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Indices of inseparability in towers of field extensions

Kevin Keating

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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A B S T R A C T

Let K be a local field whose residue field has characteristic p and let L/K be a finite separable totally ramified extension of degree $n = ap^\nu$. The indices of inseparability $i_0, i_1, \dots, i_\nu$ of L/K were defined by Fried in the case $\mathrm{char}(K) = p$ and by Heiermann in the case $\mathrm{char}(K) = 0$; they give a refinement of the usual ramification data for L/K. The indices of inseparability can be used to construct "generalized Hasse–Herbrand functions" $\phi^j_{L/K}$ for $0 \le j \le \nu$. In this paper we give an interpretation of the values $\phi^j_{L/K}(c)$ for nonnegative integers c. We use this interpretation to study the behavior of generalized Hasse–Herbrand functions in towers of field extensions.

1. Introduction

Let K be a local field whose residue field $\overline{K}$ is a perfect field of characteristic p, and let K^{sep} be a separable closure of K. Let L/K be a finite totally ramified subextension of K^{sep}/K. The *indices of inseparability* of L/K were defined by Fried [2] in the case $\mathrm{char}(K) = p$, and by Heiermann [5] in the case $\mathrm{char}(K) = 0$. The indices of inseparability of L/K determine the ramification data of L/K (as defined for instance in Chapter IV of [7]), but the ramification data does not always determine the indices of inseparability.

E-mail address: keating@ufl.edu.

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Therefore the indices of inseparability of L/K may be viewed as a refinement of the usual ramification data of L/K.

Let π_K, π_L be uniformizers for K, L. The most natural definition of the ramification data of L/K is based on the valuations of $\sigma(\pi_L) - \pi_L$ for K-embeddings $\sigma : L \to K^{sep}$; this is the approach taken in Serre's book [7]. The ramification data can also be defined in terms of the relation between the norm map $\mathrm{N}_{L/K}$ and the filtrations of the unit groups of L and K, as in Fesenko–Vostokov [1]. This approach can be used to derive the well-known relation between higher ramification theory and class field theory. Finally, the ramification data can be computed by expressing π_K as a power series in π_L with coefficients in the set R of Teichmüller representatives for $\overline{K}$. This third approach, which is used by Fried and Heiermann, makes clear the connection between ramification data and the indices of inseparability.

Heiermann [5] defined "generalized Hasse–Herbrand functions" $\phi^j_{L/K}$ for $0 \le j \le \nu$. In Section 2 we give an interpretation of the values $\phi^j_{L/K}(c)$ of these functions at non-negative integers c. This leads to an alternative definition of the indices of inseparability which is closely related to the third method for defining the ramification data. In Section 3 we consider a tower of finite totally ramified separable extensions $M/L/K$. We use our interpretation of the values $\phi^j_{L/K}(c)$ to study the relations between the generalized Hasse–Herbrand functions of L/K, M/L, and M/K.

Notation

$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$
v_p = p-adic valuation on $\mathbb{Z}$
K = local field with perfect residue field $\overline{K}$ of characteristic $p > 0$
K^{sep} = separable closure of K
v_K = valuation on K^{sep} normalized so that $v_K(K^\times) = \mathbb{Z}$
$\mathcal{O}_K = \{\alpha \in K : v_K(\alpha) \ge 0\}$ = ring of integers of K
π_K = uniformizer for K
$\mathcal{M}_K = \pi_K \mathcal{O}_K$ = maximal ideal of $\mathcal{O}_K$
R = set of Teichmüller representatives for $\overline{K}$
L/K = finite totally ramified subextension of K^{sep}/K of degree $n > 1$, with $v_p(n) = \nu$
M/L = finite totally ramified subextension of K^{sep}/L of degree $m > 1$, with $v_p(m) = \mu$
v_K, $\mathcal{O}_K$, π_K, and $\mathcal{M}_K$ have natural analogs for L and M

2. Generalized Hasse–Herbrand functions

We begin by recalling the definition of the indices of inseparability i_j $(0 \le j \le \nu)$ for a nontrivial totally ramified separable extension L/K of degree $n = ap^\nu$, as formulated by Heiermann [5]. Let $R \subset \mathcal{O}_K$ be the set of Teichmüller representatives for $\overline{K}$. Then there is a unique series $\hat{\mathcal{F}}(X) = \sum_{h=0}^{\infty} a_h X^{h+n}$ with coefficients in R such that $\pi_K = \hat{\mathcal{F}}(\pi_L)$. For $0 \le j \le \nu$ set

$$\tilde{\imath}_j = \min\{h \geq 0 : v_p(h+n) \leq j,\ a_h \neq 0\}. \tag{2.1}$$

If $\mathrm{char}(K) = 0$ it may happen that $a_h = 0$ for all $h \geq 0$ such that $v_p(h+n) \leq j$, in which case we set $\tilde{\imath}_j = \infty$. The indices of inseparability are defined recursively in terms of $\tilde{\imath}_j$ by $i_\nu = \tilde{\imath}_\nu = 0$ and $i_j = \min\{\tilde{\imath}_j, i_{j+1} + v_L(p)\}$ for $j = \nu - 1, \dots, 1, 0$. Thus

$$i_j = \min\{\tilde{\imath}_{j_1} + (j_1 - j)v_L(p) : j \leq j_1 \leq \nu\}. \tag{2.2}$$

It follows from the definitions that $0 = i_\nu < i_{\nu-1} \leq i_{\nu-1} \leq \cdots \leq i_0$. If $\mathrm{char}(K) = p$ then $v_L(p) = \infty$, so $i_j = \tilde{\imath}_j$ in this case. If $\mathrm{char}(K) = 0$ then $\tilde{\imath}_j$ can depend on the choice of π_L, and it is not obvious that i_j is a well-defined invariant of the extension L/K. We will have more to say about this issue in Remark 2.5.

Following [5, (4.4)], for $0 \leq j \leq \nu$ we define functions $\tilde{\phi}^j_{L/K} : [0,\infty) \to [0,\infty)$ by $\tilde{\phi}^j_{L/K}(x) = i_j + p^j x$. The generalized Hasse–Herbrand functions $\phi^j_{L/K} : [0,\infty) \to [0,\infty)$ are then defined by

$$\phi^j_{L/K}(x) = \min\{\tilde{\phi}^{j_0}_{L/K}(x) : 0 \leq j_0 \leq j\}. \tag{2.3}$$

Hence we have $\phi^j_{L/K}(x) \leq \phi^{j'}_{L/K}(x)$ for $0 \leq j' \leq j$. Let $\phi_{L/K} : [0,\infty) \to [0,\infty)$ be the usual Hasse–Herbrand function, as defined for instance in Chapter IV of [7]. Then by [5, Cor. 6.11] we have $\phi^\nu_{L/K}(x) = n\phi_{L/K}(x)$.

In order to reformulate the definition of $\phi^j_{L/K}(x)$ we will use the following elementary fact about binomial coefficients, which is proved in [5, Lemma 5.6].

Lemma 2.1. *Let $b \geq c \geq 1$. Then $v_p(\binom{b}{c}) \geq v_p(b) - v_p(c)$, with equality if $v_p(b) \geq v_p(c)$ and c is a power of p.*

Proposition 2.2. *For $0 \leq j \leq \nu$ and $x \geq 0$ we have*

$$\phi^j_{L/K}(x) = \min\left\{h + v_L\left(\binom{h+n}{p^{j_0}}\right) + p^{j_0}x : 0 \leq j_0 \leq j,\ a_h \neq 0\right\}.$$

Proof. Using (2.1)–(2.3) we get

$$\begin{aligned}\phi^j_{L/K}(x) = \min\{h + (j_1 - j_0)v_L(p) + p^{j_0}x : 0 \leq j_0 \leq j,\ j_0 \leq j_1 \leq \nu,\\ v_p(h+n) \leq j_1,\ a_h \neq 0\}.\end{aligned}$$

If $j_0 > v_p(h+n)$ then we can replace j_0 with j_0-1 and j_1 with j_1-1 without increasing the value of $h+(j_1-j_0)v_L(p)+p^{j_0}x$. Hence we may assume $j_0 \leq v_p(h+n)$ and $j_1 = v_p(h+n)$. It follows that

$$\begin{aligned}\phi^j_{L/K}(x) &= \min\big\{h + \big(v_p(h+n) - j_0\big)v_L(p) + p^{j_0}x : 0 \le j_0 \le j,\ j_0 \le v_p(h+n),\ a_h \ne 0\big\} \\ &= \min\left\{h + v_L\left(\binom{h+n}{p^{j_0}}\right) + p^{j_0}x : 0 \le j_0 \le j,\ j_0 \le v_p(h+n),\ a_h \ne 0\right\} \\ &= \min\left\{h + v_L\left(\binom{h+n}{p^{j_0}}\right) + p^{j_0}x : 0 \le j_0 \le j,\ a_h \ne 0\right\},\end{aligned}$$

where the second and third equalities follow from Lemma 2.1. □

For $d \ge 0$ set $B_d = \mathcal{O}_L/\mathcal{M}_L^{n+d}$ and let $A_d = (\mathcal{O}_K + \mathcal{M}_L^{n+d})/\mathcal{M}_L^{n+d}$ be the image of $\mathcal{O}_K$ in B_d. For $0 \le j \le \nu$ set $B_d[\epsilon_j] = B_d[\epsilon]/(\epsilon^{p^{j+1}})$, so that $\epsilon_j = \epsilon + (\epsilon^{p^{j+1}})$ satisfies $\epsilon_j^{p^{j+1}} = 0$.

Proposition 2.3. *Let $0 \le j \le \nu$, let $d \ge c \ge 0$, and let $u \in \mathcal{O}_L[\epsilon_j]^\times$. Choose $F(X) \in X^n \cdot \mathcal{O}_K[[X]]$ such that $F(\pi_L) = \pi_K$. Then the following are equivalent:*

1. $F(\pi_L + u\pi_L^{c+1}\epsilon_j) \equiv \pi_K \pmod{\pi_L^{n+d}}$.
2. *There exists an A_d-algebra homomorphism $s_d : B_d \to B_d[\epsilon_j]$ such that $s_d(\pi_L) = \pi_L + u\pi_L^{c+1}\epsilon_j$.*
3. *There exists an A_d-algebra homomorphism $s_d : B_d \to B_d[\epsilon_j]$ such that*

$$\begin{aligned}s_d &\equiv \mathrm{id}_{B_d} \quad \big(\mathrm{mod}\ \pi_L^{c+1}\epsilon_j\big) \\ s_d &\not\equiv \mathrm{id}_{B_d} \quad \big(\mathrm{mod}\ \pi_L^{c+1}\epsilon_j \cdot (\pi_L, \epsilon_j)\big).\end{aligned}$$

Proof. Suppose Condition 1 holds. Let $\tilde{u}(X, \epsilon_j)$ be an element of $\mathcal{O}_K[[X]][\epsilon_j]$ such that $\tilde{u}(\pi_L, \epsilon_j) = u$. Since $F(0) = 0$ the Weierstrass polynomial of $F(X) - \pi_K$ is the minimum polynomial of π_L over K. Therefore $\mathcal{O}_L \cong \mathcal{O}_K[[X]]/(F(X) - \pi_K)$. It follows that the $\mathcal{O}_K$-algebra homomorphism $\tilde{s} : \mathcal{O}_K[[X]] \to \mathcal{O}_K[[X]][\epsilon_j]$ defined by $\tilde{s}(X) = X + \tilde{u}X^{c+1}\epsilon_j$ induces an A_d-algebra homomorphism $s_d : B_d \to B_d[\epsilon_j]$ such that $s_d(\pi_L) = \pi_L + u\pi_L^{c+1}\epsilon_j$. Therefore Condition 2 holds. On the other hand, if Condition 2 holds then applying the homomorphism s_d to the congruence $F(\pi_L) \equiv \pi_K \pmod{\pi_L^{n+d}}$ gives Condition 1. Hence the first two conditions are equivalent. Suppose Condition 2 holds. Since $d \ge c$ and $n \ge 2$ we see that s_d satisfies the requirements of Condition 3. Suppose Condition 3 holds. Then $s_d(\pi_L) = \pi_L + v\pi_L^{c+1}\epsilon_j$ for some $v \in B_d[\epsilon_j]^\times$. Let $\gamma : B_d[\epsilon_j] \to B_d[\epsilon_j]$ be the B_d-algebra homomorphism such that $\gamma(\epsilon_j) = uv^{-1}\epsilon_j$, and define $s'_d : B_d \to B_d[\epsilon_j]$ by $s'_d = \gamma \circ s_d$. Then s'_d satisfies the requirements of Condition 2. □

The assumptions on $F(X)$ imply that $F(\pi_L + u\pi_L^{c+1}\epsilon_j) \equiv \pi_K \pmod{\pi_L^{n+c}}$. Therefore the conditions of the proposition are satisfied when $d = c$. On the other hand, since L/K is separable we have $F(\pi_L + u\pi_L^{c+1}\epsilon_j) \ne \pi_K$. Hence for d sufficiently large the conditions in the proposition are not satisfied. We define a function $\Phi^j_{L/K} : \mathbb{N}_0 \to \mathbb{N}_0$ by setting $\Phi^j_{L/K}(c)$ equal to the largest integer d satisfying the equivalent conditions of

Proposition 2.3. By Condition 3 we see that this definition does not depend on the choice of π_L, u, or F.

We now show that $\Phi^j_{L/K}$ and $\phi^j_{L/K}$ agree on nonnegative integers. This gives an alternative description of the restriction of $\phi^j_{L/K}$ to $\mathbb{N}_0$ which does not depend on the indices of inseparability.

Proposition 2.4. *For $c \in \mathbb{N}_0$ we have $\Phi^j_{L/K}(c) = \phi^j_{L/K}(c)$.*

Proof. Let $c \in \mathbb{N}_0$. Since $\hat{\mathcal{F}}(X)$ satisfies the hypotheses for $F(X)$ in Proposition 2.3, $\Phi^j_{L/K}(c)$ is equal to the largest $d \in \mathbb{N}_0$ such that

$$\hat{\mathcal{F}}\big(\pi_L + \pi_L^{c+1}\epsilon_j\big) \equiv \hat{\mathcal{F}}(\pi_L) \pmod{\pi_L^{n+d}}. \tag{2.4}$$

For $m \geq 0$ define

$$\big(D^m\hat{\mathcal{F}}\big)(X) = \sum_{h=0}^{\infty} \binom{h+n}{m} a_h X^{h+n-m}.$$

Then

$$\hat{\mathcal{F}}\big(X + \epsilon_j X^{c+1}\big) = \sum_{m=0}^{p^{j+1}-1} \big(D^m\hat{\mathcal{F}}\big)(X) \cdot \big(\epsilon_j X^{c+1}\big)^m.$$

Since $\epsilon_j, \epsilon_j^2, \dots, \epsilon_j^{p^{j+1}-1}$ are linearly independent over $\mathcal{O}_L$, (2.4) holds if and only if

$$\big(D^m\hat{\mathcal{F}}\big)(\pi_L) \cdot \pi_L^{(c+1)m} \in \mathcal{M}_L^{n+d} \quad \text{for } 1 \leq m < p^{j+1}. \tag{2.5}$$

Hence by Proposition 2.2 it is sufficient to prove that (2.5) is equivalent to the following:

$$h + v_L\left(\binom{h+n}{p^{j_0}}\right) + cp^{j_0} \geq d \quad \text{for all } j_0, h \text{ such that } 0 \leq j_0 \leq j \text{ and } a_h \neq 0. \tag{2.6}$$

Assume first that (2.6) holds. Choose m such that $1 \leq m < p^{j+1}$ and write $m = rp^{j_0}$ with $p \nmid r$ and $j_0 \leq j$. Choose $h \geq 0$ such that $a_h \neq 0$ and set $l = v_p(h+n)$. If $m > h+n$ then $\binom{h+n}{m} = 0$, so we have

$$\binom{h+n}{m} a_h \pi_L^{h+n-m} \cdot \pi_L^{(c+1)m} \in \mathcal{M}_L^{n+d}. \tag{2.7}$$

Suppose $m \leq h+n$ and $l \geq j_0$. Using Lemma 2.1 we get

$$v_p\left(\binom{h+n}{m}\right) \geq l - j_0 = v_p\left(\binom{h+n}{p^{j_0}}\right).$$

Combining this with (2.6) we get

$$h + v_L\left(\binom{h+n}{m}\right) + cm + n \geq h + v_L\left(\binom{h+n}{p^{j_0}}\right) + cp^{j_0} + n \geq n + d.$$

Hence (2.7) holds in this case. Finally, suppose $m \leq h+n$ and $l < j_0 \leq j$. It follows from Lemma 2.1 that $v_L\left(\binom{h+n}{p^l}\right) = 0$, so by (2.6) we have $h + cp^l \geq d$. Since $m \geq p^{j_0} > p^l$ we get

$$h + v_L\left(\binom{h+n}{m}\right) + cm + n \geq h + cp^l + n \geq n + d.$$

Therefore (2.7) holds in this case as well. It follows that every term in $(D^m\hat{\mathcal{F}})(\pi_L)$ lies in $\mathcal{M}_L^{n+d}$, so (2.5) holds.

Assume conversely that (2.5) holds. Among all the nonzero terms that occur in any of the series

$$\left(D^{p^i}\hat{\mathcal{F}}\right)(\pi_L) \cdot \pi_L^{(c+1)p^i} = \sum_{h=0}^{\infty} a_h \binom{h+n}{p^i} \pi_L^{h+n+cp^i}$$

for $0 \leq i \leq j$ let $a_h\binom{h+n}{p^i}\pi_L^{n+h+cp^i}$ be a term whose L-valuation w is minimum. If $\mathrm{char}(K) = p$ then for each $m \geq 1$ the nonzero terms of $(D^m\hat{\mathcal{F}})(\pi_L)$ have distinct L-valuations, so it follows from (2.5) that $w \geq n + d$. Suppose $\mathrm{char}(K) = 0$ and set $l = v_p(h+n)$. If $i > l$ then since $v_L\left(\binom{h+n}{p^l}\right) = 0$ we have

$$v_L\left(\binom{h+n}{p^l}\pi_L^{n+h+cp^l}\right) \leq v_L\left(\binom{h+n}{p^i}\pi_L^{n+h+cp^i}\right) = w.$$

Therefore we may assume $i \leq l$. Since $v_p\left(\binom{n}{p^i}\right) = \nu - i$ and $a_0 \neq 0$ we have $l \leq \nu$. Suppose $w < n + d$. Then it follows from (2.5) that there is $h' \neq h$ such that $a_{h'} \neq 0$ and

$$v_L\left(\binom{h'+n}{p^i}\pi_L^{n+h'+cp^i}\right) = v_L\left(\binom{h+n}{p^i}\pi_L^{n+h+cp^i}\right). \tag{2.8}$$

Since $n \mid v_L(p)$ this implies $h' \equiv h \pmod{n}$. Since $v_p(h+n) \leq \nu$ and $v_p(h'+n) \leq \nu$ we get $v_p(h'+n) = v_p(h+n) = l$. Therefore by Lemma 2.1 we have

$$v_p\left(\binom{h'+n}{p^i}\right) = v_p\left(\binom{h+n}{p^i}\right) = l - i.$$

Combining this with (2.8) gives $h' = h$, a contradiction. Therefore $w \geq n + d$ holds in general. Hence by the minimality of w we get (2.6). $\square$

Remark 2.5. If $\mathrm{char}(K) = 0$ then the value of $\tilde{i}_j$ may depend on the choice of uniformizer π_L for L. It was proved in [5, Th. 7.1] that i_j is a well-defined invariant of the extension L/K. This can also be deduced from Proposition 2.4 by setting $c = 0$.

Remark 2.6. Let $0 \le j \le \nu$. Even though the function $\phi^j_{L/K} : [0, \infty) \to [0, \infty)$ may not be determined by its restriction to $\mathbb{N}_0$, it is determined by the sequence $(i_0, i_1, \dots, i_j)$. Since $i_{j_0} = \phi^{j_0}_{L/K}(0)$ this implies that the collection consisting of the restrictions of $\phi^{j_0}_{L/K}$ to $\mathbb{N}_0$ for $0 \le j_0 \le j$ determines $\phi^j_{L/K}$.

For $0 \le j \le \nu$ let $B_d[\bar{\epsilon}_j] = B_d[\epsilon]/(\epsilon^{p^j+1})$, so that $\bar{\epsilon}_j = \epsilon + (\epsilon^{p^j+1})$ satisfies $\bar{\epsilon}_j^{p^j+1} = 0$. Define $\bar{\Phi}^j_{L/K} : \mathbb{N}_0 \to \mathbb{N}_0$ analogously to $\Phi^j_{L/K}$, using $\bar{\epsilon}_j$ in place of ϵ_j. Then the arguments in this section remain valid with ϵ_j, $\Phi^j_{L/K}$ replaced by $\bar{\epsilon}_j$, $\bar{\Phi}^j_{L/K}$. (In particular, note that the proof that (2.5) implies (2.6) only uses the fact that (2.5) holds with $m = p^i$ for $0 \le i \le j$.) Hence by Propositions 2.3 and 2.4 and their analogs for $\bar{\epsilon}_j, \bar{\Phi}^j_{L/K}$ we get the following:

Corollary 2.7. *Let $c, d \in \mathbb{N}_0$, let $u \in \mathcal{O}_L[\epsilon_j]^\times$, and let $\bar{u} \in \mathcal{O}_L[\bar{\epsilon}_j]^\times$. Choose $F(X) \in X^n \cdot \mathcal{O}_K[[X]]$ such that $F(\pi_L) = \pi_K$. Then the following are equivalent:*

1. $\phi^j_{L/K}(c) \ge d$,
2. $F(\pi_L + u\pi_L^{c+1}\epsilon_j) \equiv F(\pi_L) \pmod{\pi_L^{n+d}}$,
3. $F(\pi_L + \bar{u}\pi_L^{c+1}\bar{\epsilon}_j) \equiv F(\pi_L) \pmod{\pi_L^{n+d}}$.

Some of the proofs in Section 3 depend on "tame shifts":

Lemma 2.8. *Let π_L be a uniformizer for L and choose a uniformizer π_K for K such that $\pi_K \equiv \pi_L^n \pmod{\pi_L^{n+1}}$. Let $e \ge 1$ be relatively prime to $p[L : K] = pn$ and let $\pi_{K_e} \in K^{sep}$ be a root of $X^e - \pi_K$. Set $K_e = K(\pi_{K_e})$ and $L_e = LK_e$. Then*

1. *K_e/K and L_e/L are totally ramified extensions of degree e.*
2. *There is a uniformizer π_{L_e} for L_e such that $\pi_{L_e}^e = \pi_L$ and $\pi_{L_e}^n \equiv \pi_{K_e} \pmod{\pi_{L_e}^{n+1}}$.*
3. *Let $F(X) \in X^n \cdot \mathcal{O}_K[[X]]$ be such that $F(\pi_L) = \pi_K$. Then we can define a series $F_e(X) = F(X^e)^{1/e}$ with coefficients in $\mathcal{O}_K$ such that $F_e(\pi_{L_e}) = \pi_{K_e}$.*

Proof. Statement 1 is clear. Since e and n are relatively prime there are $s, t \in \mathbb{Z}$ such that $es + nt = 1$. Then $\tilde{\pi}_{L_e} = \pi_L^s \pi_{K_e}^t$ is a uniformizer for L_e with $\tilde{\pi}_{L_e}^e \equiv \pi_L \pmod{\tilde{\pi}_{L_e}^{e+1}}$ and $\tilde{\pi}_{L_e}^n \equiv \pi_{K_e} \pmod{\tilde{\pi}_{L_e}^{n+1}}$. Hence there is a 1-unit $v \in \mathcal{O}_{L_e}^\times$ such that $\pi_{L_e} = v\tilde{\pi}_{L_e}$ satisfies the requirements of Statement 2. To prove Statement 3 we note that since $\pi_{L_e}^n \equiv \pi_{K_e} \pmod{\pi_{L_e}^{n+1}}$, the coefficient a_0 in the series $F(X) = a_0X^n + a_1X^{n+1} + \dots$ is a 1-unit. Therefore we may define

$$F_e(X) = F\big(X^e\big)^{1/e} = \big(a_0X^{ne} + a_1X^{ne+e} + \dots\big)^{1/e} = a_0^{1/e}X^n\big(1 + a_0^{-1}a_1X^e + \dots\big)^{1/e},$$

where $a_0^{1/e}$ is the unique 1-unit in $\mathcal{O}_K$ whose eth power is a_0. Since $\pi_{K_e}^e = F(\pi_{L_e}^e)$ and $\pi_{L_e}^n \equiv \pi_{K_e} \pmod{\pi_{L_e}^{n+1}}$ we get $F_e(\pi_{L_e}) = \pi_{K_e}$. □

Lemma 2.9. *Let K_e, L_e be as in Lemma 2.8. Then for $x \geq 0$ and $0 \leq j \leq \nu$ we have*

$$\tilde{\phi}^j_{L_e/K_e}(x) = e\tilde{\phi}^j_{L/K}(x/e)$$
$$\phi^j_{L_e/K_e}(x) = e\phi^j_{L/K}(x/e).$$

Proof. It suffices to show that $ei_0, ei_1, \dots, ei_\nu$ are the indices of inseparability of L_e/K_e. By Proposition 2.4 this is equivalent to showing that $\Phi^j_{L_e/K_e}(0) = e\Phi^j_{L/K}(0)$. Let π_K, π_L, π_{K_e}, π_{L_e}, $F(X)$, $F_e(X)$ satisfy the conditions of Lemma 2.8. If $\Phi^j_{L/K}(0) \geq d$ then

$$\begin{aligned} F_e(\pi_{L_e} + \pi_{L_e}\epsilon_j)^e &= F\big(\pi_L(1+\epsilon_j)^e\big) \\ &\equiv F(\pi_L) \quad \big(\text{mod } \pi_L^{n+d}\big) \\ &\equiv F_e(\pi_{L_e})^e \quad \big(\text{mod } \pi_L^{n+d}\big). \end{aligned}$$

Since $F_e(X) = a_0^{1/e}X^n + \dots$ with $a_0^{1/e}$ a 1-unit, it follows that

$$F_e(\pi_{L_e} + \pi_{L_e}\epsilon_j) \equiv F_e(\pi_{L_e}) \quad \big(\text{mod } \pi_{L_e}^{n+de}\big).$$

Therefore $\Phi^j_{L_e/K_e}(0) \geq de$. Conversely, if $\Phi^j_{L_e/K_e}(0) \geq d$ then

$$\begin{aligned} F(\pi_L + \pi_L\epsilon_j) &= F_e\big(\pi_{L_e}(1+\epsilon_j)^{1/e}\big)^e \\ &\equiv F_e(\pi_{L_e})^e \quad \big(\text{mod } \pi_K \cdot \pi_{L_e}^d\big) \\ &\equiv F(\pi_L) \quad \big(\text{mod } \pi_L^{n+\lceil d/e\rceil}\big), \end{aligned}$$

and hence $\Phi^j_{L/K}(0) \geq \lceil d/e\rceil$. By combining these results we get $\Phi^j_{L_e/K_e}(0) = e\Phi^j_{L/K}(0)$. □

3. Towers of extensions

In this section we consider a tower $M/L/K$ of finite totally ramified subextensions of K^{sep}/K. Our goal is to determine relations between the generalized Hasse–Herbrand functions $\phi^l_{M/K}$ of the extension M/K and the corresponding functions for L/K and M/L. It is well-known that the indices of inseparability of L/K and M/L do not always determine the indices of inseparability of M/K (see for instance Example 5.8 in [3] or Remark 7.8 in [5]). Therefore we cannot expect to obtain a general formula which expresses $\phi^l_{M/K}$ in terms of $\phi^j_{L/K}$ and $\phi^k_{M/L}$. However, we do get a lower bound for $\phi^l_{M/K}(x)$, and we are able to show that this lower bound is equal to $\phi^l_{M/K}(x)$ in certain cases.

Set $[L:K]=n$, $[M:L]=m$, $\nu=v_p(n)$, and $\mu=v_p(m)$. Let π_K, π_L, π_M be uniformizers for K, L, M. Choose $F(X)\in X^n\cdot\mathcal{O}_K[[X]]$ such that $F(\pi_L)=\pi_K$ and define

$$F^*(\epsilon)=\pi_K^{-1}\big(F(\pi_L+\pi_L\epsilon)-\pi_K\big).$$

Then $F^*(\epsilon)\in\mathcal{O}_L[[\epsilon]]$ is uniquely determined by L/K up to multiplication by an element of $\mathcal{O}_L[[\epsilon]]^\times$.

Write $F^*(\epsilon)=c_1\epsilon+c_2\epsilon^2+\cdots$ and define the "valuation function" of F^* with respect to v_K by

$$\Psi^K_{F^*(\epsilon)}(x)=\min\big\{v_K(c_i)+ix:i\geq 1\big\} \tag{3.1}$$

for $x\in[0,\infty)$. The graph of $\Psi^K_{F^*(\epsilon)}$ is the Newton copolygon of $F^*(\epsilon)$ with respect to v_K. Gross [4, Lemma 1.5] attributes the following observation to Tate:

Proposition 3.1. *For $x\geq 0$ we have $\phi_{L/K}(x)=\Psi^K_{F^*(\epsilon)}(x)$.*

Suppose we also have $G(X)\in X^m\cdot\mathcal{O}_K[[X]]$ such that $G(\pi_M)=\pi_L$. Set $H(X)=F(G(X))$. Then $H(X)\in X^{nm}\cdot\mathcal{O}_K[[X]]$ satisfies $H(\pi_M)=\pi_K$. It follows that we can use the series

$$G^*(\epsilon)=\pi_L^{-1}\big(G(\pi_M+\pi_M\epsilon)-\pi_L\big)$$
$$H^*(\epsilon)=\pi_K^{-1}\big(H(\pi_M+\pi_M\epsilon)-\pi_K\big)$$

to compute the Hasse–Herbrand functions for the extensions M/L and M/K. As Lubin points out in [6, Th. 1.6], by applying Proposition 3.1 to the relation $H^*(\epsilon)=F^*(G^*(\epsilon))$, we obtain the well-known composition formula $\phi_{M/K}=\phi_{L/K}\circ\phi_{M/L}$.

We wish to extend the results above to apply to the generalized Hasse–Herbrand functions $\phi^j_{L/K}$. For $0\leq j\leq\nu$ let $F^*(\epsilon_j)$ denote the image of $F^*(\epsilon)$ in $\mathcal{O}_L[[\epsilon]]/(\epsilon^{p^{j+1}})\cong\mathcal{O}_L[\epsilon_j]$. Alternatively, we may view $F^*(\epsilon_j)$ as the polynomial obtained by discarding all the terms of $F^*(\epsilon)$ of degree $\geq p^{j+1}$. Therefore it makes sense to consider the valuation function $\Psi^L_{F^*(\epsilon_j)}(x)$ of $F^*(\epsilon_j)$.

Proposition 3.2. *$\phi^j_{L/K}(x)=\Psi^L_{F^*(\epsilon_j)}(x)$ for all $x\in[0,\infty)$.*

Proof. We first prove that $\phi^j_{L/K}$ and $\Psi^L_{F^*(\epsilon_j)}$ agree on $\mathbb{N}_0$. Let $d\geq b\geq 0$. Then $\Phi^j_{L/K}(b)\geq d$ if and only if $F^*(\pi_L^b\epsilon_j)\equiv 0\ (\mathrm{mod}\,\pi_L^d)$. By (3.1) this is equivalent to $\Psi^L_{F^*(\epsilon_j)}(b)\geq d$. Since $\Phi^j_{L/K}$ and $\Psi^L_{F^*(\epsilon_j)}$ map $\mathbb{N}_0$ to $\mathbb{N}_0$, this implies $\Phi^j_{L/K}(c)=\Psi^L_{F^*(\epsilon_j)}(c)$ for all $c\in\mathbb{N}_0$. Using Proposition 2.4 we deduce that $\phi^j_{L/K}(c)=\Psi^L_{F^*(\epsilon_j)}(c)$ for $c\in\mathbb{N}_0$.

Now choose $e \geq 1$ relatively prime to $p[L:K] = pn$. Let K_e, L_e, π_K, π_{K_e}, π_L, π_{L_e} satisfy the conditions of Lemma 2.8, and choose $F(X) \in X^n \cdot \mathcal{O}_K[[X]]$ such that $F(\pi_L) = \pi_K$. Then $F_e(X) = F(X^e)^{1/e}$ satisfies $F_e(\pi_{L_e}) = \pi_{K_e}$. Let

$$\begin{aligned} F_e^*(\epsilon) &= \pi_{K_e}^{-1}\big(F_e(\pi_{L_e} + \pi_{L_e}\epsilon) - \pi_{K_e}\big) \\ &= \big(1 + F^*\big((1+\epsilon)^e - 1\big)\big)^{1/e} - 1. \end{aligned}$$

Then $F_e^*(\epsilon) = \eta^{-1}(F^*(\eta(\epsilon)))$, where $\eta(\epsilon) = (1+\epsilon)^e - 1$ and $\eta^{-1}(\epsilon) = (1+\epsilon)^{1/e} - 1$ have coefficients in $\mathcal{O}_K$. It follows that for $0 \leq j \leq \nu$ we have $F_e^*(\epsilon_j) = \eta^{-1}(F^*(\eta(\epsilon_j)))$, so for $c \in \mathbb{N}_0$ we get

$$\Psi^{L_e}_{F_e^*(\epsilon_j)}(c) = \Psi^{L_e}_{F^*(\epsilon_j)}(c) = e\Psi^{L}_{F^*(\epsilon_j)}(c/e).$$

By Lemma 2.9 we have $\phi^j_{L/K}(c/e) = e^{-1}\phi^j_{L_e/K_e}(c)$. Since the proposition holds for the extension L_e/K_e with $x = c$ this implies

$$\phi^j_{L/K}(c/e) = e^{-1}\Psi^{L_e}_{F_e^*(\epsilon_j)}(c) = \Psi^{L}_{F^*(\epsilon_j)}(c/e).$$

Since the set $\{c/e : c, e \in \mathbb{N},\ \gcd(e, pn) = 1\}$ is dense in $[0,\infty)$, and $\phi^j_{L/K}$, $\Psi^L_{F^*(\epsilon_j)}$ are continuous on $[0,\infty)$, we conclude that $\phi^j_{L/K}(x) = \Psi^L_{F^*(\epsilon_j)}(x)$ for all $x \in [0,\infty)$. $\square$

Following [5, (4.4)], for $0 \leq j \leq \nu$ and $m \in \mathbb{N}$ we define functions on $[0,\infty)$ by

$$\begin{aligned} \tilde{\phi}^{j,m}_{L/K}(x) &= m\tilde{\phi}^{j}_{L/K}(x/m) = mi_j + p^j x \\ \phi^{j,m}_{L/K}(x) &= m\phi^{j}_{L/K}(x/m) = \min\big\{\tilde{\phi}^{j_0,m}_{L/K}(x) : 0 \leq j_0 \leq j\big\}. \end{aligned}$$

For $0 \leq l \leq \nu + \mu$ let

$$\Omega_l = \big\{(j,k) : 0 \leq j \leq \nu,\ 0 \leq k \leq \mu,\ j + k = l\big\},$$

and for $x \geq 0$ define

$$\begin{aligned} \lambda^l_{M/K}(x) &= \min\big\{\phi^{j,m}_{L/K}\big(\phi^k_{M/L}(x)\big) : (j,k) \in \Omega_l\big\} \\ &= \min\big\{\tilde{\phi}^{j,m}_{L/K}\big(\tilde{\phi}^k_{M/L}(x)\big) : (j,k) \in \Omega_{l_0} \text{ for some } 0 \leq l_0 \leq l\big\}. \end{aligned}$$

For $0 \leq a \leq l$ set

$$S^a_l(x) = \big\{(j,k) \in \Omega_a : \tilde{\phi}^{j,m}_{L/K}\big(\tilde{\phi}^k_{M/L}(x)\big) = \lambda^l_{M/K}(x)\big\}.$$

Theorem 3.3. *Let $0 \leq l \leq \nu + \mu$ and $x \in [0,\infty)$. Then*

(a) $\phi^l_{M/K}(x) \geq \lambda^l_{M/K}(x)$.
(b) *Suppose there exists $l_0 \leq l$ such that $|S^{l_0}_l(x)| = 1$. Then $\phi^l_{M/K}(x) = \lambda^l_{M/K}(x)$.*

The rest of the paper is devoted to proving this theorem. We first consider the cases where $x = c \in \mathbb{N}_0$. The proof in these cases is based on Proposition 2.4. To get information about $\Phi^l_{M/K}(c)$ we compute the most significant terms of $\hat{\mathcal{F}}(\hat{\mathcal{G}}(\pi_M + \pi_M^{c+1}\epsilon))$.

It follows from Proposition 2.4 that for $0 \le j \le \nu$ we have

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) \equiv \pi_K \pmod{\big(\pi_L^{n+i_j}, \epsilon^{p^{j+1}}\big)}.$$

In addition, since X^n divides $\hat{\mathcal{F}}(X)$ we have

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) \equiv \pi_K \pmod{\pi_L^n \epsilon}. \tag{3.2}$$

Hence

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) \equiv \pi_K \pmod{\pi_L^n \cdot \big(\pi_L^{i_j}, \epsilon^{p^{j+1}}\big)}. \tag{3.3}$$

Define an ideal in $\mathcal{O}_L[[\epsilon]]$ by

$$\begin{aligned} I_{\mathcal{F}} &= \big(\pi_L^{i_0}, \epsilon^{p^1}\big) \cap \big(\pi_L^{i_1}, \epsilon^{p^2}\big) \cap \cdots \cap \big(\pi_L^{i_\nu}, \epsilon^{p^{\nu+1}}\big) \cap (\epsilon) \\ &= \big(\pi_L^{i_0}\epsilon^{p^0}, \pi_L^{i_1}\epsilon^{p^1}, \ldots, \pi_L^{i_\nu}\epsilon^{p^\nu}\big). \end{aligned}$$

It follows from (3.2) and (3.3) that

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) \equiv \pi_K \pmod{\pi_L^n \cdot I_{\mathcal{F}}}. \tag{3.4}$$

Let $i'_0, i'_1, \ldots, i'_\mu$ be the indices of inseparability of M/L. As above we find that

$$\hat{\mathcal{G}}\big(\pi_M(1+\epsilon)\big) \equiv \pi_L \pmod{\pi_M^m \cdot I_{\mathcal{G}}},$$

where $I_{\mathcal{G}}$ is the ideal in $\mathcal{O}_M[[\epsilon]]$ defined by

$$I_{\mathcal{G}} = \big(\pi_M^{i'_0}\epsilon^{p^0}, \pi_M^{i'_1}\epsilon^{p^1}, \ldots, \pi_M^{i'_\mu}\epsilon^{p^\mu}\big).$$

By replacing ϵ with $\pi_M^c\epsilon$ we get

$$\hat{\mathcal{G}}\big(\pi_M\big(1+\pi_M^c\epsilon\big)\big) \equiv \pi_L \pmod{\pi_M^m \cdot I'_{\mathcal{G}}}, \tag{3.5}$$

where $I'_{\mathcal{G}}$ is the ideal in $\mathcal{O}_M[[\epsilon]]$ defined by

$$I'_{\mathcal{G}} = \big(\pi_M^{\tilde{\phi}^0_{M/L}(c)}\epsilon^{p^0}, \pi_M^{\tilde{\phi}^1_{M/L}(c)}\epsilon^{p^1}, \ldots, \pi_M^{\tilde{\phi}^\mu_{M/L}(c)}\epsilon^{p^\mu}\big).$$

It follows from (3.4) and (3.5) that there are $r_j, s_k \in R$, $\delta_{\mathcal{F}} \in (\pi_L, \epsilon) \cdot I_{\mathcal{F}}$, and $\delta_{\mathcal{G}} \in (\pi_M, \epsilon) \cdot I'_{\mathcal{G}}$ such that

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) = \pi_K \cdot \left(1 + \sum_{j=0}^{\nu} r_j \pi_L^{i_j} \epsilon^{p^j} + \delta_{\mathcal{F}}\right) \tag{3.6}$$

$$\hat{\mathcal{G}}\big(\pi_M\big(1+\pi_M^c\epsilon\big)\big) = \pi_L \cdot \left(1 + \sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}}\right). \tag{3.7}$$

Define an ideal in $\mathcal{O}_M[[\epsilon]]$ by

$$\begin{aligned} I_{\mathcal{F}\mathcal{G}} &= \big(\pi_M^{\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^k_{M/L}(c))} \epsilon^{p^{j+k}} : 0 \le j \le \nu,\ 0 \le k \le \mu\big) \\ &= \big(\pi_M^{\lambda^g_{M/K}(c)} \epsilon^{p^g} : 0 \le g \le \nu + \mu\big). \end{aligned}$$

Hence for $d \ge 0$ and $0 \le g \le \nu + \mu$ we have $\pi_M^d \epsilon^{p^g} \in I_{\mathcal{F}\mathcal{G}}$ if and only if $d \ge \lambda^g_{M/K}(c)$. We also define $u = \pi_L/\pi_M^m \in \mathcal{O}_M^\times$.

Lemma 3.4. *Let $0 \le j \le \nu$. Then*

$$\pi_L^{i_j} \left(\sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}}\right)^{p^j} \equiv u^{i_j} \sum_{k=0}^{\mu} s_k^{p^j} \pi_M^{\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^k_{M/L}(c))} \epsilon^{p^{j+k}} \pmod{(\pi_M, \epsilon) \cdot I_{\mathcal{F}\mathcal{G}}}.$$

Proof. For $0 \le j \le \nu$ define ideals in $\mathbb{Z}[X_0, X_1, \dots, X_\mu]$ by

$$H_j = \big(p^h X_k^{p^{j-h}} : 1 \le h \le j,\ 0 \le k \le \mu\big).$$

By induction on j we get

$$(X_0 + X_1 + \dots + X_\mu)^{p^j} \equiv X_0^{p^j} + X_1^{p^j} + \dots + X_\mu^{p^j} \pmod{H_j}.$$

Since both sides of this congruence are homogeneous polynomials of degree p^j, it follows that

$$(X_0 + X_1 + \dots + X_\mu)^{p^j} \equiv X_0^{p^j} + X_1^{p^j} + \dots + X_\mu^{p^j} \pmod{H_j'}, \tag{3.8}$$

where

$$H_j' = \big(p^h X_k^{p^{j-h}} X_w : 1 \le h \le j,\ 0 \le k \le \mu,\ 0 \le w \le \mu\big).$$

Since $\delta_{\mathcal{G}} \in (\pi_M, \epsilon) \cdot I_{\mathcal{G}}'$ there are $\tilde{s}_k \in \mathcal{O}_M[[\epsilon]]$ such that $\tilde{s}_k \equiv s_k \pmod{(\pi_M, \epsilon)}$ and

$$\sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}} = \sum_{k=0}^{\mu} \tilde{s}_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k}.$$

Hence by replacing X_k with $\tilde{s}_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k}$ for $0 \le k \le \mu$ in (3.8) we get

$$\left(\sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}}\right)^{p^j} \equiv \sum_{k=0}^{\mu} \tilde{s}_k^{p^j} \pi_M^{p^j \tilde{\phi}^k_{M/L}(c)} \epsilon^{p^{j+k}} \pmod{\epsilon \cdot A},$$

where A is the ideal in $\mathcal{O}_M[[\epsilon]]$ defined by

$$A = \left(p^h \left(\pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k}\right)^{p^{j-h}} : 1 \le h \le j,\ 0 \le k \le \mu\right).$$

Let $1 \le h \le j$ and $0 \le k \le \mu$. Since $i_j + h v_L(p) \ge i_{j-h}$ we have

$$\begin{aligned} v_M\left(\pi_L^{i_j} \cdot p^h \pi_M^{p^{j-h} \tilde{\phi}^k_{M/L}(c)}\right) &\ge m i_{j-h} + p^{j-h} \tilde{\phi}^k_{M/L}(c) \\ &= \tilde{\phi}^{j-h,m}_{L/K}\left(\tilde{\phi}^k_{M/L}(c)\right) \\ &\ge \lambda^{j-h+k}_{M/K}(c). \end{aligned}$$

It follows that $\pi_L^{i_j} \epsilon \cdot p^h (\pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k})^{p^{j-h}} \in \epsilon \cdot I_{\mathcal{FG}}$, and hence that $\pi_L^{i_j} \epsilon \cdot A \subset \epsilon \cdot I_{\mathcal{FG}}$. Therefore

$$\begin{aligned} \pi_L^{i_j} \left(\sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}}\right)^{p^j} &\equiv \pi_L^{i_j} \sum_{k=0}^{\mu} \tilde{s}_k^{p^j} \pi_M^{p^j \tilde{\phi}^k_{M/L}(c)} \epsilon^{p^{j+k}} \pmod{\epsilon \cdot I_{\mathcal{FG}}} \\ &\equiv u^{i_j} \sum_{k=0}^{\mu} \tilde{s}_k^{p^j} \pi_M^{\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^k_{M/L}(c))} \epsilon^{p^{j+k}} \pmod{\epsilon \cdot I_{\mathcal{FG}}}. \end{aligned}$$

Since $\tilde{s}_k \equiv s_k \pmod{(\pi_M, \epsilon)}$ the lemma follows. $\square$

We now replace ϵ with $\sum_{k=0}^{\mu} s_k \pi_M^{\tilde{\phi}^k_{M/L}(c)} \epsilon^{p^k} + \delta_{\mathcal{G}}$ in (3.6). With the help of Lemma 3.4 we get

$$\begin{aligned} \hat{\mathcal{F}}\left(\hat{\mathcal{G}}\left(\pi_M\left(1 + \pi_M^c \epsilon\right)\right)\right) &= \pi_K \cdot \left(1 + \sum_{j=0}^{\nu} r_j u^{i_j} \sum_{k=0}^{\mu} s_k^{p^j} \pi_M^{\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^k_{M/L}(c))} \epsilon^{p^{j+k}} + \delta_{\mathcal{FG}}\right) \\ &= \pi_K \cdot \left(1 + \sum_{g=0}^{\nu+\mu} \left(\sum_{(j,k) \in \Omega_g} u^{i_j} r_j s_k^{p^j} \pi_M^{\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^k_{M/L}(c))}\right) \epsilon^{p^g} + \delta_{\mathcal{FG}}\right) \end{aligned} \tag{3.9}$$

for some $\delta_{\mathcal{FG}} \in (\pi_M, \epsilon) \cdot I_{\mathcal{FG}}$.

To prove (a) in the case $x = c \in \mathbb{N}_0$ we define an ideal $J_l = (\pi_M^{nm + \lambda^l_{M/K}(c)}, \epsilon^{p^{l+1}})$ in $\mathcal{O}_M[[\epsilon]]$. Since $\pi_K \cdot I_{\mathcal{FG}} \subset J_l$, by (3.9) we get

$$\hat{\mathcal{F}}\left(\hat{\mathcal{G}}\left(\pi_M\left(1 + \pi_M^c \epsilon\right)\right)\right) \equiv \pi_K \pmod{J_l}.$$

It follows from Corollary 2.7 that $\phi^l_{M/K}(c) \ge \lambda^l_{M/K}(c)$.

Now let $e \geq 1$ be relatively prime to $p[M:K] = pnm$. Let π_M be a uniformizer for M, and choose uniformizers π_L, π_K for L, K such that $\pi_L \equiv \pi_M^m \pmod{\pi_M^{m+1}}$ and $\pi_K \equiv \pi_L^n \pmod{\pi_L^{n+1}}$; then $\pi_K \equiv \pi_M^{nm} \pmod{\pi_M^{nm+1}}$. Let $\pi_{K_e} \in K^{sep}$ be a root of $X^e - \pi_K$ and set $K_e = K(\pi_{K_e})$, $L_e = LK_e$, and $M_e = MK_e$. Let $0 \leq h \leq \nu$, $0 \leq i \leq \mu$, and $0 \leq l \leq \nu + \mu$. Then by Lemma 2.9 we get

$$\tilde{\phi}^i_{M/L}(x) = e^{-1}\tilde{\phi}^i_{M_e/L_e}(ex) \tag{3.10}$$

$$\tilde{\phi}^{h,m}_{L/K}(x) = e^{-1}\tilde{\phi}^{h,m}_{L_e/K_e}(ex) \tag{3.11}$$

$$\phi^l_{M/K}(x) = e^{-1}\phi^l_{M_e/K_e}(ex) \tag{3.12}$$

$$\lambda^l_{M/K}(x) = e^{-1}\lambda^l_{M_e/K_e}(ex). \tag{3.13}$$

We know from the preceding paragraph that $\phi^l_{M_e/K_e}(c) \geq \lambda^l_{M_e/K_e}(c)$ for every $c \in \mathbb{N}_0$. By applying (3.12) and (3.13) with $x = c/e$ we get $\phi^l_{M/K}(c/e) \geq \lambda^l_{M/K}(c/e)$. It follows that (a) holds whenever $x = c/e$ with $c \geq 0$, $e \geq 1$, and $\gcd(e, pnm) = 1$. Since numbers of this form are dense in $[0, \infty)$, by continuity we get $\phi^l_{M/K}(x) \geq \lambda^l_{M/K}(x)$ for all $x \geq 0$. This proves (a).

To facilitate the proof of (b) we define a subset of the nonnegative reals by

$$T_l(M/K) = \left\{t \geq 0 : \exists\, l_0 \leq l \text{ with } \left|S_l^{l_0}(t)\right| = 1 \text{ and } \left|S_l^a(t)\right| = 0 \text{ for } 0 \leq a < l_0\right\}. \tag{3.14}$$

Suppose $t > 0$ and $(t, \lambda^l_{M/K}(t))$ is not a vertex of the graph of $\lambda^l_{M/K}$. Then there is a unique $0 \leq l_0 \leq l$ such that $|S_l^{l_0}(t)| \geq 1$; in fact, l_0 is determined by the condition $(\lambda^l_{M/K})'(t) = p^{l_0}$. Hence if the hypotheses of (b) are satisfied with $x = t$ then $t \in T_l(M/K)$.

Lemma 3.5. *Suppose the hypotheses of (b) are satisfied with $x = 0$. Then $0 \in T_l(M/K)$.*

Proof. Suppose $0 \notin T_l(M/K)$, and let l_0 be the minimum integer satisfying the hypotheses of (b) with $x = 0$. Also let $l_1 < l_0$ be maximum such that $|S_l^{l_1}(0)| \neq 0$. Then $|S_l^{l_1}(0)| \geq 2$. Hence there is $(j,k) \in S_l^{l_1}(0)$ such that $k < \mu$. Since

$$\tilde{\phi}^{k+1}_{M/L}(0) = i'_{k+1} \leq i'_k = \tilde{\phi}^k_{M/L}(0)$$

we get

$$\lambda^l_{M/K}(0) \leq \tilde{\phi}^{j,m}_{L/K}\big(\tilde{\phi}^{k+1}_{M/L}(0)\big) \leq \tilde{\phi}^{j,m}_{L/K}\big(\tilde{\phi}^{k}_{M/L}(0)\big) = \lambda^l_{M/K}(0).$$

It follows that $\tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^{k+1}_{M/L}(0)) = \tilde{\phi}^{j,m}_{L/K}(\tilde{\phi}^{k}_{M/L}(0))$, so we have $i'_k = i'_{k+1}$ and $(j, k+1) \in S_l^{l_1+1}(0)$. Hence by the maximality of l_1 we get $l_1 = l_0 - 1$. Since $|S_l^{l_0}(0)| = 1$ we must have $|S_l^{l_0-1}(0)| = 2$ and $(l_0 - \mu - 1, \mu) \in S_l^{l_0-1}(0)$. Since $\tilde{\phi}^{\mu}_{M/L}(0) = 0$ we have

$$mi_{l_0-\mu} \le mi_{l_0-\mu-1} = \tilde{\phi}_{L/K}^{l_0-\mu-1,m}\big(\tilde{\phi}_{M/L}^{\mu}(0)\big) = \lambda_{M/K}^{l}(0) \le \tilde{\phi}_{L/K}^{l_0-\mu,m}\big(\tilde{\phi}_{M/L}^{\mu}(0)\big) = mi_{l_0-\mu}$$

and hence $\lambda_{M/K}^{l}(0) = \tilde{\phi}_{L/K}^{l_0-\mu,m}(\tilde{\phi}_{M/L}^{\mu}(0))$. Thus $(l_0-\mu,\mu) \in S_l^{l_0}(0)$. Since $(j,k+1) \in S_l^{l_0}(0)$, and $|S_l^{l_0}(0)| = 1$, we get $k+1 = \mu$, and hence $i'_{\mu-1} = i'_k = i'_{k+1} = i'_\mu = 0$. Since $i'_{\mu-1} > i'_\mu = 0$, this is a contradiction. Therefore $0 \in T_l(M/K)$. $\square$

Lemma 3.6. *Let $c \in \mathbb{N}_0 \cap T_l(M/K)$, let l_0 be the integer specified by (3.14) for $t = c$, and let (j,k) be the unique element of Ω_{l_0} such that $\lambda_{M/K}^{l}(c) = \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^{k}(c))$. Then r_j and s_k are nonzero.*

Proof. Since $c \in T_l(M/K)$, for $0 \le j' < j$ we have $\tilde{\phi}_{L/K}^{j',m}(\tilde{\phi}_{M/L}^{k}(c)) > \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^{k}(c))$. It follows that $i_{j'} > i_j$, and hence that $\pi_K \cdot (\pi_L, \epsilon) \cdot I_{\mathcal{F}} \subset (\pi_L^{n+i_j+1}, \epsilon^{p^j+1})$. Therefore by (3.6) we get

$$\hat{\mathcal{F}}\big(\pi_L(1+\epsilon)\big) \equiv \pi_K \cdot \big(1 + r_j \pi_L^{i_j} \epsilon^{p^j}\big) \quad \big(\mathrm{mod}\ \big(\pi_L^{n+i_j+1}, \epsilon^{p^j+1}\big)\big).$$

If $r_j = 0$ then by Corollary 2.7 we have $i_j = \phi_{L/K}^{j}(0) \ge i_j + 1$, a contradiction. It follows that $r_j \ne 0$.

Suppose there is $0 \le k' < k$ such that $\tilde{\phi}_{M/L}^{k'}(c) \le \tilde{\phi}_{M/L}^{k}(c)$. Since $c \in T_l(M/K)$ we have $(j,k') \notin S_l^{j+k'}(c)$, and hence

$$\lambda_{M/K}^{l}(c) < \tilde{\phi}_{L/K}^{j}\big(\tilde{\phi}_{M/L}^{k'}(c)\big) \le \tilde{\phi}_{L/K}^{j}\big(\tilde{\phi}_{M/L}^{k}(c)\big) = \lambda_{M/K}^{l}(c).$$

This is a contradiction, so we must have $\tilde{\phi}_{M/L}^{k'}(c) > \tilde{\phi}_{M/L}^{k}(c)$ for $0 \le k' < k$. Hence $\phi_{M/L}^{k}(c) = \tilde{\phi}_{M/L}^{k}(c)$. Set $d = \phi_{M/L}^{k}(c)$. Then $\pi_L \cdot (\pi_M, \epsilon) \cdot I'_{\mathcal{G}} \subset (\pi_M^{m+d+1}, \epsilon^{p^k+1})$. Using (3.7) we get

$$\hat{\mathcal{G}}\big(\pi_M\big(1+\pi_M^c\epsilon\big)\big) \equiv \hat{\mathcal{G}}(\pi_M)\big(1 + s_k \pi_M^d \epsilon^{p^k}\big) \quad \big(\mathrm{mod}\ \big(\pi_M^{m+d+1}, \epsilon^{p^k+1}\big)\big).$$

If $s_k = 0$ then by Corollary 2.7 we have $\phi_{M/L}^{k}(c) \ge d+1$, a contradiction. It follows that $s_k \ne 0$. $\square$

We now prove (b) for $x = c \in \mathbb{N}_0 \cap T_l(M/K)$. Let l_0 be the minimum integer satisfying the hypotheses of (b) for $x = c$. Then there is a unique pair $(j,k) \in \Omega_{l_0}$ such that $\lambda_{M/K}^{l}(c) = \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^{k}(c))$. Furthermore, we have $\lambda_{M/K}^{l_0}(c) = \lambda_{M/K}^{l}(c)$ and $\lambda_{M/K}^{l_1}(c) > \lambda_{M/K}^{l}(c)$ for $l_1 < l_0$. Define $J'_{l_0} = (\pi_M^{nm+\lambda_{M/K}^{l}(c)+1}, \epsilon^{p^{l_0}+1})$. Then $\pi_L^n \cdot (\pi_M, \epsilon) \cdot I_{\mathcal{F}\mathcal{G}} \subset J'_{l_0}$, so by (3.9) we get

$$\hat{\mathcal{F}}\big(\hat{\mathcal{G}}\big(\pi_M\big(1+\pi_M^c\epsilon\big)\big)\big) \equiv \pi_K \cdot \big(1 + u^{i_j} r_j s_k^{p^j} \pi_M^{\lambda_{M/K}^{l}(c)} \epsilon^{p^{l_0}}\big) \quad \big(\mathrm{mod}\ J'_{l_0}\big).$$

It follows from Lemma 3.6 that $r_j, s_k \in R \setminus \{0\}$ are units. Therefore we have

$$\hat{\mathcal{F}}\big(\hat{\mathcal{G}}\big(\pi_M\big(1+\pi_M^c\epsilon\big)\big)\big) \not\equiv \pi_K \pmod{J'_{l_0}}.$$

Hence by (a) and Corollary 2.7 we get

$$\lambda_{M/K}^{l_0}(c) \le \phi_{M/K}^{l_0}(c) < \lambda_{M/K}^{l}(c)+1 = \lambda_{M/K}^{l_0}(c)+1.$$

Since $\lambda_{M/K}^{l_0}(c)$ and $\phi_{M/K}^{l_0}(c)$ are integers this implies that $\lambda_{M/K}^{l_0}(c) = \phi_{M/K}^{l_0}(c)$. Using (a) we get

$$\lambda_{M/K}^{l}(c) \le \phi_{M/K}^{l}(c) \le \phi_{M/K}^{l_0}(c) = \lambda_{M/K}^{l_0}(c) = \lambda_{M/K}^{l}(c),$$

and hence $\lambda_{M/K}^{l}(c) = \phi_{M/K}^{l}(c)$. Thus (b) holds for $x \in \mathbb{N}_0 \cap T_l(M/K)$. In particular, it follows from Lemma 3.5 that (b) holds for $x = 0$.

As in the proof of (a) let $e \ge 1$ be relatively prime to pnm, let π_M be a uniformizer for M, and choose uniformizers π_L, π_K for L, K such that $\pi_L \equiv \pi_M^m \pmod{\pi_M^{m+1}}$ and $\pi_K \equiv \pi_L^n \pmod{\pi_L^{n+1}}$. Let $\pi_{K_e} \in K^{sep}$ be a root of $X^e - \pi_K$ and set $K_e = K(\pi_{K_e})$, $L_e = LK_e$, and $M_e = MK_e$. Let $c \in \mathbb{N}_0$ be such that $c/e \in T_l(M/K)$ and the hypotheses of (b) are satisfied for the extensions $M/L/K$ with $x = c/e$. Then it follows from (3.10)–(3.13) that $c \in T_l(M_e/K_e)$ and the hypotheses of (b) are satisfied for the extensions $M_e/L_e/K_e$ with $x = c$. Hence by the preceding paragraph we get $\phi_{M_e/K_e}^{l}(c) = \lambda_{M_e/K_e}^{l}(c)$. Using (3.12) and (3.13) we deduce that $\phi_{M/K}^{l}(c/e) = \lambda_{M/K}^{l}(c/e)$.

Now let r be any positive real number such that the hypotheses of (b) are satisfied with $x = r$, and let l_0 be the minimum integer which satisfies the hypotheses. Then there is a unique element $(j,k) \in \Omega_{l_0}$ such that $\tilde{\phi}_{L/K}^{j,m} \circ \tilde{\phi}_{M/L}^{k}(r) = \lambda_{M/K}^{l}(r)$. Let $0 \le a \le l_0$ and let $(u,v) \in \Omega_a$. Then the graph of $\tilde{\phi}_{L/K}^{u,m} \circ \tilde{\phi}_{M/L}^{v}$ is a line of slope $p^{u+v} = p^a \le p^{l_0}$. Hence if $(u,v) \ne (j,k)$ and $0 \le t < r$ then $\tilde{\phi}_{L/K}^{u,m} \circ \tilde{\phi}_{M/L}^{v}(t) > \tilde{\phi}_{L/K}^{j,m} \circ \tilde{\phi}_{M/L}^{k}(t)$. It follows that $S_{l_0}^{l_0}(t) = \{(j,k)\}$ and $S_{l_0}^{a}(t) = \varnothing$ for $0 \le a < l_0$. Hence $t \in T_{l_0}(M/K)$ and the hypotheses of (b) are satisfied with $x = t$ and l replaced by l_0.

Suppose $\phi_{M/K}^{l}(r) > \lambda_{M/K}^{l}(r)$. Then there are $c, e \ge 1$ such that $\gcd(e,pnm) = 1$ and

$$0 < r - \frac{c}{e} < \frac{\phi_{M/K}^{l}(r) - \lambda_{M/K}^{l}(r)}{p^{\nu+\mu}}. \tag{3.15}$$

Since $\lambda_{M/K}^{l_0}(r) = \lambda_{M/K}^{l}(r)$ we get

$$\phi_{M/K}^{l_0}(r) - \lambda_{M/K}^{l_0}(r) \ge \phi_{M/K}^{l}(r) - \lambda_{M/K}^{l}(r) > 0. \tag{3.16}$$

Since $\phi_{M/K}^{l_0}$ and $\lambda_{M/K}^{l_0}$ are continuous increasing piecewise linear functions with derivatives at most $p^{\nu+\mu}$ it follows from (3.15) and (3.16) that $\phi_{M/K}^{l_0}(c/e) - \lambda_{M/K}^{l_0}(c/e) > 0$. On the other hand, by the preceding paragraph we know that $c/e \in T_{l_0}(M/K)$ and the hypotheses of (b) are satisfied with $x = c/e$ and l replaced by l_0. Hence $\phi_{M/K}^{l_0}(c/e) = \lambda_{M/K}^{l_0}(c/e)$. This is a contradiction, so we must have $\phi_{M/K}^{l}(r) \le \lambda_{M/K}^{l}(r)$.

By combining this inequality with (a) we get $\phi^l_{M/K}(r) = \lambda^l_{M/K}(r)$. This completes the proof of (b).

By setting $x = 0$ in Theorem 3.3 we get the following. A special case of this result is given in [3, Prop. 5.10].

Corollary 3.7. *For $0 \le l \le \nu + \mu$ let i''_l denote the lth index of inseparability of M/K. Then*

$$i''_l \le \min\{mi_j + p^j i'_k : (j,k) \in \Omega_{l_0} \text{ for some } 0 \le l_0 \le l\},$$

with equality if there exists $0 \le l_0 \le l$ such that there is a unique pair $(j,k) \in \Omega_{l_0}$ which realizes the minimum.

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