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Addendum

Addendum to "Distribution of points on the circle" [J. Number Theory 127 (1) (2007) 127–135]

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Abstract

In an earlier paper we gave a result relating the value of an exponential sum with the distribution of summands in the arcs of the unit circle in the complex plane. In a marginal case, this result is superseded by a result of Lev. In this addendum we use an idea of Lev to give a new proof of our earlier result and to improve upon the marginal result.
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Let n and N be non-negative integers, and let $\mathbb{U}$ be the unit circle in the complex plane. Let $\varphi \in (0, \pi]$ be a real number and suppose that the (not necessarily distinct) complex numbers $z_1, \ldots, z_N \in \mathbb{U}$ have the property that any open arc of $\mathbb{U}$ of length φ contains at most n of them. By [2, Theorem 4], we then have

$$|z_1 + \cdots + z_N| \leqslant (kn - N)\frac{\sin((k-1)\varphi/2)}{\sin(\varphi/2)} + \big(N - (k-1)n\big)\frac{\sin(k\varphi/2)}{\sin(\varphi/2)} \tag{1}$$

for any positive integer $k \leqslant 2\pi/\varphi$.

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Let $L_k(\varphi, n, N)$ denote the right-hand side of (1). A routine verification shows that if φ, n, and N satisfy $n \leqslant N \leqslant \lfloor 2\pi/\varphi \rfloor n$, then the minimum value of $L_k(\varphi, n, N)$ in the range $1 \leqslant k \leqslant 2\pi/\varphi$ is attained for $k = \lceil N/n \rceil$; consequently, in this case

$$|z_1 + \cdots + z_N| \leqslant L_\kappa(\varphi, n, N), \quad \kappa = \lceil N/n \rceil. \tag{2}$$

As shown in [2], this estimate is sharp. However, the assumptions only imply $N \leqslant (2\pi/\varphi)n$, and if $\lfloor 2\pi/\varphi \rfloor n < N \leqslant (2\pi/\varphi)n$, then [1, Theorem 1] is stronger than (2). In this note we use an idea of Lev [1] to give a (partially) new and simpler proof of (1) and to improve (1) for $\lfloor 2\pi/\varphi \rfloor n < N \leqslant (2\pi/\varphi)n$.

Without loss of generality, we can assume that $S := z_1 + \cdots + z_N$ is real and non-negative. Following Lev [1], for $\theta \in [-\pi, \pi]$, we denote by $K(\theta)$ the number of values of $j \in [1, N]$ for which there is a value of $\arg z_j$ satisfying $|\arg z_j - \theta| < \varphi/2$. Then we have (cf. Lev [1]),

$$\int_{-\pi}^{\pi} K(\theta)\, d\theta = \varphi N \tag{3}$$

and

$$\int_{-\pi}^{\pi} K(\theta) \cos\theta\, d\theta = 2S \sin(\varphi/2). \tag{4}$$

We have, of course, $0 \leqslant K(\theta) \leqslant n$, and the inequality $N \leqslant (2\pi/\varphi)n$ is immediate by (3).

We assume that there exist sequences $z_1, \ldots, z_N \in \mathbb{U}$ which satisfy the assumptions, but violate (1) for some positive integer $k \leqslant 2\pi/\varphi$. In [2, Section 4] we showed that in this case there is such a sequence with ($S = |S|$ and) all its points on the closed arc of $\mathbb{U}$ of length $(k-1)\varphi$ centered around 1. For this sequence we thus have

$$K(\theta) = 0 \quad \text{if } \frac{k\varphi}{2} \leqslant |\theta| \leqslant \pi. \tag{5}$$

In addition,

$$K(\theta) + (k-1)n \geqslant N \quad \text{if } \frac{(k-1)\varphi}{2} \leqslant |\theta| < \frac{k\varphi}{2}. \tag{6}$$

Now we have, using (3) and (4),

$$2S \sin(\varphi/2) = \int_{-\pi}^{\pi} K(\theta)\big(\cos\theta - \cos\big((k-1)\varphi/2\big)\big)\, d\theta + \varphi N \cos\big((k-1)\varphi/2\big).$$

We use (5), and split the integral into three:

$$\int_{-k\varphi/2}^{-(k-1)\varphi/2} + \int_{-(k-1)\varphi/2}^{(k-1)\varphi/2} + \int_{(k-1)\varphi/2}^{k\varphi/2}.$$

On the three integration intervals, $\cos\theta - \cos((k-1)\varphi/2)$ is non-positive, non-negative, and non-positive, respectively. Using the inequalities $K(\theta) \leqslant n$ and (6), we obtain

$$\begin{aligned} 2S\sin(\varphi/2) \leqslant & \int_{-k\varphi/2}^{-(k-1)\varphi/2} \big(N-(k-1)n\big)\big(\cos\theta - \cos\big((k-1)\varphi/2\big)\big)\,d\theta \\ & + \int_{-(k-1)\varphi/2}^{(k-1)\varphi/2} n\big(\cos\theta - \cos\big((k-1)\varphi/2\big)\big)\,d\theta \\ & + \int_{(k-1)\varphi/2}^{k\varphi/2} \big(N-(k-1)n\big)\big(\cos\theta - \cos\big((k-1)\varphi/2\big)\big)\,d\theta \\ & + \varphi N\cos\big((k-1)\varphi/2\big). \end{aligned}$$

It is now easy to see that

$$2S\sin(\varphi/2) \leqslant 2(kn-N)\sin\big((k-1)\varphi/2\big) + 2\big(N-(k-1)n\big)\sin(k\varphi/2),$$

so that (1) holds, a contradiction. This completes the new proof of (1).

Next, assume that $z_1, \ldots, z_N \in \mathbb{U}$ satisfy the assumptions, and that $k = \lfloor 2\pi/\varphi \rfloor$. Then, for any real θ, one can cover $\mathbb{U}$ by $k+1$ open arcs of length φ with one of the arcs centered around $\exp(i\theta)$. Hence,

$$K(\theta) + kn \geqslant N. \tag{7}$$

Moreover, we have $N \leqslant 2\pi n/\varphi < (k+1)n$, and, putting

$$c = \frac{\pi n - \varphi N/2}{(k+1)n - N}, \tag{8}$$

we see that $0 \leqslant c < \varphi/2$. Using (3) and (4), we get

$$2S\sin(\varphi/2) = \int_{-\pi}^{\pi} K(\theta)(\cos\theta + \cos c)\,d\theta - \varphi N\cos c.$$

We split the integral into three:

$$\int_{-\pi}^{-\pi+c} + \int_{-\pi+c}^{\pi-c} + \int_{\pi-c}^{\pi}.$$

On the three integration intervals, $\cos\theta + \cos c$ is non-positive, non-negative, and non-positive, respectively. Consequently, the inequalities $K(\theta) \leqslant n$ and (7) give

$$2S\sin(\varphi/2) \leqslant \int_{-\pi}^{-\pi+c} (N-kn)(\cos\theta+\cos c)\,d\theta + \int_{-\pi+c}^{\pi-c} n(\cos\theta+\cos c)\,d\theta + \int_{\pi-c}^{\pi} (N-kn)(\cos\theta+\cos c)\,d\theta - \varphi N\cos c,$$

and it follows easily that

$$2S\sin(\varphi/2) \leqslant 2\big((k+1)n-N\big)\sin c.$$

Thus we have the result

$$|z_1+\cdots+z_N| \leqslant \big((k+1)n-N\big)\frac{\sin c}{\sin(\varphi/2)},$$

for c given by (8). It can be shown that this bound is stronger than [1, Theorem 1] on the interval $\lfloor 2\pi/\varphi\rfloor n < N < (2\pi/\varphi)n$.

References

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