Available online at www.sciencedirect.com
ScienceDirect
Journal of Number Theory 123 (2007) 241–253
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

On conditional irrationality measures for values of the digamma function

Kh. Hessami Pilehrood [a,b,*,1], T. Hessami Pilehrood [a,b,2]

[a] *Mathematics Department, Shahrekord University, Shahrekord, PO Box 115, Iran*
[b] *Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran*

Received 7 January 2005; revised 3 December 2005

Available online 10 August 2006

Communicated by David Goss

Abstract

The aim of this work is to obtain the so-called standard lemmas on irrationality bases using the principles of Chudnovsky and then apply them to obtain conditional irrationality measures for values of the digamma function.

Keywords: Irrationality measure; Irrationality base; Digamma function; Euler's constant; Diophantine approximations

0. Introduction

In this note we answer the question posed by Sondow [5] and obtain the so-called standard lemmas on irrationality bases using the principles of Chudnovsky. We then apply them to prove conditional irrationality measures for values of the digamma function $\gamma_\alpha = -\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ using a Diophantine approximation construction from [4].

* Corresponding author.
E-mail addresses: hessamik@ipm.ir (Kh. Hessami Pilehrood), hessamit@ipm.ir (T. Hessami Pilehrood).
1 This research was in part supported by a grant from IPM (N 85110020).
2 This research was in part supported by a grant from IPM (N 85110021).

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2006.07.008

Let us recall some definitions. Let θ be an irrational real number, the irrationality of which is usually measured by determining the lower bounds,

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{q^{\mu(\theta)+\varepsilon}},$$

which hold for all $\varepsilon > 0$ and all integers p, q, with $q \geqslant q_0(\varepsilon)$. Then the least such $\mu(\theta)$ is called the irrationality exponent of θ. If $\mu(\theta) = \infty$, then θ is called a Liouville number.

If θ has the irrationality measure $1/\beta^q$ (according to Sondow's definition [5]), so that $\beta = \beta(\theta)$ is the least number with the property that for any $\varepsilon > 0$ there exists $q_0(\varepsilon) > 0$, such that

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{(\beta+\varepsilon)^q} \quad \text{for all integers } p, q \text{ with } q \geqslant q_0(\varepsilon),$$

then $\beta(\theta)$ is called the irrationality base of θ. Otherwise, if no such β exists, we define $\beta(\theta) = \infty$ and say that θ is a super Liouville number. Note that $\beta(\theta) = 1$ if $\mu(\theta)$ is finite (see Lemma 2 in [5]).

Explicit formulas for $\mu(\theta)$ and $\beta(\theta)$ in terms of the continued fraction expansion of θ were proved by Sondow [6].

In practice, to obtain the upper bounds of $\mu(\theta)$, one of the following two standard lemmas is normally used. (For proofs, see Lemma 3.5 in [2] and Remark 2.1 in [3], respectively.)

Lemma 1 *(G. Chudnovsky). Let θ be a real number satisfying*

$$R_n = A_n\theta - B_n, \quad n = 1, 2, \ldots,$$

for some $A_n, B_n \in \mathbb{Z}$. Suppose that

$$\limsup_{n\to\infty} \frac{1}{n}\log|A_n| \leqslant \sigma, \qquad \lim_{n\to\infty} \frac{1}{n}\log|R_n| = -\tau$$

for some positive numbers σ, τ. Then θ is irrational and has irrationality exponent $\mu(\theta) \leqslant 1 + \frac{\sigma}{\tau}$.

Lemma 2 *(M. Hata). Let θ be a real irrational number satisfying*

$$R_n = A_n\theta - B_n, \quad n = 1, 2, \ldots,$$

for some $A_n, B_n \in \mathbb{Z}$. Suppose that

$$\lim_{n\to\infty} \frac{1}{n}\log|A_n| = \sigma, \qquad \limsup_{n\to\infty} \frac{1}{n}\log|R_n| \leqslant -\tau$$

for some positive numbers σ, τ. Then the irrationality exponent $\mu(\theta) \leqslant 1 + \frac{\sigma}{\tau}$.

In Sections 1 and 3, we prove analogous statements for the irrationality base $\beta(\theta)$. Then in Sections 2 and 3 we apply them to obtain the conditional upper bounds for $\beta(\gamma_\alpha)$. Section 4 is devoted to the conditional upper bounds of the irrationality exponent $\mu(\gamma_\alpha)$.

1. Some lemmas on irrationality measures

We need the following lemmas concerning rational approximations.

Lemma 3. *Let θ be a real number satisfying $R_n = A_n\theta - B_n$, $n = 1, 2, \ldots$, for some $A_n, B_n \in \mathbb{Z}$. Suppose that f, g are positive real functions defined in $(0, +\infty)$, f is monotonically increasing, and $\lim_{n\to\infty} f(n) = +\infty$. Suppose further that*

$$\limsup_{n\to\infty} \frac{1}{n} \log|A_n| \leqslant \sigma, \quad \sigma \in \mathbb{R},\ \sigma \geqslant 0,$$

and

$$\|nR_{[f(n)]}\| \geqslant g(n) \tag{1}$$

for all n sufficiently large, where $\|x\|$ denotes the distance from x to the nearest integer, and $[x]$ is the integer part of x. Then, for any $\varepsilon > 0$ there exists $q_0(\varepsilon) > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \geqslant \frac{g(q)}{qe^{(\sigma+\varepsilon)f(q)}}$$

for all integers p, q with $q > q_0(\varepsilon)$.

Proof. Let $p \in \mathbb{Z}, q \in \mathbb{N}$ be arbitrarily fixed numbers. Then, for any positive integer n we have

$$R_n = A_n\left(\theta - \frac{p}{q}\right) - B_n + A_n\frac{p}{q}. \tag{2}$$

Set $n = [f(q)]$. Then, multiplying both sides of (2) by q, we have

$$\left|qA_n\left(\theta - \frac{p}{q}\right)\right| = \bigl|M + \|qR_n\|\bigr|, \tag{3}$$

where $M \in \mathbb{Z}$. From (1) it follows that

$$g(q) \leqslant \|qR_n\| \leqslant \frac{1}{2} \tag{4}$$

for all $q > q_1$. Now if $|M| \geqslant 1$, then the right-hand side of (3) is not less than $1/2$. Hence, according to (4), for any integer M we have

$$\left|qA_n\left(\theta - \frac{p}{q}\right)\right| \geqslant g(q),$$

or

$$\left|\theta - \frac{p}{q}\right| \geqslant \frac{g(q)}{q|A_n|}.$$

Since for any $\varepsilon > 0$, $|A_n| \leqslant e^{(\sigma+\varepsilon)f(q)}$ for all $q \geqslant q_2(\varepsilon)$, then

$$\left|\theta - \frac{p}{q}\right| \geqslant \frac{g(q)}{qe^{(\sigma+\varepsilon)f(q)}},$$

as required. □

Corollary 4. *Let θ be a real number satisfying $R_n = A_n\theta - B_n$, $n = 1, 2, \ldots,$ for some $A_n, B_n \in \mathbb{Z}$. Let f, g_1, g_2 be positive real functions defined in $(0, +\infty)$, with f monotonically increasing, and $\lim_{n\to\infty} f(n) = +\infty$. Suppose that*

$$\limsup_{n\to\infty} \frac{1}{n} \log |A_n| \leqslant \sigma, \quad \sigma \in \mathbb{R},\ \sigma \geqslant 0,$$

and

$$g_1(n) \leqslant \{nR_{[f(n)]}\} \leqslant 1 - g_2(n) \tag{5}$$

for all n sufficiently large (where $\{x\}$ denotes the fractional part of x). Then for any $\varepsilon > 0$ there exists $q_0(\varepsilon) > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \geqslant \frac{\min(g_1(q), g_2(q))}{qe^{(\sigma+\varepsilon)f(q)}} \tag{6}$$

for all integers p, q with $q > q_0(\varepsilon)$.

Proof. Inequality (6) follows from the proof of Lemma 3 if we replace $\|\cdot\|$ by $\{\cdot\}$ in (3) and use inequality (5) in place of (4). □

Lemma 5. *Let θ be a real number satisfying $R_n = A_n\theta - B_n$, $n = 1, 2, \ldots,$ for some $A_n, B_n \in \mathbb{Z}$, and*

$$\limsup_{n\to\infty} \frac{1}{n} \log |A_n| \leqslant \sigma, \quad \sigma \in \mathbb{R},\ \sigma \geqslant 0.$$

Suppose that $f(x), \psi(x)$ are positive real continuous functions defined in $(0, +\infty)$, strictly increasing for $x > x_0 > 0$, with $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \psi(n) = +\infty$. Suppose further that

$$f\big(\psi(n)\big) - f(n) \geqslant 1$$

for all n sufficiently large and

$$\lim_{n\to\infty} f^{-1}(n) \cdot |R_n| = 1, \tag{7}$$

where f^{-1} is the inverse of f (which exists by the above). Then for any $\varepsilon > 0$ there exists a positive integer $q_0(\varepsilon)$ such that

$$\left|\theta - \frac{p}{q}\right| > \begin{cases} \frac{1}{qe^{(\sigma+\varepsilon)f(\psi((1+\varepsilon)q))}}, & \text{if } \lim_{q\to\infty}\frac{q}{\psi(q)} > 0, \\ \frac{1}{\psi((1+\varepsilon)q)e^{(\sigma+\varepsilon)f(\psi((1+\varepsilon)q))}}, & \text{if } \lim_{q\to\infty}\frac{q}{\psi(q)} = 0 \end{cases}$$

for all integers p, q with $q \geqslant q_0(\varepsilon)$.

Proof. It follows from (7) that for any $\varepsilon > 0$ there exists a positive integer $k_0(\varepsilon)$ such that

$$\frac{1-\varepsilon/2}{f^{-1}(k)} \leqslant |R_k| \leqslant \frac{1+\varepsilon/2}{f^{-1}(k)} \tag{8}$$

for all $k \geqslant k_0(\varepsilon)$. Set $k = [f(\psi((1+\varepsilon)n))]$, $n \geqslant n_0(\varepsilon)$. Then the following inequalities are valid:

$$f\big((1+\varepsilon)n\big) \leqslant f\big(\psi\big((1+\varepsilon)n\big)\big) - 1 < k \leqslant f\big(\psi\big((1+\varepsilon)n\big)\big),$$

or

$$(1+\varepsilon)n < f^{-1}(k) \leqslant \psi\big((1+\varepsilon)n\big).$$

Hence, from (8) we have

$$\frac{1-\varepsilon/2}{\psi((1+\varepsilon)n)} \leqslant |R_k| < \frac{1+\varepsilon/2}{(1+\varepsilon)n},$$

i.e.,

$$\frac{(1-\varepsilon/2)n}{\psi((1+\varepsilon)n)} \leqslant \big\{n|R_k|\big\} \leqslant \frac{1+\varepsilon/2}{1+\varepsilon}.$$

Thus, according to Corollary 4 (applying inequality (6) with $\varepsilon/2$), for all $q > q_1(\varepsilon)$ we obtain

$$\left|\theta - \frac{p}{q}\right| > \begin{cases} \frac{1}{qe^{(\sigma+\varepsilon)f(\psi((1+\varepsilon)q))}}, & \text{if } \frac{(1-\varepsilon/2)q}{\psi((1+\varepsilon)q)} > \frac{\varepsilon/2}{1+\varepsilon}, \\ \frac{1}{\psi((1+\varepsilon)q)e^{(\sigma+\varepsilon)f(\psi((1+\varepsilon)q))}}, & \text{if } \frac{(1-\varepsilon/2)q}{\psi((1+\varepsilon)q)} \leqslant \frac{\varepsilon/2}{1+\varepsilon}, \end{cases}$$

from which the lemma follows. □

Now, using the fact that the irrationality base $\beta(\theta) \geqslant 1$, we can obtain the following corollaries.

Corollary 6. *If under the conditions of Lemma* 5 *we have*

$$\lim_{n\to\infty}\frac{n}{\psi(n)} > 0 \quad \textit{and} \quad \lim_{n\to\infty}\frac{f(\psi(n))}{n} = 0,$$

or

$$\lim_{n\to\infty}\frac{n}{\psi(n)}=0,\qquad \lim_{n\to\infty}\frac{f(\psi(n))}{n}=0,\quad \textit{and}\quad \lim_{n\to\infty}\frac{1}{n}\ln\bigl(\psi(n)\bigr)=0,$$

then $\beta(\theta)=1$.

Proposition 7. *Let θ be a real number satisfying $R_n=A_n\theta-B_n$, $n=1,2,\dots,$ for some $A_n,B_n\in\mathbb{Z}$, with*

$$\limsup_{n\to\infty}\frac{1}{n}\log|A_n|\leqslant\sigma,\quad \sigma\in\mathbb{R},\ \sigma\geqslant 0,$$

and

$$\lim_{n\to\infty}|nR_n|=\tau,\quad \tau\in\mathbb{R},\ \tau\geqslant 0.$$

If $\tau=0$, suppose also that $R_n\neq 0$ for all $n\geqslant n_0$. Then θ is irrational and

$$\beta(\theta)\leqslant e^{\sigma\tau}. \tag{9}$$

In particular, if σ or τ equals zero, then $\beta(\theta)=1$.

Proof. The irrationality of θ follows from the standard argument. Suppose, on the contrary, that $\theta=p/q$, $p\in\mathbb{Z}$, $q\in\mathbb{N}$. Then, for an arbitrarily small $\varepsilon>0$, there exists a positive number $n_1(\varepsilon)>n_0$ such that

$$\max\left(0,\frac{\tau-\varepsilon}{n}\right)<|qR_n|=|A_np-qB_n|<q\frac{\tau+\varepsilon}{n} \tag{10}$$

for all $n>n_1(\varepsilon)$. Since A_np-qB_n is a non-zero integer, we have that $|A_np-qB_n|\geqslant 1$ and this contradicts the right-hand side of inequality (10), which tends to zero as $n\to\infty$.

If $\tau>0$, set $f(n)=\tau n$ and $\psi(n)=n+1/\tau$. Then the required inequality immediately follows from Lemma 5. Letting τ tend to zero and using Corollary 6, we obtain that (9) holds for all $\tau\geqslant 0$. □

2. Conditional bounds on the irrationality base for values of the digamma function

In [4] we gave irrationality criteria for the values of the digamma function (or the generalized Euler constant)

$$\gamma_\alpha=-\frac{\Gamma'(\alpha)}{\Gamma(\alpha)},$$

where $\alpha=a/b$ is a rational number, $1\leqslant a\leqslant b$, $(a,b)=1$. The proof was based on the representation

$$I_{\mathbf{m}}(\alpha)=\binom{m_1+m_2}{m_1}\gamma_\alpha+L_{\mathbf{m}}(\alpha)-A_{\mathbf{m}}(\alpha), \tag{11}$$

where for $\mathbf{m}=(m_1,m_2)\in\mathbb{N}^2, m_1\leqslant m_2$,

$$I_{\mathbf{m}}(\alpha)=\iint\limits_{[0,1]^2}-\frac{(xy)^{m_1+\alpha-1}(1-x)^{m_1}(1-y)^{m_2}}{(1-xy)\log xy}\,dx\,dy, \tag{12}$$

and $L_{\mathbf{m}}(\alpha)$ is the $\mathbb{Q}$-linear form in logarithms

$$\begin{aligned}L_{\mathbf{m}}(\alpha)=&\sum_{l=1}^{m_1}\sum_{k=0}^{l-1}\binom{m_1}{k}\binom{m_2}{k}(H_{m_1-k}+H_{m_2-k}-2H_k)\log(l+m_1+\alpha-1)\\&+\sum_{l=m_1+1}^{m_2}\sum_{k=l}^{m_2}\frac{(-1)^{k-1-m_1}}{k}\binom{m_2}{k}\Big/\binom{k-1}{m_1}\log(l+m_1+\alpha-1),\end{aligned} \tag{13}$$

$$A_{\mathbf{m}}(\alpha)=\sum_{k=0}^{m_1}\binom{m_1}{k}\binom{m_2}{k}H_{m_1+k-1}(\alpha)\in\mathbb{Q},\qquad d_{2m_1}(a,b)A_{\mathbf{m}}(\alpha)\in\mathbb{Z},$$

where $d_m(a,b)$ denotes the least common multiple of the numbers $a, a+b,\dots,a+(m-1)b$, $L_{\mathbf{m}}=L_{\mathbf{m}}(1)$, $d_m=d_m(1,1)$ and $H_m(\alpha)=\sum_{l=0}^{m}(l+\alpha)^{-1}$, $H_m=H_{m-1}(1)$. Let n, r_1, r_2 be positive integers, with $m_1=r_1n, m_2=r_2n, \mathbf{m}=(m_1,m_2)$. From [4] and [1], and Stirling's formula, we have

$$I_{(r_1n,r_2n)}(\alpha)=\left(\frac{r_1^{r_1}r_2^{r_2}}{4^{r_1}(r_1+r_2)^{r_1+r_2}}\right)^{n(1+o(1))}\quad\text{as } n\to\infty, \tag{14}$$

$$\binom{(r_1+r_2)n}{r_1n}=\left(\frac{(r_1+r_2)^{r_1+r_2}}{r_1^{r_1}r_2^{r_2}}\right)^{n(1+o(1))}\quad\text{as } n\to\infty, \tag{15}$$

$$d_n(a,b)=e^{h(b)\cdot n(1+o(1))}\quad\text{as } n\to\infty, \tag{16}$$

where

$$h(b)=\frac{b}{\varphi(b)}\sum_{\substack{k=1\\(k,b)=1}}^{b}\frac{1}{k},$$

and φ is the Euler function.

Theorem 8 *(Conditional bounds on $\beta(\gamma_\alpha)$). Let $r_1\leqslant r_2$ be positive integers, and $\alpha=a/b$, $a,b\in\mathbb{Z}$, $1\leqslant a\leqslant b$, $(a,b)=1$, satisfying the inequality*

$$2r_1\log 2+(r_1+r_2)\log(r_1+r_2)-r_1\log r_1-r_2\log r_2>2r_1h(b).$$

Suppose that there exists a sequence of positive integers n_k, $k=1,2,\dots$, such that

$$\limsup_{k\to\infty}\frac{n_k}{k}\leqslant\sigma,\qquad\limsup_{k\to\infty}\frac{\log k}{n_k}<\log\left(\frac{4^{r_1}(r_1+r_2)^{r_1+r_2}}{e^{2r_1h(b)}r_1^{r_1}r_2^{r_2}}\right), \tag{17}$$

and

$$\lim_{k\to\infty} k\left|u - v\left\{d_{2r_1n_k}(a,b)L_{(r_1n_k,r_2n_k)}(\alpha)\right\}\right| = \tau, \tag{18}$$

for some integers u, v and non-negative numbers σ, τ. If $\tau = 0$, then suppose also that

$$\lim_{k\to\infty} \frac{1}{n_k}\log\left|\frac{u}{v} - \left\{d_{2r_1n_k}(a,b)L_{(r_1n_k,r_2n_k)}(\alpha)\right\}\right| \neq \log\left(\frac{e^{2r_1h(b)}r_1^{r_1}r_2^{r_2}}{4^{r_1}(r_1+r_2)^{r_1+r_2}}\right). \tag{19}$$

Then γ_α is irrational and the irrationality base of γ_α satisfies

$$\beta(\gamma_\alpha) \leqslant \left(e^{2h(b)r_1}\frac{(r_1+r_2)^{r_1+r_2}}{r_1^{r_1}r_2^{r_2}}\right)^{\sigma\tau}.$$

In particular, if σ or τ equals zero, then $\beta(\gamma_\alpha) = 1$.

Proof. We define the integers A_k, B_k, for $k = 1, 2, \ldots,$ by the formulas

$$B_k = vd_{2r_1n_k}(a,b)A_{(r_1n_k,r_2n_k)}(\alpha) - v\left[d_{2r_1n_k}(a,b)L_{(r_1n_k,r_2n_k)}(\alpha)\right] - u, \tag{20}$$

$$A_k = vd_{2r_1n_k}(a,b)\binom{(r_1+r_2)n_k}{r_1n_k}. \tag{21}$$

According to (11), we have $A_k\gamma_\alpha - B_k = R_k$, where

$$R_k = vd_{2r_1n_k}(a,b)I_{(r_1n_k,r_2n_k)}(\alpha) + u - v\left\{d_{2r_1n_k}(a,b)L_{(r_1n_k,r_2n_k)}(\alpha)\right\}.$$

The asymptotics (14), (15) and (16), together with (17) and (18), imply that

$$\limsup_{k\to\infty} \frac{1}{k}\log|A_k| \leqslant \sigma\left(2h(b)r_1 + (r_1+r_2)\log(r_1+r_2) - r_1\log r_1 - r_2\log r_2\right)$$

and $\lim_{k\to\infty} k|R_k| = \tau$. Thus, by Proposition 7, the theorem follows. $\square$

Remark. Note that the theorem remains valid if we replace the fractional parts in (18) and (19) by the distances to the nearest integers.

Setting $\alpha = a = b = r_1 = r_2 = 1$ in Theorem 8, we obtain the conditional bound on the irrationality base of Euler's constant γ.

Corollary 9. *Suppose that there exists a sequence of positive integers n_k, $k = 1, 2, \ldots,$ such that*

$$\limsup_{k\to\infty}\frac{n_k}{k} \leqslant \sigma, \qquad \limsup_{k\to\infty}\frac{\log k}{n_k} < 2\log\left(\frac{4}{e}\right),$$

and

$$\lim_{k\to\infty} k\left|u - v\{d_{2n_k}L_{(n_k,n_k)}\}\right| = \tau$$

for some integers u, v and non-negative numbers σ, τ. If $\tau = 0$, suppose also that

$$\lim_{k\to\infty} \frac{1}{n_k} \log\left|\frac{u}{v} - \{d_{2n_k} L_{(n_k,n_k)}\}\right| \neq -2\log\left(\frac{4}{e}\right).$$

Then Euler's constant γ is irrational and has irrationality base

$$\beta(\gamma) \leqslant (2e)^{2\sigma\tau}.$$

In particular, if σ or τ equals zero, then $\beta(\gamma) = 1$.

3. Another approach to the estimation of irrationality bases

Generalizing Sondow's arguments in proving conditional upper bounds for the irrationality base $\beta(\gamma)$, we can obtain the following statement that is weaker than Lemma 3.

Lemma 10. *Let θ be a real number satisfying $R_n = A_n\theta - B_n$, $n = 1, 2, \dots,$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Q}$. Let f, g be positive real functions defined in $(0, +\infty)$, with f monotonically increasing and $\lim_{n\to\infty} f(n) = \infty$. Suppose that $d_n B_{[f(n)]} \in \mathbb{Z}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log|A_n| \leqslant \sigma, \quad \sigma \in \mathbb{Z},\ \sigma \geqslant 0,$$

and

$$\|d_n R_{[f(n)]}\| \geqslant g(n).$$

Then for any $\varepsilon > 0$ there exists $q_0(\varepsilon) > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{g(q)}{e^{q(1+\varepsilon)} e^{f(q)(\sigma+\varepsilon)}}$$

for all integers p, q with $q \geqslant q_0(\varepsilon)$.

Proof. The argument is as for Lemma 3, except that here we multiply both sides of (2) by d_q instead of q. □

Directly from this lemma we have:

Proposition 11. *Let θ be a real number satisfying $R_n = A_n\theta - B_n$, $n = 1, 2, \dots,$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Q}$. Suppose that $d_{[cn]} B_n \in \mathbb{Z}$ for some positive real constant c and*

$$\limsup_{n\to\infty} \frac{1}{n} \log|A_n| \leqslant \sigma, \quad \sigma \in \mathbb{R},\ \sigma \geqslant 0.$$

If

$$\liminf_{n\to\infty} \frac{1}{n} \log\|d_{[cn]} R_n\| \geqslant -\delta \tag{22}$$

for some non-negative number δ, then θ is irrational and

$$\beta(\theta) \leqslant e^{1+\frac{\sigma+\delta}{c}}.$$

In particular, if $\delta = 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log \|d_{[cn]} R_n\| = 0,$$

then θ is irrational and $\beta(\theta) \leqslant e^{1+\frac{\sigma}{c}}$.

Proof. We show that θ is irrational. We first note that (22) implies

$$\|d_{[cn]} R_n\| > e^{(-\delta-\varepsilon)n} > 0 \quad \text{for all } n > n_0(\varepsilon). \tag{23}$$

Now, if $\theta = p/q$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we can choose $n > n_0(\varepsilon)$ such that $[cn] > q$. Then

$$d_{[cn]} R_n = d_{[cn]} A_n \theta - d_{[cn]} B_n \in \mathbb{Z}$$

and, therefore, $\|d_{[cn]} R_n\| = 0$, which contradicts (23). Thus, θ is irrational. The upper bound for $\beta(\theta)$ easily follows from the proof of Lemma 10. $\square$

Applying Proposition 11 to the Diophantine approximation construction (11), we can obtain the following.

Theorem 12. *Let $r_1 \leqslant r_2$ be positive integers and $\alpha = a/b$, $a, b \in \mathbb{Z}$, $1 \leqslant a \leqslant b$, $(a, b) = 1$. If*

$$\liminf_{n\to\infty} \frac{1}{n} \log \bigl\| d_{[cn]} L_{(r_1 n, r_2 n)}(\alpha) \bigr\| \geqslant -\delta \tag{24}$$

for some real constant $c \geqslant 2br_1$ and non-negative number δ, such that

$$\delta < (r_1 + r_2)\log(r_1 + r_2) + 2r_1 \log 2 - r_1 \log r_1 - r_2 \log r_2 - c,$$

then γ_α is irrational and

$$\beta(\gamma_\alpha) \leqslant e^{1+\frac{\delta}{c}} \left(\frac{(r_1 + r_2)^{r_1+r_2}}{r_1^{r_1} r_2^{r_2}} \right)^{\frac{1}{c}}.$$

In particular, if $\delta = 0$, i.e., if

$$\lim_{n\to\infty} \frac{1}{n} \log \bigl\| d_{[cn]} L_{(r_1 n, r_2 n)}(\alpha) \bigr\| = 0,$$

where $2br_1 \leqslant c < (r_1 + r_2)\log(r_1 + r_2) + 2r_1 \log 2 - r_1 \log r_1 - r_2 \log r_2$, then γ_α is irrational and

$$\beta(\gamma_\alpha) \leqslant e \left(\frac{(r_1 + r_2)^{r_1+r_2}}{r_1^{r_1} r_2^{r_2}} \right)^{\frac{1}{c}}.$$

Proof. From (11) we have $R_n = A_n\gamma_\alpha - B_n$, where

$$A_n = -\binom{(r_1+r_2)n}{r_1 n}, \qquad B_n = A_{(r_1 n, r_2 n)}(\alpha),$$
$$R_n = L_{(r_1 n, r_2 n)}(\alpha) - I_{(r_1 n, r_2 n)}(\alpha),$$

$d_{[cn]}B_n \in \mathbb{Z}$ if $c \geqslant 2br_1$, and

$$\lim_{n\to\infty}\frac{1}{n}\log|A_n| = (r_1+r_2)\log(r_1+r_2) - r_1\log r_1 - r_2\log r_2. \tag{25}$$

According to (24) and (14), and the Prime Number Theorem, for any $0 < 2\varepsilon < (r_1+r_2)\log(r_1+r_2) + 2r_1\log 2 - r_1\log r_1 - r_2\log r_2 - \delta - c$, there exists an integer $n_0(\varepsilon)$ such that

$$\left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\right\| \geqslant e^{(-\delta-\varepsilon)n} \tag{26}$$

and

$$\left|d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)\right| < \left(\frac{e^c r_1^{r_1} r_2^{r_2}}{4^{r_1}(r_1+r_2)^{r_1+r_2}}\right)^{n(1+\varepsilon)} \tag{27}$$

for all $n \geqslant n_0(\varepsilon)$. Setting $\nu = 2 - c - \log(\frac{r_1^{r_1} r_2^{r_2}}{4^{r_1}(r_1+r_2)^{r_1+r_2}}) > \delta + 2 \geqslant 2$, from (27) and (26), for all $n \geqslant \max(n_0(\varepsilon), (\log 2)/\varepsilon)$ we have

$$\left|d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)\right| < e^{(-\delta-\nu\varepsilon)n} \leqslant \frac{1}{2}e^{(-\delta-\varepsilon)n} \leqslant \frac{1}{2}\left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\right\|, \tag{28}$$

from which it follows that

$$\left|\pm\left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\right\| - d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)\right| < 1. \tag{29}$$

Applying inequalities (28) and (29), we obtain that

$$\|d_{[cn]}R_n\| = \left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha) - d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)\right\|$$

is equal to $\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\| \pm d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)$ or $1 - \|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\| \pm d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)$, which is not less than $\|d_{[cn]}L_{(r_1 n, r2n)}(\alpha)\| - |d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)|$ (we use here that for any real x, $1 - \|x\| \geqslant \|x\|$). Finally, by (28),

$$\|d_{[cn]}R_n\| \geqslant \left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\right\| - \left|d_{[cn]}I_{(r_1 n, r_2 n)}(\alpha)\right| > \frac{1}{2}\left\|d_{[cn]}L_{(r_1 n, r_2 n)}(\alpha)\right\|.$$

And therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\log\|d_{[cn]}R_n\| \geqslant -\delta. \tag{30}$$

Thus, from (25) and (30), and Proposition 11, the theorem follows. □

For example, from Theorem 12, we have:

Corollary 13. *If*

$$\lim_{n\to\infty} \frac{1}{n} \log \| d_{[cn]} L_{(n,n)} \| = 0$$

for some $2 \leqslant c < 4\log 2$*, then Euler's constant* γ *is irrational and* $\beta(\gamma) \leqslant 4^{\frac{1}{c}} \cdot e$.

Remark. From Theorem 12, for the case $\alpha = r_1 = r_2 = 1$ and $c = 2$ we obtain Sondow's conditional bounds (Theorems 1–3 in [5]) for the irrationality base of Euler's constant γ.

4. Conditional bounds for the irrationality exponent of γ_α

Here we obtain conditional upper bounds for the irrationality exponent of γ_α in the same way as in [5, §4] using Lemmas 1 and 2 (see the introduction).

Theorem 14 *(Conditional bounds on* $\mu(\gamma_\alpha)$*). Let* $r_1 \leqslant r_2$ *be positive integers, and* $\alpha = a/b$, $a, b \in \mathbb{Z}$, $1 \leqslant a \leqslant b$, $(a,b) = 1$, *satisfying the inequality*

$$2r_1 \log 2 + (r_1 + r_2)\log(r_1 + r_2) - r_1 \log r_1 - r_2 \log r_2 > 2r_1 h(b).$$

Suppose that there exists a sequence of positive integers n_k, $k = 1, 2, \ldots,$ *such that*

$$\lim_{k\to\infty} \frac{n_k}{k} = \sigma, \qquad \lim_{k\to\infty} \frac{1}{k} \log \left| \left\{ d_{2r_1 n_k}(a,b) L_{(r_1 n_k, r_2 n_k)}(\alpha) \right\} - \frac{u}{v} \right| = -\tau$$

for some integers u, v *and positive numbers* σ, τ*, with*

$$\tau \neq \sigma\big((r_1 + r_2)\log(r_1 + r_2) + 2r_1 \log 2 - r_1 \log r_1 - r_2 \log r_2 - 2r_1 h(b)\big).$$

Then γ_α *is irrational and has irrationality exponent* $\mu(\gamma_\alpha) \leqslant \mu_{h(b),\sigma,\tau}$*, where*

$$\mu_{h(b),\sigma,\tau} = \begin{cases} 1 + \frac{\sigma}{\tau}(\lambda + 2r_1 h(b)), & \text{if } \frac{\tau}{\sigma} < \lambda + 2r_1(\log 2 - h(b)), \\ 1 + \frac{\lambda + 2r_1 h(b)}{\lambda + 2r_1(\log 2 - h(b))}, & \text{if } \frac{\tau}{\sigma} > \lambda + 2r_1(\log 2 - h(b)), \end{cases} \tag{31}$$

and $\lambda = \log\left(\frac{(r_1+r_2)^{r_1+r_2}}{r_1^{r_1} r_2^{r_2}}\right)$.

Theorem 15. *Let* $r_2 \geqslant r_1$ *be positive integers, and* $\alpha = a/b$, $a, b \in \mathbb{Z}$, $1 \leqslant a \leqslant b$, $(a,b) = 1$, *satisfying the inequality* $\lambda > 2r_1(b - \log 2)$*. Suppose that there exists a sequence of positive integers* n_k *such that* $\lim_{k\to\infty} n_k/k = \sigma$ *and*

$$\limsup_{k\to\infty} \frac{1}{k} \log \left| \left\{ d_{2br_1 n_k} L_{(r_1 n_k, r_2 n_k)}(\alpha) \right\} - \frac{u}{v} \right| \leqslant -\tau \tag{32}$$

for some integers u, v *and positive numbers* σ, τ*. Suppose further that* $\sigma(2r_1(b - \log 2) - \lambda)$ *is not the limit of any subsequence of* $(1/k)\log|\{d_{2br_1 n_k} L_{(r_1 n_k, r_2 n_k)}(\alpha)\} - u/v|$*. Then* γ_α *has irrationality exponent* $\mu(\gamma_\alpha) \leqslant \mu_{b,\sigma,\tau}$*, where* $\mu_{b,\sigma,\tau}$ *and* λ *are given by* (31).

Remark. Note that the assertions of Theorems 14 and 15 are valid if we replace the fractional parts in their hypotheses by the distances to the nearest integers.

References

[1] K. Alladi, M. Robinson, Legendre polynomials and irrationality, J. Reine Angew. Math. 318 (1980) 137–155.
[2] G. Chudnovsky, Hermite–Padé approximations to exponential functions and elementary estimates of the measure of irrationality of π, in: Lecture Notes in Math., vol. 925, 1982, pp. 299–325.
[3] M. Hata, Rational approximations to π and some other numbers, Acta Arith. 63 (1993) 335–349.
[4] T. Hessami Pilehrood, Kh. Hessami Pilehrood, Criteria for irrationality of generalized Euler's constant, J. Number Theory 108 (2004) 169–185.
[5] J. Sondow, An irrationality measure for Liouville numbers and conditional measures for Euler's constant, arXiv.org: math.NT/0307308, July 2003.
[6] J. Sondow, Irrationality measures, irrationality bases, and a theorem of Jarník, arXiv.org: math.NT/0406300, June 2004.