map $j : \Gamma \setminus \overline{\Omega}^r \xrightarrow{\cong} \text{Proj } \mathbf{Mod} = \text{Proj } C_\infty[g_1, \dots, g_r]$ of (2.2.4) is a strong homeomorphism, i.e., with respect to the strong topologies on both sides. Similarly, $\tilde{j} : \Gamma \setminus \overline{\Psi}^r \xrightarrow{\cong} \widetilde{M}^r$ is a strong homeomorphism.*

Proof. The proof for j will also show the statement for \tilde{j} .

- (i) By construction, j is continuous as a map from $\overline{\Omega}^r$, and thus as a map from $\Gamma \setminus \overline{\Omega}^r$ supplied with the quotient topology. Therefore we must show that j^{-1} is continuous.

- (ii) Let $(\phi^{(n)})_{n \in \mathbb{N}}$ be a series of Drinfeld modules of rank $\leq r$, given by their T -division polynomials $\phi_T^{(n)}(X) = \sum_{0 \leq i \leq r} g_i^{(n)} X^{q^i}$ and converging to ϕ with $\phi_T(X) = \sum g_i X^{q^i}$. This means that $\mathbf{g}^{(n)} = (g_1^{(n)} : \dots : g_r^{(n)})$ converges to $\mathbf{g} = (g_1 : \dots : g_r)$. Let s be the rank of ϕ , i.e., $g_s \neq 0, g_{s+1} = \dots = g_r = 0$. We may suppose that $g_s = \lim_{n \rightarrow \infty} g_s^{(n)} = 1$. Let $\Lambda^{(n)}$ (resp. Λ) be the lattice associated to $\phi^{(n)}$ (resp. ϕ), each provided with an SMB $\{\omega_r^{(n)}, \omega_{r-1}^{(n)}, \dots, \omega_1^{(n)}\}$ (resp. $\{\omega_r, \dots, \omega_{r-s+1}\}$), where we have put $\omega_i^{(n)} = 0$ for $i \leq r - \text{rk}(\phi^{(n)}) = r - \text{rk}_A(\Lambda^{(n)})$. Put $\boldsymbol{\omega} := (0 : \dots : 0 : \omega_{r-s+1} : \dots : \omega_r)$ and $\boldsymbol{\omega}^{(n)} := (\omega_1^{(n)} : \dots : \omega_r^{(n)})$, and let $[\boldsymbol{\omega}]$ resp. $[\boldsymbol{\omega}^{(n)}]$ be the corresponding class in $\Gamma \backslash \overline{\Omega}^r$. Then we must show that $\lim_{n \rightarrow \infty} [\boldsymbol{\omega}^{(n)}] = [\boldsymbol{\omega}]$. Note that we suppress here our usual assumption $\omega_r = 1$, which would conflict with the normalization $g_s = \lim_{n \rightarrow \infty} g_s^{(n)} = 1$.
- (iii) If $s = r$, we are done. This follows from the fact that $j : \Gamma \backslash \Omega^r \xrightarrow{\cong} M^r$ is an isomorphism of analytic spaces, thus a strong homeomorphism. Hence we may suppose that $s < r$.
- (iv) Consider the Newton polygon $\text{NP}(\phi)$ of ϕ_T , i.e., the lower convex hull of the vertices $(q^i, v(g_i))$ with $0 \leq i \leq s$ in the plane (see [31] II Sect. 6). If $\text{rk}(\phi^{(n)}) \leq s$ for $n \gg 0$, then in fact $\text{rk}(\phi^{(n)}) = s$ for $n \gg 0$, and we are ready as in (iii). Therefore, possibly restricting to a subsequence, we may assume that $\text{rk}(\phi^{(n)}) > s$ for $n \gg 0$. Then if $\phi^{(n)}$ is sufficiently close to ϕ , the Newton polygon $\text{NP}(\phi^{(n)})$ agrees with $\text{NP}(\phi)$ from the leftmost vertex $(1, -1)$ up to $(q^s, 0)$ and, since $g_{s+1}^{(n)}, \dots, g_r^{(n)}$ tend to zero:

(2.3.1) The slope of $\text{NP}(\phi^{(n)})$ right to $(q^s, 0)$ tends to infinity if $n \rightarrow \infty$.

- (v) Considering [20] (3.3), (3.4), (3.5), the assertion (2.3.1) implies that the quotient $|\omega_{r-s}^{(n)}|/|\omega_{r-s+1}^{(n)}|$ (i.e., the quotient of absolute values of the $(s+1)$ -th divided by the s -th element of our SMB $\{\omega_r^{(n)}, \omega_{r-1}^{(n)}, \dots\}$) tends to infinity with $n \rightarrow \infty$.
- (vi) Let ${}^s\phi^{(n)}$ be the rank- s Drinfeld module that corresponds to the lattice ${}^s\Lambda^{(n)} = A\omega_r^{(n)} + \dots + A\omega_{r-s+1}^{(n)}$, with

$${}^s\phi_T^{(n)}(X) = \sum_{0 \leq i \leq s} {}^s g_i^{(n)} X^{q^i}, \quad {}^s\mathbf{g}^{(n)} := ({}^s g_1^{(n)} : \dots : {}^s g_s^{(n)} : 0 : \dots : 0).$$

Then $\lim_{n \rightarrow \infty} (g_i^{(n)} - {}^s g_i^{(n)}) = 0$, as follows from (v). (The analogous statement for the Eisenstein series E_{q^i-1} , to wit

$$\lim_{n \rightarrow \infty} (E_{q^i-1}(\Lambda^{(n)}) - E_{q^i-1}({}^s\Lambda^{(n)})) = 0,$$

is obvious; then we use the fact (2.1.6) that the g_i are polynomials in the E_k .)

As $g_i^{(n)} \rightarrow g_i$ for $1 \leq i \leq r$, we find ${}^s\mathbf{g}^{(n)} \rightarrow \mathbf{g}$ and therefore ${}^s\phi^{(n)} \rightarrow \phi$ in $\overline{M}^s = \text{Proj } C_\infty[g_1, \dots, g_s]$ with respect to the strong topology.

- (vii) Denote by ${}^s\omega^{(n)}$ the point $(0 : \dots : 0 : \omega_{r-s+1}^{(n)} : \dots : \omega_r^{(n)})$ in $\Omega_{V_s} \hookrightarrow \overline{\Omega}^r$. Applying (iii) with r replaced by s and using the identification $\Omega^s \xrightarrow{\cong} \Omega_{V_s}$,

$$\lim_{n \rightarrow \infty} [{}^s\omega^{(n)}] = [\omega] \quad (2.3.2)$$

holds in $\mathrm{GL}(s, A) \setminus \Omega_{V_s} \hookrightarrow \Gamma \setminus \overline{\Omega}^r$, where $[\cdot]$ is the class modulo Γ . But (2.3.2) together with (v) means that $[\omega^{(n)}]$ tends strongly to $[\omega]$ in $\Gamma \setminus \overline{\Omega}^r$. \square

(2.4). We want to give similar descriptions for the quotients $\Gamma' \setminus \overline{\Omega}^r$ and $\Gamma' \setminus \overline{\Psi}^r$, where $\Gamma' \subset \Gamma$ is a congruence subgroup and $r \geq 2$. This turns out, however, to be much more difficult. We restrict to deal with the case where $\Gamma' = \Gamma(N)$, the full congruence subgroup of level N .

Fix a monic $N \in A$ of degree $d \geq 1$, and write the N -th division polynomial of the Drinfeld module ϕ^ω ($\omega \in \Psi^r$) as

$$\phi_N^\omega(X) = \sum_{0 \leq i \leq rd} \ell_i(N, \omega) X^{q^i}$$

with $\ell_0(N, \omega) = N$, $\ell_{rd}(N, \omega) = \Delta(\omega)^{(q^{rd}-1)/(q^r-1)}$, where $\Delta(\omega) = g_r(\omega)$ is the discriminant function and, more generally, all the coefficient functions $\ell_i(N, \cdot)$ lie in $\mathbf{Mod} = C_\infty[g_1, \dots, g_r]$. It satisfies

$$\phi_N^\omega(X) = \Delta(\omega)^{(q^{rd}-1)/(q^r-1)} \prod_{(\mathbf{u} \in N^{-1}A/A)^r} (X - e_\omega(\mathbf{u}\omega)). \quad (2.4.1)$$

Here \mathbf{u} runs through a system of representatives of the finite A -module $(N^{-1}A/A)^r$ and $\mathbf{u}\omega = \sum_{1 \leq i \leq r} u_i \omega_i$. That is, the

$$d_{\mathbf{u}}(\omega) := e_\omega(\mathbf{u}\omega) \quad (2.4.2)$$

are the N -division points of ϕ^ω . It is known (see [23] Proposition 2.7 or [13] 3.3.5) that for $\mathbf{u} \neq 0$,

$$d_{\mathbf{u}}(\omega) = E_{\mathbf{u}}(\omega)^{-1} \quad (2.4.3)$$

with the partial Eisenstein series of weight 1 (see (1.12))

$$E_{\mathbf{u}} := E_{1, \mathbf{u}}. \quad (2.4.4)$$

We draw the conclusions

(2.4.5) $E_{\mathbf{u}}$ never vanishes on Ψ^r and Ω^r ;

(2.4.6) The coefficient $\ell_i(N, \omega)$ may be expressed as a homogeneous polynomial in the $E_{\mathbf{u}}$ ($\mathbf{u} \neq 0$); more precisely,

$$\ell_i(N, \omega) = N s_{q^i-1}(E_{\mathbf{u}}(\omega) \mid 0 \neq \mathbf{u} \in (N^{-1}A/A)^r),$$

where s_k is the k -th elementary symmetric polynomial.

We let $\mathcal{T}(N)$ be the index set

$$\mathcal{T}(N) := (N^{-1}A/A)^r \setminus \{0\}. \quad (2.4.7)$$

(2.5). As the definition of fields of meromorphic functions on non-complete analytic spaces requires some boundary conditions, we make the following *ad hoc* definitions. They are motivated from the fact that the analytic spaces \widetilde{M}^r , $\widetilde{M}^r(N)$, M^r , $M^r(N)$ appearing below are actually C_{∞} -varieties (see (2.2.1), (2.2.5) and Remark 2.7) and that by GAGA the algebraic and the analytic function fields of projective C_{∞} -varieties agree.

(2.5.1) The function field of $\widetilde{M}^r = \Gamma \setminus \Psi^r$ is

$$\widetilde{\mathcal{F}}_r = \widetilde{\mathcal{F}}_r(1) := C_{\infty}(g_1, \dots, g_r);$$

(2.5.2) The function field of $\widetilde{M}^r(N) := \Gamma(N) \setminus \Psi^r$ is $\widetilde{\mathcal{F}}_r(N)$, the field of those meromorphic functions on $\widetilde{M}^r(N)$ which are algebraic over $\widetilde{\mathcal{F}}_r$;

(2.5.3) The function field of $M^r = \Gamma \setminus \Omega^r$ is

$$\mathcal{F}_r = \mathcal{F}_r(1) := C_{\infty}(g_1, \dots, g_r)_0,$$

the subfield of isobaric elements of weight 0 of $\widetilde{\mathcal{F}}_r$;

(2.5.4) The function field of $M^r(N) := \Gamma(N) \setminus \Omega^r$ is $\mathcal{F}_r(N)$, the field of meromorphic functions on $M^r(N)$ algebraic over \mathcal{F}_r .

2.6 Proposition.

(i) The field $\widetilde{\mathcal{F}}_r(N)$ is generated over C_{∞} by the Eisenstein series $E_{\mathbf{u}} = E_{1,\mathbf{u}}$ ($\mathbf{u} \in \mathcal{T}(N)$). It is galois over $\widetilde{\mathcal{F}}_r$ with Galois group

$$\widetilde{G}(N) := \{\gamma \in \mathrm{GL}(r, A/M) \mid \det \gamma \in \mathbb{F}^*\}.$$

(ii) The field $\mathcal{F}_r(N)$ is generated over C_{∞} by the functions $E_{\mathbf{u}}/E_{\mathbf{v}}$ ($\mathbf{u}, \mathbf{v} \in \mathcal{T}(N)$). It is galois over \mathcal{F}_r with group $G(N) := \widetilde{G}(N)/Z$. Here $Z \cong \mathbb{F}^*$ is the subgroup of $\widetilde{G}(N)$ of scalar matrices with entries in \mathbb{F}^* .

- Proof.** (i) As Γ acts without fixed points on Ψ^r , $\widetilde{M}^r(N) = \Gamma(N) \setminus \Psi^r$ is an étale Galois cover of $\widetilde{M}^r = \Gamma \setminus \Psi^r$ with group $\Gamma/\Gamma(N) \xrightarrow{\cong} \widetilde{G}(N)$. Now $E_{\mathbf{u}}$ is $\Gamma(N)$ -invariant and, as (2.4.1) and (2.4.3) show, algebraic over \widetilde{F}_r , i.e., $E_{\mathbf{u}} \in \widetilde{\mathcal{F}}_r(N)$. Furthermore, the relation $E_{\mathbf{u}}(\gamma\omega) = E_{\mathbf{u}\gamma}(\omega)$ for $\omega \in \Psi^r$, $\gamma \in \Gamma$ implies $\gamma = 1$ if $\gamma \in \widetilde{G}(N)$ fixes all the $E_{\mathbf{u}}$. Therefore, $\widetilde{\mathcal{F}}_r(N) = \widetilde{\mathcal{F}}_r(E_{\mathbf{u}} \mid \mathbf{u} \in \mathcal{T}(N))$ by Galois theory. In view of (2.4.6) the coefficient functions $\ell_i(N, \cdot)$ and therefore (by the well-known commutation relations between the $g_i(\cdot)$ and the $\ell_i(N, \cdot)$) also the g_i are polynomials in the $E_{\mathbf{u}}$. Thus in fact $\widetilde{\mathcal{F}}_r(N) = C_{\infty}(E_{\mathbf{u}} \mid \mathbf{u} \in \mathcal{T}(N))$.
- (ii) The argument for $\mathcal{F}_r(N)$ is similar. The quotient $G(N) = \widetilde{G}(N)/Z$ by Z as a Galois group comes from the fact that Z acts trivially on Ω^r . \square

Remark. By (2.4.3) we may also write $\widetilde{F}_r(N) = C_{\infty}(d_{\mathbf{u}} \mid \mathbf{u} \in \mathcal{T}(N))$ and $\mathcal{F}_r(N) = C_{\infty}(d_{\mathbf{u}}/d_{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathcal{T}(N))$.

2.7 Remark. As is well known, the smooth analytic space $M^r(N) = \Gamma(N) \setminus \Omega^r$ is strongly related with the moduli scheme $M^r(N)/K$ of Drinfeld A -modules of rank r with a structure of level N ([11], [10], [15]). Let $K(N) \subset C_{\infty}$ be the field extension of K generated by the N -division points of the Carlitz module. Then $K(N)/K$ is finite abelian with group $(A/N)^*$ and ramification properties similar to those of cyclotomic extensions of \mathbb{Q} [27]. Let $K_+(N) \subset K(N)$ be the fixed field of $\mathbb{F}^* \hookrightarrow (A/N)^*$, the “maximal real subextension” of $K(N)|K$. Then $K_+(N)$ is contained in $\mathcal{K}_r(N) = K(E_{\mathbf{u}}/E_{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathcal{T}(N))$, and is actually the algebraic closure of K in $\mathcal{K}_r(N)$. Now $M^r(N)/K$ is a smooth K -scheme with function field $\mathcal{K}_r(N)$, whose set of C_{∞} -points (in fact, its analytification over C_{∞}) is given by

$$(M^r(N)/K)(C_{\infty}) \xrightarrow{\cong} \bigcup_{\sigma}^{\bullet} M^r(N)_{\sigma} = \bigcup_{\sigma}^{\bullet} (\Gamma(N) \setminus \Omega^r)_{\sigma},$$

where σ runs through the set of K -embeddings of $K_+(N)$ into C_{∞} , i.e., the Galois group $\text{Gal}(K_+(N)|K) = (A/N)^*/\mathbb{F}^*$. Correspondingly, the analytification of $M^r(N)/K \times_{K_+(N)} C_{\infty}$ is $M^r(N)$, which justifies our notation $M^r(N)$ for $\Gamma(N) \setminus \Omega^r$. In the language of pre-Grothendieck algebraic geometry, $M^r(N)/K$ is “defined over $K_+(N)$ ”.

The group $\text{GL}(r, A/N)$ acts naturally on the set of N -level structures of a fixed Drinfeld module of rank r , thus on $M^r(N)/K$, which identifies $M^r/K = (\text{Proj } K[g_1, \dots, g_r])_{(g_r \neq 0)}$ with the quotient of $M^r(N)/K$ by this group. Moreover, the action is compatible with that of $G(N) = \Gamma/\Gamma(N)Z \hookrightarrow \text{GL}(r, A/N)/Z$ on the components $M^r(N)_{\sigma}$. All of this may be transferred to the spaces $\widetilde{M}^r(N) = \Gamma(N) \setminus \Psi^r$ and their function fields $\widetilde{\mathcal{F}}_r(N)$. As we don’t really need it, we omit the details.

In the sequel of the paper, we will construct a compactification $\overline{M}^r(N)$ of $M^r(N)$ (and, similarly, a horizontal compactification of $\widetilde{M}^r(N)$), i.e., a projective C_{∞} -variety

$\overline{M}^r(N)$ with set of C_∞ -points $\overline{M}^r(N)(C_\infty) = \Gamma(N) \setminus \overline{\Omega}^r$, into which $M^r(N)$ embeds as a dense open subvariety, and compatible with the above-described group actions.

3. The boundary components

From now on, we assume that $r \geq 2$.

The set \mathfrak{U}_s of s -dimensional subspaces U of $V = K^r$ is in canonical bijection with $\mathrm{GL}(r, K)/P_s(K)$ through

$$\begin{aligned} \mathrm{GL}(r, K)/P_s(K) &\xrightarrow{\cong} \mathfrak{U}_s. \\ \gamma &\longmapsto V_s \gamma^{-1} \end{aligned} \quad (3.1)$$

As the action of Γ on \mathfrak{U}_s is transitive, we may replace the left hand side with $\Gamma/\Gamma \cap P_s(K)$. Let $(A/N)_{\mathrm{prim}}^r$ be the set of primitive elements of $(A/N)^r$, that is, of elements that belong to a basis of the free (A/N) -module $(A/N)^r$. Then, as is easily verified, the map

$$\Gamma(N) \setminus \Gamma/\Gamma \cap P_{r-1}(K) \longrightarrow (A/N)_{\mathrm{prim}}^r/\mathbb{F}^* =: \mathcal{C}_r(N) \quad (3.2)$$

that associates with the double class of $\gamma \in \Gamma$ the first column of Γ (evaluated modulo N , and modulo the scalar action of \mathbb{F}^*) is well-defined and bijective. Together with (3.1) we find that the space of orbits on \mathfrak{U}_{r-1} of $\Gamma(N)$ is

$$\Gamma(N) \setminus \mathfrak{U}_{r-1} \xrightarrow{\cong} \mathcal{C}_r(N). \quad (3.3)$$

This allows us to describe the components of codimension 1 of

$$\Gamma(N) \setminus \overline{\Psi}^r = \bigcup_{1 \leq s \leq r} \bigcup_{U \in \Gamma(N) \setminus \mathfrak{U}_s} \Gamma_U(N) \setminus \Psi_U \quad (3.4)$$

and, analogously, of $\Gamma(N) \setminus \overline{\Omega}^r$. Here $\Gamma_U = \{\gamma \in \Gamma \mid U\gamma = U\}$, which acts from the left on Ψ_U (see (1.5)), and $\Gamma_U(N) := \Gamma_U \cap \Gamma(N)$. We put

$$\widetilde{M}_U(N) := \Gamma_U(N) \setminus \Psi_U \text{ and } M_U(N) := \Gamma_U(N) \setminus \Omega_U \quad (3.5)$$

and call the components $\widetilde{M}_U(N)$ resp. $M_U(N)$ with $\dim(U) = r - 1$ the *cuspidal divisors* or simply the *cusps* of $\Gamma(N) \setminus \overline{\Psi}^r$ (or $\Gamma(N) \setminus \overline{\Omega}^r$, or of $\Gamma(N)$). Each of these sets is in canonical bijection with $\mathcal{C}_r(N)$. For later use we specify a system of representatives, namely the set S of monic elements of $(A/N)_{\mathrm{prim}}^r$. Here, some $\mathbf{n} = (n_1, \dots, n_r) \in (A/N)_{\mathrm{prim}}^r$ is *monic* if the first non-vanishing n_i has a monic representative $n'_i \in A$ of degree less than $d = \deg N$.

The cardinality $c_r(N)$ of $\mathcal{C}_r(N)$ is an easy arithmetic function of N and r , given by the following formula.

3.6 Lemma. Let $N = \prod_{1 \leq i \leq t} \mathfrak{p}_i^{s_i}$ be the decomposition of N into powers of different primes \mathfrak{p}_i of A . Write $q_i = q^{\deg \mathfrak{p}_i}$. Then

$$c_r(N) = (q-1)^{-1} \prod_{1 \leq i \leq t} (q_i^r - 1) q_i^{(s_i-1)r}.$$

Proof. We must determine $\#(A/N)_{\text{prim}}^r$, which by the Chinese Remainder Theorem is multiplicative. So we may assume that $t = 1$, $N = \mathfrak{p}_1^{s_1}$ with some prime \mathfrak{p}_1 , $q_1 = q^{\deg \mathfrak{p}_1}$. Then

$$\#(A/N)_{\text{prim}}^r = \#(A/\mathfrak{p}_1)_{\text{prim}}^r \#(\mathfrak{p}_1/\mathfrak{p}_1^{s_1})^r,$$

as some element of $(A/N)^r$ is primitive if and only if its reduction mod \mathfrak{p}_1 is. Now $\#(A/\mathfrak{p}_1)_{\text{prim}}^r = q_1^r - 1$ and $\#(\mathfrak{p}_1/\mathfrak{p}_1^{s_1})^r = q_1^{(s_1-1)r}$, and we are done. \square

In the case of smaller dimension $s < r - 1$ we get a similar description of $\Gamma(N) \setminus \mathfrak{U}_s$, which is in $1-1$ -correspondence with the set of $(r-s)$ -subsets of $(A/N)^r$ that are part of an (A/N) -basis of $(A/N)^r$, modulo the action of the group $\{\gamma \in \text{GL}(r-s, A/N) \mid \det \gamma \in \mathbb{F}^*\}$. We leave the details to the reader, as we will only need the case $s = 1$. Here, likewise,

$$\Gamma(N) \setminus \mathfrak{U}_1 \xrightarrow{\cong} \Gamma(N) \setminus \Gamma/\Gamma \cap P_1(K) \xrightarrow{\cong} \mathcal{C}_r(N), \quad (3.7)$$

where the double class of $\gamma \in \Gamma$ is mapped to the last row vector of γ (reduced modulo N , and modulo the action of \mathbb{F}^*). In particular,

$$\#(\Gamma(N) \setminus \mathfrak{U}_1) = \#\mathcal{C}_r(N) = c_r(N),$$

which of course could also be seen via the duality of projective spaces over the finite ring A/N .

4. Behavior of Eisenstein series at the boundary

In the whole section, N is a fixed monic element of A of degree $d \geq 1$.

(4.1). Let \mathbf{u} be an element of $\mathcal{T}(N) = (N^{-1}A/A)^r \setminus \{0\}$. We start with the relation from (1.12)

$$E_{k,\mathbf{u}}(\gamma(U, i)) = E_{k,\mathbf{u}\gamma}(U, i) \quad (4.1.1)$$

for $\gamma \in \Gamma$, $(U, i) \in \overline{\Psi}^r$. Suppose that $U = V_s \cdot \gamma^{-1}$ with some $1 \leq s < r$ and $\gamma \in \Gamma$. Now we read off from (4.1.1):

(4.1.2) The vanishing behavior of $E_{k,\mathbf{u}}$ around the boundary component Ψ_U is the same as the behavior of $E_{k,\mathbf{u}\gamma}$ around the standard component Ψ_{V_s} .

We say that

(4.1.3) \mathbf{u} belongs to U if $\mathbf{u} \in \mathcal{T}(N) \subset K^r/A^r = V/L_V$ is represented by an element of $U \subset V$.

In view of $\mathbf{u}\gamma = \mathbf{u}$ for $\gamma \in \Gamma(N)$, this property depends only on the $\Gamma(N)$ -orbit of U .

4.2 Proposition.

- (i) Suppose that \mathbf{u} does not belong to U . Then $E_{k,\mathbf{u}}$ vanishes identically on Ψ_U .
- (ii) If \mathbf{u} belongs to U , then $E_{k,\mathbf{u}}$ restricts to $\Psi_U \cong \Psi^s$ like an Eisenstein series $E_{k,\mathbf{u}'}$ of rank s , $\mathbf{u}' \in \mathcal{T}_s(N) = (N^{-1}A/A)^s \setminus \{0\}$. In particular, it doesn't vanish identically on Ψ_U .

Proof. In view of (4.1.2), it suffices to verify the assertions for $U = V_s$. Suppose that $\mathbf{u} = (u_1, \dots, u_r)$ does not belong to V_s , that is, $u_i \neq 0$ for some i with $1 \leq i \leq r-s$. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ be an element of the fundamental domain \tilde{F} described in (1.15). We have

$$E_{k,\mathbf{u}}(\boldsymbol{\omega}) = N^{-k} \sum_{\substack{\mathbf{a} \in A^r \\ \mathbf{a} \equiv N\mathbf{u} \pmod{N}}} (a_1\omega_1 + \dots + a_r\omega_r)^{-k} = N^{-k} \sum_{\mathbf{a}} (\mathbf{a}\boldsymbol{\omega})^{-k}.$$

In each term, $a_i \neq 0$, which by (1.8.1) forces that $(\mathbf{a}\boldsymbol{\omega})^{-k}$ tends to zero, uniformly in the \mathbf{a} , if $\boldsymbol{\omega}$ approaches Ψ_{V_s} . That is, $E_{k,\mathbf{u}}(\boldsymbol{\omega}) \rightarrow 0$, and $E_{k,\mathbf{u}} \equiv 0$ on Ψ_{V_s} . Suppose that \mathbf{u} belongs to V_s . As before, each term $(\mathbf{a}\boldsymbol{\omega})^{-k}$ tends to zero uniformly in \mathbf{a} , as long as at least one of $a_1, \dots, a_{r-s} \neq 0$. Therefore, $\lim E_{k,\mathbf{u}}(\boldsymbol{\omega}) = E_{k,\mathbf{u}'}(\boldsymbol{\omega}')$ with $\mathbf{u}' = (u_{r-s+1}, \dots, u_r)$ if $\boldsymbol{\omega}$ tends to $(0, \dots, 0, \omega'_{r-s+1}, \dots, \omega'_r) = (0, \dots, 0, \boldsymbol{\omega}')$. \square

We define the space

$$\text{Eis}_k(N) := \sum_{\mathbf{u} \in (N^{-1}A/A)^r} C_\infty E_{k,\mathbf{u}} \quad (4.3)$$

of Eisenstein series of weight k and level N .

4.4 Lemma. The vector space $\text{Eis}_k(N)$ is generated by the $E_{k,\mathbf{u}}$ with \mathbf{u} primitive of level N (i.e., $N'\mathbf{u} \neq 0$ for each proper divisor N' of N) and even by $E_{k,\mathbf{u}}$ ($\mathbf{u} \in N^{-1}S$), where S is the set of representatives for $\mathcal{C}_r(N) = (A/N)_{\text{prim}}^r/\mathbb{F}^*$ given in (3.5).

Proof. Let N' be a monic divisor of N , where $N' = 1$ is allowed. Then the distribution relation

$$(N/N')^k \sum_{(N/N')\mathbf{u}=\mathbf{v}} E_{k,\mathbf{u}} = E_{k,\mathbf{v}} \quad (4.4.1)$$

holds, where $\mathbf{v} \in (N'^{-1}A/A)^r$ and $E_{k,0} = E_k$ is the Eisenstein series without level. It shows that Eisenstein series of lower level N' may be expressed as linear combinations of those with level N . The result now follows from

$$E_{k,c\mathbf{u}} = c^{-k} E_{k,\mathbf{u}} \quad (c \in \mathbb{F}^*), \quad (4.4.2)$$

which is immediate from the definition of $E_{k,\mathbf{u}}$. \square

(4.5). We will show that these are all the relations between Eisenstein series of weight k , following the strategy of Hecke in [28], which has been introduced to the function field situation in the case of $r = 2$ by Cornelissen [9]. Let

$$F_{k,\mathbf{u}}(\boldsymbol{\omega}) := N^{-k} \sum_{\substack{\mathbf{a} \in A^r \text{ primitive} \\ \mathbf{a} \equiv N\mathbf{u} \pmod{N}}} (\mathbf{a}\boldsymbol{\omega})^{-k} \quad (4.5.1)$$

be the partial sum of $E_{k,\mathbf{u}}(\boldsymbol{\omega})$ with primitive \mathbf{a} , i.e., where $\mathbf{a} = (a_1, \dots, a_r)$ satisfies $\sum_{1 \leq i \leq r} Aa_i = A$. Then:

(4.5.2) The *restricted Eisenstein series* $F_{k,\mathbf{u}}$ is well-defined as a function on Ψ^r of weight k and invariant under $\Gamma(N)$. Like $E_{k,\mathbf{u}}$, it satisfies the functional equation

$$F_{k,\mathbf{u}}(\gamma\boldsymbol{\omega}) = F_{k,\mathbf{u}\gamma}(\boldsymbol{\omega})$$

under $\gamma \in \Gamma$.

Let $\mu : A \rightarrow \{0, \pm 1\}$ be the Möbius function: $\mu(a) = (-1)^n$ if $a = \epsilon \prod_{1 \leq j \leq n} \mathfrak{p}_j$ with n different monic primes \mathfrak{p}_j of A and $\epsilon \in \mathbb{F}^*$, and zero otherwise. (As the empty product evaluates to 1, $\mu(a) = 1$ if $a \in \mathbb{F}^*$.) Then $\sum_{b \text{ monic}, b|a} \mu(b) = 1$ if $a \in \mathbb{F}^*$ and 0 otherwise, and the usual formalism holds. Möbius inversion yields

$$F_{k,\mathbf{u}}(\boldsymbol{\omega}) = \sum_{t \in (A/N)^*} \sum_{\substack{a \in A \text{ monic} \\ at \equiv 1 \pmod{N}}} \mu(a) a^{-k} E_{k,t\mathbf{u}}(\boldsymbol{\omega}). \quad (4.5.3)$$

In particular, $F_{k,\mathbf{u}}$ lies in $\text{Eis}_k(N)$, and so has a strongly continuous extension to $\overline{\Psi}^r$. We deduce that

$$\dim(\text{Eis}_k(N)) \geq \dim\left(\sum_{\mathbf{u} \in \mathcal{T}(N)} C_\infty F_{k,\mathbf{u}}\right). \quad (4.5.4)$$

Recall that by (3.7) the set of 1-dimensional boundary components of $\Gamma(N) \backslash \overline{\Psi}^r$ is in 1-1-correspondence with $\mathcal{C}_r(N)$, or with its set S of representatives in (3.5). We let $\Psi_{\mathbf{n}}^1 \cong \Psi^1$ be the component corresponding to $\mathbf{n} \in S$.

4.6 Proposition.

- (i) Given $\mathbf{n} \in S$ there exists a unique $\mathbf{n}' \in S$ such that $F_{k,\mathbf{n}'/N}$ doesn't vanish at $\Psi_{\mathbf{n}}^1$.
(ii) The rule $\mathbf{n} \rightarrow \mathbf{n}'$ establishes a permutation of S .

Proof. First we note that $\Psi_{\mathbf{n}}^1$, where $\mathbf{n} = (0, \dots, 0, 1)$, equals Ψ_{V_1} . In view of (4.5.2) and the transitivity of Γ on the set $\{\Psi_{\mathbf{n}}^1 \mid \mathbf{n} \in S\}$, it is enough to show that there is a unique \mathbf{n}' such that $F_{k,\mathbf{n}'/N}$ doesn't vanish at Ψ_{V_1} .

Consider a term $(\mathbf{a}\boldsymbol{\omega})^{-k} = (a_1\omega_1 + \dots + a_r\omega_r)^{-k}$ of $N^k F_{k,\mathbf{n}'/N}$, where $\gcd(a_1, \dots, a_r) = 1$ and $\boldsymbol{\omega} \in \tilde{F}$ as in the proof of (4.2). If one of a_1, \dots, a_{r-1} doesn't vanish then $\lim(\mathbf{a}\boldsymbol{\omega})^{-k} = 0$, uniformly in \mathbf{a} , if $\boldsymbol{\omega}$ tends to $(0, \dots, 0, \omega_r)$. Hence

$$\lim F_{k,\mathbf{n}'/N}(\boldsymbol{\omega}) = \lim N^{-k} \sum_{\substack{\mathbf{a} \in A^r \text{ primitive} \\ a_1 \dots a_{r-1} = 0 \\ \mathbf{a} \equiv \mathbf{n}' \pmod{N}}} (\mathbf{a}\boldsymbol{\omega})^{-k},$$

which can be non-zero only if $a_r \in \mathbb{F}^*$ (and in fact $a_r = 1$, as \mathbf{n}' is required to be monic). This implies that $\mathbf{n}' = (0, \dots, 0, 1)$. Conversely, that choice of \mathbf{n}' gives $F_{k,\mathbf{n}'/N}(0, \dots, 0, 1) = N^{-k}\omega_r^{-k} \neq 0$. That is, $\mathbf{n}' = \mathbf{n} = (0, \dots, 0, 1)$ is as wanted. \square

4.7 Corollary. The restricted Eisenstein series $F_{k,\mathbf{u}}$, where $\mathbf{u} \in N^{-1}S$, are linearly independent and form a basis of $\text{Eis}_k(N)$. The dimension of $\text{Eis}_k(N)$ equals $\#(S) = c_r(N)$.

Proof. (4.4) + (4.5.4) + (4.6). \square

4.8 Proposition.

- (i) The Eisenstein series $E_{\mathbf{u}} = E_{1,\mathbf{u}}$ ($\mathbf{u} \in \mathcal{T}(N)$) of weight 1 and level N separate points of $\Gamma(N) \backslash \overline{\Psi}^r$. That is, if $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \overline{\Psi}^r$ satisfy $E_{\mathbf{u}}(\boldsymbol{\omega}) = E_{\mathbf{u}}(\boldsymbol{\omega}')$ for all $\mathbf{u} \in \mathcal{T}(N)$, then there exists $\Gamma \in \Gamma(N)$ such that $\boldsymbol{\omega}' = \gamma\boldsymbol{\omega}$.
(ii) The same statement for $\Gamma(N) \backslash \overline{\Psi}^r$ replaced with $\Gamma(N) \backslash \overline{\Omega}^r$.

Proof. We start with the identity

$$NX \prod_{\mathbf{u} \in \mathcal{T}(N)} (1 - E_{\mathbf{u}}X) = \phi_N(X) = NX + \sum_{1 \leq i \leq rd} \ell_i(N)X^{q^i}, \quad (4.8.1)$$

which comes from (2.4.1) and (2.4.3). Here the right hand side is the N -division polynomial of the general Drinfeld module ϕ of rank $\leq r$, which lives above $\overline{\Psi}^r$. The coefficients $\ell_i(N)$, where $1 \leq i \leq rd$ ($d = \deg N$), are Γ -invariant functions on $\overline{\Psi}^r$ of weights $q^i - 1$.

Hence the data $\{E_{\mathbf{u}}(\boldsymbol{\omega}) \mid \mathbf{u} \in \mathcal{T}(N)\}$ determines the values on $\boldsymbol{\omega}$ of the $\ell_i(N, \boldsymbol{\omega})$ and therefore, taking the known relations between the $\ell_i(N)$ and the g_i ($1 \leq j \leq r$) into account, the coefficients $g_1(\boldsymbol{\omega}), \dots, g_r(\boldsymbol{\omega})$ of the T -division polynomial

$$\phi_T^\omega(X) = TX + \sum_{1 \leq j \leq r} g_j(\omega) X^{q^j}$$

of the Drinfeld module ϕ^ω that corresponds to $\omega \in \overline{\Psi}^r$. Suppose that ϕ^ω has rank s ($1 \leq s \leq r$), that is, $g_s(\omega) \neq 0$, $g_{s+1}(\omega) = \cdots = g_r(\omega) = 0$.

If $[s = r]$ then ϕ^ω determines an A -lattice in C_∞ of rank r , hence a point $\omega \in \Psi^r$ up to the action of Γ . That is, if $E_u(\omega) = E_u(\omega')$ for all u then $\omega' = \gamma\omega$ with some $\gamma \in \Gamma$. The relation $E_u(\gamma\omega) = E_{u\gamma}(\omega)$ moreover shows that $u\gamma = u$ for all u , that is, γ lies in fact in $\Gamma(N)$.

If $[s < r]$, the A -lattice corresponding to ϕ^ω has rank s and is given by an embedding $i: U \hookrightarrow C_\infty$ of some s -dimensional subspace U of K^r (i.e., of $L_U = A^r \cap U \hookrightarrow C_\infty$). Now Proposition 4.2 allows to determine U up to $\Gamma(N)$ -equivalence. Choose one such U ; then $\omega \in \Psi_U$ is determined through ϕ^ω and the $E_u(\omega)$ up to an element of Γ_U and, in fact (with the same argument as in the case $[s = r]$), up to an element of $\Gamma_U(N) = \Gamma_U \cap \Gamma(N)$. This shows (i); the proof of (ii) is identical. \square

5. The projective embedding

In this section, we show that $\Gamma(N) \backslash \overline{\Omega}^r$ is the set of C_∞ -points of a closed subvariety of some projective space. This allows us to endow it with the structure of projective variety, which then will be labelled with the symbol $\overline{M}^r(N)$, the *Eisenstein compactification* of $M^r(N)$.

Throughout, $N \in A$ of degree $d \geq 1$ is fixed.

(5.1). We define the *Eisenstein ring of level N* , $\mathbf{Eis}(N)$, as the C_∞ -subalgebra of $\tilde{\mathcal{F}}_r(N)$ generated by the Eisenstein series $E_u = E_{1,u}$ of level N and weight 1. It is graded with respect to weight: its k -th piece $\mathbf{Eis}_k(N)$ is the C_∞ -space generated by monomials of degree k in the E_u . In particular, $\mathbf{Eis}(N)$ is generated as an algebra by $\mathbf{Eis}_1(N) = \mathbf{Eis}_1(N)$, a vector space of dimension $c_r(N)$ (see (4.7)). We also let $\mathbf{Mod} = \mathbf{Mod}(1) = C_\infty[g_1, \dots, g_r]$ be the graded algebra of modular forms of type zero for Γ ([21] 1.7).

5.2 Proposition.

- (i) $\mathbf{Eis}(N)$ contains the algebra \mathbf{Mod} ;
- (ii) $\mathbf{Eis}(N)$ is integral over \mathbf{Mod} ;
- (iii) $\mathbf{Eis}(N)$ contains all the Eisenstein series $E_{k,u}$ of arbitrary weight k .

Proof. (i) The argument in the proof of Proposition 4.8 shows that $g_1, \dots, g_r \in \mathbf{Eis}(N)$.
(ii) The E_u are the zeroes of the monic polynomial $N^{-1}X^{q^{rd}}\phi_N(X^{-1})$ with coefficients $N^{-1}\ell_i(N) \in \mathbf{Mod}$, where $\phi_N(X)$ is as in (4.8.1).
(iii) Let Λ be any rank- r A -lattice in C_∞ and $G_{k,\Lambda}(X)$ be its k -th Goss polynomial ([23] 2.17, [16] 3.4). It is of shape

$$G_{k,\Lambda}(X) = \sum_{0 \leq i \leq k} a_i(\Lambda) X^{k-i}, \quad (5.2.1)$$

where a_i is a modular form of weight i and type 0, that is, $a_i \in \mathbf{Mod}$. (This follows from [16] 3.4(ii).) The characteristic property of Goss polynomials ([16] 3.4(i)) implies

$$E_{k,\mathbf{u}} = G_{k,\Lambda}(E_{1,\mathbf{u}}). \quad (5.2.2)$$

Now (iii) is a consequence of (5.2.1) and (5.2.2). \square

(5.3). We define $\mathbb{P} = \mathbb{P}(N)$ as the projective space $\mathbb{P}(\mathrm{Eis}_1(N)^\wedge)$ associated with the dual vector space $\mathrm{Eis}_1(N)^\wedge$ of $\mathrm{Eis}_1(N)$. As a scheme,

$$\mathbb{P} = \mathrm{Proj} R, \quad (5.3.1)$$

where $R := \mathrm{Sym}(\mathrm{Eis}_1(N))$ is the symmetric algebra on $\mathrm{Eis}_1(N)$. Consider the map

$$j_N : \overline{\Omega}^r \longrightarrow \mathbb{P} \quad (5.3.2)$$

to the C_∞ -valued points of \mathbb{P} that with the class of (U, i) associates the class (up to scalars) of the linear form $E_{\mathbf{u}} \longmapsto E_{\mathbf{u}}(U, i)$. Then:

(5.3.3) j_N is well-defined, as the $E_{\mathbf{u}}$ have weight 1 and for each (U, i) there exists \mathbf{u} such that $E_{\mathbf{u}}(U, i) \neq 0$.

(5.3.4) $j_N(\gamma(U, i)) = j_N(U, i)$ for $\gamma \in \Gamma(N)$, as the $E_{\mathbf{u}}$ are $\Gamma(N)$ -invariant. Here and in the following, we write $j_N(U, i)$ for j_N (class of (U, i)).

(5.3.5) As a map from $\Gamma(N) \setminus \overline{\Omega}^r$ to \mathbb{P} , j_N is injective, due to Proposition 4.8.

(5.4). The graded algebra R is supplied with a canonical homomorphism

$$\epsilon : R \longrightarrow \mathbf{Eis}(N), \quad (5.4.1)$$

which is the identity on $R_1 = \mathrm{Eis}_1(N) = \mathbf{Eis}_1(N)$ and surjective, since $\mathbf{Eis}_1(N)$ generates $\mathbf{Eis}(N)$. Let J be the kernel of ϵ . Since $\mathbf{Eis}(N)$ is a domain, J is a (homogeneous) prime ideal of R , and in particular, saturated ([26] p. 125). Then j_N maps $\Gamma(N) \setminus \overline{\Omega}^r$ to the vanishing variety $V(J) \subset \mathbb{P}$ of J .

(5.5). Let $x \in V(J)$ be given. The proof of Proposition 4.8 shows that there exists an element (U, i) of $\overline{\Psi}^r$, well-defined up to the action of $\Gamma(N)$, such that $j_N(U, i) = x$. Viz., for simplicity choose a representative $\tilde{x} \in \text{Eis}_1(N)^\wedge = \text{Hom}_{C_\infty}(\text{Eis}_1(N), C_\infty)$ and put $\tilde{x}_u := \tilde{x}(E_u)$. Interpreting \tilde{x}_u as a value of E_u , the \tilde{x}_u determine the values of the coefficient forms g_1, \dots, g_r as in (4.8), thus (if $g_r \neq 0$) a point $\tilde{\omega} \in \Psi^r$ up to the action of $\Gamma(N)$. The corresponding point $\omega \in \Omega^r$ is independent of the choice of \tilde{x} and serves the purpose. If $g_s \neq 0, g_{s+1} = \dots = g_r = 0$ then, as in (4.8), the s -dimensional K -space U and its boundary component Ψ_U is determined up to $\Gamma(N)$ -equivalence by the (non-) vanishing of the \tilde{x}_u . Choosing one such U , there exists an embedding $i : U \hookrightarrow C_\infty$, unique up to $\Gamma_U(N)$, that fits the given data. Then the class of (U, i) in $\overline{\Omega}^r$ is as wanted. That is,

$$j_N : \Gamma(N) \backslash \overline{\Omega}^r \xrightarrow{\cong} V(J) \quad (5.6)$$

is in fact bijective. Furthermore, the restriction of j_N to a stratum $\Gamma_U(N) \backslash \Omega_U$ of $\Gamma(N) \backslash \overline{\Omega}^r$ is analytic with respect to the analytification of $V(J)$, as the E_u are.

(5.7). Next, we consider the canonical morphism $\kappa : V(J) \rightarrow \overline{M}^r$ defined as follows: Choose elements $G_i \in R = \text{Sym}(\text{Eis}_1(N))$ such that $\epsilon(G_i) = g_i$ ($1 \leq i \leq r$, see (5.4)). For given $x \in V(J)$, G_i may be evaluated on \tilde{x} (notation as in (5.5)), and we put

$$\kappa(x) = (G_1(\tilde{x}) : \dots : G_r(\tilde{x})),$$

which is independent of the choice of \tilde{x} above x , and of the choices of the G_i . Furthermore, the diagram

$$\begin{array}{ccc} \Gamma(N) \backslash \overline{\Omega}^r & \xrightarrow{j_N} & V(J) \\ \downarrow \pi & & \downarrow \kappa \\ \Gamma \backslash \overline{\Omega}^r & \xrightarrow{j} & \overline{M}^r = \text{Proj } C_\infty[g_1, \dots, g_r] \end{array} \quad (5.7.1)$$

commutes, where the left vertical arrow is the canonical projection π .

5.8 Proposition. j_N is a strong homeomorphism.

Proof. This follows essentially from Theorem 2.3, that is, from the corresponding property of j . As in the proof of (2.3), j_N is strongly continuous, so we must show that it is also an open map. Consider diagram (5.7.1), where π and therefore $j \circ \pi = \kappa \circ j_N$ are open. As $\kappa : V(J) \rightarrow \overline{M}^r$ is set-theoretically the quotient map of the finite group $G(N) = \Gamma/\Gamma(N) \cdot Z$, which acts through homeomorphisms on $V(J)$, the openness of $\kappa \circ j_N$ implies the openness of j_N . \square

By (5.6) and (5.8), we may use j_N to endow $\Gamma(N) \backslash \overline{\Omega}^r$ with the structure of (the set of C_∞ -points of) the projective subvariety $V(J)$ of \mathbb{P} , compatible with the analytic

structures and the strong topologies on both sides. By construction, $V(J)$ equals the projective variety associated with the graded algebra $\mathbf{Eis}(N)$. We collect what has been shown.

5.9 Theorem. *Let N be a non-constant monic element of A .*

- (i) *The set $\Gamma(N) \backslash \overline{\Omega}^r$ is the set of C_∞ -points of an irreducible projective variety $\overline{M}^r(N)$ over C_∞ , the Eisenstein compactification of $M^r(N)$, which may be described as the variety $\text{Proj } \mathbf{Eis}(N)$ associated with the Eisenstein ring $\mathbf{Eis}(N)$. It is a closed subvariety of the projective space $\mathbb{P} = \mathbb{P}(\text{Eis}_1(N)^\wedge)$ attached to the dual of the vector space $\text{Eis}_1(N)$ of Eisenstein series of level N and weight 1, which has dimension $c_r(N)$. The open subvariety $M^r(N) = \Gamma(N) \backslash \Omega^r$ of $\overline{M}^r(N)$ is characterized as $\{x \in \overline{M}^r(N) \mid E_{\mathbf{u}}(x) \neq 0 \quad \forall \mathbf{u} \in \mathcal{T}(N)\}$.*
- (ii) *The set $\Gamma(N) \backslash \overline{\Psi}^r$ is the set of C_∞ -points of an irreducible variety $\widetilde{M}^r(N)$ over C_∞ , which may be described as the variety $\text{Spec } \mathbf{Eis}(N) \backslash \{I\}$, where I is the irrelevant ideal of the graded ring $\mathbf{Eis}(N)$. It is a subvariety of the affine space attached to $\text{Eis}_1(N)^\wedge$, endowed with an action of the multiplicative group \mathbb{G}_m , and such that*

$$\mathbb{G}_m \backslash \widetilde{M}^r(N) \xrightarrow{\cong} \overline{M}^r(N).$$

Proof. (i) has been shown above (see Proposition 4.2 for the last assertion), and the proof of (ii) is - mutatis mutandis - identical. \square

5.10 Remark. We point out the following functorial properties of the construction of $\overline{M}^r(N)$ and $\widetilde{M}^r(N)$.

- (i) It is compatible with level changes, to wit: Let N' be a multiple of N and $G(N, N')$ the quotient group $\Gamma(N)/\Gamma(N')$. The action of $\Gamma(N)$ on Ψ^r induces an action of $G(N, N')$ on $\Gamma(N') \backslash \Psi^r = \widetilde{M}^r(N')$ such that $G(N, N') \backslash \widetilde{M}^r(N') = \widetilde{M}^r(N)$. Further, the fixed space of $\Gamma(N)$ in $\text{Eis}_1(N')$ is $\text{Eis}_1(N)$; hence $G(N, N')$ acts effectively on $\text{Eis}_1(N')$ with fixed space $\text{Eis}_1(N)$. Let $\tilde{j}_N : \Gamma(N) \backslash \overline{\Psi}^r \hookrightarrow \text{Eis}_1(N)^\wedge$ be the morphism analogous to j_N and implicitly referred to in Theorem 5.9(ii). Then the diagram

$$\begin{array}{ccc} \Gamma(N') \backslash \overline{\Psi}^r & \xrightarrow{\tilde{j}_{N'}} & \text{Eis}_1(N')^\wedge \\ \downarrow & & \downarrow \\ \Gamma(N) \backslash \overline{\Psi}^r & \xrightarrow{\tilde{j}_N} & \text{Eis}_1(N)^\wedge \end{array} \quad (5.10.1)$$

is commutative and compatible with the action of $G(N, N')$, where the vertical arrows are the canonical projections. In particular, the action of $G(N, N')$ on $\widetilde{M}^r(N')$ with quotient $\widetilde{M}^r(N)$ extends to $\widetilde{M}^r(N')$ with quotient $\overline{M}^r(N)$. Factoring out the

multiplicative group \mathbb{G}_m , we find similarly that $G(N, N')$ acts on $\overline{M}^r(N')$ with quotient $\overline{M}^r(N)$.

- (ii) The construction of the Eisenstein compactification $\overline{M}^r(N)$ (and likewise of $\overline{M}^r(N)$) is hereditary in the following sense. Let Ω_U be a boundary component ($U \in \mathfrak{U}_s$, $s < r$) and

$$M_U(N) := \Gamma_U(N) \backslash \Omega_U \xrightarrow{\cong} \{\gamma \in \mathrm{GL}(s, A) \mid \gamma \equiv 1 \pmod{N}\} \backslash \Omega^s$$

its image in $\overline{M}^r(N)$. Then the Zariski closure $\overline{M}_U(N)$ of $M_U(N)$ in $\overline{M}^r(N)$ is composed of the $M_{U'}(N)$, where $U' \in \mathfrak{U}$ and $U' \subset U$, and is isomorphic with the variety $\overline{M}^s(N)$. This is seen by assuming, without restriction, that $U = V_s$, in which case the description of $\overline{M}_{V_s}(N)$ is identical with that of $\overline{M}^s(N)$.

5.11 Remark. The idea of using Eisenstein series for a projective embedding of $M^r(N)$ is taken from [29]. However, Kapranov's construction has the drawback that it fails to be canonical (it depends on the choice of a certain bound m_0 , see [29] Proposition 1.12). Instead, our Proposition 4.8 assures that it suffices to consider Eisenstein series of weight 1, which culminates in the canonical description $\overline{M}^r(N) = \mathrm{Proj} \mathbf{Eis}(N) \hookrightarrow \mathbb{P}(\mathrm{Eis}_1(N)^\wedge)$ with its functorial properties.

6. Tubular neighborhood of cuspidal divisors

In this section we show that each point x on a cuspidal divisor, i.e., on a boundary component $M_U(N)$ of $\overline{M}^r(N)$ of codimension 1, possesses a neighborhood Z isomorphic with $B \times W$, where W is an open admissible affinoid neighborhood of x on $M_U(N)$ and B a ball, and such that the map $\pi : Z \rightarrow W$ derived from the canonical projection $\pi_U : \overline{M}^r(N) \rightarrow M_U(N)$ is the projection to the second factor.

(6.1). As usual, it suffices to treat the case where x is represented by $\omega^{(0)} \in \Omega_{V_{r-1}} \xrightarrow{\cong} \Omega^{r-1}$. For simplicity, we use the canonical isomorphism as an identification. We may further assume that $\omega^{(0)}$ belongs to the fundamental domain \mathbf{F}' of $\Gamma' = \mathrm{GL}(r-1, A)$ in Ω^{r-1} , that is, $\omega^{(0)} = (0 : \omega_2^{(0)} : \dots : \omega_r^{(0)})$, where $\{1 = \omega_r^{(0)}, \dots, \omega_2^{(0)}\}$ is an SMB of its lattice. Let $X \subset \Omega^{r-1}$ be the subspace

$$X = \{\omega' = (\omega'_2 : \dots : \omega'_r = 1) \mid |\omega'_i| = |\omega_i^{(0)}|, 2 \leq i \leq r\}. \quad (6.1.1)$$

Then, in fact, $X \subset \mathbf{F}'$ and X is an admissible open affinoid subspace, whose structure has been investigated in [21], Theorem 2.4. (All of this collapses for $r = 2$ to $X = \mathbf{F}_1 = \Omega^1 = \{\text{point}\}$.)

We next put

(6.1.2) $Y_c = \{\omega \in \Omega^r \mid \omega = (\omega_1 : \dots : \omega_r) \mid (\omega_2 : \dots : \omega_r) \in X \mid d(\omega_1, \langle \omega_2, \dots, \omega_r \rangle_{K_\infty}) \geq c\}$ for some large c in the value group $q^\mathbb{Q}$ of C_∞ . Here $d(\omega, \langle \cdot \rangle_{K_\infty})$ is the distance

function to the K_∞ -space generated by $\omega_2, \dots, \omega_r = 1$. It is an admissible open subspace of Ω^r . Note that $\omega \in Y_c$ in particular implies $|\omega_1| \geq c$.

6.2 Lemma. *Suppose that $c > |\omega_2^{(0)}|$. Then:*

- (i) *If $\gamma \in \Gamma$ satisfies $\gamma(Y_c) \cap Y_c \neq \emptyset$ then $\gamma \in \Gamma \cap P_{r-1}$;*
- (ii) *If $\gamma \in \Gamma(N)$ is such that $\gamma(Y_c) \cap Y_c \neq \emptyset$ then γ has the shape*

$$\gamma = \begin{array}{|c|c|} \hline 1 & u_2, \dots, u_r \\ \hline 0 & \gamma' \\ \vdots & \\ 0 & \\ \hline \end{array}$$

where $u_2, \dots, u_r \in NA$ and γ' runs through a finite subgroup of $\Gamma'(N) = \Gamma' \cap \Gamma(N)$ consisting of strictly upper triangular matrices (i.e., with ones on the diagonal). On the other hand, each γ of this form with $\gamma' = 1$ stabilizes Y_c .

Proof. (i) Let $\omega = (\omega_1 : \dots : \omega_r) \in Y_c$ be such that $\gamma\omega = (\omega'_1 : \dots : \omega'_r) \in Y_c$ with $\gamma = (\gamma_{i,j}) \in \Gamma$ (recall that $\omega_r = \omega'_r = 1$). Let further Λ be the lattice $\Lambda_\omega = \langle \omega_1, \dots, \omega_r \rangle_A$ and $\alpha := \text{aut}(\gamma, \omega)$. Now $\{\omega'_r, \dots, \omega'_1\}$ is a basis of $\alpha^{-1}\Lambda$ and, since $\gamma\omega \in Y_c$, $\omega'_r, \dots, \omega'_2$ are the first $r-1$ elements of an SMB of $\alpha^{-1}\Lambda$, so $\alpha\omega'_r, \dots, \alpha\omega'_2$ are the first $r-1$ elements of an SMB of Λ . Then $|\sum_{1 \leq j \leq r} \gamma_{i,j}\omega_j| = |\alpha\omega'_i| = |\omega_i|$ holds for $i = 2, \dots, r$ in view of (1.8.2). If $\gamma_{i,1} \neq 0$ then $|\sum_{1 \leq j \leq r} \gamma_{i,j}\omega_j| \geq |\omega_1| > |\omega_i|$, contradiction.

- (ii) The entry $\gamma_{1,1} = 1$ is obvious, as is the fact that each γ with $\gamma' = 1$ stabilizes Y_c . The possible γ' are those that fix X . The stabilizer Γ'_X of X in Γ' equals

$$\{\gamma' = (\gamma_{i,j})_{2 \leq i,j \leq r} \in \Gamma' \mid |\gamma_{i,j}| \leq |\omega_i^{(0)} / \omega_j^{(0)}|\},$$

matrices with a block structure

$$\begin{array}{|c|c|c|} \hline B_1 & & * \\ \hline & B_2 & \\ \hline 0 & & \ddots \\ \hline \end{array}$$

with zeroes below the blocks, and each block B_k an invertible matrix over \mathbb{F} . The number of such γ' is finite, and the congruence condition $\gamma' \equiv 1 \pmod{N}$ forces each block to equal 1. Hence γ' is strictly upper triangular. \square

(6.3). We write G for the group that occurs in (6.2)(ii), i.e., $G := \{\gamma \in \Gamma(N) \mid \gamma(Y_c) \cap Y_c \neq \emptyset\}$, $G_1 := \{\gamma \in G \mid \gamma' = 1\}$ and $G' := G/G_1 = \Gamma'_X$ for the group of possible γ' .

(6.4). Consider the function $\omega \mapsto t(\omega) := e_{N\Lambda}^{-1}(\omega_1)$ on Ω^r , where now $\Lambda = \langle \omega_2, \dots, \omega_r \rangle_A$. Its most important properties are:

(6.4.1) t is well-defined, as ω_1 does not belong to $N\Lambda$;

(6.4.2) it is holomorphic and has a unique strongly continuous extension to $\Omega^r \cup \Omega_{V_{r-1}}$, where $t \equiv 0$ on $\Omega_{V_{r-1}}$;

(6.4.3) $t(\gamma\omega) = t(\omega)$ for $\gamma \in G$;

(6.4.4) Fix $\omega' = (\omega_2 : \dots, \omega_r) \in X$. Then the image of the map

$$\begin{aligned} t_{\omega'} : \{ \omega \in C_\infty \mid (\omega : \omega') \in Y_c \} &\longrightarrow C_\infty \\ \omega &\longmapsto t(\omega : \omega') \end{aligned}$$

is a pointed ball $B_\rho^* := \{z \in C_\infty \mid 0 < |z| \leq \rho\}$ for some $\rho = \rho(c) \in q^\mathbb{Q}$, and is independent of the choice of $\omega' \in X$. The function $c \mapsto \rho(c)$ is strictly monotonically decreasing with $\lim_{c \rightarrow \infty} \rho(c) = 0$.

As for proofs, (6.4.1) is obvious, and (6.4.2) comes from trivial estimates.

Proof of (6.4.3). As $\text{aut}(\gamma, \omega) = 1$ for $\gamma \in G$, $(\gamma\omega)_2, \dots, (\gamma\omega)_r$ generate the same lattice $\Lambda = \langle \omega_2, \dots, \omega_r \rangle_A$. Hence

$$t(\gamma\omega) = e_{N\Lambda}((\gamma\omega)_1)^{-1} = e_{N\Lambda}(\omega_1)^{-1} = t(\omega),$$

in view of $(\gamma\omega)_1 \equiv \omega_1 \pmod{N\Lambda}$ and the $N\Lambda$ -invariance of $e_{N\Lambda}$. \square

Proof of (6.4.4). The assertion that $\text{im}(t_{\omega'})$ is a pointed ball B_ρ^* for some ρ is a general fact of rigid analysis (see e.g. [22] Lemma 10.9.1). Expansion of the product for $|e_{N\Lambda}(\omega)|$ shows that it depends only on the distance $d(\omega, K_\infty\Lambda)$ and the values $|\omega_2|, \dots, |\omega_r|$, as $\{\omega_r, \dots, \omega_2\}$ is an SMB of Λ . Since the $|\omega_i|$ are constant on X , the independence of $\rho(c)$ of the choice of ω' follows. The last statement is obvious. \square

6.5 Proposition. Let π be the projection $\omega = (\omega_1 : \dots : \omega_r) \mapsto \omega' = (\omega_2 : \dots : \omega_r)$ from Y_c to X and ρ as in (6.4.4). Then $t \times \pi$ induces an isomorphism

$$G \setminus Y_c \xrightarrow{\cong} B_\rho^* \times (G' \setminus X)$$

of analytic spaces.

Proof. For each $\omega' \in X$, $t_{\omega'}$ provides an isomorphism

$$t_{\omega'} : (N\Lambda) \setminus \{ \omega \in C_\infty \mid (\omega : \omega') \in Y_c \} \xrightarrow{\cong} B_\rho^*,$$

as it is bijective and

$$\frac{d}{d\omega} t_{\omega'}(\omega) = -t_{\omega'}(\omega)^2 \neq 0 \quad (\text{since } \frac{d}{d\omega} e_{N\Lambda}(\omega) = 1). \quad (6.5.1)$$

(Here Λ always denotes the lattice $\langle \omega_2, \dots, \omega_r \rangle_A$ associated with $\omega!$) Therefore, also

$$(t, \pi) : G_1 \setminus Y_c \xrightarrow{\cong} B_\rho^* \times X$$

is bijective, and is in fact an isomorphism, as, due to (6.5.1), its Jacobian matrix is invertible in each point. The group $G' = G/G_1$ acts on both sides (trivially on B_ρ^*) and due to (6.4.3), the map (t, π) is G' -equivariant. Therefore

$$G \setminus Y_c = G' \setminus (G_1 \setminus Y_c) \xrightarrow{\cong} G' \setminus (B_\rho^* \times X) = B_\rho^* \times (G' \setminus X). \quad \square$$

(6.6). As an admissible open affinoid in the smooth space $\Gamma'(N) \setminus \Omega^{r-1}$, the space $W := G' \setminus X$ is itself smooth and affinoid. Let $B = B_\rho$ be the unpunctured ball of radius ρ and $Z := B \times W$.

Eisenstein series extend uniquely (as strongly continuous functions and therefore, as Z is smooth, by Bartenwerfer's criterion [1] also as analytic functions) from $G \setminus Y_c \xrightarrow{\cong} B^* \times W$ to Z . The following result is a generalization of [14] Korollar 2.2.

6.7 Proposition. *Let $\mathbf{u} = (u_1, \dots, u_r) \in \mathcal{T}(N)$, $u_1 = a/N$ with $\deg a < d = \deg N$. Then the Eisenstein series $E_{\mathbf{u}} = E_{1, \mathbf{u}}$, regarded as a function on Z , has a zero of order $|a|^{r-1}$ along the divisor $(t = 0)$ of Z .*

Proof. The proof has been given for the case $r = 2$ in [14]. There, equivalently, the pole order of $E_{\mathbf{u}}(\omega)^{-1} = d_{\mathbf{u}}(\omega)' = e_{\omega}(\mathbf{u}\omega)$ has been determined, where the product expansion of e_{ω} was used. The proof generalizes without difficulty to the case of higher rank r . For another approach, see [29], Lemma 1.23. \square

(6.8). Again by Bartenwerfer's criterion, the open embedding

$$B^* \times W \xrightarrow{\cong} G \setminus Y_c \hookrightarrow \Gamma(N) \setminus \overline{\Omega}^r$$

extends to an injective map

$$i : Z = B \times W \hookrightarrow \Gamma(N) \setminus \overline{\Omega}^r,$$

since Z is smooth, thus normal. Its image is

$$\text{im}(i) = G \setminus Y_c \cup G' \setminus X = G \setminus (Y_c \cup X).$$

We will show that i is in fact an open embedding, i.e., an isomorphism of $Z = B \times W$ with $\text{im}(i)$. That is, given the class $[\omega'] \in W = G' \setminus X$ of $\omega' \in X$ with corresponding point $[\omega] = (0, [\omega']) \in Z$, we must show that the canonical map

$$\mathcal{O}_{G \setminus (Y_c \cup X), [\omega]} \longrightarrow \mathcal{O}_{Z, [\omega]}$$

of analytic local rings is an isomorphism. In fact, it suffices to show the corresponding isomorphism without dividing out the group G . In other words, we must show that the canonical injection

$$\mathcal{O}_{Y_c \cup X, \omega} \hookrightarrow \mathcal{O}_{B \times X, \omega} = \mathcal{O}_{B, 0} \hat{\otimes} \mathcal{O}_{X, \omega'}$$

is bijective. ($\hat{\otimes}$ is the topological tensor product of the two local rings, and we use “ ω ” both for the point $(\omega_1 = 0, \omega')$ of $Y_c \cup X$ and the point $(t = 0, \omega')$ of $B \times X$.) Now the left hand side contains $\mathcal{O}_{X, \omega'}$ via the projection $\pi : Y_c \cup X \longrightarrow X$, and it suffices to show that it also contains a uniformizer u of $Y_c \cup X$ along X , i.e., some u that as a germ of a function on $B \times X$ presents a zero of order 1 along $(t = 0)$. Then $\mathcal{O}_{Y_c \cup X, \omega}$ encompasses the local ring $\text{Sp}(C_\infty \langle u \rangle)$ at $u = 0$, that is, $\mathcal{O}_{B, 0}$, and we are done. By Proposition 6.7, we can take as u the Eisenstein series $E_{\mathbf{u}}$ with $\mathbf{u} = (\frac{1}{N}, 0, \dots, 0) \in \mathcal{T}(N)$.

Therefore we have shown the following result.

6.9 Theorem. *Let $M_U(N) = \Gamma(N) \setminus \Omega_U$ with $U \in \mathfrak{U}_{r-1}$ be a cuspidal divisor on $\overline{M}^r(N)$, $\omega^{(0)}$ a point on Ω_U with class $[\omega^{(0)}]$ in $M_U(N)$, and $\pi_U : \Omega^r \longrightarrow \Omega_U$ the canonical projection. There exists an admissible open affinoid neighborhood X of $\omega^{(0)}$ in Ω_U and an admissible open subspace Y of Ω^r characterized by*

$$Y = \{\omega \in \Omega^r \mid \pi_U(\omega) \in X \text{ and } \omega \text{ sufficiently close to } \pi_U(\omega)\}$$

such that $Y \cup X$ is isomorphic with a product $B \times X$ and the subspace $Z := \Gamma(N) \setminus (Y \cup X)$ of $\Gamma(N) \setminus \overline{\Omega}^r$ is isomorphic with $B \times W$, where B is a ball and W is an admissible open affinoid neighborhood of $[\omega^{(0)}]$ in $M_U(N)$, quotient of X by a finite group G' of automorphisms. The second projection $Z \longrightarrow W$ comes from π_U , divided out by the action of $\Gamma(N)$, and the first projection $Z \longrightarrow B$ is given by an explicit uniformizer.

(If $U = V_{r-1}$ then X, Y and t are specified in (6.1) and (6.4), and ‘sufficiently close’ means $d(\omega_1, \langle \omega_2, \dots, \omega_r \rangle_{K_\infty}) \geq c$ for some constant c , or equivalently, $|t(\omega)| \leq \rho$ for some ρ depending on c .) \square

6.10 Corollary. $[\omega^{(0)}]$ is a smooth point of $\overline{M}^r(N)$.

Proof. B and W are smooth. \square

6.11 Remark. Theorem 6.9 allows to expand holomorphic functions on Ω^r of weight k for $\Gamma(N)$ (so-called weak modular forms, see (7.1)) as Laurent series in t , where the

coefficients are holomorphic functions on Ω_U . This is crucial for the theory of modular forms. For the case $r = 2$, see e.g. [24] or [15]; for higher rank r , Basson and Breuer have started investigations in this direction in [2] and [3]. See also [4], [5], [6].

6.12 Remark. Analogous tubular neighborhoods along cuspidal divisors may also be constructed for $\widetilde{M}^r(N)$. The proof for $\overline{M}^r(N)$ given in (6.4)–(6.8) may easily be adapted.

7. Modular forms

In this section, we define the ring of modular forms for $\Gamma(N)$ and relate it with the ring of sections of the very ample line bundle of $\overline{M}^r(N)$ given by the embedding j_N .

Theorem 7.9 gives several different descriptions of modular forms. The assumptions of the preceding sections remain in force. Thus $r \geq 2$ and $N \in A$ is monic of degree $d \geq 1$.

(7.1). Let $\mathcal{O}(1)$ be the usual twisting line bundle on $\mathbb{P} = \mathbb{P}(\text{Eis}_1(N)^\wedge)$ and $\mathfrak{M} := j_N^*(\mathcal{O}(1))$ its restriction to the subvariety $j_N : \overline{M}^r(N) \hookrightarrow \mathbb{P}$. Tracing back the definitions, one sees that the sections of $\mathfrak{M}^{\otimes k}$ restricted to the open analytic subspace $M^r(N) = \Gamma(N) \backslash \Omega^r$ are just the functions f on Ω^r subject to

- (i) f is holomorphic;
- (ii) for $\omega \in \Omega^r$ and $\gamma \in \Gamma(N)$, the rule

$$f(\gamma\omega) = \text{aut}(\gamma, \omega)^k f(\omega)$$

holds ($\text{aut}(\gamma, \omega) = \sum_{1 \leq i \leq r} \gamma_{r,i} \omega_i$, ω normalized such that $\omega_r = 1$).

For further use, we baptize functions f satisfying (i) and (ii) as *weak modular forms* of weight k for $\Gamma(N)$. Later we will define modular forms as weak modular forms that additionally satisfy certain boundary conditions discussed below.

(7.2). First, we recall the fundamental domain \mathbf{F} for Γ on Ω^r :

$$\mathbf{F} = \{\omega \in \Omega^r \mid \{\omega_r = 1, \omega_{r-1}, \dots, \omega_1\} \text{ is an SMB of its lattice } \Lambda_\omega\}. \quad (7.2.1)$$

The group $\Gamma/\Gamma(N)$ acts on $\overline{M}^r(N)$ and thus on weak modular forms of weight k and level N through

$$f \mapsto f_{[\gamma]_k}, \text{ where } f_{[\gamma]_k}(\omega) = \text{aut}(\gamma, \omega)^{-k} f(\gamma\omega). \quad (7.2.2)$$

(I.e., the formula is valid for $\gamma \in \Gamma$, and $\gamma \in \Gamma(N)$ acts trivially.) For simplicity, we choose and fix a system RS of representatives for $\Gamma/\Gamma(N)$.

(7.3). Let now f be a weak modular form of some weight $k \in \mathbb{N}$. Consider the conditions on f :

- (ast) f extends to a holomorphic section of $\mathfrak{M}^{\otimes k}$ over $\overline{M}^r(N)$;
- (a) f extends to a strongly continuous section of $\mathfrak{M}^{\otimes k}$ over $\overline{M}^r(N)$ (that is, f regarded as a $\Gamma(N)$ -invariant homogeneous function of weight k on Ψ^r has a strongly continuous extension to $\overline{\Psi}^r$);
- (b) f is integral over the ring $\mathbf{Mod} = C_\infty[g_1, \dots, g_r]$ of modular forms of type 0 for Γ ;
- (c) f along with all its conjugates $f_{[\gamma]_k}$ ($\gamma \in RS$) is bounded on \mathbf{F} .

7.4 Theorem. *For weak modular forms f of weight k and level N , we have the following implications: (ast) \Rightarrow (a) \Leftrightarrow (b) \Leftrightarrow (c). If the variety $\overline{M}^r(N)$ happens to be normal, then the four conditions are equivalent.*

Proof. (ast) \Rightarrow (a) is trivial.

(a) \Rightarrow (b) The elementary symmetric functions in the $f_{[\gamma]_k}$ ($\gamma \in RS$) are invariant under $\Gamma/\Gamma(N)$, that is, under Γ and have a strongly continuous extension to $\overline{\Psi}^r$. As $\Gamma \setminus \overline{\Psi}^r = \overline{M}^r$ is normal, these extensions are in fact holomorphic by [1] and therefore modular forms for Γ . Hence f satisfies an integral equation

$$f^n + a_1 f^{n-1} + \dots + a_n = 0 \text{ with } a_i \in \mathbf{Mod}. \quad (7.4.1)$$

(b) \Rightarrow (c) Suppose that f is subject to an equation (7.4.1). Then it also holds for f replaced by $f_{[\gamma]_k}$, and f along with all its conjugates is bounded on \mathbf{F} , as the a_i are ([21] Proposition 1.8).

(c) \Rightarrow (b) As in (a) \Rightarrow (b), f satisfies an equation of type (7.4.1) with the elementary symmetric functions in the $f_{[\gamma]_k}$ ($\gamma \in RS$) as coefficients a_i up to sign. From the boundedness of the $f_{[\gamma]_k}$ we conclude the boundedness of the a_i on \mathbf{F} , which in turn implies $a_i \in \mathbf{Mod}$ ([21] Proposition 1.8).

(b) \Rightarrow (a) Suppose that $P(f) = 0$ with $P(X) = X^n + a_1 X^{n-1} + \dots + a_n$ and coefficients $a_1, \dots, a_n \in \mathbf{Mod}$. Regarding f as a homogeneous and $\Gamma(N)$ -invariant function of weight k on Ψ^r , we must show that it extends to a strongly continuous function on $\overline{\Psi}^r$.

Let $\omega = (U, i)$ be a boundary point, $\omega \in \overline{\Psi}^r \setminus \Psi^r$, and let $(\omega_\ell)_{\ell \in \mathbb{N}}$ be a sequence of elements of Ψ^r that tends to ω . Without restriction, replacing f with some transform $f_{[\gamma]_k}$ if necessary, we may assume that $\omega \in \Psi_{V_s}$ for some $1 \leq s < r$, $\omega = (0, \dots, 0, \omega_{r-s+1}, \dots, \omega_r)$, where ω lies in the corresponding fundamental domain \mathbf{F}_s , see (1.15.4). We will show:

(A) $(f(\omega_\ell))_{\ell \in \mathbb{N}}$ converges in C_∞ .

Therefore, $\overline{f}(\omega) := \lim_{\ell \rightarrow \infty} f(\omega_\ell) = \lim_{\substack{\omega' \in \Psi^r \\ \omega' \rightarrow \omega}} f(\omega')$ exists;

(B) The so-defined extension \bar{f} of f to $\bar{\Psi}^r$ is strongly continuous, of weight k and $\Gamma(N)$ -invariant.

Put $\bar{a}_i := \lim_{\ell \rightarrow \infty} a_i(\omega_\ell)$, which exists as a_i is a modular form for Γ , and let $\bar{P}(X) = X^n + \sum_{1 \leq i \leq n} \bar{a}_i X^{n-i}$ be the limit polynomial. Elementary estimates show that $f(\omega_\ell)$ is close to a zero of \bar{P} if $\ell \gg 0$. Hence the set of limit points of $(f(\omega_\ell))_{\ell \in \mathbb{N}}$ is contained in the set $Z(\bar{P})$ of zeroes of \bar{P} , and each $f(\omega_\ell)$ is close to one of them for $\ell \gg 0$. Let Y be a small neighborhood (w.r.t. the strong topology) of ω in $\bar{\Psi}^r$. Then for Y and $\epsilon > 0$ small enough,

$$\Psi^r \cap Y = \bigcup_{x \in Z(\bar{P})} Y_x, \quad (7.4.2)$$

where $Y_x := \{\omega' \in \Psi^r \cap Y \mid |f(\omega') - x| < \epsilon\}$. We may further choose Y such that

$$\begin{aligned} \Psi^r \cap Y &= \{\omega' \in \Psi \mid (\omega'_{r-s+1}, \dots, \omega'_r) \\ &\text{lies in a fixed connected open affinoid neighborhood } X \text{ of} \\ &\omega \text{ in } \Psi_{V_s} \text{ and } d(\omega'_i, \langle \omega'_{r-s+1}, \dots, \omega'_r \rangle_{K_\infty}) \geq c \text{ for } 1 \leq i \leq r-s\} \end{aligned} \quad (7.4.3)$$

for sufficiently large $c \in q^{\mathbb{Q}}$. Such a set is connected as an analytic space.

Now the occurrence of at least two different zeroes x in (7.4.2) would contradict the connectedness of $\Psi^r \cap Y$. Hence there exists only one limit point x , which equals

$$\bar{f}(\omega) := \lim_{\ell \rightarrow \infty} f(\omega_\ell) = \lim_{\substack{\omega' \in \Psi^r \\ \omega' \rightarrow \omega}} f(\omega'),$$

and (A) is established. The fact (B) that \bar{f} is strongly continuous is seen by a modification of the above argument, working now with approximating sequences $(\omega_\ell)_{\ell \in \mathbb{N}}$ for ω with $\omega_\ell \in \bar{\Psi}^r$. Also the properties of weight k and $\Gamma(N)$ -invariance turn over from f to \bar{f} .

Finally, suppose $\bar{M}^r(N)$ (and thus $\widetilde{\bar{M}}^r(N)$) is normal. Then the existence of a holomorphic extension \bar{f} of f follows, again by Bartenwerfer's criterion, from the existence of a strongly continuous extension. Hence in this case, (a) implies in fact (ast), and all four conditions are equivalent. \square

7.5 Definition. We define the Satake compactification $M^r(N)^{\text{Sat}}$ of $M^r(N)$ as the normalization of $\bar{M}^r(N)$ in its function field $\mathcal{F}_r(N)$. It is a normal projective C_∞ -variety provided with an embedding $\iota : M^r(N) \hookrightarrow M^r(N)^{\text{Sat}}$ and a finite birational morphism $\nu : M^r(N)^{\text{Sat}} \rightarrow \bar{M}^r(N)$ such that $\nu \circ \iota$ is the identity on $M^r(N)$. Likewise, we define $\widetilde{M}^r(N)^{\text{Sat}}$ as the normalization of $\widetilde{\bar{M}}^r(N)$ in its function field $\widetilde{\mathcal{F}}_r(N)$. It has similar properties and is supplied with an action of \mathbb{G}_m such that $\mathbb{G}_m \backslash \widetilde{M}^r(N)^{\text{Sat}} \xrightarrow{\cong} M^r(N)^{\text{Sat}}$.

7.6 Corollary (to the proof of Theorem 7.4; see also [29], proof of Proposition 1.23). The varieties $\widetilde{M}^r(N)$ and $\overline{M}^r(N)$ are unbranched, that is, the canonical maps

$$\widetilde{\nu} : \widetilde{M}^r(N)^{\text{Sat}} \longrightarrow \widetilde{M}^r(N) \text{ and } \nu : M^r(N)^{\text{Sat}} \longrightarrow \overline{M}^r(N)$$

are bijective.

Proof. Since the open subspace $\widetilde{M}^r(N)$ of $\widetilde{M}(N)$ is smooth, it suffices to consider boundary points $[\omega]$ of $\widetilde{M}(N)$ represented by ω as in the proof of $\boxed{(b) \Rightarrow (a)}$. Then we have to show that $[\omega]$ has at most one pre-image in the normalization $\widetilde{M}^r(N)^{\text{Sat}}$. However, this follows from the connectedness of the sets $\Psi^r \cap Y$ in (7.4.3). (If there were several pre-images of $[\omega]$ then $Y \setminus \{\omega\}$ and also $\Psi^r \cap Y$ had to split into several components for Y sufficiently small.) The argument for $\overline{M}^r(N)$ follows the same lines. \square

7.7 Remark. We know from (6.10) that the singular locus of $\overline{M}^r(N)$ is contained in

$$\overline{M}_{\leq r-2}^r(N) := \bigcup_{\substack{U \in \mathfrak{U} \\ \dim U \leq r-2}} M_U(N)$$

and therefore has codimension ≥ 2 , as expected for a normal variety. Together with unbranchedness this suggests that $\overline{M}^r(N)$ should itself be normal, i.e., $M^r(N)^{\text{Sat}} = \overline{M}^r(N)$. However, this is not a formal implication, and the question of normality of $\overline{M}^r(N)$ is still open.

(7.8). At least, $\nu : M^r(N)^{\text{Sat}} \longrightarrow \overline{M}^r(N)$ is bijective by (7.6) and therefore (since it is a finite morphism) a strong homeomorphism of the sets of C_∞ -points. As we don't know whether ν is always an isomorphism (see Section 8 for examples), we make the following double definition.

(7.8.1) A *strong modular form* of weight k and level N is a weak modular form f that satisfies condition (7.3) (ast), that is, f extends to a holomorphic section of $\mathfrak{M}^{\otimes k}$ over $\overline{M}^r(N)$. A *modular form* of weight k and level N is a weak modular form f that satisfies (7.3) (a), i.e., the boundary condition is relaxed to: f extends to a strongly continuous section of $\mathfrak{M}^{\otimes k}$, or equivalently (by (7.6) and Bartenwerfer's criterion), f extends holomorphically to a section of $\nu^*(\mathfrak{M}^{\otimes k})$ over the Satake compactification $M^r(N)^{\text{Sat}}$.

(7.8.2) We let $\mathbf{Mod}_k^{\text{st}}(N)$ resp. $\mathbf{Mod}_k(N)$ be the C_∞ -spaces of (strong) modular forms of weight k and

$$\mathbf{Mod}^{\text{st}}(N) = \bigoplus_{k \geq 0} \mathbf{Mod}_k^{\text{st}}(N), \quad \mathbf{Mod}(N) = \bigoplus_{k \geq 0} \mathbf{Mod}_k(N)$$

the corresponding graded C_∞ -algebras. Then

$$\mathbf{Eis}(N) \subset \mathbf{Mod}^{\text{st}}(N) \subset \mathbf{Mod}(N). \quad (7.8.3)$$

The common quotient field of the three rings is the field $\widetilde{\mathcal{F}}_r(N)$. By (7.4) the following criterion holds.

7.9 Theorem. *Let f be a weak modular form of weight k and level N . Then the following three conditions are equivalent:*

- (a) $f \in \mathbf{Mod}(N)$;
- (b) f is integral over $\mathbf{Mod} = C_\infty[g_1, \dots, g_r]$;
- (c) f and all its conjugates $f_{[\gamma]_k}$ ($\gamma \in RS$) are bounded on the fundamental domain \mathbf{F} .

(7.10). Let J be the ideal of Eisenstein relations in $R = \text{Sym}(\mathbf{Eis}_1(N))$, see (5.4). To the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow \mathbf{Eis}(N) \longrightarrow 0$$

corresponds an exact sequence of sheaves (in the algebraic sense) on the variety $\mathbb{P} = \text{Proj}(R)$

$$0 \longrightarrow \mathfrak{J} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\overline{M}^r(N)} \longrightarrow 0, \quad (7.10.1)$$

where we regard the structure sheaf $\mathcal{O}_{\overline{M}^r(N)}$ of $\overline{M}^r(N)$ as a sheaf on \mathbb{P} with support in $\overline{M}^r(N) \hookrightarrow \mathbb{P}$. It remains exact upon tensoring with the sheaf $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ over \mathbb{P} , where $k > 0$. As $\mathcal{O}(1)$ restricted to $\overline{M}^r(N)$ is the sheaf \mathfrak{M} of strong modular forms, we find the exact sequence

$$0 \longrightarrow \mathfrak{J}(k) \longrightarrow \mathcal{O}_{\mathbb{P}}(k) \longrightarrow \mathfrak{M}(k) \longrightarrow 0. \quad (7.10.2)$$

The first part of its exact cohomology sequence reads:

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}, \mathfrak{J}(k)) &\longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \xrightarrow{\alpha} H^0(\mathbb{P}, \mathfrak{M}(k)) \longrightarrow \\ &\longrightarrow H^1(\mathbb{P}, \mathfrak{J}(k)) \longrightarrow H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \dots \end{aligned}$$

Now,

- $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$ vanishes (see e.g. [26] III Theorem 5.1);
- $H^0(\mathbb{P}, \mathfrak{M}(k)) = H^0(\overline{M}^r(N), \mathfrak{M}^{\otimes k}) = \mathbf{Mod}_k^{\text{st}}(N)$;
- $\text{im}(\alpha)$ is the subspace $\mathbf{Eis}_k(N)$ of strong modular forms that belong to the Eisenstein algebra $\mathbf{Eis}(N)$.

Hence $H^1(\mathbb{P}, \mathfrak{J}(k))$ measures the difference between $\mathbf{Eis}_k(N)$ and $\mathbf{Mod}^{\text{st}}(N)$. By standard properties ([26] III Theorem 5.2), $H^1(\mathbb{P}, \mathfrak{J}(k))$ vanishes for large k . As it is always finite-dimensional, we see:

7.11 Corollary. For k sufficiently large, $\mathbf{Mod}_k^{\text{st}}(N)$ agrees with its subspace $\mathbf{Eis}_k(N)$. In particular, the Eisenstein algebra $\mathbf{Eis}(N)$ has finite codimension in the algebra $\mathbf{Mod}^{\text{st}}(N)$.

7.12 Corollary. $\overline{M}^r(N) = \text{Proj}(\mathbf{Eis}(N)) = \text{Proj}(\mathbf{Mod}^{\text{st}}(N))$.

Proof. The first equality has been shown in Section 5, the second is a formal consequence of the definition of $\mathbf{Mod}^{\text{st}}(N)$, but follows also from $\dim(\mathbf{Mod}^{\text{st}}(N)/\mathbf{Eis}(N)) < \infty$. \square

(7.13). Consider the projective variety $\text{Proj}(\mathbf{Mod}(N))$ attached to the algebra of modular forms. It is normal (as $\mathbf{Mod}(N)$ is integrally closed), provided with a natural map to $\text{Proj}(\mathbf{Mod}^{\text{st}}(N)) = \overline{M}^r(N)$, and birational with $\overline{M}^r(N)$, and thus agrees with the normalization, i.e.,

$$M^r(N)^{\text{Sat}} = \text{Proj}(\mathbf{Mod}(N)). \quad (7.13.1)$$

7.14 Corollary. The three assertions are equivalent:

- (i) $\overline{M}^r(N)$ is normal;
- (ii) $\mathbf{Mod}^{\text{st}}(N) = \mathbf{Mod}(N)$;
- (iii) $\mathbf{Mod}^{\text{st}}(N)$ has finite codimension in $\mathbf{Mod}(N)$.

Proof. (i) \Rightarrow (ii) is the last assertion of Theorem 7.4, and (ii) \Rightarrow (iii) is trivial. Suppose that (iii) holds. Then $\text{Proj}(\mathbf{Mod}(N)) = \text{Proj}(\mathbf{Mod}^{\text{st}}(N)) = \overline{M}^r(N)$, and it follows that the latter is normal. \square

7.15 Corollary. Suppose that $\overline{M}^r(N)$ fails to be normal. Then there exist arbitrarily large weights k such that $\mathbf{Mod}_k(N)$ is strictly larger than its subspace $\mathbf{Eis}_k(N)$.

Proof. As all the $\mathbf{Mod}_k(N)$ have finite dimension, this follows from the last corollary. \square

(7.16). We conclude this section with an observation about $\mathbf{Mod}^{\text{st}}(N)$. Given $\mathbf{u} \in \mathcal{T}(N)$, we let $\mathbf{Eis}(N)_{E_{\mathbf{u}}}$ be the localization w.r.t. $E_{\mathbf{u}}$, i.e., $\text{Spec}(\mathbf{Eis}(N)_{E_{\mathbf{u}}})$ is the open subvariety of $\overline{M}^r(N)$ where $E_{\mathbf{u}}$ doesn't vanish. Hence

$$\overline{M}^r(N) = \bigcup_{\mathbf{u} \in \mathcal{T}(N)} \text{Spec}(\mathbf{Eis}(N)_{E_{\mathbf{u}}}) = \bigcup_{\mathbf{u} \in N^{-1}S} \dots,$$

where S is the set of representatives of $(A/N)_{\text{prim}}^r/\mathbf{F}^*$ given in (3.5). Similarly,

$$\overline{M}^r(N) = \bigcup_{\mathbf{u} \in N^{-1}S} \overline{M}^r(N)_{(E_{\mathbf{u}} \neq 0)}.$$

A weak modular form of weight k extends to a section of $\mathfrak{M}^{\otimes k}$ if and only if its restriction to each $\overline{M}^r(N)_{(E_u \neq 0)}$ has the corresponding property, that is, belongs to $\mathbf{Eis}(N)_{E_u}$. Therefore we may describe $\mathbf{Mod}^{\text{st}}(N)$ as the intersection

$$\mathbf{Mod}^{\text{st}}(N) = \bigcap_{u \in N^{-1}S} \mathbf{Eis}(N)_{E_u} \quad (7.16.1)$$

in $\widetilde{\mathcal{F}}(N)$.

8. Examples and concluding remarks

The preceding immediately raises a number of important questions and desiderata.

8.1 Question. Do the Eisenstein and Satake compactifications $\overline{M}^r(N)$ and $M^r(N)^{\text{Sat}}$ always coincide, i.e., is $\overline{M}^r(N)$ always normal?

(8.2). Describe the singularities of both compactifications and construct natural desingularizations together with a modular interpretation! (See [32] for some results.)

(8.3). How far do the algebras $\mathbf{Eis}(N)$, $\mathbf{Mod}^{\text{st}}(N)$, $\mathbf{Mod}(N)$ differ, if at all? Describe their Hilbert functions, that is, the dimensions of their pieces in dimension k , and find presentations for these algebras!

Almost nothing about these questions is known when the rank r is larger than 2. We will briefly present the state-of-the-art in the case where $\boxed{r=2}$, which we assume until (8.9). Here the $\overline{M}^2(N)$ are smooth curves [11], so the Eisenstein and Satake compactification agree, and therefore $\mathbf{Mod}^{\text{st}}(N) = \mathbf{Mod}(N)$. The genera of the $\overline{M}^2(N)$ have been determined by Goss [23] and, with a different method, by the author [13].

(8.4). Let $d \geq 1$ be the degree of N , and suppose that

$$N = \prod_{1 \leq i \leq t} \mathfrak{p}_i^{s_i}$$

is the prime decomposition, where the \mathfrak{p}_i are different monic prime polynomials. As in (3.6), write $q_i = q^{\deg \mathfrak{p}_i}$. We define

$$\lambda(N) := \prod_{1 \leq i \leq t} q_i^{2s_i-2} (q_i^2 - 1), \quad (8.4.1)$$

which appears in the formulas below. (Note that $\lambda(N) = \varphi(N)\epsilon(N)$ with the arithmetic functions φ, ϵ defined in [17] 1.5.) Then the numbers $g(N)$ = genus of the modular curve $\overline{M}^2(N)$, $c(N) = c_2(N)$ = number of cusps of $M^2(N)$, $\deg(\mathfrak{M})$ = degree of the line bundle of modular forms over $\overline{M}^2(N)$ and $\dim \mathbf{Mod}_k(N) = \dim \mathbf{Mod}_k^{\text{st}}(N)$ are given by

$$(8.4.2) \quad g(N) = 1 + \lambda(N)(q^d - q - 1)/(q^2 - 1);$$

$$(8.4.3) \quad c(N) = \lambda(N)/(q - 1);$$

$$(8.4.4) \quad \deg(\mathfrak{M}) = \lambda(N)q^d/(q^2 - 1);$$

$$(8.4.5) \quad \dim \mathbf{Mod}_k(N) = ((k - 1)q^d + q + 1)\lambda(N)/(q^2 - 1).$$

Here (8.4.2) and (8.4.3) may be found in [23] and [13] (such data for other Drinfeld modular curves are collected in [17]) and (8.4.4) is from [15] VII 6.1. The last formula (8.4.5) is an immediate consequence of the Riemann-Roch theorem provided that $k \geq 2$; for $k = 1$, Riemann-Roch and Serre duality yield only

$$\dim \mathbf{Mod}_1(N) = c(N) + \dim H^1(\overline{M}^2(N), \mathfrak{M}), \quad (8.4.6)$$

where $c(N) = \dim \mathbf{Eis}_1(N) = \dim \text{Eis}_1(N)$ and $\dim H^1(\overline{M}^2(N), \mathfrak{M}) = \dim \mathbf{Mod}_1^2(N)$ with the space $\mathbf{Mod}_1^2(N)$ of double cuspidal (double zeroes at the cusps) modular forms of weight 1 ([15] p. 92). However by the next result, (8.4.5) is valid for $k = 1$, too.

8.5 Proposition. *For $r = 2$ we have $\text{Eis}_1(N) = \mathbf{Mod}_1(N)$, of dimension $c(N)$.*

As a proof of this basic fact so far has not been published, we give a brief sketch here.

Proof. By Corollary 4.7, $\dim \text{Eis}_1(N) = c(N)$. Since moreover the space of cusp forms $\mathbf{Mod}_1^1(N)$ is a complement of $\text{Eis}_1(N)$ in $\mathbf{Mod}_1(N)$ (this is a consequence of Proposition 4.6), it suffices to show that there are no non-trivial cusp forms of weight 1.

Assume that $f \in \mathbf{Mod}_1^1(N)$. Then f^p is a cusp form of weight $p \geq 2$ for $\Gamma(N)$, where $p = \text{char}(\mathbf{F})$. For $0 \leq i \leq p - 2$, the residue $\text{res}_e \omega^i f^p(\omega) d\omega$ of the differential form $\omega^i f^p(\omega) d\omega$ on $\Omega = \Omega^2$ at the oriented edge e of the Bruhat-Tits tree of $\text{PGL}(2, K_\infty)$ vanishes for each e , as is immediate from the definition of res_e , see [35] Definition 9. Hence the image $\text{res}(f^p)$ under Teitelbaum's isomorphism ([35] Theorem 16) of cusp forms of weight p with the space of cocycles of a certain type vanishes, and so do f^p and f itself. \square

8.6 Remark. All the formulas and results in (8.4) and (8.5) have generalizations

- to other congruence subgroups of Γ , e.g., Hecke congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, etc., see [17];
- to more general Drinfeld rings A than $A = \mathbb{F}[T]$, e.g., A the affine ring of an elliptic curve over \mathbb{F} , see [15] pp. 92–93.

As to the relationship between $\mathbf{Eis}(N)$ and $\mathbf{Mod}(N)$, there is the following result of Cornelissen [9].

8.7 Theorem (Cornelissen). Let $r = 2$. The algebra $\mathbf{Mod}(N)$ of modular forms for $\Gamma(N)$ is generated by $\mathbf{Mod}_1(N) = \mathbf{Eis}_1(N)$ and the space $\mathbf{Mod}_2^1(N)$ of cusp forms of weight 2.

In fact, it is not difficult (using [14] Korollar 2.2) to show that $\mathbf{Mod}_2^1(N)$ above may be replaced with the space $\mathbf{Mod}_2^2(N)$ of double cuspidal forms of weight 2, which under $f(\omega) \mapsto f(\omega)d\omega$ corresponds to the $g(N)$ -dimensional space of holomorphic differentials on $\overline{M}^2(N)$. Still, this doesn't completely answer the question (8.3) of whether $\mathbf{Eis}(N) = \mathbf{Mod}(N)$ in this case. The only positive results in this direction seem to be the following two examples.

8.8 Example. Suppose that $r = 2$ and $d = \deg N = 1$. Then $g(N) = 0$, that is, $\overline{M}^2(N)$ is a projective line, and $\deg(\mathfrak{M}) = q$. Therefore, $\mathbf{Eis}(N) = \mathbf{Mod}(N)$ and $\dim \mathbf{Mod}_k(N) = 1 + kq$ in this case. From this, a presentation of $\mathbf{Mod}(N)$ may be derived ([9], see also [36], [37], which also study the spaces $\mathbf{Mod}_k(N)$ as modules under the action of $\Gamma/\Gamma(N) = \mathrm{GL}(2, \mathbb{F})$).

8.9 Example (Cornelissen [8]). Let again $r = 2$, $q = 2$, and $d = \deg N = 2$. Then $\mathbf{Eis}(N) = \mathbf{Mod}(N)$.

As the possible genera $g(N)$ here are positive (they may take the values 4, 5 and 6), this case is less trivial than (8.8). The equality of the two rings, i.e., the projective normality of the Eisenstein embedding $\overline{M}^2 \hookrightarrow \mathbb{P}$, is based on a numerical criterion of Castelnuovo. Unfortunately, the validity of this argument is strictly limited to the requirements of Example 8.9.

For the next example, we return to the general case, where $r \geq 2$ is arbitrary.

8.10 Example. Suppose that $d = \deg N = 1$. After a coordinate change, we may assume $N = T$. This case has been extensively studied by Pink and Schieder [33]. Actually, they consider a certain \mathbb{F} -variety Q_V , but which after base extension with C_∞ and some translational work may also be seen as our $\overline{M}^r(T)$. Their results (overlapping with (8.8) if $r = 2$) give

- $\mathbf{Eis}(T) = \mathbf{Mod}(T)$ (Theorem 1.7 in [33]), i.e., the normality of $R_V = \mathbf{Eis}(T)$;
- a presentation through generators and relations (Theorem 1.6);
- the Hilbert function of $\mathbf{Eis}(T)$ (Theorem 1.10).

Further, they construct and discuss a desingularization B_V of Q_V (Section 10). Hence, concerning our questions (8.1)–(8.3), nothing is left to desire. But note that these satisfactory and complete results refer only to the (isolated?) case where $\deg N = 1$. None of our current knowledge excludes the possibility that always $\mathbf{Eis}(N) = \mathbf{Mod}(N)$, or the sheer opposite possibility that the two rings differ almost always and (8.8), (8.9), (8.10) are just extreme boundary cases.

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