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Journal of Number Theory 119 (2006) 307–314

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

On a conjecture of Erdős, Graham and Spencer ☆

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Received 10 October 2004; revised 26 April 2005

Available online 10 January 2006

Communicated by David Goss

Abstract

It is conjectured by Erdős, Graham and Spencer that if $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $\sum_{i=1}^{s} 1/a_i < n - 1/30$, then this sum can be decomposed into n parts so that all partial sums are $\leqslant 1$. This is not true for $\sum_{i=1}^{s} 1/a_i = n - 1/30$ as shown by $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = \cdots = a_{5n-3} = 5$. In 1997, Sándor proved that Erdős–Graham–Spencer conjecture is true for $\sum_{i=1}^{s} 1/a_i \leqslant n - 1/2$. In this paper, we reduce Erdős–Graham–Spencer conjecture to finite calculations and prove that Erdős–Graham–Spencer conjecture is true for $\sum_{i=1}^{s} 1/a_i \leqslant n - 1/3$. Furthermore, it is proved that Erdős–Graham–Spencer conjecture is true if $\sum_{i=1}^{s} 1/a_i < n - 1/(\log n + \log\log n - 2)$ and no partial sum (certainly not a single term) is the inverse of an positive integer.

MSC: 11B75

Keywords: Erdős–Graham–Spencer conjecture; Erdős problem; Partition; Prime

1. Introduction

Erdős [2, p. 41] asked the following question: is it true that if a_i's are positive integers with $1 < a_1 < a_2 < \cdots < a_s$ and $\sum_{i=1}^{s} 1/a_i < 2$, then there exist $\varepsilon_i = 0$ or 1 such that

$$\sum_{i=1}^{s} \frac{\varepsilon_i}{a_i} < 1, \qquad \sum_{i=1}^{s} \frac{1-\varepsilon_i}{a_i} < 1?$$

☆ Supported by the National Natural Science Foundation of China, Grant No. 10471064 and the Teaching and Research Award Program for Outstanding Young Teachers in Nanjing Normal University.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2005.11.003

Sándor [3] gave a simple construction to show that the answer is negative: let $\{a_i\} = \{\text{divisors of } 120 \text{ with the exception of } 1 \text{ and } 120\}$. Furthermore, Sándor [3] proved the following nice results:

Theorem A. *For every $n \geqslant 2$ there exist integers $1 < a_1 < a_2 < \cdots < a_s$ such that $\sum_{i=1}^{s} 1/a_i < n$ and this sum cannot be split into n parts so that all partial sums are $\leqslant 1$.*

Theorem B. *Let $n \geqslant 2$. If $1 < a_1 < a_2 < \cdots < a_s$ with $\sum_{i=1}^{s} 1/a_i < n(1 - e^{1-n})$, then this sum can be decomposed into n parts so that all partial sums are $\leqslant 1$.*

If we allow repetition of integers, then it is conjectured by Erdős, Graham and Spencer [2, p. 41] that if $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $\sum_{i=1}^{s} 1/a_i < n - 1/30$, then this sum can be decomposed into n parts so that all partial sums are $\leqslant 1$. This is not true for $\sum_{i=1}^{s} 1/a_i = n - 1/30$ as shown by $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = \cdots = a_{5n-3} = 5$. Sándor [3] proved the following weaker assertion.

Theorem C. *Let $n \geqslant 2$. If $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $\sum_{i=1}^{s} 1/a_i \leqslant n - 1/2$, then this sum can be decomposed into n parts so that all partial sums are $\leqslant 1$.*

Sándor [3] noted that $n - 1/2 = n - 0.5$ can be improved to $n - 3/7 = n - 0.428\ldots$ by similar arguments but much longer calculation (no proof is included in [3]). In this paper, it is improved the number to $n - 1/3 = n - 0.333\ldots$. In order to prove or disprove Erdős–Graham–Spencer conjecture, it is natural to consider only those sequences for which each term is more than 1 and no partial sum (certainly not a single term, the same meaning for late) is the inverse of a positive integer, otherwise, we may replace the partial sum by the inverse of the integer. We call a sequence $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ *primitive* if there is no partial sum of $\sum_{i=1}^{s} 1/a_i$ is the inverse of a positive integer. It is clear that if a sequence $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ is primitive, then each integer a repeats at most $p(a) - 1$ times, where $p(a)$ is the least prime divisor of a. Hence, the following Theorem 1 reduces Erdős–Graham–Spencer conjecture to finite calculations. In this paper, the following main results are proved.

Theorem 1. *If for $1 \leqslant n < e^{28}$ and any primitive sequence $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $a_1 > 1$ and $a_s < 30n$, Erdős–Graham–Spencer conjecture is true, then Erdős–Graham–Spencer conjecture is true in general.*

Theorem 2. *There exists an effective constant n_0 such that if $n \geqslant n_0$ and $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ is primitive with*

$$\sum_{i=1}^{s} \frac{1}{a_i} \leqslant n - \frac{1}{\log n + \log\log n - 2},$$

then the sum $\sum_{i=1}^{s} 1/a_i$ can be decomposed into n parts with each partial sum $\leqslant 1$.

Theorem 3. *Let n be a positive integer. If $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $\sum_{i=1}^{s} 1/a_i \leqslant n - 1/3$, then this sum can be decomposed into n parts with each partial sum $\leqslant 1$.*

Remark. If we require each partial sum < 1, then the problem becomes an easy one. It is clear that for $a_1 = a_2 = \cdots = a_{n+1} = 2$ the sum $\sum_{i=1}^{s} 1/a_i$ $(= (n+1)/2)$ cannot be decomposed into n parts with each partial sum < 1. On the other hand, we can prove that if $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ with $\sum_{i=1}^{s} 1/a_i < (n+1)/2$, then this sum can be decomposed into n parts with each partial sum < 1. $n = 1$ is clear. We assume that $n \geqslant 2$. First we take n boxes $A_1, A_2, \ldots, A_n$. Since $\sum_{i=1}^{s} 1/a_i < (n+1)/2$, there are at most n index i with $a_i = 2$. Then we put these a_i $(= 2)$ into these boxes. Each box contains at most one such a_i. Then put each of remaining a_i into one of n boxes $A_1, A_2, \ldots, A_n$ such that the partial sum corresponding to each A_i is < 1. Write $T(A_i)$ for the partial sum corresponding to A_i. If some a_j fails to be put into any of n boxes $A_1, A_2, \ldots, A_n$, then $a_j \geqslant 3$ and

$$T(A_i) \geqslant 1 - \frac{1}{a_j}, \quad i = 1, 2, \ldots, n.$$

Hence

$$n\left(1 - \frac{1}{a_j}\right) \leqslant \sum_{i=1}^{n} T(A_i) \leqslant \sum_{i=1}^{s} \frac{1}{a_i} < \frac{n+1}{2}.$$

Thus $3 \leqslant a_j < (2n)/(n-1)$. Hence $n = 2$ and $a_j = 3$. Since $\sum_{i=1}^{s} 1/a_i < 3/2$, we have $\sum_{a_i=2,3} 1/a_i$ is a part of $\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3}$ or $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$. Thus we can decompose $\sum_{a_i=2,3} 1/a_i$ into 2 parts with each partial sum < 1. This contradicts the definition of $a_j = 3$. The above assertion is proved.

2. Notations

In this paper, we consider finite sets of positive integers with *repetitions*. For example, $\{3, 3, 4\} \neq \{3, 4\}$. We call such a set A *multiset*. For a multiset A and a positive real number x, let $m_A(a)$ denote the multiplicity of a in A, $m(A)$ denote the cardinality of A and let

$$T(A) = \sum_{a \in A} \frac{1}{a}, \qquad S(A) = \sum_{a \in A} a, \quad A(x) = \{a\colon a \in A,\ a < x\}.$$

For example, if $A = \{2, 3, 3, 4, 5, 5, 5\}$ and $B = \{4, 5, 5\}$, then $m_A(1) = 0$, $m_A(2) = 1$, $m_A(3) = 2$, $m_A(4) = 1$, $m_A(5) = 3$, $m(A) = 7$ and

$$T(A) = \frac{1}{2} + \frac{2}{3} + \frac{1}{4} + \frac{3}{5}, \qquad S(A) = 2 + 3 + 3 + 4 + 5 + 5 + 5,$$
$$A(5) = \{2, 3, 3, 4\}, \qquad A \setminus B = \{2, 3, 3, 5\}.$$

With these terms, a multiset A is *primitive* if there is no any multisubset A_1 of A with $m(A_1) \geqslant 2$ and $T(A_1)^{-1}$ being an integer. We say that A has a *n-quasiunit-partition* if A can be decomposed into n multisubsets $A_1, A_2, \ldots, A_n$ with $T(A_i) \leqslant 1$ $(1 \leqslant i \leqslant n)$ and $m_{A_1}(a) + \cdots + m_{A_n}(a) = m_A(a)$ for all integers a.

3. Proofs

Lemma 1. *Let A be a finite multiset of positive integers. Then there exists an effective constructible finite primitive multiset A' and a nonnegative integer k such that $T(A) = k + T(A')$.*

Proof. If there exists a multisubset B of A such that $m(B) \geqslant 2$ and $T(B)^{-1}$ is an integer b, then

$$T(A) = T\big((A \setminus B) \cup \{b\}\big), \qquad S(A) > S\big((A \setminus B) \cup \{b\}\big).$$

Let $A_1 = (A \setminus B) \cup \{b\}$. We continue this procedure and obtain $A_1, A_2, \ldots$. Noting that $S(A) > S(A_1) > \cdots$ and $S(A_i)$ are positive integers, the procedure must be terminated. This completes the proof of Lemma 1. $\square$

From Lemma 1 we immediately obtain the following Lemma 2.

Lemma 2. *Let η be a positive real number and n a positive integer. If for any positive integer $k \leqslant n$, any finite primitive multiset A with $T(A) \leqslant k - \eta$ (respectively $T(A) < k - \eta$) has a k-quasiunit-partition, then any finite multiset A with $T(A) \leqslant n - \eta$ (respectively $T(A) < n - \eta$) has a n-quasiunit-partition.*

The idea of Lemma 3 is due to Sándor [3]. But Sándor [3] did not formulate a lemma.

Lemma 3. *Let η be a positive real number and let A be a multiset with $T(A) = n - \eta$. Then A has a n-quasiunit-partition if and only if $A(\frac{1}{\eta}n)$ has a n-quasiunit-partition.*

Proof. It is clear that if A has a n-quasiunit-partition, then $A(\frac{1}{\eta}n)$ has a n-quasiunit-partition. Now we assume that $A(\frac{1}{\eta}n)$ has a n-quasiunit-partition:

$$A\left(\frac{1}{\eta}n\right) = B_1 \cup B_2 \cup \cdots \cup B_n,$$

$$T(B_i) \leqslant 1, \quad i = 1, 2, \ldots, n, \qquad \sum_{i=1}^{n} m_{B_i}(a) = m_A(a) \quad \text{for all integers } a.$$

We add each $a \in A \setminus A(\frac{1}{\eta}n)$ to one of $B_1, B_2, \ldots, B_n$ to keep all $T(B_i) \leqslant 1$. Suppose we stuck at $b \in A \setminus A(\frac{1}{\eta}n)$. Then

$$T(B_i) > 1 - \frac{1}{b}, \quad i = 1, 2, \ldots, n.$$

Thus

$$n\left(1 - \frac{1}{b}\right) < \sum_{i=1}^{n} T(B_i) \leqslant T(A) = n - \eta.$$

Hence $b < \frac{1}{\eta}n$, a contradiction with $b \notin A(\frac{1}{\eta}n)$. This completes the proof of Lemma 3. $\square$

Lemma 4. *Let $n \geqslant 2$ and $L(n)$ be a real number with $L(n) \geqslant 1$ and*

$$\frac{L(n)}{\log(nL(n))} + \frac{1.2762L(n)}{(\log(nL(n)))^2} + \frac{2\sqrt{L(n)}}{\sqrt{n}} \leqslant 1. \tag{1}$$

If A is a primitive multiset with

$$T(A) \leqslant n - \frac{1}{L(n)},$$

then A has a n-quasiunit-partition.

Proof. By Lemma 3 and [3, Theorem 3] (that is, Theorem C), we need only to prove that $T(A(nL(n))) \leqslant n - \frac{1}{2}$. Since A is primitive, we have $1 \notin A$ and $m_A(a) \leqslant p(a) - 1$. Hence, if a is composite, then $m_A(a) \leqslant p(a) - 1 < \sqrt{a}$. Thus ($p$ denotes a prime)

$$\begin{aligned} T\big(A\big(nL(n)\big)\big) &\leqslant \sum_{a=2}^{nL(n)} \frac{m_A(a)}{a} \leqslant \sum_{a=2}^{nL(n)} \frac{p(a)-1}{a} \\ &\leqslant \sum_{2 \leqslant p \leqslant nL(n)} \frac{p-1}{p} + \sum_{a=2}^{nL(n)} \frac{1}{\sqrt{a}} \leqslant \pi\big(nL(n)\big) + \sum_{a=2}^{nL(n)} \frac{1}{\sqrt{a}} \\ &\leqslant \frac{nL(n)}{\log(nL(n))}\left(1 + \frac{1.2762}{\log(nL(n))}\right) + 2\sqrt{nL(n)} - 2 \leqslant n - 2. \end{aligned}$$

Here we employ a result of prime distribution of Dusart [1]:

$$\pi(x) \leqslant \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \quad x > 1.$$

This completes the proof of Lemma 4. □

Lemma 5. *Let η be a positive real number with $0 < \eta < 1$. Suppose that any finite multiset B with $T(B) \leqslant n - 1 - \eta$ has a $(n-1)$-quasiunit-partition. Let A be a multiset with $T(A) \leqslant n - \eta$, $A_1 \subseteq A$ with $T(A_1) = 1 - \delta$, $0 \leqslant \delta < \eta$. If $\delta = 0$ or if there exists $a \in A \setminus A_1$ with $(n-1)/(\eta - \delta) \leqslant a \leqslant 1/\delta$, then A has a n-quasiunit-partition.*

Proof. By $T(A_1) = 1 - \delta$ and $T(A) \leqslant n - \eta$, we have

$$T(A \setminus A_1) \leqslant n - 1 - (\eta - \delta). \tag{2}$$

If $\delta \neq 0$ and there exists $a \in A \setminus A_1$ with $(n-1)/(\eta - \delta) \leqslant a \leqslant 1/\delta$, then

$$T\left((A \setminus A_1)\left(\frac{n-1}{\eta - \delta}\right)\right) \leqslant n - 1 - (\eta - \delta) - \frac{1}{a} \leqslant n - 1 - \eta.$$

By the assumption we have that $(A \setminus A_1)(\frac{n-1}{\eta-\delta})$ has a $(n-1)$-quasiunit-partition. By (2) and Lemma 3, $A \setminus A_1$ has a $(n-1)$-quasiunit-partition. Therefore, A has a n-quasiunit-partition. This completes the proof of Lemma 5. $\square$

Proof of Theorem 1. Take $L(n) = 30$. By calculation for $n \geqslant e^{28}$ we have (1). Hence, by Lemma 4, if $n \geqslant e^{28}$ and A is a finite primitive multiset with

$$T(A) \leqslant n - \frac{1}{30},$$

then A has a n-quasiunit-partition. The assumption in Theorem 1 and Lemma 3 imply that if $n < e^{28}$ and A is a finite primitive multiset with

$$T(A) < n - \frac{1}{30},$$

then A has a n-quasiunit-partition. Now Theorem 1 follows from Lemma 2. $\square$

Proof of Theorem 2. Take $L(n) = \log n + \log\log n - 2$. Then

$$\begin{aligned}
&\frac{L(n)}{\log(nL(n))} + \frac{1.2762L(n)}{(\log(nL(n)))^2} + \frac{2\sqrt{L(n)}}{\sqrt{n}} \\
&\quad\leqslant \frac{\log n + \log\log n - 2}{\log n + \log\log n} + \frac{1.2762}{\log n + \log\log n} + \frac{2\sqrt{\log n + \log\log n}}{\sqrt{n}} \\
&\quad= 1 - \frac{0.7238}{\log n + \log\log n} + \frac{2\sqrt{\log n + \log\log n}}{\sqrt{n}} \\
&\quad\leqslant 1
\end{aligned}$$

for all sufficiently large n. Now Theorem 2 follows from Lemma 4. $\square$

Proof of Theorem 3. By Lemma 2 we may assume that A is primitive. Take $L(n) = 3$. By calculation for $n \geqslant 100$ we have (1). By Lemma 4 we may further assume that $n < 100$. By Lemma 3 we need only to prove that $A(3n)$ has a n-quasiunit-partition. Since A is primitive and $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$, we have $m_A(3) = 0$ or $m_A(6) = 0$. For $12 \leqslant n \leqslant 99$, by directly calculation, we have

$$T\big(A(3n)\big) \leqslant \sum_{a=2}^{3n-1} \frac{p(a)-1}{a} - \frac{1}{6} \leqslant n - \frac{1}{2}.$$

By [3, Theorem 3], for $12 \leqslant n \leqslant 99$, we have that $A(3n)$ has a n-quasiunit-partition.

Now we prove Theorem 3 for $2 \leqslant n \leqslant 11$. First we consider the case $n = 2$. Let A be a primitive multiset with $T(A) \leqslant 2 - \frac{1}{3}$. Then $T(A(6))$ is the partial sum of

$$\frac{1}{2} + \frac{2}{3} + \frac{1}{4} + \frac{4}{5}.$$

It is clear that if $m_A(2)=0$ or $m_A(3)=0$, then $T(A(6))$ has a 2-quasiunit-partition. If $2\in A$ and $3\in A$, then

$$T(A)-\left(\frac{1}{2}+\frac{1}{3}\right)\leqslant 2-\frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)<1.$$

So $T(A(6))$ has a 2-quasiunit-partition.

In the following, we assume that $3\leqslant n\leqslant 11$. Let

$$S_1=\frac{1}{3}+\frac{1}{3}\left(\frac{1}{6}\right)+\frac{1}{4}\left(\frac{1}{12}\right)+\frac{1}{16},\qquad S_2=\frac{1}{5}+\frac{1}{5}\left(\frac{1}{10}\right)+\frac{1}{5}\left(\frac{1}{15}\right)+\frac{1}{5}\left(\frac{1}{20}\right)+\frac{2}{11},$$
$$S_3=\frac{4}{7}+\frac{1}{7}\left(\frac{1}{14}\right)+\frac{1}{7}\left(\frac{1}{21}\right)+\frac{1}{8},\qquad S_4=\frac{1}{9}+\frac{1}{9}\left(\frac{1}{18}\right)+\frac{8}{11},\qquad S_5=\frac{12}{13}+\frac{1}{15},$$
$$S_6=\frac{16}{17}+\frac{1}{21},\qquad S_7=\frac{18}{19}+\frac{1}{22},\qquad S_8=\frac{22}{23}+\frac{1}{24},$$
$$S_9=\frac{1}{2}+\frac{4}{25}+\frac{1}{26}+\frac{2}{27}+\frac{1}{28},\qquad S_{10}=\frac{28}{29}+\frac{1}{30},\qquad S_{11}=\frac{30}{31}+\frac{1}{32},$$
$$S_1'=\frac{1}{2}+\frac{1}{4}\left(\frac{1}{12}\right)+\frac{1}{6}+\frac{1}{16},\qquad S_3'=\frac{1}{2}+\frac{1}{7}+\frac{1}{8}+\frac{1}{14}+\frac{1}{21},$$
$$S_4'=\frac{1}{2}+\frac{1}{9}+\frac{1}{9}\left(\frac{1}{18}\right)+\frac{2}{11},\qquad S_5'=\frac{1}{2}+\frac{5}{13}+\frac{1}{15},$$
$$S_6'=\frac{1}{2}+\frac{7}{17}+\frac{1}{21},\qquad S_7'=\frac{1}{2}+\frac{8}{19}+\frac{1}{22},\qquad S_8'=\frac{1}{2}+\frac{10}{23}.$$

In the above constructions, $\frac{1}{3}(\frac{1}{6})$ denotes that we can only choose one of $\frac{1}{3}$ and $\frac{1}{6}$. The others $a(b)$ have the similar meanings. The reasons are that A is primitive and

$$\frac{1}{3}+\frac{1}{6}=\frac{1}{2},\qquad \frac{1}{4}+\frac{1}{12}=\frac{1}{3},\qquad \frac{2}{5}+\frac{1}{10}=\frac{1}{2},\qquad \frac{1}{5}+\frac{2}{15}=\frac{1}{3},\qquad \frac{1}{5}+\frac{1}{20}=\frac{1}{4},$$
$$\frac{3}{7}+\frac{1}{14}=\frac{1}{2},\qquad \frac{2}{7}+\frac{1}{21}=\frac{1}{3},\qquad \frac{1}{9}+\frac{1}{18}=\frac{1}{6}.$$

For every possible, each of S_i and S_i' does not exceed 1. For the convenience of reader, we give more explanations here. For example, we consider the integer 15. Since A is primitive, we have $m_A(15)\leqslant 2$. If $m_A(15)\leqslant 1$, then we put $1/15$ in S_5 and $\frac{1}{5}(\frac{1}{15})=\frac{1}{5}$ in S_2. If $m_A(15)=2$, then by

$$\frac{2}{15}+\frac{1}{5}=\frac{1}{3}$$

we know that $m_A(5)=0$. Then we put $1/15$ in S_5 and $\frac{1}{5}(\frac{1}{15})=\frac{1}{15}$ in S_2.

If $n=9,10,11$, then $S_1,S_2,\dots,S_n$ imply a n-quasiunit-partition of $A(3n)$. To see this, we explain why $S_1,S_2,\dots,S_{11}$ imply a 11-quasiunit-partition of $A(33)$. The others are similar. Let

$$A_i=\left\{a\colon\ \frac{1}{a}\text{ appears in }S_i\right\}.$$

For $a \in A(33)$, we have $2 \leqslant a \leqslant 32$. Since $\frac{1}{a}$ appears in $S_1, S_2, \ldots, S_{11}$ exactly $p(a)-1$ times and $m_A(a) \leqslant p(a)-1$, where $p(a)$ is the least prime factor of a, we have

$$A(33) \subseteq A_1 \cup A_2 \cup \cdots \cup A_{11}.$$

Since

$$T(A \cap A_i) \leqslant S_i < 1, \quad i = 1, 2, \ldots, 11,$$

we have that $S_1, S_2, \ldots, S_{11}$ imply a 11-quasiunit-partition of $A(33)$.

If $3 \leqslant n \leqslant 8$ and $m_A(2) = 0$, then $S_1, S_2, \ldots, S_n$ imply a n-quasiunit-partition of $A(3n)$. If $3 \leqslant n \leqslant 8$ and $m_A(3) = 0$, then $S_1', S_2, \ldots, S_n$ imply a n-quasiunit-partition of $A(3n)$. If $3 \leqslant n \leqslant 8$ and $m_A(7) \leqslant 1$, then $S_1, S_2, S_3', S_4, \ldots, S_n$ imply a n-quasiunit-partition of $A(3n)$. Hence, for $3 \leqslant n \leqslant 8$, we may assume that

$$m_A(2) \geqslant 1, \qquad m_A(3) \geqslant 1, \qquad m_A(7) \geqslant 2.$$

Similarly, by using $S_4', S_5', S_6', S_7', S_8'$, we may assume that $m_A(11) \geqslant 3$, for $4 \leqslant n \leqslant 8$; $m_A(13) \geqslant 6$, for $5 \leqslant n \leqslant 8$; $m_A(17) \geqslant 8$, for $6 \leqslant n \leqslant 8$; $m_A(19) \geqslant 9$, for $7 \leqslant n \leqslant 8$; $m_A(23) \geqslant 11$, for $n = 8$.

Now we apply Lemma 5 to complete the proof.

For $3 \leqslant n \leqslant 5$, let $\eta = \frac{1}{3}$, $A_1 = \{2, 3, 7\}$ and $\delta = \frac{1}{42}$.

For $6 \leqslant n \leqslant 8$, let $\eta = \frac{1}{3}$, $A_1 = \{2, 3, 13, 13\}$ and $\delta = \frac{1}{78}$.

Then

$$\frac{n-1}{\eta - \delta} < 3n - 2 < 3n - 1 < \frac{1}{\delta}.$$

By Lemma 5, if Theorem 3 is true for $n-1$ and $A \setminus A_1$ contains $3n-1$ or $3n-2$, then A has a n-quasiunit-partition. In fact, by $m_A(7) \geqslant 2$ we have $7 \in A \setminus A_1$ for $n = 3$. Similarly, $11 \in A \setminus A_1$ for $n = 4$; $13 \in A \setminus A_1$ for $n = 5$; $17 \in A \setminus A_1$ for $n = 6$; $19 \in A \setminus A_1$ for $n = 7$; $23 \in A \setminus A_1$ for $n = 8$. This completes the proof of Theorem 3. $\square$

Acknowledgment

I thank the referee for his/her useful suggestions.

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