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Journal of Number Theory 110 (2005) 136–163

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

On the Euler characteristic of the discrete spectrum

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Received 20 August 2003; revised 15 January 2004

Communicated by D. Goss

Available online 29 September 2004

To the memory of Arnold Ross

Abstract

This paper, which is largely expository in nature, seeks to illustrate some of the advances that have been made on the trace formula in the past 15 years. We review the basic theory of the trace formula, then introduce some ideas of Arthur and Kottwitz that allow one to calculate the Euler characteristic of the S-cohomology of the discrete spectrum. This Euler characteristic is first expressed as a trace of a certain test function on the space of automorphic forms, and then, by the stable trace formula, is converted into a sum of orbital integrals. A result on global measures allows us to calculate these integrals in terms of the values of certain Artin L-functions at negative integers.

Our intention is to show how advances in the theory have allowed one to render such calculations completely explicit. As a byproduct of this calculation, we obtain the existence of automorphic representations with certain local behavior at the places in S.

Keywords: Automorphic form; Trace formula; Orbital integrals; Artin L-function

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0022-314X/$ - see front matter
doi:10.1016/j.jnt.2004.03.008

1. Introduction

For a smooth, compactly supported function f on $\mathbb{R}$, with Fourier transform $\hat{f}$, the Poisson summation formula asserts that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

This formula (and its generalization to functions f of rapid decay) has had broad application in many areas of mathematics. In number theory, for instance, it can be used to prove the modularity of the theta function of a Euclidean lattice.

We can give a representation theoretic interpretation of the right-hand side of the summation formula. The function f acts on the Hilbert space $L^2(\mathbb{Z}\backslash\mathbb{R})$ by the linear operator sending $F \in L^2(\mathbb{Z}\backslash\mathbb{R})$ to

$$(R(f)F)(x) = \int_{\mathbb{R}} f(y)F(x+y)\,dy.$$

This is just an averaging of the right regular representation of the additive group $\mathbb{R}$ on $L^2(\mathbb{Z}\backslash\mathbb{R})$. Now $L^2(\mathbb{Z}\backslash\mathbb{R})$ is well understood as a representation of $\mathbb{R}$: it has a Hilbert space basis consisting of the functions $v_n(x) = e^{-inx}$, and $y \in \mathbb{R}$ acts on v_n by multiplication by the character e^{-iny}. Then we see that

$$(R(f)v_n)(x) = \int_{\mathbb{R}} f(y)e^{-in(x+y)}\,dy = e^{-inx}\int_{\mathbb{R}} f(y)e^{-iny}\,dy = \hat{f}(n)v_n(x).$$

Hence $\hat{f}(n)$ is the eigenvalue of $R(f)$ on the vector v_n, and the right-hand side of the Poisson summation formula is the trace of $R(f)$ on $L^2(\mathbb{Z}\backslash\mathbb{R})$. On the other hand, the left-hand side of the formula is a sum over the elements (or conjugacy classes) of the discrete subgroup $\mathbb{Z}$ of $\mathbb{R}$.

In his 1956 paper [23], Selberg introduced his trace formula for SL(2), which gives a non-abelian generalization of the Poisson summation formula. We'll start by looking at the trace formula in an abstract setting.

Let $\mathcal{G}$ be a locally compact topological group, and Γ a subgroup of $\mathcal{G}$ which is both discrete and co-compact. In the case of the Poisson summation formula, $\mathcal{G}$ is the additive group of real numbers and Γ is the subgroup of integers. A Haar measure dg on $\mathcal{G}$ induces a measure on the coset space $\Gamma\backslash\mathcal{G}$, taking counting measure on Γ. Again, in the case of Poisson summation we take dg to be Lebesgue measure; the induced measure of $\Gamma\backslash\mathcal{G} \cong S^1$ is the Haar measure of volume 1.

Now right translation gives a representation of $\mathcal{G}$ on $L^2(\Gamma\backslash\mathcal{G})$:

$$gF(x) = F(xg)$$

for $g \in \mathcal{G}$ and $F \in L^2(\Gamma\backslash\mathcal{G})$. If f is a compactly supported measurable function on $\mathcal{G}$, then we can average this representation according to the measure $\varphi = f\,dg$. So φ gives an endomorphism of the Hilbert space $L^2(\Gamma\backslash\mathcal{G}, dg)$, mapping F to the function

$$\varphi F(x) = \int_{\mathcal{G}} F(xg)\ f(g)\,dg.$$

We assume further that f satisfies a regularity condition. If $\mathcal{G}$ is a Lie group, this regularity condition is exactly that f be infinitely differentiable. In general, the regularity condition is that given by Bruhat [6].

We compute, using Fubini's theorem and the Γ-invariance of F:

$$\begin{aligned}\int_{\mathcal{G}} F(xg)f(g)\ dg &= \int_{\Gamma\backslash\mathcal{G}} \sum_{\gamma\in\Gamma} F(\gamma h) f(x^{-1}\gamma h)\,dh \\ &= \int_{\Gamma\backslash\mathcal{G}} F(h) \sum_{\gamma\in\Gamma} f(x^{-1}\gamma h)\,dh.\end{aligned}$$

Thus we see that the endomorphism φ is given by integration against the compact kernel

$$K(x, g) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma g).$$

For fixed x the sum is finite, as Γ is discrete and f has compact support. Note that K is a function on $\Gamma\backslash\mathcal{G} \times \Gamma\backslash\mathcal{G}$. Since the kernel is compact, the endomorphism φ has a trace, namely,

$$\mathrm{Tr}(\varphi) = \int_{\Gamma\backslash\mathcal{G}} K(g, g)\,dg.$$

Note that $K(g, g) = \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)$. We would like to exchange the order of the sum and the integral in our formula for $\mathrm{Tr}(\varphi)$. This motivates the following definition.

For γ in Γ, let Γ_γ be its centralizer in Γ and let $\mathcal{G}_\gamma$ be its centralizer in $\mathcal{G}$. Define the orbital integral

$$O_\gamma(\varphi, dg_\gamma) = \int_{\mathcal{G}_\gamma\backslash\mathcal{G}} f(g^{-1}\gamma g)\ \frac{dg}{dg_\gamma}.$$

This depends on the choice of a Haar measure dg_γ on $\mathcal{G}_\gamma$. The orbital measure

$$dg_\gamma(\varphi) = O_\gamma(\varphi, dg_\gamma)\,dg_\gamma$$

on $\mathcal{G}_\gamma$ is invariant and depends only on γ.

We then have

$$\begin{aligned}\mathrm{Tr}(\varphi|L^2(\Gamma\backslash\mathcal{G})) &= \sum_\gamma \int_{\Gamma_\gamma\backslash\mathcal{G}_\gamma} dg_\gamma(\varphi)\\ &= \sum_\gamma \int_{\Gamma_\gamma\backslash\mathcal{G}_\gamma} dg_\gamma \cdot O_\gamma(\varphi, dg_\gamma),\end{aligned}$$

where γ runs through a set of representatives of the conjugacy classes of Γ. The sum, moreover, is absolutely convergent [19].

On the other hand, since $\Gamma\backslash\mathcal{G}$ is compact, the representation $L^2(\Gamma\backslash\mathcal{G})$ of $\mathcal{G}$ is completely reducible. That is,

$$L^2(\Gamma\backslash\mathcal{G}) \cong \bigoplus_\pi m_\pi \pi,$$

where the π are irreducible representations of $\mathcal{G}$. Then φ acts on each π, again by averaging. The trace of φ on $L^2(\Gamma\backslash\mathcal{G})$ is then the sum of the traces on the π's and thus we get the abstract trace formula

$$\sum_\gamma \int_{\Gamma_\gamma\backslash\mathcal{G}_\gamma} dg_\gamma \cdot O_\gamma(\varphi, dg_\gamma) = \mathrm{Tr}(\varphi|L^2(\Gamma\backslash\mathcal{G})) = \sum_\pi m_\pi \mathrm{Tr}(\varphi|\pi).$$

The left-hand side of the trace formula is called the geometric side, as it involves the geometry of integrals over conjugacy classes, whereas the right-hand side of the trace formula is called the spectral side, as it involves the spectral decomposition of the Hilbert space $L^2(\Gamma\backslash\mathcal{G})$ as a representation of $\mathcal{G}$.

One wants to apply the trace formula to situations where the quotient $\Gamma\backslash\mathcal{G}$ is not compact. Quite a number of difficulties arise here, not the least of which is that the operator is the endomorphism given by φ. In his 1956 paper, Selberg employed the theory of Eisenstein series to study the case $\mathcal{G} = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is directly related to the study of modular forms on the upper half plane.

In a more modern language, we let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ and consider $\mathcal{G} = \mathrm{GL}_2(\mathbb{A})$ and $\Gamma = \mathrm{GL}_2(\mathbb{Q})$. Then the representation $V = L^2(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ of $\mathrm{GL}_2(\mathbb{A})$ encodes information about classical modular forms (holomorphic or not) on the upper half-plane. Indeed, knowing the irreducible constituents of V tells us the dimensions of the spaces of classical cusp forms, as well their Hecke eigenvalues.

More generally, if G is any reductive algebraic group over $\mathbb{Q}$, we can again look at $V = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. This representation again encodes important arithmetic information. Since $G(\mathbb{Q})\backslash G(\mathbb{A})$ need not be compact, the version of the trace formula given above does not always apply. However, if we restrict our attention to a suitable subspace of V and to suitable φ then results of Arthur give a version of the trace formula that does apply. In Sections 2 and 3, we will present a simple version of

Arthur's trace formula. For a discussion of Arthur's proof of this formula we refer the reader to the books by Gelbart [11] or Shokranian [24].

Our goal for using the trace formula here will be to explicitly determine multiplicities m_π appearing on the spectral side. To do this we will have to pick good test functions f that will let us isolate certain π. We discuss the choice of these functions in Section 4.

Another important application of the trace formula is the comparison of the spectra of two different groups. Langlands's theory of functoriality predicts that a map between the L-groups of two groups G_1 and G_2 allows one to transfer certain automorphic representations between the groups. A major tool in proving instances of this functoriality is to choose suitable test functions f_1 and f_2 on G_1 and G_2 as above and then to prove the corresponding geometric sides of the trace formula agree. We will not go into this matter here; for the first important case, the reader could consult [15].

Even Arthur's version of the trace formula we give in Section 2 is still too difficult to use, since it requires an enumeration of the conjugacy classes in $G(\mathbb{Q})$. In Section 5, we discuss the "stabilization" by Kottwitz that rewrites the trace formula in terms of stable conjugacy rather than $G(\mathbb{Q})$ conjugacy (in our case, stable conjugacy is just conjugacy in $G(\overline{\mathbb{Q}})$). In Section 6, we explain how to compare the various local measures that come up in the orbital integrals with a global measure so that we can make use of special values of L-series. In Sections 7 and 8, we relate the results of our trace formula calculations to modular forms. There is an amusing subtlety that arises here: our final version of the trace formula contains some local quantities whose computation is quite difficult. We use direct calculations modular forms to obtain these values.

Finally, in Section 9 we make some conjectures related to our computations.

2. The trace formula

Let G be a simply connected, semi-simple algebraic group defined over $\mathbb{Q}$. We will keep this condition on G throughout, unless otherwise noted. For example, G could be SL_2, the group Sp_4 of 4×4 symplectic matrices, or the group G_2 of automorphisms of the octonions.

Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$. The group $G(\mathbb{A})$ is locally compact and unimodular; let dg be a fixed Haar measure on $G(\mathbb{A})$. The subgroup $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$, so dg induces a measure on the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$, which has finite volume [4].

The group $G(\mathbb{A})$ acts unitarily, by right translation, on the Hilbert space

$$L^2 = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), dg).$$

If $G(\mathbb{R})$ is compact, then $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact and the abstract trace formula as presented in the introduction applies. If, as is the case for $G = \mathrm{SL}_2$, $G(\mathbb{R})$ is not compact, then we need instead to look at a subspace of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), dg)$.

Let

$$L = L^2_{\text{disc}} \subset L^2$$

be the sum of all irreducible $G(\mathbb{A})$-subspaces of L^2. L is called the discrete spectrum and decomposes as a Hilbert direct sum of irreducible unitary representations π of $G(\mathbb{A})$, with finite multiplicities $m(\pi)$:

$$L = \oplus\, m(\pi)\pi.$$

Each irreducible π is a restricted tensor product

$$\pi = \otimes\, \pi_v,$$

with π_v an irreducible, unitary representation of $G(\mathbb{Q}_v)$ [10].

We need a modification of the trace formula which gives the trace of φ only on the discrete spectrum L. This modification will exist for measures $\varphi = \Pi\varphi_v$ on $G(\mathbb{A})$, satisfying certain local conditions. In order to state these local conditions, we will first need a few definitions.

If $\varphi = \varphi_v$ is a smooth, compactly supported measure on $G(\mathbb{Q}_v)$, and π is an irreducible, complex representation of $G(\mathbb{Q}_v)$, then the endomorphism

$$\varphi(w) = \int_{G(\mathbb{Q}_v)} g \cdot w \; d\varphi(g)$$

of π has a trace, which we denote $\mathrm{Tr}(\varphi|\pi)$. Similarly, if γ is a conjugacy class in $G(\mathbb{Q}_v)$, we define the orbital integral

$$O_\gamma(\varphi, dg_\gamma) = \int_{G_\gamma(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} f(g^{-1}\gamma g)\, \frac{dg}{dg_\gamma},$$

which depends on the choice of an invariant measure dg_γ on the centralizer $G_\gamma(\mathbb{Q}_v)$. For the convergence of this integral, see [22]. The orbital measure

$$dg_\gamma(\varphi) = O_\gamma(\varphi, dg_\gamma)\, dg_\gamma$$

on G_γ is again well-defined, independent of the choice of dg_γ.

Before stating the trace formula in this context we note that when $G(\mathbb{R})$ is compact, G is anisotropic over $\mathbb{Q}$ (that is G does not contain a split torus over $\mathbb{Q}$). It follows that every conjugacy class in $G(\mathbb{Q})$ is semi-simple and elliptic over $\mathbb{Q}$. (Recall that γ

is elliptic over F if it is contained in a maximal anisotropic torus T of G over F. On the other hand, if $G(\mathbb{R})$ is not compact, $G(\mathbb{Q})$ will contain elements that are not elliptic semi-simple. The geometric side of Arthur's trace formula on the discrete spectrum, however, is still a sum of orbital integrals only over the elliptic semi-simple conjugacy classes of $G(\mathbb{Q})$.

Proposition (Arthur). *Assume that the smooth, compactly supported measure $\varphi = \Pi\varphi_v$ on $G(\mathbb{A})$ satisfies the following three local conditions*:

1. $\mathrm{Tr}(\varphi_\infty|\pi_\infty) = 0$, *unless the infinitesimal character of π_∞ is regular.*
2. $dg_{\gamma_\infty}(\varphi_\infty) = 0$, *unless the class γ_∞ is both elliptic and semi-simple.*
3. $dg_{\gamma_p}(\varphi_p) = 0$, *unless the class γ_p is both elliptic and semi-simple*, *for some finite p.*

Then φ is of trace class on the discrete spectrum L, *and*

$$\begin{aligned}\mathrm{Tr}(\varphi|L) &= \sum_\gamma \int_{G_\gamma(\mathbb{Q})\backslash G\gamma(\mathbb{A})} dg_\gamma(\varphi)\\ &= \sum_\gamma \int_{G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})} dg_\gamma \cdot O_\gamma(\varphi, dg_\gamma),\end{aligned}$$

where the sum is taken over representatives for the elliptic, *semi-simple conjugacy classes in $G(\mathbb{Q})$*, *only finitely many of which have a non-zero orbital integral for φ.*

We now sketch the proof, which follows from Arthur's general theory, [1,2]. Hypotheses (2) and (3) above imply that the contributions of non-elliptic terms to Arthur's trace formula all vanish. Thus the geometric side $I(f)$ of the trace formula is given by the sum of orbital integrals over elliptic, semi-simple conjugacy classes in $G(\mathbb{Q})$:

$$I(f) = \sum_\gamma \tau(G_\gamma)O_\gamma(f).$$

Here we have used the fact that G is simply connected, so by a result of Borel Steinberg, G_γ is connected. This allows us to identify Arthur's weighting factor a^G with the Tamagawa number $\tau(G_\gamma)$, which is the integral over $G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})$ of Tamagawa measure.

The spectral side $J(f)$ of trace formula is given by a sum over conjugacy classes of Levi subgroups M of G. However, if $M \neq G$, each of these terms will be a linear combination of traces of representations whose real component has singular infinitesimal character. Since hypothesis (1) implies that these terms vanish for the test measure φ, one is left with the term for $M = G$, which is just

$$J(f) = \mathrm{Tr}(\varphi|L).$$

3. The cohomology of the discrete spectrum (cf. [5])

We are going to use the trace formula to compute a certain Euler characteristic on $L \otimes V$ for an irreducible, finite-dimensional representation V of the real Lie group $G(\mathbb{R})$. We'll see in Section 7 that this is tantamount to counting the number of irreducible subrepresentations $\pi = \otimes\pi_v$ of L satisfying certain prescribed conditions on the π_v.

We say a group G is split at the prime p if G splits over $\mathbb{Q}_p$, that is, if $G(\mathbb{Q}_p)$ contains a split maximal torus. The group G need not be split at every prime p. Indeed if, for example, $G = SU_3(\mathbb{Q}(i)/\mathbb{Q})$ is the special unitary group in three variables attached to the extension $\mathbb{Q}(i)/\mathbb{Q}$ then G is split only at those primes congruent to 1 to mod 4. However, for almost all primes p, G must split over an unramified extension of $\mathbb{Q}_p$ and must contain a Borel subgroup defined over $\mathbb{Q}_p$ [26, 3.9.1]. If p is such a prime, we say G is unramified at p.

If S is a finite set of places of $\mathbb{Q}$ which contains the real place and all finite primes p where G is ramified, we may choose an integral model $\underline{G}$ for G over the ring $\mathbb{Z}_S$ of S-integers, with $\underline{G}$ having good reduction at all primes p outside of S. For such a good prime p, $\underline{G}(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\underline{G}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$ (see [26, 1.10] for the definition of hyperspecial). The product

$$G_S(\mathbb{A}) = \prod_{v \in S} G(\mathbb{Q}_v) \times \prod_{p \notin S} \underline{G}(\mathbb{Z}_p)$$

is locally compact, and is open in $G(\mathbb{A})$. Moreover,

$$G(\mathbb{A}) = \varinjlim_{S} G_S(\mathbb{A}).$$

Fix such a finite set S and an integral model $\underline{G}$ for G over $\mathbb{Z}_S$, as well as an irreducible, finite-dimensional representation V of the real Lie group $G(\mathbb{R})$, such that V has trivial central character. The tensor product $L \otimes V$ is a continuous, complex representation of the locally compact group $G_S(\mathbb{A})$, and we may define the continuous cohomology groups

$$H^i(G_S(\mathbb{A}), L \otimes V)$$

following [5, Chapter IX]. These complex vector spaces are finite dimensional, and are zero for $i \gg 0$. Indeed, the subgroup

$$K = \underline{G}(\hat{\mathbb{Z}}_S) = \prod_{p \notin S} \underline{G}(\mathbb{Z}_p)$$

of $G_S(\mathbb{A})$ is compact, so only contributes to H^0, and we find

$$H^i(G_S(\mathbb{A}), L \otimes V) \simeq H^i\left(\prod_{v \in S} G(\mathbb{Q}_v), L^{\underline{G}(\hat{\mathbb{Z}}_S)} \otimes V\right),$$

by the Künneth formula. The local continuous cohomology groups are known to be finite dimensional [5, Proposition X.6.3].

We define the Euler characteristic of the discrete spectrum tensored with V by the formula

$$\chi = \chi(G, S, V) = \sum_{i \geqslant 0} (-1)^i \dim H^i(G_S(\mathbb{A}), L \otimes V).$$

Our goal is to give an explicit formula for χ, under the following two hypotheses:

- $\mathrm{Card}(S) \geqslant 2$, so S contains a finite prime,
- $G(\mathbb{R})$ contains a maximal compact torus.

The first hypothesis is essential to allow us to use the version of the trace formula in the previous section, as well as results of Kottwitz, to rewrite the geometric side in terms of stable conjugacy classes rather than rational conjugacy classes. In our setting, two elements will be stably conjugate if and only if they are conjugate over $G(\overline{\mathbb{Q}})$. The second hypothesis is not essential, but one finds that $\chi = 0$ for local reasons if it is not met.

When $G(\mathbb{R})$ contains a maximal compact torus T, we let $W^{\mathrm{c}} = N(T)/T$ be its Weyl group in $G(\mathbb{R})$ (the compact Weyl group) and $W = N(T_{\mathbb{C}})/T_{\mathbb{C}}$ be its Weyl group in $G(\mathbb{C})$. We will see that

$$\chi = (W : W^{\mathrm{c}}) \cdot \chi^*$$

with χ^* equal to the Euler characteristic $\chi(G^*, S, V)$ of any inner form G^* of G which is compact over $\mathbb{R}$ and unramified outside of S. (A form G^* of G is called an *inner* form if the actions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Dynkin diagrams of G and G^* are the same.) Our formula will express the integer χ^* as a sum of rational numbers. The terms in the sum will be indexed by the rational stable torsion conjugacy classes in G (or equivalently, in G^*). If S is sufficiently large (for example, if S contains all of the torsion primes for G) the global contribution of each torsion class γ to the sum will be

$$\frac{1}{2^{\ell}} L_S(M_\gamma) \operatorname{Tr}(\gamma | V).$$

Here $\ell = \dim(T)$ is the rank of G over $\mathbb{C}$, and M_γ is the Artin-Tate motive of rank l which is associated to the centralizer G_γ in [12]. This motive is well-defined by the stable class of γ, as G_γ is determined up to inner twisting over $\mathbb{Q}$. The term $L_S(M_\gamma)$

is the value of the Artin L-function of M_γ, with the Euler factors at S removed, at the point $s = 0$. This special value is known to be a rational number, by results of Siegel [25].

4. A test function to compute $\chi(G,S,V)$

To use the trace formula to compute

$$\chi(G, S, V) = \chi(G_S(\mathbb{A}), L \otimes V),$$

we will construct a measure φ on $G(\mathbb{A})$ such that

$$\chi(G_S(\mathbb{A}), L \otimes V) = \mathrm{Tr}(\varphi | L).$$

To this end, write L as a Hilbert direct sum

$$L = \hat{\bigoplus} \, m(\pi)\pi$$

with finite multiplicities. Then

$$\chi(G_S(\mathbb{A}), L \otimes V) = \sum m(\pi)\chi(G_S(\mathbb{A}), \pi \otimes V).$$

The group $G_S(\mathbb{A})$ is a direct product, and the representation $\pi \otimes V$ of $G_S(\mathbb{A})$ is a restricted tensor product: $\pi = \otimes \, \pi_v$. Since the Euler characteristic is multiplicative, we have

$$\chi(G_S(\mathbb{A}), \pi \otimes V) = \chi(G(\mathbb{R}), \pi_\infty \otimes V) \cdot \prod_{p \in S} \chi(G(\mathbb{Q}_p), \pi_p) \prod_{p \notin S} \chi(\underline{G}(\mathbb{Z}_p), \pi_p).$$

The term $\chi(\underline{G}(\mathbb{Z}_p), \pi_p) = \dim \pi_p^{\underline{G}(\mathbb{Z}_p)}$ is either 0 or 1, so the product of Euler characteristics is either 0 or finite.

Since $\mathrm{Tr}(\varphi|\pi) = \prod \mathrm{Tr}(\varphi_v|\pi_v)$, our task is to find local measures φ_v, such that for all irreducible representations π_v of $G(\mathbb{Q}_v)$:

$$\begin{aligned} \mathrm{Tr}(\varphi_\infty|\pi_\infty) &= \chi(G(\mathbb{R}), \pi_\infty \otimes V), \\ \mathrm{Tr}(\varphi_p|\pi_p) &= \chi(G(\mathbb{Q}_p), \pi_p), \quad p \in S \\ \mathrm{Tr}(\varphi_p|\pi_p) &= \chi(\underline{G}(\mathbb{Z}_p), \pi_p), \quad p \notin S. \end{aligned}$$

Then we will have

$$\chi(G_S(\mathbb{A}), \pi \otimes V) = \mathrm{Tr}(\varphi|\pi) \qquad \text{for all irreducible } \pi, \text{ and hence}$$
$$\chi(G_S(\mathbb{A}), L \otimes V) = \mathrm{Tr}(\varphi|L).$$

Of course, to calculate $\mathrm{Tr}(\varphi|L)$ using the trace formula, we will have to verify that φ_∞ and φ_p satisfy the local conditions of the proposition. We will also need to calculate orbital measures $dg_\gamma(\varphi)$ of the test measure φ. For this last calculation we will ultimately use the fact that the global orbital measure factors as a product of local orbital measures,

$$dg_\gamma(\varphi) = \prod_v dg_\gamma(\varphi_v).$$

However, as we'll see in Sections 5 and 6, some complications will arise from the fact that the natural measure to take on $G_\gamma(\mathbb{A})$ doesn't factor easily as a product of local measures. In the meantime, we will carry out the local computations below.

We now proceed to construct the desired local measures φ_v. At primes p which are not in S, the measure

$$\varphi_p = \frac{\mathrm{ch}(\underline{G}(\mathbb{Z}_p))}{\int_{\underline{G}(\mathbb{Z}_p)} dg_p}\, dg_p$$

has the desired property, where ch is the characteristic function of the open compact subset $\underline{G}(\mathbb{Z}_p)$. Indeed, the endomorphism φ_p of π_p is

$$\begin{aligned}\varphi_p(w) &= \int_{G(\mathbb{Q}_p)} g(w)\ \varphi(g)\\ &= \int_{\underline{G}(\mathbb{Z}_p)} g(w)\ dg_p \Bigg/ \int_{\underline{G}(\mathbb{Z}_p)} dg_p.\end{aligned}$$

This is just the projection of w to the $\underline{G}(\mathbb{Z}_p)$–fixed space in π_p, so

$$\mathrm{Tr}(\varphi_p|\pi_p) = \dim \pi_p^{\underline{G}(\mathbb{Z}_p)}.$$

The calculation of the orbital integrals of the local measure φ_p specified above is a fundamental problem in local harmonic analysis. Clearly this orbital integral is zero unless the conjugacy class $C(\gamma)$ of γ in $G(\mathbb{Q}_p)$ meets $\underline{G}(\mathbb{Z}_p)$. In this case, we say γ is integral. There are finitely many $\underline{G}(\mathbb{Z}_p)$ orbits on $C(\gamma) \cap \underline{G}(\mathbb{Z}_p)$, and their stabilizers are open compact subgroups K_i of $G_\gamma(\mathbb{Q}_p)$. The orbital measure is then

$$dg_\gamma(\varphi_p) = \sum_i \frac{1}{\int_{K_i} dg_\gamma}\, dg_\gamma.$$

We say an integral, semi-simple class γ has good reduction (mod p) if, for every root α of G, the p-adic integer $(\alpha(\gamma)-1)$ is either 0 or a unit. In other words, the class of γ has good reduction if it has no excess intersection (mod p) with the discriminant divisor, in the variety of conjugacy classes. In this case, Kottwitz has shown [17, Proposition 7.1] that the group scheme $\underline{G}_\gamma$ over $\mathbb{Z}_p$ has good reduction (mod p), so $\underline{G}_\gamma(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup in $G_\gamma(\mathbb{Q}_p)$. Moreover, if γ has good reduction (mod p), the group $\underline{G}(\mathbb{Z}_p)$ has a single orbit on $C(\gamma)\cap\underline{G}(\mathbb{Z}_p)$, with stabilizer $\underline{G}_\gamma(\mathbb{Z}_p)$. Hence, in this case, $dg_\gamma(\varphi_p)$ is the unique Haar measure with

$$\int_{\underline{G}_\gamma(\mathbb{Z}_p)} dg_\gamma(\varphi_p) = 1.$$

If the class of γ has bad reduction (mod p), the calculation is much more difficult. We discuss this further in Section 6.

At finite primes p in S, we need a locally constant, compactly supported measure φ_p such that

$$\mathrm{Tr}(\varphi_p|\pi_p) = \sum (-1)^i \dim H^i(G(\mathbb{Q}_p), \pi_p).$$

Let $\mathcal{F}$ be a facet of maximal dimension in the building of $G(\mathbb{Q}_p)$, and let $\mathcal{F}_j$ be the facets of $\mathcal{F}$. The dimension of $\mathcal{F}$ is the rank ℓ of G over $\mathbb{Q}_p$. Let $K_j \subset G(\mathbb{Q}_p)$ be the parahoric subgroup fixing the facet $\mathcal{F}_j$. Then Kottwitz has shown that the measure

$$\varphi_p = \sum_j (-1)^{\dim \mathcal{F}_j} \cdot \frac{\mathrm{ch}(K_j)}{\int_{K_j} dg_p}\, dg_p$$

has the desired traces. In particular, we have

$$\sum_i (-1)^i \dim H^i(G(\mathbb{Q}_p), \pi_p) = \sum_j (-1)^{\dim \mathcal{F}_j} \dim(\pi_p^{K_j}).$$

For example, the Steinberg representation St of $G(\mathbb{Q}_p)$ has a line fixed by the Iwahori subgroup K fixing $\mathcal{F}$ pointwise, and has no fixed vectors under any larger parahoric subgroup. Hence $\chi(\mathrm{St}) = (-1)^\ell$; this agrees with the calculation of $H^i(G(\mathbb{Q}_p), \mathrm{St})$ by Casselman, as the cohomology is zero for $i \neq \ell$, and one-dimensional for $i = \ell$.

Kottwitz also calculated the orbital integrals of φ_p. For $\gamma = 1$, we have

$$dg_\gamma(\varphi_p) = \sum_j (-1)^{\dim \mathcal{F}_j} \cdot \frac{1}{\int_{K_j} dg_p} dg_p,$$

which is Serre's formula for Euler–Poincaré measure on $G(\mathbb{Q}_p)$. This is the unique invariant measure μ such that

$$\int_{\Gamma\backslash G(\mathbb{Q}_p)} d\mu = \chi(\Gamma) = \sum_i (-1)^i \dim H^i(\Gamma, \mathbb{Q})$$

for each discrete, co-compact, torsion-free subgroup Γ. More generally, Kottwitz has shown that for any γ

$$dg_\gamma(\varphi_p) = d\mu_\gamma = \text{Euler–Poincaré measure on } G_\gamma(\mathbb{Q}_p).$$

This measure is zero, unless γ is elliptic and semi-simple.

At the real place, we need to construct a smooth, compactly supported measure φ_∞ on $G(\mathbb{R})$ such that

$$\mathrm{Tr}(\varphi_\infty|\pi_\infty) = \sum (-1)^i \dim H^i(G(\mathbb{R}), \pi_\infty \otimes V).$$

When $G(\mathbb{R})$ is compact, we have $H^i = 0$ for $i \geqslant 1$ and the Euler characteristic is equal to

$$\dim(\pi_\infty \otimes V)^{G(\mathbb{R})}.$$

In this case, we may take the test measure

$$\varphi_\infty = \frac{\mathrm{Tr}(g_\infty|V)}{\int_{G(\mathbb{R})} dg_\infty} dg_\infty.$$

Indeed, the endomorphism φ_∞ of π_∞ is just $1/\dim V^*$ times the projection onto the V^*-isotypical space. In the case when $G(\mathbb{R})$ is not compact, a suitable measure φ_∞ was constructed by Clozel and Delorme [9], who also calculated its orbital integrals. We have

$$dg_\gamma(\varphi_\infty) = \mathrm{Tr}(\gamma|V) \cdot \text{Euler–Poincaré measure on } G_\gamma(\mathbb{R}).$$

This is zero, unless γ is semi-simple and elliptic. Also, since any π_∞ with cohomology has the same infinitesimal character as V^*, which is regular, we have $\mathrm{Tr}(\varphi_\infty|\pi_\infty) = 0$ unless π_∞ has a regular infinitesimal character.

Since $\#S \geqslant 2$, with these choices of φ_v the test measure $\varphi = \prod \varphi_v = f\,dg$ satisfies all the conditions of the proposition. Hence

$$\chi(G,S,V) = \mathrm{Tr}(\varphi|L) = \sum_{\gamma} \int_{G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})} dg_\gamma \cdot \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\ \frac{dg}{dg_\gamma},$$

where again the sum is taken over representatives for the elliptic, semi-simple conjugacy classes in $G(\mathbb{Q})$. Moreover, since the support of φ_p is the union of compact open subgroups for all p, the class γ must be lie in a compact subgroup of each $G(\mathbb{Q}_p)$ to contribute a non-vanishing orbital integral. Since γ is also elliptic over $\mathbb{R}$, it is contained in a compact subgroup K of $G(\mathbb{A})$. But $K \cap G(\mathbb{Q})$ is finite, so γ is a torsion conjugacy class. Finally, if γ is not elliptic at some finite prime p in S, then we've seen that $dg_\gamma(\varphi_p)$ is zero, and hence γ doesn't contribute to the sum. Hence, the above sum is over torsion classes which are also elliptic at the finite primes in S.

We now fix this choice of test measure φ for the rest of the paper.

5. The stable trace formula

The problem in using the trace formula as just obtained to calculate $\chi(G,S,W)$ is that semi-simple conjugacy classes γ in $G(\mathbb{Q})$ are difficult to describe. For example, when $G = \mathrm{SL}_2$, there are infinitely many conjugacy classes of order 4, all conjugate over $\overline{\mathbb{Q}}$. Using the Euler–Poincaré test measure φ_p, Kottwitz was able to convert the above expression into a sum over *stable* conjugacy classes in the quasi-split inner form G' of G over $\mathbb{Q}$. (A group over $\mathbb{Q}$ is called quasi-split if it contains a Borel subgroup defined over $\mathbb{Q}$. Every group G has a unique quasi-split inner form.) Recall that two semi-simple elements of $G'(\mathbb{Q})$ are stably conjugate if and only if they are conjugate in $G'(\overline{\mathbb{Q}})$ since G' is simply connected.

We describe Kottwitz's formula below, and use it to compute χ in the next section. To carry out the stabilization, Kottwitz takes dg_γ to be the Tamagawa measure on the adèlic group $G_\gamma(\mathbb{A})$, so

$$\int_{G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})} dg_\gamma = \tau(G_\gamma)$$

is, by definition, the Tamagawa number. We henceforth fix this choice of dg_γ. For a discussion of Tamagawa measure see [8]. The trace formula then reads

$$\chi(G,S,V) = \mathrm{Tr}(\varphi|L) = \sum_{\gamma} \tau(G_\gamma) O_\gamma(\varphi, dg_\gamma).$$

The sum is over torsion classes γ of $G(\mathbb{Q})$ which are elliptic in $G(\mathbb{Q}_v)$ for all $v \in S$.

Let T denote a set of representatives for the (finitely many) torsion stable conjugacy classes in $G'(\mathbb{Q})$. Fix an inner twisting $\psi : G' \to G$ over $\overline{\mathbb{Q}}$. The geometric side of the stable trace formula will be a sum over those γ in $G(\mathbb{A})$ that, for some $t \in T$, are conjugate to $\psi(t)$ in $G(\overline{\mathbb{A}})$. For each such γ we have the adèlic centralizer $G_\gamma(\mathbb{A})$, but in general G_γ is not defined over $\mathbb{Q}$. If γ is conjugate to an element in $G(\mathbb{Q})$, then $G_\gamma(\mathbb{A})$ contains the discrete subgroup $G_\gamma(\mathbb{Q})$ and so we have the usual notion of Tamagawa measure on $G_\gamma(\mathbb{A})$.

Even if G_γ is not defined over $\mathbb{Q}$, we can still define Tamagawa measure dg_γ on $G_\gamma(\mathbb{A})$, using the inner twisting. Indeed, let dg'_t be Tamagawa measure on $G'_t(\mathbb{A})$, and fix a product decomposition: $dg'_t = \otimes (dg'_t)_v$. For each place v, G_{γ_v} is an inner twist of G'_{t_v} over $\mathbb{Q}_v$, so we may transfer the measure $(dg'_t)_v$ to a measure $(dg)_{\gamma_v}$ on $G_{\gamma_v}(\mathbb{Q}_v)$. We then define

$$dg_\gamma = \otimes\, (dg)_{\gamma_v}.$$

If γ is in $G(\mathbb{Q})$, this agrees with usual Tamagawa measure, and we can define $\tau(G_\gamma)$. In general, there is no Tamagawa number, but we can still define the adèlic orbital integral

$$O_\gamma(\varphi, dg_\gamma) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\frac{dg}{dg_\gamma}.$$

We may also attach a sign $e(\gamma) = \pm 1$ to the adèlic class γ, by the formula

$$e(\gamma) = \prod_v e(G_{\gamma_v}),$$

where the local invariants $e(G_{\gamma_v}) = \pm 1$ are defined in [16]. If γ is in $G(\mathbb{Q})$, $e(\gamma) = +1$.

Proposition (Kottwitz).

$$\chi(G, S, V) = \sum_T \sum_\gamma e(\gamma) O_\gamma(\varphi, dg_\gamma),$$

where the first sum is over representatives t of the stable torsion classes in $G'(\mathbb{Q})$, and the second is over representatives γ of the $G(\mathbb{A})$-conjugacy classes $G(\mathbb{A})$ which are conjugate to $\psi(t)$ in $G(\overline{\mathbb{A}})$.

We sketch the proof. As usual, there are an infinite number of γ in the inner sum, but only finitely many have a non-zero orbital integral.

For each t, Kottwitz defines a finite abelian group $\mathfrak{R}$, and for $\gamma \in G(\mathbb{A})$ conjugate to $\psi(t)$ in $G(\overline{\mathbb{A}})$ he defines an invariant $\mathrm{obs}(\gamma)$ in the dual of $\mathfrak{R}$. This invariant gives an obstruction to the existence of an element of $G(\mathbb{Q})$ in the $G(\mathbb{A})$-conjugacy class of γ. He then [17, 9.6.5] writes the geometric side of the trace formula as a triple sum

$$\sum_T \sum_\gamma \sum_\kappa \langle \mathrm{obs}(\gamma), \kappa\rangle e(\gamma) O_\gamma(\varphi, dg_\gamma),$$

where κ runs over $\mathfrak{R}$.

Actually, Kottwitz only states this triple sum formula for the contributions of the non-central classes $\eta \in G(\mathbb{Q})$. This restriction was needed at the time since he used Weil's conjecture on Tamagawa numbers for G_η and he was only assuming Weil's conjecture for groups of smaller dimension than G. He later used this formula to prove Weil conjecture [18, Theorem 3], and so his original derivation gives the triple sum expansion of the entire geometric side.

We switch the inner sums, and exploit the fact that φ_p is the Euler–Poincaré function at a finite prime in S.

Then Kottwitz shows [18, p. 641] that for $\kappa \neq 1$:

$$\sum_\gamma \langle \mathrm{obs}(\gamma), \kappa\rangle e(\gamma) O_\gamma(\varphi, dg_\gamma) = 0.$$

Hence we obtain the simple stable formula in the proposition.

6. A comparison of measures

The stable formula for $\chi(G, S, V)$ is still not readily computable, as we have only evaluated the *local* orbital measures for our test measure φ, while the trace formula involves the global term $O_\gamma(\varphi, dg_\gamma)$. To convert $O_\gamma(\varphi, dg_\gamma)$ into a product of local integrals, we need to express Tamagawa measure dg_γ on $G_\gamma(\mathbb{A})$ as a product of local measures.

To do this, we use the results of [12]. Again let G' be the quasi-split inner form of G over $\mathbb{Q}$ with fixed inner twisting ψ over $\overline{\mathbb{Q}}$ and let $t \in G'(\mathbb{Q})$ be a torsion element (in particular, an element appearing in the outer sum of the stable trace formula). Let $\gamma = (\gamma_v) \in G(\mathbb{A})$ be an element conjugate to $\psi(t)$ in $G(\overline{\mathbb{A}})$ (in particular, an element appearing in the inner sum of the stable trace formula).

For $v \in S$, we let $d\mu_{\gamma_v}$ be Euler–Poincaré measure on $G_{\gamma_v}(\mathbb{Q}_v)$. For p not in S, the group G'_t is the quasi-split inner form of G_{γ_p} over $\mathbb{Q}_p$, and we let $d\mu_{\gamma_p}$ be the measure on $G_{\gamma_p}(\mathbb{Q}_p)$ transferred from the Haar measure on $G'_t(\mathbb{Q}_p)$ which gives the connected component of a certain special compact subgroup volume 1. This measure on $G_{\gamma_p}(\mathbb{Q}_p)$ is denoted $L(M^\vee_{G_{\gamma_p}}(1)) \cdot |\omega_{G_{\gamma_p}}|$ in [12, Section 4]. When G_{γ_p} is unramified at p and $\underline{G}_{\gamma_p}$ is a model over $\mathbb{Z}_p$ with good reduction, we have $\int_{\underline{G}_{\gamma_p}(\mathbb{Z}_p)} d\mu_{\gamma_p} = 1$. Hence we can form the product measure $d\mu_\gamma = \otimes\, d\mu_{\gamma_v}$ on $G_\gamma(\mathbb{A})$.

The main global result of [12] then gives the ratio of measures on $G_\gamma(\mathbb{A})$:

$$d\mu_\gamma/dg_\gamma = L_S(M_t)/\prod_{v\in S} e(\gamma_v)c(\gamma_v).$$

Here $L_S(M_t)$ is the value of the Artin L-series of the motive of G_γ at $s=0$, which only depends on the stable class $\psi(t)$ of γ, and the sign $e(\gamma_v) = e(G_{\gamma_v}) = \pm 1$ is the local invariant defined by Kottwitz [16]. The invariant $c(\gamma_v)$ is defined as follows.

For finite primes p in S,

$$c(\gamma_p) = \#H^1(\mathbb{Q}_p, G_\gamma).$$

This depends only on the stable class of $\psi(t_p)$ over $\mathbb{Q}_p$, and gives the number of classes γ_p in the stable class (as $H^1(\mathbb{Q}_p, G) = 1$).

At the real place, we have

$$c(\gamma_\infty) = \frac{\#H^1(\mathbb{R}, T)}{\#\ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G_\gamma))},$$

where $T \subset G_\gamma \subset G$ is a maximal anisotropic torus, so $\#H^1(\mathbb{R}, T) = 2^\ell$, with $\ell = \dim T$.

We now replace the measure dg/dg_γ on $G_\gamma(\mathbb{A})\backslash G(\mathbb{A})$ by the equivalent term

$$dg/d\mu_\gamma \cdot L_S(M_t)/\prod_{v\in S} e(\gamma_v)c(\gamma_v).$$

This allows us to write the adèlic orbital integral as a product of local integrals

$$e(\gamma)O_\gamma(\varphi, dg_\gamma) = L_S(M_t)\cdot \prod_{v\in S} O_{\gamma_v}(\varphi_v, d\mu_{\gamma_v})/c(\gamma_v)\cdot \prod_{p\notin S} O_{\gamma_p}(\varphi_p, d\mu_{\gamma_p})e(\gamma_p).$$

For a fixed $t = (t_v)$, each adèlic class γ in the stable class of $\psi(t)$ is the product of local classes γ_v in the stable classes of the $\psi(t_v)$. We define the local stable orbital integrals by

$$\mathrm{SO}_t(\varphi_v) = \sum_{\gamma_v} e(\gamma_v)O_{\gamma_v}(\varphi_v, d\mu_{\gamma_v}),$$

and for $v \in S$ the modified local stable orbital integrals by

$$\mathrm{SO}_t^*(\varphi_v) = \sum_{\gamma_v} c(\gamma_v)^{-1}O_{\gamma_v}(\varphi_v, d\mu_{\gamma_v}),$$

where the sums are taken over the finitely many classes γ_v in $G(\mathbb{Q}_v)$ which are in the stable class of $\psi(t_v)$ in $G(\overline{\mathbb{Q}}_v)$. If $v \notin S$ we let $\mathrm{SO}_t^*(\varphi_v) = \mathrm{SO}_t(\varphi_v)$. Then summing over the classes γ in the stable class of $\psi(t)$ we see

$$\sum_{\gamma} e(\gamma) O_\gamma(\varphi, dg_\gamma) = L_S(M_t) \prod_v \mathrm{SO}_t^*(\varphi_v),$$

and so

$$\chi(G, S, V) = \sum_T L_S(M_t) \cdot \prod_v \mathrm{SO}_t^*(\varphi_v).$$

We now turn to the evaluation of the stable local terms SO_t^*. Let $v = p$ be a finite prime in S. If γ_v is elliptic then we have $O_{\gamma_v}(\varphi_v, d\mu_{\gamma_v}) = 1$. If not, $L_S(M_t) = 0$. The constant $c(\gamma_v) = c(t_v)$ is the number of local classes in the stable class of $\psi(t_v)$. Hence either the contribution of the stable class t is killed off by the $L_S(M_t)$ term, or $\mathrm{SO}_t^*(\varphi_v) = 1$.

When $v = \infty$ and γ_v is elliptic, we have $O_{\gamma_v}(\varphi_v, d\mu_{\gamma_v}) = \mathrm{Tr}(\gamma_v|V)$. This depends only on the stable class $\psi(t_v)$ of γ_v. Using the formula for $c(\gamma_v)$ above, we get

$$\begin{aligned}\mathrm{SO}_t^*(\varphi_v) &= \frac{\mathrm{Tr}(t|V)}{2^\ell} \cdot \sum_{\gamma_v} \#\ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G_{\gamma_v})) \\ &= \frac{\mathrm{Tr}(t|V)}{2^\ell} \cdot \#\ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)).\end{aligned}$$

The latter kernel has cardinality $(W : W^{\mathrm{c}})$. Hence we have shown

$$\chi(G, S, V) = (W : W^{\mathrm{c}}) \sum_T \frac{1}{2^\ell} L_S(M_t) \mathrm{Tr}(t|V) \cdot \prod_{p \notin S} \mathrm{SO}_t(\varphi_p).$$

Finally, we consider the stable orbital integrals at the primes p not in S. For each class t, almost all of these terms are equal to 1. For example, if p does not divide the order of t, then there is a single class γ_p in the stable class over $\mathbb{Q}_p$ which meets $\underline{G}(\mathbb{Z}_p)$, and for this class we have seen that $O_{\gamma_p}(\varphi_p, d\mu_{\gamma_p}) = 1$. Since G_{γ_p} is unramified in this case, $e(\gamma_p) = 1$ and hence $\mathrm{SO}_t(\varphi_p) = 1$. We are left with the formula

$$\chi(G, S, V) = (W : W^{\mathrm{c}}) \sum_T \frac{1}{2^\ell} L_S(M_t) \mathrm{Tr}(t|V) \cdot \prod_{\substack{p \,|\mathrm{order}(t) \\ p \notin S}} \mathrm{SO}_t(\varphi_p). \tag{1}$$

If, for example, the torsion primes for G are all contained in S, we have a complete formula (as the product is empty). In all cases, the primary contribution of the stable torsion class t to χ is

$$(W : W^{\mathrm{c}}) \cdot \frac{1}{2^{\ell}} L_S(M_t) \mathrm{Tr}(t|V),$$

as claimed earlier.

The remaining calculation of $\mathrm{SO}_t(\varphi_p)$ is a central local problem. For each γ_p in $G(\mathbb{Q}_p)$ which is stably conjugate to $\psi(t_p)$, we must write

$$C\ell(\gamma_p) \cap \underline{G}(\mathbb{Z}_p) = \coprod_i K_i \backslash \underline{G}(\mathbb{Z}_p).$$

Then,

$$\mathrm{SO}_t(\varphi_p) = \sum_{\gamma_p} e(\gamma_p) \cdot \sum_i \frac{1}{\int_{K_i} d\mu_{\gamma_p}}. \tag{2}$$

Unfortunately, even the first step of decomposing the integral elements of $C\ell(\gamma_p)$ into integral conjugacy classes is not readily computable. Our approach to computing the stable orbital integrals $\mathrm{SO}_t(\varphi_p)$ in the next section of this paper is rather roundabout. We will see in the next section that the Euler characteristic $\chi(G, S, V)$ can be computed directly for certain G and small S, V. We may use these values in Eq. (1) to get a system of equations in the unknowns $\mathrm{SO}_t(\varphi_p)$. We are able to compute enough values of $\chi(G, S, V)$ to solve for all of the remaining $\mathrm{SO}_t(\varphi_p)$ when G is SL_2, Sp_4, or G_2. We give these values in Section 7 and use them to compute more values of $\chi(G, S, V)$ via (1).

Before going on, we note that from expression (2), it follows that SO_t is a rational number, which is positive whenever t is regular. In the regular case, $e(\gamma_p) = 1$ and $d\mu_{\gamma_p}$ has volume 1 on the connected component $\underline{T}^0(\mathbb{Z}_p)$ of the Néron model of $T = G_{\gamma_p}$. Hence

$$\mathrm{SO}_t(\varphi_p) = \sum_{\gamma_p} \sum_i (\underline{\mathrm{T}}^0(\mathbb{Z}_p) : K_i).$$

These "indices" can have denominators $(\underline{\mathrm{T}}(\mathbb{Z}_p) : \underline{\mathrm{T}}^0(\mathbb{Z}_p))$. However, in all cases where we have been able to determine SO_t, it turns out to be an integer (which can be negative for non-regular t).

7. Algebraic modular forms

For this section we drop the requirement that G be simply connected, but insist that $G(\mathbb{R})$ be compact. This guarantees that $G(\mathbb{Q})$ is discrete and co-compact in $G(\mathbb{A})$. For a given representation V of G over $\mathbb{Q}$ and an open compact subgroup K of $G(\hat{\mathbb{Q}})$

(where $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adéles) we define the space of (algebraic) modular forms on G of weight V and level K to be the rational vector space [13]:

$$M_G(V, K) = \{F\colon G(\mathbb{A})/(G(\mathbb{R})_+ \times K) \to V \ :$$
$$F(\gamma g) = \gamma F(g), \ \text{for all } \gamma \in G(\mathbb{Q})\},$$

where $G(\mathbb{R})_+$ is the connected component of the identity in $G(\mathbb{R})$.

If K is a product $K = \prod_p K_p$, with each K_p open and compact in $G(\mathbb{Q}_p)$, then the Hecke algebras $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ each act on $M(V, K)$, and commute with each other in End(M(V, K)). We will fix a finite set S of places of $\mathbb{Q}$ containing those for which G is ramified, and an integral model $\underline{G}$ for G over the ring $\mathbb{Z}_S$ with good reduction at all p not in S. For p not in S, we let $K_p = \underline{G}(\mathbb{Z}_p)$. For primes p in S, we let K_p be an Iwahori subgroup of $G(\mathbb{Q}_p)$, which fixes a maximal facet in the Bruhat–Tits building pointwise.

The Steinberg representation of $G(\mathbb{Q}_p)$ has a vector fixed by the Iwahori subgroup, so gives rise to a one-dimensional representation of the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), K_p)$. We call a character of this algebra *special* if it is the twist of the Steinberg character by a character of the fundamental group Ω of G. We may twist by such characters as $\Omega \cong G(\mathbb{Q}_p)/G(\mathbb{Q}_p)_s$, where $G(\mathbb{Q}_p)_s \supset K_p$ is the normal subgroup of elements of $G(\mathbb{Q}_p)$ that preserve the types of vertices in the building. Thus, special representations are those representations of $G(\mathbb{Q}_p)$ with an Iwahori-fixed vector on which the standard generators of the simply-connected Hecke algebra act by -1. We denote by $M_G(V, K)^{\mathrm{St}}$ the subspace of $M_G(V, K)$ on which the Hecke algebras $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ act by special characters for all p in S.

Proposition (Padowitz [21]). *Assume that G is absolutely simple and simply connected, and let* $r_s = \sum_{p\in S} \operatorname{rank} \mathrm{G}(\mathbb{Q}_\mathrm{p})$. *Let V be an absolutely irreducible representation of G over* $\mathbb{Q}$ *with trivial central character, and define* $K = \prod K_p$ *as above.*

Then

$$\chi(G, S, V) = (-1)^{r_s} \dim M_G(V^*, K)^{\mathrm{St}},$$

except in the case when V is the trivial representation and $r_s > 0$. *In the exceptional case,*

$$\chi(G, S, V) = 1 + (-1)^{r_s} \dim M_G(V^*, K)^{\mathrm{St}}.$$

Proof. The dimension of $M_G(V^*, K)^{\mathrm{St}}$ is the number of irreducible automorphic representations π (counted with their multiplicities in the discrete spectrum) which satisfy:

- $\pi_\infty \cong V^*$,

- π_p is the Steinberg representation for $p \in S$,
- π_p has a vector fixed by $\underline{G}(\mathbb{Z}_p)$ for $p \notin S$.

Each such representation contributes a space of dimension $m(\pi)$ in $H^{r_S}(G_S(\mathbb{A}), L \otimes V)$ where $m(\pi)$ is the multiplicity of π occurring in L. Moreover, by results of Casselman [7], these are the *only* unitary representations contributing to cohomology (except when V is trivial and $r_S > 0$, in which case $\pi = \mathbb{C}$ contribute a line to $H^0(G_S(\mathbb{A}), L)$). This completes the proof. $\square$

Since we will actually compute the spaces $M_G(V, K)^{\mathrm{St}}$ for groups G of adjoint type, we need a lemma to compare spaces for isogenous groups. Let G be a reductive group (such as GL_n or GSp_{2n}) with the following property: the derived subgroup G_0 is simply connected, and the center C of G is a split torus. Put $\bar{G} = G/C$, which is a group of adjoint type, and let $f: G_0 \to \bar{G}$ be the corresponding isogeny.

Let V be an irreducible representation of $\bar{G}$, which we may also view as a representation of G_0 with trivial central character. Let K_0 be an open compact subgroup of $G_0(\hat{\mathbb{Q}})$, defined as above, and let $\bar{K}$ be such a subgroup of $\bar{G}(\hat{\mathbb{Q}})$ which contains $f(K_0)$.

The map $f: G_0 \to \bar{G}$ then induces a linear map of $\mathbb{Q}$-vector spaces $M_{\bar{G}}(V, \bar{K}) \to M_{G_0}(V, K_0)$ which is equivariant for the action of the Hecke algebras. The comparison lemma we need is the following easily proved fact.

Lemma. *The induced map*

$$M_{\bar{G}}(V, \bar{K})^{\mathrm{St}} \to M_{G_0}(V, K_0)^{\mathrm{St}}$$

is an isomorphism.

The proposition and the lemma together allow us to use the calculations of $M_G(V, K)^{\mathrm{St}}$ in [20] to get the values of

$$\chi^* = \frac{1}{(W : W^{\mathrm{c}})}\, \chi(G, S, V).$$

8. Examples

We now give some examples. By interpreting $G(\mathbb{A})/K$ geometrically, and making heavy use of a computer, the spaces $M_G(V, K)$ and $M_G(V, K)^{\mathrm{St}}$ are worked out for certain G, V, K in [20]. In particular the calculations there work with the (unique) form of G_2 which is compact over $\mathbb{R}$ and with the forms of PGSp_4 which are ramified at $\{2, \infty\}$ and at $\{3, \infty\}$.

The calculation of the $M(V, K)$ is computationally intensive and so has only been carried out for small weights and levels. We now tabulate the values of χ^* we derive from these direct calculations. The corresponding values when G is the split form of SL_2 are well known.

Directly computed values of $\chi^*(G, S, V)$ for $G = \mathrm{Sp}_4$

S	$V = V_\lambda$ $\lambda =$ $\dim V =$	(0,0) 1	(0,1) 5
$\{\infty, 2\}$		1	0
$\{\infty, 2, 3\}$		1	−1
$\{\infty, 2, 5\}$		−1	
$\{\infty, 2, 7\}$		−4	
$\{\infty, 2, 11\}$		−33	
$\{\infty, 3, 5\}$		−8	

Directly computed values of $\chi^*(G, S, V)$ for $G = G_2$

S	$V = V_\lambda$ $\lambda =$ $\dim V =$	(0,0) 1	(1,0) 7	(0,1) 14	(2,0) 27	(1,1) 64
$\{\infty, 2\}$		1	0	0	0	1
$\{\infty, 3\}$		1	0	0	2	
$\{\infty, 5\}$		2	7	11	31	
$\{\infty, 7\}$		13	54	120		
$\{\infty, 11\}$		135				
$\{\infty, 13\}$		386				
$\{\infty, 2, 3\}$		2				
$\{\infty, 2, 7\}$		253				

For the three split, simply connected groups SL_2, Sp_4, and G_2 over $\mathbb{Q}$, we will now tabulate the rational stable torsion classes. Since our groups are simply connected, these are just the stable torsion classes that meet the group of rational points. We group the classes t and zt, for z in the center, as these have the same contribution to the stable trace formula for χ. There are 3 groups for SL_2, 12 groups for Sp_4, and 14 rational stable torsion classes for G_2. Similarly, one can show there are 102 rational stable torsion classes for F_4, and 785 rational stable torsion classes for E_8.

The stable class of an element t in SL_2, Sp_4, or G_2 is determined by its characteristic polynomial on the fundamental representation of dimension 2, 4, or 7 respectively. Since t is torsion, this is a product of cyclotomic polynomials ϕ_m. We tabulate this polynomial, as well as the value $L(M_t)$.

Using Eq. (2), the data in the two preceding tables, and a separate calculation of $\chi(\mathrm{Sp}_4, \{p\}, V)$ for p prime and V trivial, we are able to solve for the values of $\mathrm{SO}_t(\varphi_p)$.

Recall that we know that all but finitely many of these values are equal to 1. We include in our tables only those values of $\mathrm{SO}_t(\varphi_p)$ which are *not* equal to 1. With these values computed, we are then able to tabulate the integers

$$\chi^* = \frac{1}{(W : W^{\mathrm{c}})}\, \chi(G, S, V)$$

for many pairs (S, V) beyond those values obtained directly from looking at modular forms. The value of χ^* depends only on the inner class of G over $\mathbb{Q}$.

Torsion classes in SL_2

order t	char poly t	$L(M_t)$	SO_t
1,2	ϕ_1^2, ϕ_2^2	$-\frac{1}{12}$	
3,6	ϕ_3, ϕ_6	$\frac{1}{3}$	
4	ϕ_4	$\frac{1}{2}$	

Torsion Classes in Sp_4

order t	char poly t	$L(M_t)$	SO_t
1,2	ϕ_1^4, ϕ_2^4	$-\frac{1}{1440}$	
2	$\phi_1^2\phi_2^2$	$\frac{1}{144}$	$\mathrm{SO}_t(\varphi_2) = 7$
3,6	$\phi_1^2\phi_3, \phi_1^2\phi_6$	$-\frac{1}{36}$	
3,6	ϕ_3^2, ϕ_6^2	$-\frac{1}{36}$	
4	ϕ_4^2	$-\frac{1}{24}$	
4,4	$\phi_1^2\phi_4^2, \phi_2^2\phi_4^2$	$-\frac{1}{24}$	
6,6	$\phi_1^2\phi_6, \phi_2^2\phi_3$	$-\frac{1}{36}$	
5,10	ϕ_5, ϕ_{10}	$\frac{2}{6}$	
6	$\phi_3\phi_6$	$\frac{1}{9}$	$\mathrm{SO}_t(\varphi_2) = 4$
8	ϕ_8	$\frac{1}{2}$	
12	ϕ_{12}	$\frac{1}{6}$	
12,12	$\phi_3\phi_4, \phi_6\phi_4$	$\frac{1}{6}$	

Torsion Classes in G_2

order t	char poly t	$L(M_t)$	SO_t
1	ϕ_1^7	$\frac{1}{3024}$	
2	$\phi_1^3\phi_2^4$	$\frac{1}{144}$	$\mathrm{SO}_t(\varphi_2) = 31$
3	$\phi_1\phi_3^3$	$\frac{1}{54}$	
3	$\phi_1^3\phi_3$	$-\frac{1}{36}$	
4	$\phi_1\phi_2^2\phi_4^2$	$-\frac{1}{24}$	
4	$\phi_1^3\phi_4^2$	$-\frac{1}{24}$	
6	$\phi_1\phi_3\phi_6^2$	$-\frac{1}{36}$	$\mathrm{SO}_t(\varphi_2) = -2$
6	$\phi_1^3\phi_6^2$	$-\frac{1}{36}$	
6	$\phi_1\phi_2^2\phi_3\phi_6$	$\frac{1}{9}$	$\mathrm{SO}_t(\varphi_2) = 4$
7	$\phi_1\phi_7$	$\frac{4}{7}$	
8	$\phi_1\phi_2^2\phi_8$	$\frac{1}{2}$	
8	$\phi_1\phi_4\phi_8$	$\frac{1}{2}$	
12	$\phi_1\phi_2^2\phi_{12}$	$\frac{1}{6}$	
12	$\phi_1\phi_3\phi_{12}$	$\frac{1}{6}$	$\mathrm{SO}_t(\varphi_2) = 4$

Values of $\chi^*(G, S, V)$ for $G = \mathrm{SL}_2$, using the trace formula

	$V = V_\lambda$						
	$\lambda =$	0	2	4	6	8	10
S	$\dim V =$	1	3	5	7	9	11
$\{\infty, 2\}$		1	0	0	1	1	0
$\{\infty, 3\}$		1	0	1	1	2	1
$\{\infty, 5\}$		1	1	1	3	3	3
$\{\infty, 7\}$		1	1	3	3	5	5
$\{\infty, 11\}$		2	2	4	6	8	8
$\{\infty, 13\}$		1	3	5	7	9	11
$\{\infty, 2, 3\}$		1	–1	–1	–1	–1	–3
$\{\infty, 2, 5\}$		1	–1	–3	–1	–3	–5
$\{\infty, 3, 5\}$		0	–2	–4	–4	–6	–8

Values of $\chi^*(G, S, V)$ for $G = Sp_4$, using the trace formula

S	$V = V_\lambda$ $\lambda =$ dim $V =$	(0,0) 1	(0,1) 5	(2,0) 10	(0,2) 14	(0,3) 30	(2,1) 35	(4,0) 35	(0,4) 55	(2,2) 81	(6,0) 84	(0,5) 91
$\{\infty, 2\}$		1	0	0	0	0	0	0	0	0	0	0
$\{\infty, 3\}$		1	0	0	0	–1	–1	0	0	–1	–1	–2
$\{\infty, 5\}$		1	–1	–1	–1	–7	–6	–5	–7	–12	–12	–20
$\{\infty, 7\}$		1	–5	–6	–8	–26	–27	–23	–31	–55	–58	–73
$\{\infty, 11\}$		–1	–25	–42	–56	–150	–167	–155	–235	–365	–378	–445
$\{\infty, 13\}$		–7	–51	–88	–118	–292	–329	–315	–477	–725	–762	–869
$\{\infty, 17\}$		–22	–144	–264	–362	–848	–968	–944	–1456	–2182	–2274	–2550
$\{\infty, 19\}$		–37	–225	–420	–578	–1326	–1521	–1485	–2295	–3439	–3584	–3979
$\{\infty, 2, 3\}$		1	–1	–2	–2	–4	–5	–5	–7	–9	–12	–11
$\{\infty, 2, 5\}$		–1	–7	–14	–18	–38	–43	–43	–65	–97	–104	–109
$\{\infty, 2, 7\}$		–4	–26	–50	–70	–150	–174	–176	–274	–402	–420	–456
$\{\infty, 2, 11\}$		–33	–165	–328	–452	–974	–1135	–1135	–1775	–2615	–2722	–2945
$\{\infty, 2, 13\}$		–63	–321	–640	–896	–1924	–2243	–2241	–3519	–5185	–5380	–5833
$\{\infty, 3, 5\}$		–8	–48	–90	–122	–278	–318	–312	–480	–718	–752	–830
$\{\infty, 3, 7\}$		–36	–192	–368	–508	–1128	–1304	–1296	–2016	–2980	–3108	–3412

Values of $\chi^*(G, S, V)$ for $G = G_2$, using the trace formula

S	$V = V_\lambda$ $\lambda =$ dim $V =$	(0,0) 1	(1,0) 7	(0,1) 14	(2,0) 27	(1,1) 64	(3,0) 77	(0,2) 77	(4,0) 182	(2,1) 189	(0,3) 273
$\{\infty, 2\}$		1	0	0	0	1	0	0	1	1	0
$\{\infty, 3\}$		1	0	0	2	3	3	4	9	7	9
$\{\infty, 5\}$		2	7	11	31	71	76	77	198	194	261
$\{\infty, 7\}$		13	54	120	231	523	642	670	1520	1570	2302
$\{\infty, 11\}$		135	938	1826	3613	8569	10212	10200	24308	25150	36140
$\{\infty, 13\}$		386	2552	5188	9968	23500	28386	28532	67020	69594	100784
$\{\infty, 17\}$		1871	13176	26160	50753	120375	144472	144384	342056	354928	511984
$\{\infty, 19\}$		3733	25716	51702	99539	235579	283818	284226	670506	696348	1006692
$\{\infty, 2, 3\}$		2	8	17	33	79	95	96	225	234	340
$\{\infty, 2, 5\}$		35	218	460	863	2029	2476	2498	5810	6050	8814
$\{\infty, 2, 7\}$		253	1822	3584	6977	16593	19864	19806	47080	48844	70350
$\{\infty, 2, 11\}$		4157	28832	57922	111437	263927	317948	318206	750992	780080	1127636
$\{\infty, 3, 5\}$		505	3494	6998	13509	31991	38492	38530	91012	94488	136506
$\{\infty, 3, 7\}$		4039	28240	56456	108961	258247	310640	310680	734392	762552	1101360

For groups of higher rank, one can enumerate the classes t and determine the motives M_t of their centralizers. The local stable orbital integrals $\mathrm{SO}_t(\varphi_p)$ at primes p dividing the order of t are difficult to calculate. However, a good estimate for χ^* comes from the central terms in the trace formula, which together contribute the rational number

$$\#Z \cdot \frac{1}{2^{\ell}}\, L_S(M_G) \dim V.$$

For $G = F_4$, this estimate suggests that $\chi^* > 10^3$ whenever $S \neq \{\infty, 2\}$, and for $G = E_8$, this estimate suggests that $\chi^* > 10^{30}$ for *all* pairs (S, V).

9. Discrete series and a conjecture

How can one account for the term $(W : W^{\mathrm{c}})$, which is the only non-stable factor in the formula for $\chi(G, S, V)$:

$$\chi(G, S, V) = (W : W^{\mathrm{c}}) \cdot \chi^*(G, S, V)?$$

On one hand, $(W : W^{\mathrm{c}})$ is the Euler characteristic of the trivial representation $\mathbb{C}$ of $G_S(\mathbb{A})$, arising from the cohomology of the trivial representation of $G(\mathbb{R})$. Indeed, if K is a maximal compact subgroup of $G(\mathbb{R})$ and $\mathfrak{p} = \mathrm{Lie}(G)/\mathrm{Lie}(K)$, then:

$$H^\bullet(G(\mathbb{R}), \mathbb{C}) = (\dot{\Lambda}\mathfrak{p})^K.$$

On the other hand, $(W : W^{\mathrm{c}})$ is the number of discrete series representations π_∞ of $G(\mathbb{R})$ with a fixed central and infinitesimal character. This leads us to make the following optimistic prediction.

Conjecture. *Let π be an irreducible representation of $G(\mathbb{A})$ which occurs in $L = L^2_{\mathrm{disc}}$ and has non-zero $G_S(\mathbb{A})$-cohomology $H^\bullet(G_S(\mathbb{A}), \pi \otimes V)$ when tensored with the finite-dimensional representation V of $G(\mathbb{R})$.*

Then either:

1. *π is the trivial representation of $G(\mathbb{A})$ and $V = \mathbb{C}$, or*
2. *π_∞ is a discrete series representation of $G(\mathbb{R})$ with trivial central character and the same infinitesimal character as V^*, and for all finite places $v \in S$, π_v is the Steinberg representation.*

Note that this conjecture is true when the highest weight of V is regular, since then the only unitary representations that have cohomology when tensored with V are the discrete series representations.

Even more should be true. Let G' be *any* inner form of G, with good reduction outside of S. Let $\pi = \pi_\infty \otimes \bigotimes_{v \in S} \mathrm{St}_v \otimes \pi^S$ be the local factorization of a representation of type 2) in L, with π^S unramified. If π'_∞ is *any* discrete series for $G'(\mathbb{R})$ with the same infinitesimal and central character as π_∞, then we would expect that:

$$\dim \mathrm{Hom}_{G'(\mathbb{A})}\left(\pi'_\infty \otimes \bigotimes_{v\in S} \mathrm{St}'_v \otimes \pi^S, L'\right) = 1.$$

If this is true, we can use the fact that discrete series representations of $G(\mathbb{R})$ and the Steinberg representation of $G(\mathbb{Q}_p)$ contribute cohomology of dimension 1 in a single degree, to count the *number* of distinct automorphic representations of a fixed local type.

Conjecture. *Let d_∞ be a fixed discrete series for $G(\mathbb{R})$, with infinitesimal character equal to the infinitesimal character of V^*. Then the number of distinct irreducible representations $\pi = \otimes'_v \pi_v$ of $G(\mathbb{A})$ with local components*

$$\begin{cases} \pi_\infty \simeq d_\infty, & \\ \pi_v \simeq \mathrm{St}_v & \text{for all } v \in S, \\ \pi_p^{G(\mathbb{Z}_p)} \neq 0 & \text{for all } p \notin S \end{cases}$$

which appear in the discrete spectrum L of G is equal to the absolute value of the integer $\chi^*(G, S, V)$ (except in the case when $V = \mathbb{C}$ and the group $G_S(\mathbb{A})$ is non-compact, when this number is the absolute value of the integer $\chi^*(G, S, V) - 1$).

For example, when $G = G_2$, $S = \{\infty, 5\}$, and $V = \mathbb{C}$, we saw that $\chi^*(G, S, V) = 2$. Hence, for any discrete series representation d_∞ of $G_2(\mathbb{R})$ with infinitesimal character ρ, there should be a *unique* automorphic irreducible representation π of the form

$$\pi = d_\infty \otimes \mathrm{St}_5 \otimes \bigotimes_{p\neq 5} \pi_p$$

with π_p unramified for all $p \neq 5$. For the anisotropic form G' of G_2, this is true by calculations of Lansky and Pollack (who also determined π_2 and π_3). The representation π' of $G'(\mathbb{A})$ lifts to $\mathrm{PGSp}_6(\mathbb{A})$ via an exceptional theta correspondence, and yields a holomorphic Siegel modular form F of weight 4, whose level is the Iwahori subgroup at 5 in $\mathrm{PGSp}_6(\mathbb{Z})$ [14, Proposition 5.8].

Acknowledgments

We would like to thank Jim Arthur and Bob Kottwitz, who explained important points of their work; they have since gone far beyond the techniques used here. We would also like to thank Wee-Teck Gan, Joshua Lansky, and Seth Padowitz for their help. The first author would like to thank his hosts at the IMS of the National University of Singapore, where a preliminary version of this paper was written.

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Further reading

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