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Sums of three squares under congruence condition modulo a prime

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A B S T R A C T

Let p be an odd prime. We show that the integral points on the sphere with radius n are equidistributed modulo p as $n \longrightarrow \infty$ where n is not of the shape $4^l(8m+7)$ and its 2-adic valuation is bounded. In particular if n is sufficiently large and if n satisfies a congruence equation $\alpha_1^2+\alpha_2^2+\alpha_3^2 \equiv n \pmod p$ where $p^2|n$ if all $\alpha_i \equiv 0 \pmod p$, then there are integers x_i with $x_i \equiv \alpha_i \pmod p$ $(i=1,2,3)$ satisfying $x_1^2+x_2^2+x_3^2=n$. The similar result holds also in the case modulo 8.

1. Introduction

Let $r_3(n)$ be the number of representations of a natural number n as a sum of three squares. Legendre showed that $r_3(n)$ is positive for every natural number n not of the shape $4^l(8m+7)$, and Gauss gave formula for $r_3(n)$ when n is square-free.

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Let $\mathbf{1}_N$ denote the trivial character modulo N for which $\mathbf{1}_N(n)$ is 1 or 0 according as the greatest common divisor (n, N) is 1 or not. If D is a discriminant of a quadratic field, then χ_D denotes the Kronecker–Jacobi–Legendre symbol associated with D. If $m \in \mathbf{Z}, \neq 0$, then χ_{m^2D} denotes $\chi_D \mathbf{1}_m$. We define $\sigma_{k,\chi}^{\chi'}$ by

$$\sigma_{k,\chi}^{\chi'}(n) := \sum_{d|n} \chi(d)\chi'(n/d)d^k \quad (n \in \mathbf{N})$$

for Dirichlet characters χ, χ'. We omit χ or χ' from the notation if it is the trivial character $\mathbf{1}$, for which $\mathbf{1}(n) = 1$ for all $n \in \mathbf{Z}$. We define $\sigma_{k,\chi}^{\chi'}(n)$ to be 0 if n is negative or not integral. Bateman [1] (see also Corollary 5.3 later) showed the formula

$$r_3(an^2) = 12L(0, \chi_{-4a}) \sum_{d|n} \mu(d)\chi_{-4a}(d)\sigma_{1,\mathbf{1}_2}(n/d)$$

for a square-free a where μ denotes Möbius function. Here we note that $L(0, \chi_{-4a}) = 2L(0, \chi_{-a})$ if $a \equiv 3 \pmod 8$, and $L(0, \chi_{-4a}) = 0$ if $a \equiv 7 \pmod 8$.

The $r_3(n)$ is regarded as the number of lattice points on the sphere with the radius $\sqrt{n}$. Linnik [12] showed under the generalised Riemann hypothesis, that the projection of lattice points to the unit sphere S is equidistributed as $n \longrightarrow \infty$ with $n \not\equiv 0, 4, 7 \pmod 8$, namely if Ω_n is the set of the points on the unit sphere, then

$$\frac{1}{r_3(n)} \sum_{\mathbf{x} \in \Omega_n} f(\mathbf{x}) \longrightarrow \int_S f(\mathbf{x}) d\mathbf{x} \quad (n \longrightarrow \infty)$$

for any continuous function f on the unit sphere where $d\mathbf{x}$ is the normalised measure on the sphere so that the area of the sphere be 1. The equidistribution property is unconditionally proved by Duke [3], Golubeva and Fomenko [5].

The purpose of the present paper is to show that the lattice points are equidistributed modulo 8 or modulo any odd prime p. Let

$$|\boldsymbol{x}|^2 := x_1^2 + x_2^2 + x_3^2 \quad \text{for } \boldsymbol{x} = (x_1, x_2, x_3).$$

For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3$, let

$$r_{\boldsymbol{\alpha}}^{(p)}(n) := \#\{\boldsymbol{x} \in \mathbf{Z}^3 \mid x_i \equiv \pm\alpha_i \pmod p \ (i = 1, 2, 3),\ |\boldsymbol{x}|^2 = n\}.$$

If n is not integral, then $r_3(n)$ and $r_{\boldsymbol{\alpha}}^{(p)}(n)$ are obviously 0. There holds $r_{\mathbf{0}}^{(p)}(n) = r_3(n/p^2)$ with $\mathbf{0} = (0, 0, 0)$, whereas $r_3(n) = r_3(2^2n)$ and $r_{\boldsymbol{\alpha}}^{(p)}(n) = r_{2\boldsymbol{\alpha}}^{(p)}(2^2n)$. We make the similar definition for vectors in $\mathbf{F}_p^3$. Let

$$\bar{r}_3^{(p)}(n) := \#\{\boldsymbol{x} \in \mathbf{F}_p^3 \mid |\boldsymbol{x}|^2 = n,\ \boldsymbol{x} \neq \mathbf{0}\},$$
$$\bar{r}_{\boldsymbol{\alpha}}^{(p)}(n) := \#\{\boldsymbol{x} \in \mathbf{F}_p^3 \mid x_i = \pm\alpha_i \ (i = 1, 2, 3),\ |\boldsymbol{x}|^2 = n \text{ in } \mathbf{F}_p\}$$

for $\boldsymbol{\alpha} \in \mathbf{F}_p^3, \neq \mathbf{0}$, where the former is always positive (see (6) later).

Suppose that $\boldsymbol{\alpha} \in \mathbf{Z}^3$ is not congruent to $\mathbf{0}$ modulo p. We show for a square-free or for $a = 1$,

$$\begin{aligned} & r_{\boldsymbol{\alpha}}^{(p)}(an^2) \\ &= \frac{12\,\bar{r}_{\boldsymbol{\alpha}}^{(p)}(an^2)L(0,\chi_{-4a})}{\bar{r}_3^{(p)}(a)} \sum_{d|n} \mu(d)\chi_{-4a}(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d) + O((an^2)^{13/28+\varepsilon}) \end{aligned} \tag{1}$$

where ε is any fixed positive number (Theorem 8.1). As a consequence of the formula, it follows that for a fixed odd prime p, sufficiently large n not of the shape $4^l(8m+7)$ with bounded 2-adic valuation and with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \equiv n \pmod p$ has the integral solution of

$$x_i \equiv \alpha_i \pmod p, \quad x_1^2 + x_2^2 + x_3^2 = n \tag{2}$$

(Corollary 8.2). In the case $\boldsymbol{\alpha} \equiv \mathbf{0} \pmod p$, the equations (2) have the solution if and only if $p^2|n$.

Let $S(n) := \{\boldsymbol{x} \in \mathbf{Z}^3 \mid |\boldsymbol{x}|^2 = n\}$ be the integral points on the sphere with radius $\sqrt{n}$, and let $\overline{S}_p(n) := \{\boldsymbol{x} \in \mathbf{F}_p^3 \mid |\boldsymbol{x}|^2 = n\}$ be the sphere in $\mathbf{F}_p^3$. If π_p denotes the natural reduction map of $S(n)$ to $\overline{S}_p(n)$, then our result shows that π_p is surjective for a sufficiently large n not of the shape $4^l(8m+7)$ with bounded 2-adic valuation. Moreover for any two nonsingular points P, P' on $\overline{S}_p(n)$, the ratio $\#\pi_p^{-1}(P)/\#\pi_p^{-1}(P')$ of the numbers of elements of fibres is tending to 1 as $n \longrightarrow \infty$, which we are calling the *equidistribution property modulo p*. The similar assertion holds also when the modulus is 8. Further assuming the weak Birch–Swinnerton–Dyer conjecture, we give some criterion that a square free natural number be a congruent number in connection with numbers of lattice points on a sphere under congruence conditions modulo 8.

In Hsia and Jöchner [7], or in Jöchner and Kitaoka [10], the representations of positive definite integral quadratic forms with congruence condition are discussed in much more general context. To have desired integral solutions, it is required in [7,10], that the quadratic forms also satisfy appropriate local conditions at primes. The present paper shows that the local conditions are not necessary about (2), and the condition that n is large enough in terms of p, is only required.

Let N be the natural number, and let $(\mathbf{Z}/N)^*$ denote the group of the Dirichlet characters modulo N. We define the 0-th power ρ^0 of $\rho \in (\mathbf{Z}/N)^*$, to be $\mathbf{1}_N$. A character ρ is called *even* or *odd* according as $\rho(-1)$ is 1 or -1. We denote by $\mathfrak{f}_\rho$, the conductor of ρ. A theta series $\theta(z)$ is defined by

$$\theta(z) := \sum_{n=-\infty}^{\infty} \mathbf{e}(n^2 z), \tag{3}$$

and a theta series $\theta_\rho(z)$ with an even Dirichlet character ρ is defined by

$$\theta_\rho(z) := \sum_{n=-\infty}^{\infty} \rho(n)\mathbf{e}(n^2 z). \tag{4}$$

The outline of the argument proving (1) is as follows. Let an^2 be any natural number satisfying $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \equiv an^2 \pmod p$ where a is 1 or a square free integer. Then $r_{\boldsymbol{\alpha}}^{(p)}(an^2)$ is equal to the an^2-th Fourier coefficient of $\theta(z)\sum_{m\equiv\pm\alpha_2 (\operatorname{mod} p)} \mathbf{e}(m^2 z) \times \sum_{m\equiv\pm\alpha_3 (\operatorname{mod} p)} \mathbf{e}(m^2 z)$. Let ω be a generator of the cyclic group $(\mathbf{Z}/p)^*$. Then the equality

$$\sum_{m\equiv\pm\alpha (\operatorname{mod} p)} \mathbf{e}(m^2 z) = \frac{2}{p-1}\sum_{i=0}^{p-2} \overline{\omega}(\alpha)^{2i}\theta_{\omega^{2i}}(z) \tag{5}$$

holds true, and our problem is reduced to evaluating the an^2-th Fourier coefficient of $\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$ $(0 \le i, j < p)$. There is the Shimura lift f associated with a, of $\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$, whose n/d-th Fourier coefficients with $d|n$ give the an^2-th Fourier coefficient of $\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$ (see [20], or (20) of the present paper). The modular form f is expressed as a sum of an Eisenstein series and a cusp form, where the former is dominant with respect to the magnitude of Fourier coefficients. The values of the Shimura lift f at cusps are worked out with the aid of Hilbert modular forms, by which the Eisenstein series is completely determined.

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2. Jacobi sum

In this section, we derive some fundamental properties of the Jacobi sum for later use.

For $a \in \mathbf{Z}$, a^* denotes a or $4a$ according as $a \equiv 0, 1 \pmod 4$ or not. If a is square-free, then a^* is a discriminant of a quadratic field except for the case $a = 1$. If a is odd, then $a^\vee$ denotes a or $-a$ according as $a \equiv 1 \pmod 4$ or not. If a is odd and square-free, then $\chi_{a^\vee}(\ ) = \left(\frac{\ }{a}\right)$ where $\left(\frac{\ }{a}\right)$ is the Legendre symbol. Further $\chi_{-4}(d) = (-1)^{d-1}$ for d odd, and $\chi_{-4}(d) = 0$ for d even.

Let χ be a primitive Dirichlet character with conductor $\mathfrak{f}_\chi$. Then the Gauss sum $\tau(\chi)$ is define by

$$\tau(\chi) = \sum_{i=1}^{\mathfrak{f}_\chi} \chi(i)\mathbf{e}(i/p)$$

where $\mathbf{e}(z)$ stands for $e^{2\pi\sqrt{-1}z}$. We have $\tau(\chi_{-4}) = 2\sqrt{-1}, \tau(\chi_{\pm 8}) = 2\sqrt{2}\sqrt{\pm 1}$ and $\tau(\chi_{p^\vee}) = \iota_p\sqrt{p}$ for an odd prime p by the Gauss theorem, where $\iota_d = \sqrt{\chi_{-4}(d)}$ for an odd d, namely ι_d denotes 1 or $\sqrt{-1}$ according as $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$.

Let $\chi, \phi, \psi \in (\mathbf{Z}/p)^*$ for an odd prime p. Then we define the Jacobi sum by

$$J(\chi, \phi) := \sum_{\alpha \in \mathbf{F}_p} \chi(\alpha)\phi(1-\alpha),$$

$$J(\chi, \phi, \psi) := \sum_{\alpha, \beta \in \mathbf{F}_p} \chi(\alpha)\phi(\beta)\psi(1-\alpha-\beta).$$

If any of χ, ϕ, $\chi\phi$ is not equal to $\mathbf{1}_p$, then $J(\chi, \phi) = \tau(\chi)\tau(\phi)/\tau(\chi\phi)$. If any of χ, ϕ, ψ, $\chi\phi\psi$ is not equal to $\mathbf{1}_p$, then $J(\chi, \phi, \psi) = \tau(\chi)\tau(\phi)\tau(\psi)/\tau(\chi\phi\psi)$. The following lemma is easy to see, and we skip the proof.

Lemma 2.1. *Let p be an odd prime.*

(i) *We have $J(\mathbf{1}_p, \mathbf{1}_p) = p - 2$, and if $\chi \neq \mathbf{1}_p$, then $J(\chi, \mathbf{1}_p) = -1$ and $J(\chi, \overline{\chi}) = -\chi(-1)$.*

(ii) *We have $J(\mathbf{1}_p, \mathbf{1}_p, \mathbf{1}_p) = p^2 - 3p + 3$, and $J(\chi, \psi, \mathbf{1}_p) = -J(\chi, \psi)$ for χ, $\psi \neq \mathbf{1}_p$, and $J(\chi, \psi, \rho) = -p^{-1}\tau(\chi)\tau(\psi)\tau(\rho)$ for χ, ψ, $\rho \neq \mathbf{1}_p$ with $\chi\psi\rho = \mathbf{1}_p$, and $J(\chi, \psi, \chi_{p^\vee}) = -\chi_{-4}(p)J(\chi, \psi)$ for χ, $\psi \neq \mathbf{1}_p$ with $\chi\psi = \chi_{p^\vee}$.*

As one of the typical application of Jacobi sums, the number of solutions of $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = n$ in $\mathbf{F}_p$ for an odd p is obtained (cf. Ireland and Rosen [8], Chap. 8, §6, or Berndt, Evans and Williams [2], Chap. 10). From this the number $\bar{r}_3^{(p)}(n)$ of nontrivial representation is given as

$$\bar{r}_3^{(p)}(n) = \begin{cases} p(p + \chi_{-4}(p)\chi_{p^\vee}(n)) & (p \nmid n) \\ p^2 - 1 & (p|n) \end{cases}. \tag{6}$$

On the other hand, $\bar{r}_{\boldsymbol{\alpha}}^{(p)}(n)$ for $\boldsymbol{\alpha} \in \mathbf{F}_p^3$ is simply obtained by $\bar{r}_{\boldsymbol{\alpha}}^{(p)}(n) = 2^t$ where t is the number of nonzero elements of the vector $\boldsymbol{\alpha}$ when $\bar{r}_{\boldsymbol{\alpha}}^{(p)}(n)$ is positive.

Proposition 2.2. *Let p be an odd prime. Let ω be a generator of the cyclic group $(\mathbf{Z}/p)^*$. Let $\alpha_1, \alpha_2, m \in \mathbf{F}_p$ with $\alpha_1\alpha_2 \neq 0$.*

(i) *Let $\chi_{p^\vee}(m) = 1$. Then*

$$\begin{aligned}\sum_{i,j=0}^{(p-1)/2-1} & \overline{\omega}(\alpha_1^2)^i \overline{\omega}(\alpha_2^2)^j \omega(2^2 m)^{i+j} \{J(\omega^i, \omega^j, \omega^{i+j}) + J(\chi_{p^\vee}\omega^i, \chi_{p^\vee}\omega^j, \omega^{i+j}) \\ & + J(\chi_{p^\vee}\omega^i, \omega^j, \chi_{p^\vee}\omega^{i+j}) + J(\omega^i, \chi_{p^\vee}\omega^j, \chi_{p^\vee}\omega^{i+j})\}\end{aligned} \tag{7}$$

is equal to 0 if there is no α_3 with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = m$. Suppose that such α_3 exists. Then it is equal to $2(p-1)^2$ if $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\alpha_3 \neq 0$, and it is equal to $(p-1)^2$ if otherwise.

(ii) *Let $\chi_{p^\vee}(m) = -1$. Then*

$$\sum_{i,j=0}^{(p-1)/2-1} \overline{\omega}(\alpha_1^2)^i \overline{\omega}(\alpha_2^2)^j \omega(2^2 m)^{i+j} \{-J(\omega^i, \omega^j, \omega^{i+j}) - J(\chi_{p^\vee}\omega^i, \chi_{p^\vee}\omega^j, \omega^{i+j}) + J(\chi_{p^\vee}\omega^i, \omega^j, \chi_{p^\vee}\omega^{i+j}) + J(\omega^i, \chi_{p^\vee}\omega^j, \chi_{p^\vee}\omega^{i+j})\} \tag{8}$$

is equal to $-2(p-1)^2$ *if there is no* α_3 *with* $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = m$. *Suppose that such* α_3 *exists. Then it is equal to* 0 *if* $\alpha_1^2 + \alpha_2^2 \neq 0$ *and* $\alpha_3 \neq 0$, *and it is equal to* $-(p-1)^2$ *otherwise.*

(iii) *Let* $\chi_{p^\vee}(m) = 1$. *Then*

$$\sum_{i=0}^{(p-1)/2-1} \overline{\omega}(\alpha_1^2)^i \omega(2^2 m)^i \{J(\omega^i, \omega^i) + J(\chi_{p^\vee}\omega^i, \chi_{p^\vee}\omega^i)\}$$

is equal to 0 *if there is no* $\alpha_3 \in \mathbf{F}_p$ *with* $\alpha_1^2 + \alpha_3^2 = m$. *Suppose that such* α_3 *exists. Then it is equal to* $2(p-1)$ *if* $\alpha_3 \neq 0$, *and it is equal to* $p-1$ *if* $\alpha_3 = 0$.

(iv) *Let* $\chi_{p^\vee}(m) = -1$. *Then*

$$\sum_{i=0}^{(p-1)/2-1} \overline{\omega}(\alpha_1^2)^i \omega(2^2 m)^i \{-J(\omega^i, \omega^i) + J(\chi_{p^\vee}\omega^i, \chi_{p^\vee}\omega^i)\}$$

is equal to $-2(p-1)$ *if there is no* $\alpha_3 \in \mathbf{F}_p$ *with* $\alpha_1^2 + \alpha_3^2 = m$. *Suppose that such* α_3 *exists. Then it is equal to* 0 *if* $\alpha_3 \neq 0$, *and it is equal to* $-(p-1)$ *if* $\alpha_3 = 0$.

(v) *There holds*

$$\sum_{i=0}^{(p-1)/2-1} \overline{\omega}(\alpha_1^2)^i \omega(\alpha_2^2)^i \{J(\omega^i, \chi_{p^\vee}\overline{\omega}^i) + J(\chi_{p^\vee}\omega^i, \overline{\omega}^i)\} = (p-1)\chi_{p^\vee}(\alpha_1^2 + \alpha_2^2).$$

Proof. (i) The summation (7) is equal to

$$\sum_{i,j=0}^{p-2} \overline{\omega}(\alpha_1)^{2i} \overline{\omega}(\alpha_2)^{2j} \omega(2^2 m)^{i+j} \sum_{\alpha,\beta \in \mathbf{F}_p} \omega(\alpha(1-\alpha-\beta))^i \omega(\beta(1-\alpha-\beta))^j$$
$$= \sum_{\alpha,\beta \in \mathbf{F}_p} \sum_{i=0}^{p-2} \omega^i(2^2 m \alpha_1^{-2} \alpha(1-\alpha-\beta)) \sum_{j=0}^{p-2} \omega^j(2^2 m \alpha_2^{-2} \beta(1-\alpha-\beta)).$$

Put $w = m^2 - m\alpha_1^2 - m\alpha_2^2$. Then the equations over $\mathbf{F}_p$ for α, β

$$\begin{cases} 2^2 m \alpha_1^{-2} \alpha(1-\alpha-\beta) = 1, \\ 2^2 m \alpha_2^{-2} \beta(1-\alpha-\beta) = 1 \end{cases} \tag{9}$$

have common solutions in $\mathbf{F}_p$ if and only if w is square. In such a case, the solution is

$$\begin{cases} \alpha = \frac{\alpha_1^2(m+s\sqrt{w})}{2m(\alpha_1^2+\alpha_2^2)} \\ \beta = \frac{\alpha_2^2(m+s\sqrt{w})}{2m(\alpha_1^2+\alpha_2^2)} \end{cases} (\alpha_1^2+\alpha_2^2 \neq 0,\ s=\pm 1), \qquad \begin{cases} \alpha = \frac{\alpha_1^2}{4m} \\ \beta = \frac{\alpha_2^2}{4m} \end{cases} (\alpha_1^2+\alpha_2^2 = 0).$$

Since $\chi_{p^\vee}(m) = 1$, there is $\sqrt{m} \in \mathbf{F}_p$ and hence there is $\alpha_3 \in \mathbf{F}_p$ satisfying $\sqrt{w} = \sqrt{m}\alpha_3$. This shows (i).

(ii) The right hand side of (8) is equal to

$$-\sum_{\alpha,\beta\in\mathbf{F}_p}\sum_{i=0}^{p-2}\omega^i(2^2m\alpha_1^{-2}\alpha(1-\alpha-\beta))\sum_{j=0}^{p-2}\omega^j(2^2m\alpha_2^{-2}\beta(1-\alpha-\beta)). \tag{10}$$

If $\alpha_1^2+\alpha_2^2 = 0$ or $\alpha_1^2+\alpha_2^2 = m$, then (9) has only one solution, and hence the value of (10) is $-(p-1)^2$. Suppose $\alpha_1^2+\alpha_2^2 \neq 0$ and $\alpha_1^2+\alpha_2^2 \neq m$. By the argument of (i), the values of (10) is $-2(p-1)^2$ (resp. 0) if $w \in \{\mathbf{F}_p^\times\}^2$ (resp. $w \notin \{\mathbf{F}_p^\times\}^2$). Since m is not square, this is equivalent to $m-\alpha_1^2-\alpha_2^2 \notin \{\mathbf{F}_p^\times\}^2$ (resp. $m-\alpha_1^2-\alpha_2^2 \in \{\mathbf{F}_p^\times\}^2$), namely, there is no α_3 with $\alpha_1^2+\alpha_2^2+\alpha_3^2 = m$ (resp. there is α_3 with $\alpha_1^2+\alpha_2^2+\alpha_3^2 = m$). This shows (ii).

We omit the proof of (iii), (iv), (v), since the argument is similar. □

3. Elliptic modular forms

In this section we obtain the values at cusps, of Eisenstein series and theta series.

Let $\mathfrak{H}$ be the complex upper half-plane $\{z \in \mathbf{C} \mid \Im z > 0\}$ where $\Im z$ denotes the imaginary part of z. The group $\mathrm{SL}_2(\mathbf{Z})$ acts on $\mathfrak{H}$ by the modular substitution $z \longrightarrow Mz = (az+b)/(cz+d)$ for

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}). \tag{11}$$

For $N \in \mathbf{N}$, let $\Gamma_0(N) := \{M \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \pmod{n}\}$, which is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Let $k \in \mathbf{N}$, and let χ_0 be a Dirichlet character modulo a divisor of N with the same parity as k. A holomorphic function on $\mathfrak{H}$ is an *elliptic modular form* (or simply a *modular form*) for $\Gamma_0(N)$ of weight k with character χ_0 if it satisfies $f(Mz) = \chi_0(d)(cz+d)^k f(z)$ for any $M \in \Gamma_0(N)$, and it is holomorphic also at cusps. We denote by $\mathbf{M}_k(N,\chi_0)$ (resp. $\mathbf{S}_k(N,\chi_0)$), the vector space of such modular forms (resp. cusp forms).

The Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s,\chi)$ for a character $\chi \in (\mathbf{Z}/N)^*$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s},\ L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

By analytic continuation, these functions can be extended meromorphically to the whole complex plane. If $\widetilde{\chi}$ denotes the primitive Dirichlet character associated with χ, then

$$L(s,\chi)=L(s,\widetilde{\chi})\prod_{p|N,p\nmid \mathfrak{f}_\chi}(1-\frac{\widetilde{\chi}(p)}{p^s}).$$

In particular, for $a\equiv 3 \pmod 4$ square-free, we have $L(0,\chi_{-4a})=L(0,\chi_{-a})(1-\chi_{-a}(2))$, which is $2L(0,\chi_{-a})$ or 0 according as $a\equiv 3 \pmod 8$ or $a\equiv 7 \pmod 8$.

Let $M,N\in \mathbf{N}$ with $M|N$. We define the Eisenstein series by setting for $c_0,d_0\in \mathbf{Z}$,

$$G_k(z,c_0,d_0;M,N):=(\sum_{\substack{c\equiv c_0 \,(\mathrm{mod}\; N/M)\\ d\in M^{-1}d_0+\mathbf{Z}}}{}^{'}(cz+d)^{-k}|cz+d|^{-s})|_{s=0}$$

where $\sum'$ means that the term for $(c,d)=(0,0)$ is omitted in the summation and where $|_{s=0}$ denotes the values at $s=0$ of the analytic function of s. Let χ, χ' be primitive Dirichlet characters with conductor $\mathfrak{f}_\chi$, $\mathfrak{f}_{\chi'}$ respectively. We assume that $\chi\chi'$ has the same parity as k. We define

$$G_{k,\chi}^{\chi'}(z):=\frac{(k-1)!}{(-2\sqrt{-1}\pi)^k\tau(\chi)}\sum_{c_0:\mathbf{Z}/\mathfrak{f}_{\chi'}}\sum_{d_0:\mathbf{Z}/\mathfrak{f}_\chi}\overline{\chi}(d_0)\chi'(c_0)G_k(z,c_0,d_0;\mathfrak{f}_\chi,\mathfrak{f}_\chi\mathfrak{f}_{\chi'})$$

where $\sum_{c_0:\mathbf{Z}/\mathfrak{f}_{\chi'}}$ means that c_0 runs over a complete set of representatives of $\mathbf{Z}$ modulo $\mathfrak{f}_{\chi'}$. We omit χ or χ' from the notation $G_{k,\chi}^{\chi'}$ if it is the trivial character. It has the Fourier expansion

$$G_{k,\chi}^{\chi'}(z)=C+2\sum_{n=1}^{\infty}\sigma_{k-1,\chi}^{\chi'}(n)\mathbf{e}(nz) \tag{12}$$

where $C=L(1-k,\chi)$ for $\chi'=\mathbf{1}$, $C=L(0,\chi')$ for $k=1$ and $\chi=\mathbf{1}$, and $C=0$ for all other cases, and where there is the additional term $\sqrt{-1}/(4\pi\Im z)$ if $k=2$ and $\chi=\chi'=\mathbf{1}$. Let m be square-free. Then $G_{k,\chi\mathbf{1}_m}^{\chi'}(z)$ is defined by $G_{k,\chi\mathbf{1}_m}^{\chi'}(z):=\sum_{d|m}\mu(d)\chi(d)d^{k-1}G_{k,\chi}^{\chi'}(dz)$. Thus $G_{k,\chi}^{\chi'}(z)$ is defined also when χ is not primitive. Similarly we define $G_{k,\chi}^{\chi'\mathbf{1}_m}(z):=\sum_{d|m}\mu(d)\chi'(d)G_{k,\chi}^{\chi'}(dz)$. If χ, χ' are Dirichlet characters modulo M/N, M respectively which are not necessarily primitive, then the Eisenstein series $G_{k,\chi}^{\chi'}(z)$ is a modular form in $\mathbf{M}_k(N,\chi\chi')$ and has the Fourier expansion (12) with the constant term C except for the case that $k=2$, $\chi=\mathbf{1}$ and $\mathfrak{f}_{\chi'}=1$, where $C=L(1-k,\chi)$ for $\chi'=\mathbf{1}$, $C=L(0,\chi')$ for $k=1$ and $\chi=\mathbf{1}$, and $C=0$ for all other cases.

The theta series defined in (3), is a modular form of weight $1/2$ for $\Gamma_0(4)$. We have $\theta(Mz)=j(M,z)\theta(z)$ for $M\in\Gamma_0(4)$ where $j(M,\gamma)$ is the automorphy factor satisfying $j(M,z)^2=\chi_{-4}(d)(cz+d)$ for c, d as in (11). Let $4|N$. Suppose that k and a character χ_0 modulo N have the same parity. Then $\mathbf{M}_{k+1/2}(N,\chi_0)$ (resp. $\mathbf{S}_{k+1/2}(N,\chi_0)$) denotes the space of modular forms (resp. cusp forms) of weight $k+1/2$ satisfying

$$f(Mz)=\chi_0(d)j(M,z)(cz+d)^kf(z)\quad (M\in\Gamma_0(N)).$$

If $q \in \mathbf{N}$, then $\theta(qz)$ is in $\mathbf{M}_{1/2}(4q, \chi_{q^*})$. Let ρ be an even Dirichlet character modulo m. Then $\theta_\rho(z)$ defined in (4), is in $\mathbf{M}_{1/2}(4m^2, \rho)$. If ρ is totally even, namely, the p-part of ρ is even for any prime p, then $\theta_\rho(z)$ is not a cusp form (Serre and Stark [16]).

We define the value at a cusp, of a modular form f with character, of integral or half-integral weight l. Let a/c be a cusp where $c > 0$. For such a, c, there is a matrix M as in (11). We define the value of f at the cusp a/c by

$$\lim_{z \to \sqrt{-1}\infty} (cz + d)^{-l} f(Mz)$$

where $0 < \arg(cz + d)^{1/2} < \pi/2$.

Let

$$\mathcal{C}(N) := \left\{ \tfrac{i}{jM} \mid 0 < M|N,\ (i,j) : \{(\mathbf{Z}/M)^\times \times (\mathbf{Z}/(N/M))^\times\}/\{\pm 1\},\ \text{g.c.d}(i,j) = 1 \right\}$$

be the set of the representatives of inequivalent cusps of $\Gamma_1(N)$, and let

$$\mathcal{C}_0(N) := \{i/M \mid 0 < M|N, (i, M) = 1, i : (\mathbf{Z}/(M, N/M))^\times\}$$

be the set of representatives of inequivalent cusps of $\Gamma_0(N)$. Let χ_0 be a Dirichlet character modulo a divisor of N. Suppose that there is a positive divisor M of N and a Dirichlet character χ modulo a divisor of M satisfying

$$\chi = \chi_0 \text{ on } 1 + (N/M)\mathbf{Z}. \tag{13}$$

Then if we put $\chi' = \chi_0\overline{\chi}$, then χ' is a character whose conductor is a divisor of N/M, in other words, χ_0 is written as a product of χ and χ'. Under the condition, we consider the function $\kappa_{\chi_0,N}(\overline{\chi}, M)$ on $\mathcal{C}(N)$ which satisfies

$$\kappa_{\chi_0,N}(\overline{\chi}, M)(i'/j'M) = \overline{\chi}_0(\xi)\kappa_{\chi_0,N}(\overline{\chi}, M)(i/jM) \tag{14}$$

for any integer ξ with $i' \equiv \xi i \pmod{M}$, $\xi j' \equiv j \pmod{N/M}$. This property is satisfied by the restriction $f|_{\mathcal{C}(N)}$ to $\mathcal{C}(N)$, of a modular form for $\Gamma_0(N)$ of integral weight with character χ_0 ([20], Lemma 1). Then the restriction $\kappa_{\chi_0,N}(\overline{\chi}, M)|_{\mathcal{C}_0(N)}$ determines $\kappa_{\chi_0,N}(\overline{\chi}, M)$. Then we defines $\kappa_{\chi_0,N}(\overline{\chi}, M)$ to be a function satisfying both (14) and

$$\kappa_{\chi_0,N}(\overline{\chi}, M)(i/L) = \begin{cases} \overline{\chi}(i) & (L = M) \\ 0 & (L \neq M) \end{cases} \quad (i/L \in \mathcal{C}_0(N)). \tag{15}$$

The condition (13) assures that the conditions (14) and (15) are compatible. The restriction of Eisenstein series to the set of cusps is computed in [20] as follows:

Proposition 3.1. *Let χ, χ' be primitive Dirichlet characters with conductor $\mathfrak{f}_\chi$, $\mathfrak{f}_{\chi'}$ respectively where $\chi_0 = \chi\chi'$ has the same parity as k. Suppose $t\mathfrak{f}_\chi\mathfrak{f}_{\chi'}|N$ with $t \in \mathbf{N}$.*

(i) *Let* $k > 1$. *Then*

$$G^{\chi'}_{k,\chi}(tz)|_{\mathcal{C}(N)} = (-\sqrt{-1}\pi^{-1})^k 2^{-k+1}(k-1)!\mathfrak{f}_\chi^{k-1}\tau(\chi)L(k,\overline{\chi}\chi') \times \sum_{\substack{M|N \\ (M/(M,t),\mathfrak{f}_\chi\mathfrak{f}_{\chi'})=\mathfrak{f}_\chi}} \tfrac{(M,t)^k}{t^k}\chi'\left(\tfrac{M}{\mathfrak{f}_\chi(M,t)}\right)\overline{\chi}\left(\tfrac{t}{(M,t)}\right)\kappa_{\chi_0,N}(\overline{\chi},M).$$

In particular if $\chi' = \mathbf{1}$, *then*

$$G_{k,\chi}(tz)|_{\mathcal{C}(N)} = L(1-k,\chi)\sum_{\mathfrak{f}_\chi(M,t)|M|N} \tfrac{(M,t)^k}{t^k}\overline{\chi}\left(\tfrac{t}{(M,t)}\right)\kappa_{\chi_0,N}(\overline{\chi},M).$$

(ii) *Let* $k = 1$. *Then*

$$\begin{aligned} &G^{\chi'}_{1,\chi}(tz)|_{\mathcal{C}(N)} \\ &= -\sqrt{-1}\pi^{-1}\{\tau(\chi)L(1,\overline{\chi}\chi')\sum_{\substack{M|N \\ (M/(M,t),\mathfrak{f}_\chi\mathfrak{f}_{\chi'})=\mathfrak{f}_\chi}} \tfrac{(M,t)}{t}\chi'\left(\tfrac{M}{\mathfrak{f}_\chi(M,t)}\right)\overline{\chi}\left(\tfrac{t}{(M,t)}\right)\kappa_{\chi_0,N}(\overline{\chi},M) \\ &\quad + \tau(\chi')L(1,\chi\overline{\chi}')\sum_{\substack{M|N \\ (M/(M,t),\mathfrak{f}_\chi\mathfrak{f}_{\chi'})=\mathfrak{f}_{\chi'}}} \tfrac{(M,t)}{t}\chi\left(\tfrac{M}{\mathfrak{f}_{\chi'}(M,t)}\right)\overline{\chi}'\left(\tfrac{t}{(M,t)}\right)\kappa_{\chi_0,N}(\overline{\chi}',M)\}. \end{aligned}$$

We define another function κ_θ on $\mathbf{Q}$ by setting $\kappa_\theta(i/M) := 2^{-1/2}\mathbf{e}(-\frac{1}{8})\iota_M\chi_{M^\vee}(i)$ for M odd, $\kappa_\theta(i/M) := 2^{-1/2}\mathbf{e}(-\frac{1}{8})\iota_{M/2}\chi_{(M/2)^\vee}(i)$ for M just divisible by 2, and $\kappa_\theta(i/M) := \bar{\iota}_i\chi_{M^*}(i)$ for M divisible by 4 where $\iota_M := \sqrt{\chi_{-4}(M)}$. The following proposition is proved in [20], Section 5.

Proposition 3.2. (i) *Let t be odd with $4t|N$. If M is odd, then we put* $\varepsilon_{M,t} := \left(\frac{(M,t)}{t}\right)^{1/2}\iota_{M/(M,t)}\bar{\iota}_M\chi_{(M/(M,t))^\vee}(\frac{t}{(M,t)})$ *and* $\chi_{M,t} := \chi_{(M,t)^\vee}$, *and if* $4|M$, *then we put* $\varepsilon_{M,t} := \left(\frac{(M,t)}{t}\right)^{1/2}\bar{\iota}_{t/(M,t)}\chi_{(M/(M,t))^*}(\frac{t}{(M,t)})$ *and* $\chi_{M,t} := \chi_{-4}^{(t/(M,t)-1)/2}\chi_{(M,t)^*}$. *Then*

$$\theta(tz)|_{\mathcal{C}(N)} = \kappa_\theta\sum_{2\nmid M|N}\varepsilon_{M,t}\kappa_{\chi_{t^*},N}(\chi_{M,t},M). \tag{16}$$

(ii) *Let ρ be an even primitive character with an odd prime p as its conductor. Let ρ' denote a fixed primitive character with ${\rho'}^2 = \rho$. Let N be a multiple of $4p$. Then*

$$\begin{aligned} \theta_\rho(z)|_{\mathcal{C}(N)} = \bar{\iota}_p p^{-1/2}\kappa_\theta &\sum_{p|M|N,2\nmid M}(\chi_{p^\vee}\rho')(M/p)\{\tau(\rho')\kappa_{\rho,N}(\chi_{p^\vee}\overline{\rho}',M) \\ &+ \chi_{p^\vee}(M/p)\tau(\chi_{p^\vee}\rho')\kappa_{\rho,N}(\overline{\rho}',M)\}. \end{aligned} \tag{17}$$

As an application of the above two propositions, we have the following:

Proposition 3.3. *We have the identity*

$$\theta(z)^2 = 2G_{1,\chi_{-4}}(z).$$

Further

$$\begin{aligned}
&\theta(z)^{4k+2}(z) - L(-2k,\chi_{-4})^{-1}\{G_{2k+1,\chi_{-4}}(z) + (-1)^k 2^{2k} G^{\chi_{-4}}_{2k+1}(z)\} \in \mathbf{S}_{2k+1}(4,\chi_{-4}),\\
&\theta(z)^{4k} - (2^{2k}-1)^{-1}(2^{2k-1}-1)^{-1}\zeta(1-2k)^{-1}\\
&\quad \times [\{(-1)^k 2^{2k-1}+1-(-1)^k\}G^{\mathbf{1}_2}_{2k}(z) - G_{2k,\mathbf{1}_2}(z) - 2(2^{2k-1}-1)G_{2k,\mathbf{1}_2}(2z)] \in \mathbf{S}_{2k}(4)
\end{aligned}$$

for $k \in \mathbf{N}$.

Proof. By Proposition 3.1 and by Proposition 3.2 (i), the equality $\theta(z)^2|_{\mathcal{C}(4)} = 2G_{1,\chi_{-4}}(z)|_{\mathcal{C}(4)}$ holds and hence $\theta(z)^2 - 2G_{1,\chi_{-4}}(z)$ is a cusp form. Since $\mathbf{S}_1(4,\chi_{-4}) = \{0\}$, the identity of the proposition follows. We can apply this argument also to other modular forms. □

Since $\mathbf{S}_2(4), \mathbf{S}_3(4,\chi_{-4})$ and $\mathbf{S}_4(4)$ are a null space, the above proposition gives the equality among the powers of theta series and Eisenstein series in these cases.

Corollary 3.4. *Let* n *be a natural number. Let* $\varepsilon > 0$. *Then*

$$r_{4k+2}(n) = 2L(-2k,\chi_{-4})^{-1}\{\sigma_{2k,\chi_{-4}}(n) + (-4)^k \sigma^{\chi_{-4}}_{2k}(n)\} + O(n^{k+\varepsilon})$$

for $k \geq 0$, *where the error term vanishes when* $k = 0, 1$, *and*

$$\begin{aligned}
r_{4k}(n) = {}& \frac{2}{(2^{2k}-1)\zeta(1-2k)}[(-1)^k \sigma_{2k-1}(n)\\
&- \{1 + (-1)^k\}\sigma_{2k-1}(n/2) + 2^{2k}\sigma_{2k-1}(n/4)] + O(n^{k-1/2+\varepsilon})
\end{aligned}$$

for $k \geq 1$, *where the error term vanishes when* $k = 1, 2$.

4. Hilbert modular forms

Shimura lifting map of modular forms of half-integral weight is constructed by making use of Hilbert modular forms [18,20]. In this section we give a summary of the part of Hilbert modular forms in [20] and in [19].

Let K be a real quadratic number field. Let $\mathcal{O}_K$, $\mathfrak{d}_K$, D_K denote the ring of integers in K, the different, the discriminant respectively. We denote by tr and N, the trace map of K over $\mathbf{Q}$ and the norm map respectively. The group $\mathrm{SL}_2(\mathcal{O}_K)$ acts on $\mathfrak{H}^2$ by sending

$$\mathfrak{z} = (z_1, z_2) \in \mathfrak{H}^2 \longmapsto M\mathfrak{z} = \left(\frac{\alpha^{(1)}z_1 + \beta^{(1)}}{\gamma^{(1)}z_1 + \delta^{(1)}}, \frac{\alpha^{(2)}z_2 + \beta^{(2)}}{\gamma^{(2)}z_2 + \delta^{(2)}}\right) \quad (M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix})$$

where $\alpha = \alpha^{(1)}, \alpha^{(2)}$ denote the conjugates of α. Let $\mathfrak{N}$ be an integral ideal. Then $\mathcal{E}_{\mathfrak{N}}$ denotes the group of totally positive units congruent to 1 modulo $\mathfrak{N}$, and let $C_{\mathfrak{N}}$ denote the ideal class group modulo $\mathfrak{N}$ in the narrow sense. We denote by $C^*_{\mathfrak{N}}$, the group of characters of $C_{\mathfrak{N}}$, whose identity element is $\mathbf{1}_{\mathfrak{N}}$ for which $\mathbf{1}_{\mathfrak{N}}(\mathfrak{M})$ takes the value 1 or 0 according as $(\mathfrak{M}, \mathfrak{N}) = \mathcal{O}_K$ or not. Let $\psi \in C^*_{\mathfrak{N}}$. It is *even* or *odd* according as $\psi(\xi)$ equals 1 or $\mathrm{sgn}(\mathrm{N}(\xi))$ for $\xi \equiv 1 \pmod{\mathfrak{N}}$. Let e_ψ denote 0 or 1 according as ψ is even or odd. Then $\psi(\xi) = \mathrm{sgn}(\mathrm{N}(\xi))^{e_\psi}$ for $\xi \equiv 1 \pmod{\mathfrak{N}}$. Let $\mathfrak{f}_\psi$ denote the conductor of ψ and let $\mathfrak{e}_\psi := \mathfrak{f}_\psi \prod_{\mathfrak{P}\nmid \mathfrak{f}_\psi, \psi(\mathfrak{P})} \mathfrak{P}$, $\mathfrak{P}$ denoting a prime ideal. We denote by $\psi_{\mathfrak{M}}$, the product of characters $\widetilde{\psi}\mathbf{1}_{\mathfrak{M}}$ where $\widetilde{\psi}$ is the primitive character associated with ψ. As in the case of Dirichlet characters, the value of characters of non-integral ideals is defined to be 0.

If ψ is a primitive character of the ideal class group of K with the conductor $\mathfrak{f}_\psi$, then we define its Gauss sum $\tau_K(\psi)$ by

$$\tau_K(\psi) := \psi(\rho\mathfrak{f}_\psi\mathfrak{d}_K) \sum_{\substack{\xi \succ 0 \\ \xi : \mathcal{O}_K/\mathfrak{f}_\psi}} \psi(\xi)\mathbf{e}(\mathrm{tr}(\rho\xi))$$

with $\rho \in K, \ \succ 0, \ (\rho\mathfrak{f}_\psi\mathfrak{d}_K, \mathfrak{f}_\psi) = \mathcal{O}_K$ where $\rho \succ 0$ means that ρ is totally positive. The value $\tau_K(\psi)$ is determined up to the choices of ρ.

Let $\mathfrak{N}$, $\mathfrak{N}'$ be integral ideals. Let $\mathfrak{A}$ be an integral ideal relatively prime to $\mathfrak{N}\mathfrak{N}'$. Let k be a natural number. We define the Eisenstein series by setting for $\gamma_0 \in \mathfrak{A}\mathfrak{d}_K$ and $\delta_0 \in \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K$,

$$G_{k,\mathfrak{A}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}') := \mathrm{N}(\mathfrak{A})^k \Big(\sum_{\substack{\gamma \equiv \gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}) \\ \delta \equiv \delta_0 (\mathfrak{A}\mathfrak{d}_K^{-1}) \\ (\gamma,\delta)/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}{}^{\prime} \mathrm{N}(\gamma\mathfrak{z} + \delta)^{-k} |\mathrm{N}(\gamma\mathfrak{z} + \delta)|^{-s} \Big)\Big|_{s=0}$$

where $\mathrm{N}(\gamma\mathfrak{z} + \delta)$ denotes $\prod_{i=1}^{2}(\gamma^{(i)} z_i + \delta^{(i)})$ and where $\sum'$ means that the term for $(\gamma, \delta) = (0, 0)$ is omitted in the summation. It is a Hilbert modular form for $\Gamma_1(\mathfrak{N}\mathfrak{N}')_K$ of weight k.

Let $\psi \in C^*_{\mathfrak{N}}, \psi' \in C^*_{\mathfrak{N}'}$ which are even or odd. Let $k \in \mathbf{N}$ be so that k has the same parity as $\psi\psi'$. Then we put

$$\begin{aligned}
&\widehat{\lambda}^{\psi'}_{k,\psi}(\mathfrak{z}; \mathfrak{N}, \mathfrak{N}') \\
&:= \left(\frac{(k-1)!}{(-2\sqrt{-1}\pi)^k} \right)^2 D_K^{-1/2} [\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1} \tau_K(\overline{\widetilde{\psi}})^{-1} \sum_{\mathfrak{A} \in C_{\mathfrak{N}}} \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0}} \\
&\quad \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K) E_{k,\mathfrak{A}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')
\end{aligned}$$

where $\sum_{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0}$ implies that γ_0 runs over a complete set of totally positive representatives of $\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}$, which is a Hilbert modular form of weight k for $\Gamma_0(\mathfrak{N}\mathfrak{N}')_K$ with character $\psi\psi'$. The Fourier expansions of

$G_{k,\mathfrak{A}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$ are essentially obtained in Hecke [6] (see also [18]), and the Fourier expansion of $\widehat{\lambda}^{\psi'}_{k,\psi}(\mathfrak{z}; \mathfrak{N}, \mathfrak{N}')$ is obtained as their linear combination. Further we define the function suitable for our purpose as

$$\widetilde{\lambda}^{\psi'}_{k,\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1},\psi}(\mathfrak{z}) := \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}$$
$$\times \sum_{\substack{\mathfrak{M}|\mathfrak{N}\\ (\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K}} \left(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\right)\overline{\widetilde{\psi}}(\mathfrak{M})\widehat{\lambda}^{\psi'}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}}}(\mathfrak{z}; \mathfrak{N}\mathfrak{M}^{-1}, \mathfrak{N}').$$

We denote by μ_K, φ_K, the Möbius function on K, the Euler function on K respectively. In [19], we have shown the following:

Proposition 4.1. *We assume*

$$(\mathfrak{N}, \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}) = \mathcal{O}_K. \tag{18}$$

Let α/γ be a cusp with $\alpha, \gamma \in \mathcal{O}_K$, $(\alpha, \gamma) = \mathcal{O}_K$. The value of $\widetilde{\lambda}^{\psi'}_{k,\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1},\psi}(\mathfrak{z})$ at the cusp α/γ is 0 if there are not integral ideals $\mathfrak{M}$, $\mathfrak{M}'_\gamma$ with $\mathfrak{M}|\mathfrak{N}$, $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$, $\mathfrak{M}'_\gamma|\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}$ and with $(\gamma, \mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma\mathfrak{N}'$. Suppose otherwise and let $\mathfrak{M}$ be the largest such ideal. The value of $\widetilde{\lambda}^{\psi'}_{k,\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1},\psi}(\mathfrak{z})$ at the cusp α/γ is given by

$$\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\,\mathrm{sgn}(\mathrm{N}(\gamma))^{e_{\psi'}}\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}\mathfrak{N}'))\overline{\widetilde{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\widetilde{\psi}((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}\mathfrak{N}'))$$
$$\times \psi'(\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{M}^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}$$
$$\times \mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\overline{\widetilde{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M})^{-1}\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))$$
$$\times L_K(1-k, (\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}\mathfrak{N}')^{-1}})\prod_{\mathfrak{P}|\mathfrak{N}',\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\frac{\widetilde{\overline{\psi}\psi'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})^k}).$$

Put $\mathfrak{M}_\gamma := \gamma\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$, $\mathfrak{L}_\gamma := (\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\cap\mathcal{O}_K, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}, \mathfrak{N})^{-1})$. If $k = 1$ and if there is the divisor $\mathfrak{R}$ of $\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ such that the numerator of $\mathfrak{M}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$, then there is the additional term. Let $\widetilde{\mathfrak{R}}_\gamma$ be the divisor of $(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$ satisfying $v_\mathfrak{P}(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}) = 0$ for any prime $\mathfrak{P}|(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$. Then the additional term is

$$\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\,\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\mu_K(\widetilde{\mathfrak{R}}_\gamma)\psi(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)$$
$$\times \overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}, \mathcal{O}_K)^{-1})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}_\gamma^{-1})\mathrm{N}(\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}(\mathfrak{M}_\gamma, \widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})$$

$$\times\, \tau_K(\widetilde{\overline{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})L_K(0,(\overline{\psi}\psi')_{\mathfrak{L}_\gamma}) \prod_{\mathfrak{P}|\mathfrak{N},\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}} (1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})}).$$

Let χ, χ' be Dirichlet characters. Let

$$\psi = \chi\circ \mathrm{N},\ \ \psi' = \chi'\circ\mathrm{N}. \tag{19}$$

Then we have $L_K(s,\psi) = L(s,\chi)L(s,\chi\chi_{D_K})$ for $s\in\mathbf{C}$. The parity of ψ (resp. ψ') coincides with that of χ (resp. χ').

For a prime p, $\{\chi\}_p$ denotes the p-part of χ. If p is ramified in K, let $\mathfrak{P}_p$ denote the ramified ideal with $p=\mathfrak{P}_p^2$. If p is decomposed in K, let $\mathfrak{P}_p$, $\overline{\mathfrak{P}}_p$ denote the prime ideals with $p=\mathfrak{P}_p\overline{\mathfrak{P}}_p$.

Lemma 4.2. *Let χ be a Dirichlet character with prime power conductor $\mathfrak{f}_\chi$. Let ψ be as in (19).*

(i) *Suppose that $(\mathfrak{f}_\chi, D_K)=1$. Then $\mathfrak{f}_\psi=\mathfrak{f}_\chi$.*

(ii) *Suppose that 2 is ramified in K, namely, D_K is even. If $\chi=\chi_{-4}$, then $\mathfrak{f}_\psi|(2)$, and $\mathfrak{f}_\psi=\mathcal{O}_K$ only if $4\|D_K$. If $\chi=\chi_{\pm8}$, then $\mathfrak{f}_\psi=(4)$ for D_K with $4\|D_K$, and $\mathfrak{f}_\psi=(2)$ for D_K with $D_K/8\equiv\mp1 \pmod 4$, and $\mathfrak{f}_\psi=\mathcal{O}_K$ if $D_K/8\equiv\pm1 \pmod 4$.*

(iii) *Suppose that $\mathfrak{f}_\chi$ is an odd prime p and suppose that $p|D_K$. Let $\mathfrak{P}_p$ be the ideal of K with ${\mathfrak{P}_p}^2=p$. If $\chi=\chi_{p^\vee}$, then $\mathfrak{f}_\psi=\mathcal{O}_K$. If otherwise, then $\mathfrak{f}_\psi=\mathfrak{P}_p$.*

Lemma 4.3. *Let χ, ψ be as in (19). Let $\widetilde{\psi}$ denote the primitive character associated with ψ.*

(i) *Let $\chi=\chi_{-4}$. Then $\tau_K(\psi)=-4$ if $D_K\equiv1 \pmod 4$. Assume that $D_K\not\equiv1 \pmod 4$. Then $\tau_K(\widetilde{\psi})$ is equal to -1 or -2 according as $D_K\equiv3 \pmod 4$ or $D_K\equiv0 \pmod 8$.*

(ii) *Let $\chi=\chi_{\pm8}$. Then $\tau_K(\psi)=\pm8$ if $D_K\equiv1 \pmod 4$. Assume that $D_K\not\equiv1 \pmod 4$. Then $\tau_K(\widetilde{\psi})$ is equal to ±4 if $D_K\equiv3 \pmod 4$, and it is equal to ±1 or ±2 according as $D_K/8\equiv\pm1 \pmod 4$ or $D_K/8\equiv\mp1 \pmod 4$.*

(iii) *Let χ be a primitive character with odd prime conductor p. If $p|D_K$ and $\chi=\chi_{p^\vee}$, then $\tau_K(\widetilde{\psi})=\chi_{-4}(p)$. If otherwise, $\tau_K(\psi)=(\chi\chi_{p^\vee})(D_K)\tau(\chi)^2$.*

Let χ, ψ be as in (19). Then Lemma 4.2 implies that $\mathfrak{e}_\psi$ is the product of $\mathfrak{f}_\psi$ and of the following ideals with multiplicity 1:

$$\begin{aligned}
&\mathfrak{P}_2\ \ \text{if } 4\|D_K \text{ and } \{\chi\}_2=\chi_{-4}, \text{ or } D_K/8\equiv s \bmod 4 \text{ and } \{\chi\}_2=\chi_{8s} \text{ with } s=\pm1,\\
&\mathfrak{P}_p\ \ \text{if } 2\nmid p|D_K \text{ and } \chi(p)=0\ \text{ and if } \{\chi\}_p=\chi_{p^\vee} \text{ or } p\nmid\mathfrak{f}_\chi,\\
&(p)\ \ \text{if } 2\nmid p\nmid D_K\mathfrak{f}_\chi \text{ and } \chi(p)=0.
\end{aligned}$$

Let us fix an integral ideal $\mathfrak{T}$ whose prime factors are divisors of $\mathfrak{e}_\psi\mathfrak{e}_{\psi'}$. Then there are ideals $\mathfrak{N}$, $\mathfrak{N}'$ satisfying the condition (18) with (i) $\mathfrak{e}_\psi|\mathfrak{N}$, $\mathfrak{e}_{\psi'}|\mathfrak{N}'$ and (ii) $\mathfrak{T}=\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}'}$, which are uniquely determined. We define

$$\lambda^{\chi'}_{2k,\mathfrak{T},\chi}(z) := \widetilde{\lambda}^{\psi'}_{k,\mathfrak{N}\mathfrak{e}_{\psi}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1},\psi}(z,z)$$

which is an elliptic modular for of weight $2k$ with character $(\chi\chi')^2$. If i/M is a rational number. The value of $\lambda^{\chi'}_{2k,\mathfrak{T},\chi}(z)$ at the cusp i/M coincide with that of $\widetilde{\lambda}^{\psi'}_{k,\mathfrak{N}\mathfrak{f}_{\psi}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1},\psi}(\mathfrak{z})$ at the cusp i/M. Hence Proposition 4.1 gives the values of $\lambda^{\chi'}_{2k,\mathfrak{T},\chi}(z)$ at all the cusps.

5. Shimura lifts

In this section we explain the Shimura lifting map. As its application, we obtain approximate formulas for the number of representation as a sum of any odd number of squares, which may provide a good illustration of our method.

Let $4|N$ and $k \geq 1$. The Shimura lifting map $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}$ associated with a square-free natural number a, of $\mathbf{M}_{k+1/2}(N,\chi_0)$ to $M_{2k}(N/2,\chi_0^2)$ is defined by

$$\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(f)(z) := C + \sum_{n=1}^{\infty}\left(\sum_{0<d|n}(\chi_{a^*}\chi_0\mathbf{1}_N)(d)d^{k-1}c_{an^2/d^2}\right)\mathbf{e}(nz) \tag{20}$$

for $f(z) = \sum_{n=0}^{\infty} c_n\mathbf{e}(nz)$ provided that there is a constant C so that $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(f)$ is a modular form of weight $2k$. Shimura [17] and Niwa [13] showed that $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}$ is well-defined map of $\mathbf{S}_{3/2}(N,\chi_0)$ to $M_2(N/2,\chi_0^2)$, and is a well-defined map of $\mathbf{S}_{k+1/2}(N,\chi_0)$ to $\mathbf{S}_{2k}(N/2,\chi_0^2)$ for $k \geq 2$. In Pei [14,15], the domain of the map extends to $\mathbf{M}_{k+1/2}(N,\chi_0)$ under the condition that $N/4$ is square-free and χ_0 is real, and in Tsuyumine [18,20] unconditionally. If we put $g(z) := \mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(f)(z)$ in (20) and if $g(z) = \sum_{n=0}^{\infty} b_n\mathbf{e}(nz)$, then

$$c_{an^2} = \sum_{d|n}(\mu\chi_{a^*}\chi_0\mathbf{1}_N)(d)d^{k-1}b_{n/d}. \tag{21}$$

Let T_p, T_{p^2} denote, as usual, the Hecke operators on $M_{2k}(N/2,\chi_0^2)$ and on $\mathbf{M}_{k+1/2}(N,\chi_0)$ respectively for a prime p with $p\nmid N$. Then $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}\circ T_{p^2} = T_p\circ\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}$. Let m be a positive integer whose prime factors all divide N. We define an operator U_m by $U_m(\sum_{n=0}^{\infty} c_n\mathbf{e}(nz)) = \sum_{n=0}^{\infty} c_{mn}\mathbf{e}(nz)$. Then $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}\circ U_{p^2} = U_p\circ\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}$ for $p|N$.

We define a notation

$$f \sim g \tag{22}$$

which means that $f-g$ is a cusp form when $f,g \in \mathbf{M}_k(N,\chi_0)$, and that $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(f-g)$ is a cusp form for any square-free natural number a when $f,g \in \mathbf{M}_{k+1/2}(N,\chi_0)$. In the half-integral weight case, $f \sim g$ is equivalent to $f-g \in \mathbf{S}_{k+1/2}(N,\chi_0)$ if the weight $k+1/2$ is at least $5/2$. If the weight is $3/2$, then there are cusp forms lifted to non-cusp forms. All of them are written as linear combinations of the cusp forms

$$\Theta_\phi(tz) := \sum_{n\in\mathbf{Z}} \phi(n)n\mathbf{e}(n^2tz) \in \mathbf{S}_{3/2}(4t\mathfrak{f}_\phi^2, \phi\chi_{t^*}) \tag{23}$$

for an odd Dirichlet character ϕ and for $t \in \mathbf{N}$ (Duke and Schulze-Pillot [4]). Let p be an odd prime. If $\chi \in (\mathbf{Z}/p)^*$ is even, then $\mathbf{M}_{3/2}(4p^2, \chi_{-4}\chi)$ does not contain any cusps form such as (23). Hence $f \sim g$ for $f, g \in \mathbf{M}_{3/2}(4p^2, \chi_{-4}\chi)$ if and only if $f - g \in \mathbf{S}_{3/2}(4p^2, \chi_{-4}\chi)$. As far as we are concerned in the present paper, the notation (22) is equivalent to that the difference of the both sides is a cusp form.

In [20] Section 18, we have proved the following:

Lemma 5.1. *Let N and let χ_0 be a Dirichlet character modulo a divisor of N. Suppose that there is a prime p with $p^2|N$ and $\mathfrak{f}_{\chi_0}|N/p$. Let $f \in \mathbf{M}_k(N, \chi_0)$ and put $g := U_p(f) \in \mathbf{M}_k(N/p, \chi_0)$. We denote by $\kappa(\frac{j}{M}, f)$, the value of f at the cusp j/M. Then the value of $g(z)$ at a cusp $j/M \in \mathcal{C}_0(N/p)$ is $p^{-1}\sum_{i=1}^{p} \kappa(\frac{j}{pM} + \frac{i}{p}, f)$ if $p|M$, and it is $p^{k-1}\kappa(\frac{p'j}{M}, f) + p^{-1}\sum_{\substack{1\le i\le p\\ Mi\not\equiv -j\,(p)}} \kappa(\frac{j}{pM} + \frac{i}{p}, f)$ if $(p, M) = 1$, where p' denote the inverse of p modulo M.*

Let $a > 1$ be a square-free natural number. Let $K = \mathbf{Q}(\sqrt{a})$. Then $D_K = a^*$. For $t \in \mathbf{N}$, $\mathbf{I}_K(t)$ denotes the set of integral ideals of K whose norms equal t. Let

$$t = t_0t_1t_{-1}$$

be the decomposition such that prime factors p of t_i satisfy $\chi_{a^*}(p) = i$ $(i = 0, \pm 1)$. The set $\mathbf{I}_K(t)$ is not empty if and only if t_{-1} is square. We denote by t'_{-1}, the product of prime factors p of t_{-1} where t_{-1} has p in odd power. Then $\mathbf{I}_K(tt'_{-1}) \ne \emptyset$. We define $m\mathbf{I}_K(t) := \{m\mathfrak{T} \mid \mathfrak{T} \in \mathbf{I}_K(t)\}$ for $m \in \mathbf{Z}$.

We have proved in [18,20], the following result:

Theorem 5.2. *Let χ, χ' be a Dirichlet character. Put $\chi_0 := \chi\chi'$. We assume that $k \in \mathbf{N}$ and χ_0 have the same parity. Let t be a natural number all of whose prime factors p satisfy $\chi_0(p) = 0$. Let N be a natural number divisible by 4 with $G_{k,\chi}^{\chi'}(tz) \in \mathbf{M}_k(N, \chi_0)$. Then if $\chi(p) = \chi'(p) = 0$ for some p with $p|t'_1$, then $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(\theta(z)G_{k,\chi}^{\chi'}(tz)) = 0$. If otherwise, $\mathscr{S}_{a^*,\chi_0\mathbf{1}_N}(\theta(z)G_{k,\chi}^{\chi'}(tz))$ equals $2^{-1}U_2(\lambda_{2k,\chi}^{\chi'}(z; a^*, t)) \in \mathbf{M}_{2k}(N/2, (\chi\chi')^2)$ for $a \equiv 1 \pmod 4$, $2^{-1}\lambda_{2k,\chi}^{\chi'}(z; a^*, t) \in \mathbf{M}_{2k}(N/2, (\chi\chi')^2)$ for $a \not\equiv 1 \pmod 4$ where*

$$\lambda_{2k,\chi}^{\chi'}(z; a^*, t) := \prod_{p|t'_{-1}} (\chi(p)p^{k-1}+\chi'(p))^{-1} \sum_{m|t_1} \mu(m) \prod_{p|m} (\chi(p)p^{k-1}+\chi'(p)) \sum_{\mathfrak{T}\in m\mathbf{I}_K(tt'_{-1}/m)} \lambda_{k,\mathfrak{T},\chi}^{\chi'}(z).$$

When $a = 1$, we define

$$\lambda_{2k,\chi}^{\chi'}(z; 1, t) := \sum_{s|t} \mu(s) \prod_{p|\tilde{s}} \{\chi(p)p^{k-1} + \chi'(p)\} \sum_{\substack{s_1s_2=ts\\ s|s_1, s|s_2}} G_{k,\chi}^{\chi'}(s_1z)G_{k,\chi}^{\chi'}(s_2z).$$

Then the assertion of Theorem 5.2 holds also for $a = 1$.

As an application, we obtain approximate formulas for the number of representation as a sum of any odd number of squares. Let F be a positive integral quadratic forms with $2k+1$ $(k \geq 1)$ variables, and let $F_1 = F, F_2, \cdots, F_g$ be representatives of equivalence classes of quadratic forms in the genus $\boldsymbol{G}$ to which F belongs. Let $\mathrm{Aut}(F_i) := \{P \in \mathrm{GL}_{2k+1}(\mathbf{Z}) \mid PF_i{}^tP = F_i\}$. The mass $m(\boldsymbol{G})$ of $\boldsymbol{G}$ is defined by $m(\boldsymbol{G}) := \sum_{i=1}^{g}(\#\mathrm{Aut}(F_i))^{-1}$. Let $\theta_{F_i} \in \mathbf{M}_{k+1/2}(N, \chi_0)$ be the theta series associated with F_i, and let $\theta(z; \boldsymbol{G})$ the theta series of the genus $\boldsymbol{G}$ which is given by $\theta(z; \boldsymbol{G}) := \frac{1}{m(\boldsymbol{G})}\sum_{i=1}^{g}\frac{1}{\#\mathrm{Aut}(F_i)}\theta_{F_i}(z)$. Then $\theta(z; \boldsymbol{G}) \sim \theta_F(z)$ provided that any cusp form such as (23), is not contained in $\mathbf{M}_{k+1/2}(N, \chi_0)$ [4]. In such a case if $\theta_F(z) - \theta(z; \boldsymbol{G}) = \sum_{n=1}^{\infty} c_n \mathbf{e}(nz)$, then $c_n = O(n^{k/2-1/28+\varepsilon})$ for any fixed $\varepsilon > 0$ by Iwaniec [9] and by Duke and Schulze-Pillot [4]. The Shimura lift of $\theta(z; \boldsymbol{G})$ is an Eisenstein series [20]. If the Eisenstein series is obtained, then the Fourier coefficients of $\theta(z; \boldsymbol{G})$ are obtained by (21), to which the corresponding Fourier coefficients of $\theta_F(z)$ are close, up to $O(n^{k/2-1/28+\varepsilon})$. We consider the case that F is the identity matrix.

If $k \geq 1$ is odd, then $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_{k,\chi_{-4}}(z))$ is $2^{-1}U_2(\lambda_{2k,\mathcal{O}_K,\chi_{-4}}(z))$ or $2^{-1}\times \lambda_{2k,\mathcal{O}_K,\chi_{-4}}(z)$ according as $a \equiv 1 \pmod 4$ or not, and $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_k^{\chi_{-4}}(z))$ is $2^{-1}U_2(\lambda_{2k,\mathcal{O}_K}^{\chi_{-4}}(z))$ or $2^{-1}\lambda_{2k,\mathcal{O}_K}^{\chi_{-4}}(z)$ according as $a \equiv 1 \pmod 4$ or not. If k is even, then $\mathscr{S}_{a^*,\mathbf{1}_2}(\theta(z)G_k^{\mathbf{1}_2}(z))$ is $2^{-1}U_2(\lambda_{2k,\mathcal{O}_K}^{\mathbf{1}_2}(z))$ or $2^{-1}\lambda_{2k,\mathcal{O}_K}^{\mathbf{1}_2}(z)$ according as $a \equiv 1 \pmod 4$ or not, and $\mathscr{S}_{a^*,\mathbf{1}_2}(\theta(z)G_{k,\mathbf{1}_2}(2z))$ is $2^{-1}U_2(\lambda_{2k,\mathfrak{P}_2,\mathbf{1}_2}(z)+\lambda_{2k,\overline{\mathfrak{P}}_2,\mathbf{1}_2}(z)-\lambda_{2k,(2),\mathbf{1}_2}(z))$ (resp. $2^{-1}U_2(\lambda_{2k,(2),\mathbf{1}_2}(z))$, resp. $2^{-1}\lambda_{2k,\mathfrak{P}_2,\mathbf{1}_2}(z)$ for $a \equiv 1 \pmod 8$ (resp. $a \equiv 5 \pmod 8$, resp. $a \not\equiv 1 \pmod 4$). By Proposition 4.1 and by Lemma 5.1, we have

$$\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_{1,\chi_{-4}}(z))|_{\mathcal{C}(2)} = 2^{-2}L(0,\chi_{-4a})\{2\kappa_{\mathbf{1}_2,2}(\mathbf{1},2) - \kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}. \tag{24}$$

Let $k > 1$ be odd and put $l_a := L(1-k,\chi_{-4})L(1-k,\chi_{-4a})$ for $a \not\equiv 3 \pmod 4$ and $l_a := L(1-k,\chi_{-4})L(1-k,\chi_{-a})$ for $a \equiv 3 \pmod 4$. Then by Proposition 4.1 and Lemma 5.1, $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_{k,\chi_{-4}}(z))|_{\mathcal{C}(2)}$ is equal to $2^{-1}l_a\kappa_{\mathbf{1}_2,2}(\mathbf{1},2)$ for $a \not\equiv 3 \pmod 4$, and $2^{-2}l_a\{2(1-\chi_8(a)2^{k-1})\kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + \kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \equiv 3 \pmod 4$), and $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_k^{\chi_{-4}}(z))|_{\mathcal{C}(2)}$ is equal to $-2^{-2k}l_a\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)$ for $a \not\equiv 3 \pmod 4$, and $-2^{-1}(1-\chi_8(a)2^{-k})l_a\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)$ for $a \equiv 3 \pmod 4$.

Let k be even and put $l'_a := \zeta(1-k)L(1-k,\chi_a)$ for $a \equiv 1 \pmod 4$ and $l'_a := \zeta(1-k)L(1-k,\chi_{4a})$ for $a \not\equiv 1 \pmod 4$. Then $\mathscr{S}_{a^*,\mathbf{1}_2}(\theta(z)G_k^{\mathbf{1}_2}(z))|_{\mathcal{C}(2)}$ is equal to $2^{-2}(2^k-1)(2^k-\chi_8(a))l'_a\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \equiv 1 \pmod 4$, and $2^{-1}(1-2^{-k})l'_a\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)$ for $a \not\equiv 1 \pmod 4$, and $\mathscr{S}_{a^*,\mathbf{1}_2}(\theta(z)G_{k,\mathbf{1}_2}(z))|_{\mathcal{C}(2)}$ is equal to $l'_a[2^{-1}(1-2^{k-1})^2 \times \kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + 2^{-2}\{1-2^k+2^{2k-1}\}\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)]$ for $a \equiv 1 \pmod 8$, $l'_a\{2^{-1}(1-2^{2k-2})\times\kappa_{\mathbf{1}_2,2}(\mathbf{1},2)+2^{-2}(1+2^{2k-1})\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \equiv 5 \pmod 8$, $l'_a\{2^{-1}(1-2^{k-1})\kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + 2^{-2}\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \not\equiv 1 \pmod 4$, and $\mathscr{S}_{a^*,\mathbf{1}_2}(\theta(z)G_{k,\mathbf{1}_2}(2z))|_{\mathcal{C}(2)}$ is equal to $l'_a\{2^{-1}(1-2^{k-1})^2\kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + 2^{-3}\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \equiv 1 \pmod 8$, $l'_a\{2^{-1}(1-2^{2k-2}) \times \kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + 2^{-3}3\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \equiv 5 \pmod 8$, $l'_a\{2^{-1}(1-2^{k-1})\kappa_{\mathbf{1}_2,2}(\mathbf{1},2) + 2^{-k-2}\kappa_{\mathbf{1}_2,2}(\mathbf{1},1)\}$ for $a \not\equiv 1 \pmod 4$.

Corollary 5.3. *Let a be a square-free natural number.*

(i) *Let $k \geq 0$. Then $r_{4k+3}(an^2)$ is given up to $O(\{an^2\}^{k+13/28+\varepsilon})$ by $\frac{L(-2k,\chi_{-4a})}{\{1+(-1)^{k-1}2^{2k+1}\}\zeta(-4k-1)}\sum_{d|n}(\mu\chi_{-4a})(d)d^{2k}\{\sigma_{4k+1}(n)+(-1)^{k-1}2^{2k+1}\sigma_{4k+1}(n/2)\}$ for $a \not\equiv 3 \pmod 4$, and $\frac{L(-2k,\chi_{-a})}{\{1+(-1)^{k-1}2^{2k+1}\}\zeta(-4k-1)}\sum_{d|n}(\mu\chi_{-a}\mathbf{1}_2)(d)d^{2k}[\{2^{4k+1}+(-1)^{k-1}2^{2k+1}+1-2^{2k}\chi_8(a)\}\sigma_{4k+1}(n/d)-2^{4k+1}\{(-1)^{k-1}\chi_8(a)+1\}\sigma_{4k+1}(n/(2d))]$ for $a \equiv 3 \pmod 4$, where the error term is 0 if $k=0,1$.*

(ii) *Let $k \geq 1$. Then $r_{4k+1}(an^2)$ is given up to $O(\{an^2\}^{k+13/28+\varepsilon})$ by $\frac{(2^{2k-1}-1)L(1-2k,\chi_a)}{(2^{4k}-1)\zeta(1-4k)}\sum_{d|n}(\mu\chi_a\mathbf{1}_2)(d)d^{2k-1}[\{2^{4k-1}+1+(-1)^k2^{4k-1}(2^{2k}-1)\}\sigma_{4k-1}(n)-2^{4k}\{(-1)^k(2^{2k}-1)+3\}\sigma_{4k-1}(n/2)]$ for $a \equiv 1 \pmod 8$, and $\frac{L(1-2k,\chi_a)}{(2^{2k}-1)\zeta(1-4k)}\times\sum_{d|n}(\mu\chi_a\mathbf{1}_2)(d)d^{2k-1}[\{2^{2k-1}-1+(-1)^k2^{4k-1}\}\sigma_{4k-1}(n)+2^{4k-1}\{1-(-1)^k\}\times\sigma_{4k-1}(n/2)]$ for $a \equiv 5 \pmod 8$, and $\frac{L(1-2k,\chi_{4a})}{(2^{4k}-1)\zeta(1-4k)}\sum_{d|n}(\mu\chi_{4a})(d)d^{2k-1}[-\{1-(-1)^k2^{2k}\}\sigma_{4k-1}(n)+2^{2k}\{2^{2k}-(-1)^k\}\sigma_{4k-1}(n/2)]$ for $a \not\equiv 1 \pmod 4$, where the error term is 0 if $k=1$.*

Proof. We write $\kappa_{\mathbf{1}_2,2}(\mathbf{1},2), \kappa_{\mathbf{1}_2,2}(\mathbf{1},1)$ as $\kappa(1)$, $\kappa(2)$ respectively for short.

(i) If $k \geq 1$, then

$$\theta(z)^{4k+3}(z) \sim L(-2k,\chi_{-4})^{-1}\{\theta(z)G_{2k+1,\chi_{-4}}(z)+(-1)^k2^{2k}\theta(z)G_{2k+1}^{\chi_{-4}}(z)\}$$

by Proposition 3.3. Let us suppose $a \not\equiv 3 \pmod 4$. Then by the calculation before Corollary 5.3, $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^{4k+3}(z))|_{\mathcal{C}(2)} = L(-2k,\chi_{-4a})\{2^{-1}\kappa(2)+(-1)^{k-1}2^{-2k-2}\kappa(1)\}$, which holds also in the case $k=0$ by (24). Hence by Proposition 3.1(i), we have

$$\begin{aligned}&\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^{4k+3}(z))\\ &\quad\sim \tfrac{L(-2k,\chi_{-4a})}{(1+(-1)^{k-1}2^{2k+1})\zeta(-4k-1)}\{2^{-1}G_{4k+2}(z)+(-1)^{k-1}2^{2k}G_{4k+2}(2z)\}.\end{aligned}$$

The Fourier expansion of the right hand side is obtained from (12). The formula for the case $a \not\equiv 3 \pmod 4$ follows from this. Since $\mathbf{S}_{3/2}(4,\chi_{-4})=\{0\}$ and $\mathbf{S}_{7/2}(4,\chi_{-4})=\{0\}$, the error term vanishes if $k=0,1$. The estimate of error terms comes from the estimate of the Fourier coefficients of a cusp form of weight $2k+3/2$ by Iwaniec [9]. Suppose $a \equiv 3 \pmod 4$. By the calculation before Corollary 5.3,

$$\begin{aligned}&\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^{4k+3}(z))|_{\mathcal{C}(2)}\\ &\quad= L(-2k,\chi_{-a})[2^{-1}(1-\chi_8(a)2^{2k})\kappa(2)+2^{-2}\{1+(-1)^k\chi_8(a)+(-1)^{k-1}2^{2k+1}\}\kappa(1)],\end{aligned}$$

which holds including the case $k=0$. By Proposition 3.1(i),

$$\begin{aligned}&\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^{4k+3}(z))\\ &\quad\sim \tfrac{L(-2k,\chi_{-a})}{(1+(-1)^{k-1}2^{2k+1})\zeta(-4k-1)}[2^{-1}\{2^{4k+1}+(-1)^{k-1}2^{2k+1}+1-2^{2k}\chi_8(a)\}G_{4k+2}(z)\\ &\qquad-2^{4k}\{(-1)^{k-1}\chi_8(a)+1\}G_{4k+2}(2z)].\end{aligned}$$

This proves the formula in the case $a \equiv 3 \pmod 4$.

(ii) By Proposition 3.3, we have

$$\theta(z)^{4k+1}(z) \sim \tfrac{1}{(2^{2k}-1)(2^{2k-1}-1)\zeta(1-2k)}[\{(-1)^k 2^{2k-1} + 1 - (-1)^k\}\theta(z)G_{2k}^{\mathbf{1}_2}(z) \\ - \theta(z)G_{2k,\mathbf{1}_2}(z) - 2(2^{2k-1}-1)\theta(z)G_{2k,\mathbf{1}_2}(2z)].$$

If $a \equiv 1 \pmod 8$, then

$$\mathscr{S}_{a,\mathbf{1}_2}(\theta(z)^{4k+1}(z))|_{\mathcal{C}(2)} \\ = 2^{-2}(2^{2k-1} - 1)L(1-2k, \chi_a)\{-2\kappa(2) + \{(-1)^k(2^(2k) - 1) + 1\}\kappa(1)$$

by the calculation before Corollary 5.3. Then by Proposition 3.1(i),

$$\mathscr{S}_{a,\mathbf{1}_2}(\theta(z)^{4k+1}(z)) \sim \tfrac{(2^{2k-1}-1)L(1-2k,\chi_a)}{2^2(2^{4k}-1)\zeta(1-4k)}[\{2(2^{4k-1} + 1) + (-1)^k 2^{4k}(2^{2k} - 1)\}G_{4k}(z) \\ - 2^{4k}((-1)^k(2^{2k} - 1) + 3)G_{4k}(2z)].$$

The formula for $a \equiv 1 \pmod 8$ follows from this. If $a \equiv 5 \pmod 8$, then

$$\mathscr{S}_{a,\mathbf{1}_2}(\theta(z)^{4k+1}(z))|_{\mathcal{C}(2)} \\ = 2^{-2}L(1-2k, \chi_a)[2(2^{2k-1} + 1)\kappa(2) + \{1 + (-1)^k(2^{2k} + 1)\}\kappa(1)].$$

Then by Proposition 3.1 (i),

$$\mathscr{S}_{a,\mathbf{1}_2}(\theta(z)^{4k+1}(z)) \\ \sim \tfrac{L(1-2k,\chi_a)}{2(2^{2k}-1)\zeta(1-4k)}[\{(2^{2k-1} - 1) + (-1)^k 2^{4k-1}\}G_{4k}(z) + 2^{4k-1}\{1 - (-1)^k\}G_{4k}(2z)].$$

The formula for $a \equiv 5 \pmod 8$ follows from this. If $a \not\equiv 1 \pmod 4$, then $\mathscr{S}_{4a,\mathbf{1}_2}(\theta(z)^{4k+1}(z))|_{\mathcal{C}(2)} = 2^{-2k-1}L(1-2k, \chi_{4a})\{2^{2k}\kappa(2) + (-1)^k\kappa(1)\}$. Then

$$\mathscr{S}_{4a,\mathbf{1}_2}(\theta(z)^{4k+1}(z)) \\ \sim \tfrac{L(1-2k,\chi_{4a})}{2(2^{4k}-1)\zeta(1-4k)}[\{-1 + (-1)^k 2^{2k}\}G_{4k}(z) + 2^{2k}\{2^{2k} - (-1)^k\}G_{4k}(2z)].$$

We have proved the formula. $\square$

6. Theta series on $\Gamma_0(4p^2)$

Let p be an odd prime. The generating function of $r^{(p)}_{(\alpha_1,\alpha_2,\alpha_3)}$ is given by $\prod_{i=1}^3 \sum_{n\equiv\pm\alpha_i \pmod p} \mathbf{e}(n^2 z) = \sum_{n=0}^\infty r^{(p)}_{(\alpha_1,\alpha_2,\alpha_3)}(n)\mathbf{e}(nz)$. However if $\sum_{i=1}^3 \alpha_i^2 \equiv m \pmod p$, then the m-th Fourier coefficient of $\theta(z)\prod_{i=1}^2 \sum_{n\equiv\pm\alpha_i \pmod p} \mathbf{e}(n^2 z)$ equals $r^{(p)}_{(\alpha_1,\alpha_2,\alpha_3)}(m)$. Let ω be a generator of the group $(\mathbf{Z}/p)^*$. We have $\chi_{p^\vee} = \omega^{(p-1)/2}$. All even characters in $(\mathbf{Z}/p)^*$ are written in the form ω^{2i}. From (5), it follows that if

$\alpha_2, \alpha_3 \not\equiv 0 \pmod{p}$ and if $\sum_{i=1}^{3} \alpha_i^2 \equiv n \pmod{p}$, then $r^{(p)}_{(\alpha_1,\alpha_2,\alpha_3)}(n)$ is equal to the n-th Fourier coefficient of

$$\left(\frac{2}{p-1}\right)^2 \theta(z) \sum_{i=0}^{p-2} \overline{\omega}(\alpha_2)^{2i} \theta_{\omega^{2i}}(z) \sum_{j=0}^{p-2} \overline{\omega}(\alpha_3)^{2j} \theta_{\omega^{2j}}(z).$$

If $\alpha_2 \not\equiv 0 \pmod{p}$ and if $\alpha_1^2 + \alpha_2^2 \equiv n \pmod{p}$, then $r^{(p)}_{(\alpha_1,\alpha_2,0)}(n)$ is equal to the n-th Fourier coefficient of

$$\frac{2}{p-1} \theta(z) \sum_{i=0}^{p-2} \overline{\omega}(\alpha_2)^{2i} \theta_{\omega^{2i}}(z) \theta(p^2 z).$$

The space $\mathbf{M}(4p^2, \chi_{-4}\chi)$ does not contain cusp forms such as (23) for χ even, and hence $f \in \mathbf{M}(4p^2, \chi_{-4}\chi)$ is written as $f(z) = E_{3/2}(z) + g(z)$ where $E_{3/2}(z)$ is an Eisenstein series of weight 3/2, and g is a cusp form any of whose Shimura lift is a cusp form. By Duke and Schulze-Pillot [4], the n-th Fourier coefficients of $f(z)$ agree with that of $E_{3/2}(z)$ up to $O(n^{13/28+\varepsilon})$ for $\varepsilon > 0$. Since all the Shimura lifting maps are Hecke equivariant, the Shimura lifts of $E_{3/2}(z)$ are Eisenstein series of weight 2. If all such Eisenstein series of weight 2 are obtained, then the Fourier coefficients of $E_{3/2}(z)$ are obtained by (21), and hence the Fourier coefficients of $f(z)$ up to $O(n^{13/28+\varepsilon})$. We need to write the Shimura lifts of $\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$ $(0 \le i, j < (p-1)/2)$ and $\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2 z)$ $(0 \le i < (p-1)/2)$ by Eisenstein series up to cusp forms.

We express $\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$, $\theta_{\omega^{2i}}(z)\theta(p^2 z)$ as a linear combination of Eisenstein series up to a cusp form to make use of Theorem 5.2. By (17), $\theta_{\omega^{2i}}(z)|_{\mathcal{C}(4p^2)} = \bar{\imath}_p p^{-1/2} \kappa_\theta \{\tau(\omega^i) \kappa_{\omega^{2i},4p^2}(\chi_{p^\vee}\omega^{-i}, p) + \tau(\chi_{p^\vee}\omega^i) \kappa_{\omega^{2i},4p^2}(\omega^{-i}, p) + \omega(2)^{2i} \tau(\omega^i) \times \kappa_{\omega^{2i},4p^2}(\chi_{p^\vee}\omega^{-i}, 4p) + \omega(2)^{2i} \tau(\chi_{p^\vee}\omega^i) \kappa_{\omega^{2i},4p^2}(\omega^{-i}, 4p)\}$ for $1 \le i < (p-1)/2$. By (16), $\theta(p^2 z)|_{\mathcal{C}_0(4p^2)} = \kappa_\theta \{\kappa_{\mathbf{1},4p^2}(\mathbf{1}, p^2) + \kappa_{\mathbf{1},4p^2}(\mathbf{1}, 4p^2) + p^{-1/2} \bar{\imath}_p \kappa_{\mathbf{1},4p^2}(\chi_{p^\vee}, p) + p^{-1/2} \bar{\imath}_p \kappa_{\mathbf{1},4p^2}(\chi_{p^\vee}, 4p) + p^{-1} \kappa_{\mathbf{1},4p^2}(\mathbf{1}, 4) + p^{-1} \kappa_{\mathbf{1},4p^2}(\mathbf{1}, 1)\}$. Since $\theta_{\mathbf{1}_p}(z) = \theta(z) - \theta(p^2 z)$, $\theta_{\mathbf{1}_p}(z)|_{\mathcal{C}(4p^2)} = \kappa_\theta \{\kappa_{\mathbf{1},4p^2}(\mathbf{1}, 4p) - p^{-1/2} \bar{\imath}_p \kappa_{\mathbf{1},4p^2}(\chi_{p^\vee}, 4p) + \kappa_{\mathbf{1},4p^2}(\mathbf{1}, p) - p^{-1/2} \bar{\imath}_p \kappa_{\mathbf{1},4p^2}(\chi_{p^\vee}, p) + (1 - p^{-1}) \kappa_{\mathbf{1},4p^2}(\mathbf{1}, 4) + (1 - p^{-1}) \kappa_{\mathbf{1},4p^2}(\mathbf{1}, 1)\}$. Multiplying these and applying $J(\chi, \phi) = \tau(\chi)\tau(\phi)/\tau(\chi\phi)$ for χ, ψ with χ, ψ, $\chi\phi \ne \mathbf{1}_p$, we obtain the following lemma:

Lemma 6.1. *Let $1 \le i, j < (p-1)/2$.*

(i) *Suppose $i + j \ne (p-1)/2$. Then $\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)|_{\mathcal{C}(4p^2)}$ is equal to*

$$\begin{aligned}
&\chi_{-4}(p) p^{-1} [\tau(\omega^{i+j}) \{J(\omega^i, \omega^j) + J(\chi_{p^\vee}\omega^i, \chi_{p^\vee}\omega^j)\} \{-2^{-1}\sqrt{-1}\chi_{-4}(p) \\
&\quad \times \kappa_{\chi_{-4}\omega^{2i+2j},4p^2}(\omega^{-i-j}, p) + \omega(2)^{2i+2j} \kappa_{\chi_{-4}\omega^{2i+2j},4p^2}(\chi_{-4}\omega^{-i-j}, 4p)\} + \tau(\chi_{p^\vee}\omega^{i+j}) \\
&\quad \times \{J(\chi_{p^\vee}\omega^i, \omega^j) + J(\omega^i, \chi_{p^\vee}\omega^j)\} \{-2^{-1}\sqrt{-1}\chi_{-4}(p) \kappa_{\chi_{-4}\omega^{2i+2j},4p^2}(\chi_{p^\vee}\omega^{-i-j}, p) \\
&\quad + \omega(2)^{2i+2j} \kappa_{\chi_{-4}\omega^{2i+2j},4p^2}(\chi_{-4}\chi_{p^\vee}\omega^{-i-j}, 4p)\}].
\end{aligned}$$

(ii) *Suppose that* $i+j=(p-1)/2$. *Then* $\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)|_{\mathcal{C}(4p^2)}$ *is equal to*

$$\chi_{-4}(p)p^{-1}[\iota_p p^{1/2}\{J(\omega^i,\omega^j)+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j)\}\{-2^{-1}\sqrt{-1}\chi_{-4}(p)\kappa_{\chi_{-4},4p^2}(\chi_{p^\vee},p) + \kappa_{\chi_{-4},4p^2}(\chi_{-4}\chi_{p^\vee},4p)\}+\omega(-1)^i(\chi_{p^\vee}(-1)+1)p\{-2^{-1}\sqrt{-1}\chi_{-4}(p)\kappa_{\chi_{-4},4p^2}(\mathbf{1}_p,p) + \kappa_{\chi_{-4},4p^2}(\chi_{-4}\mathbf{1}_p,4p)\}].$$

(iii) *Then* $\theta_{\mathbf{1}_p}(z)\theta_{\omega^{2i}}(z)|_{\mathcal{C}(4p^2)}$ *is equal to*

$$\chi_{-4}(p)p^{-1}\tau(\omega^i)\{-2^{-1}\sqrt{-1}\chi_{-4}(p)\kappa_{\omega^{2i},p}(\overline{\omega}^i,p)+\omega(2)^{2i}\kappa_{\omega^{2i},4p^2}(\chi_{-4}\overline{\omega}^i,4p)\} + \chi_{-4}(p)p^{-1}\tau(\chi_{p^\vee}\omega^i)\{-2^{-1}\sqrt{-1}\chi_{-4}(p)\kappa_{\omega^{2i},p}(\chi_{p^\vee}\overline{\omega}^i,p) + \omega(2)^{2i}\kappa_{\omega^{2i},4p^2}(\chi_{-4}\chi_{p^\vee}\overline{\omega}^i,4p)\}.$$

(iv) *The function* $\theta_{\mathbf{1}_p}(z)^2|_{\mathcal{C}(4p^2)}$ *on* $\mathcal{C}(4p^2)$ *is equal to*

$$p^{-1/2}\bar\iota_p\kappa_{\mathbf{1},4p^2}(\chi_{-4}\chi_{p^\vee},4p)-\chi_{-4}(p)p^{-1}\kappa_{\mathbf{1},4p^2}(\chi_{-4},4p)-2^{-1}\sqrt{-1}\chi_{-4}(p) \times\{p^{-1/2}\bar\iota_p\kappa_{\mathbf{1},4p^2}(\chi_{p^\vee},p)-\chi_{-4}(p)p^{-1}\kappa_{\mathbf{1},4p^2}(\mathbf{1},p)\}+p^{-1}(1-p^{-1})\kappa_{\mathbf{1},4p^2}(\chi_{-4},4) -2^{-1}\sqrt{-1}p^{-1}(1-p^{-1})\kappa_{\mathbf{1},4p^2}(\mathbf{1},1).$$

(v) *The function* $\theta_{\omega^{2i}}(z)\theta(p^2z)|_{\mathcal{C}_0(4p^2)}$ *is equal to*

$$\chi_{-4}(p)p^{-1}\tau(\omega^i)\{\omega(2)^{2i}\kappa_{\omega^{2i},4p^2}(\chi_{-4}\overline{\omega}^i,4p)-2^{-1}\sqrt{-1}\chi_{-4}(p)\kappa_{\omega^{2i},4p^2}(\overline{\omega}^i,p)\} + \chi_{-4}(p)p^{-1}\tau(\chi_{p^\vee}\omega^i)\{\omega(2)^{2i}\kappa_{\omega^{2i},4p^2}(\chi_{-4}\chi_{p^\vee}\overline{\omega}^i,4p)-2^{-1}\sqrt{-1}\chi_{-4}(p) \times\kappa_{\omega^{2i},4p^2}(\chi_{p^\vee}\overline{\omega}^i,p)\}.$$

(vi) *The function* $\theta_{\mathbf{1}_p}(z)\theta(p^2z)|_{\mathcal{C}(4p^2)}$ *is equal to*

$$p^{-1/2}\bar\iota_p\kappa_{\mathbf{1},4p^2}(\chi_{-4}\chi_{p^\vee},4p)-\chi_{-4}(p)p^{-1}\kappa_{\mathbf{1},4p^2}(\chi_{-4},4p)-2^{-1}\sqrt{-1}\chi_{-4}(p) \times\{p^{-1/2}\bar\iota_p\kappa_{\mathbf{1},4p^2}(\chi_{p^\vee},p)-\chi_{-4}(p)p^{-1}\kappa_{\mathbf{1},4p^2}(\mathbf{1},p)\}+p^{-1}(1-p^{-1})\kappa_{\mathbf{1},4p^2}(\chi_{-4},4) -2^{-1}\sqrt{-1}p^{-1}(1-p^{-1})\kappa_{\mathbf{1},4p^2}(\mathbf{1},1).$$

By Proposition 3.1, we have

$$\begin{pmatrix} G_{1,\chi_{-4}}(z)|_{\mathcal{C}(4p^2)} \\ G_{1,\chi_{-4}}(pz)|_{\mathcal{C}(4p^2)} \\ G_{1,\chi_{-4}}(p^2z)|_{\mathcal{C}(4p^2)} \end{pmatrix} = \begin{pmatrix} \frac12 & -\frac{\sqrt{-1}}{4} & \frac12 & -\frac{\sqrt{-1}\chi_{-4}(p)}{4} & \frac12 & -\frac{\sqrt{-1}}{4} \\ \frac12 & -\frac{\sqrt{-1}\chi_{-4}(p)}{4} & \frac12 & -\frac{\sqrt{-1}}{4} & \frac{\chi_{-4}(p)}{2p} & -\frac{\sqrt{-1}}{4p} \\ \frac12 & -\frac{\sqrt{-1}}{4} & \frac{\chi_{-4}(p)}{2p} & -\frac{\sqrt{-1}}{4p} & \frac{1}{2p^2} & -\frac{\sqrt{-1}}{4p^2} \end{pmatrix}\mathbf{v}$$

with $\mathbf{v} = {}^t(\kappa_{\chi_{-4},4p^2}(\chi_{-4},4p^2),\kappa_{\chi_{-4},4p^2}(\mathbf{1},p^2),\kappa_{\chi_{-4},4p^2}(\chi_{-4},4p),\kappa_{\chi_{-4},4p^2}(\mathbf{1},p), \kappa_{\chi_{-4},4p^2}(\chi_{-4},4),\kappa_{\chi_{-4},4p^2}(\mathbf{1},1))$, and

$$G^{\omega}_{1,\chi_{-4}\omega}(z)|_{\mathcal{C}(4p^2)}$$
$$= \frac{\chi_{-4}(p)\tau(\omega)(p-\chi_{-4}(p))}{2p}\{-\frac{\sqrt{-1}\chi_{-4}(p)}{2}\kappa_{\chi_{-4}\omega^2,4p^2}(\overline{\omega},p)+\omega(2)^2\kappa_{\chi_{-4}\omega^2,4p^2}(\chi_{-4}\overline{\omega},4p)\}.$$

From Lemma 6.1 and from these equalities we obtain the following proposition:

Proposition 6.2. *Let p be an odd prime, and let ω be a generator of $(\mathbf{Z}/p)^*$. Let $1 \le i, j < (p-1)/2$.*

(i) *Suppose $i + j \ne (p-1)/2$. Then*

$$\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z) \sim 2(p-\chi_{-4}(p))^{-1}[\{J(\omega^i,\omega^j)+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j)\}G^{\omega^{i+j}}_{1,\chi_{-4}\omega^{i+j}}(z)$$
$$+\{J(\chi_{p^\vee}\omega^i,\omega^j)+J(\omega^i,\chi_{p^\vee}\omega^j)\}G^{\chi_{p^\vee}\omega^{i+j}}_{1,\chi_{-4}\chi_{p^\vee}\omega^{i+j}}(z)].$$

(ii) *Suppose $i + j = (p-1)/2$. Then*

$$\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z) \sim 2(p-\chi_{-4}(p))^{-1}[\{J(\omega^i,\omega^j)+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j)\}G^{\chi_{p^\vee}}_{1,\chi_{-4}\chi_{p^\vee}}(z)$$
$$-\omega(-1)^i(1+\chi_{-4}(p))\{G^{\mathbf{1}_p}_{1,\chi_{-4}}(z)-pG^{\mathbf{1}_p}_{1,\chi_{-4}}(pz)\}].$$

(iii) *There hold*

$$\theta_{\mathbf{1}_p}(z)\theta_{\omega^{2i}}(z) \sim 2(p-\chi_{-4}(p))^{-1}[\{-1+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee})\}G^{\omega^i}_{1,\chi_{-4}\omega^i}(z)$$
$$+\{-1+J(\omega^i,\chi_{p^\vee})\}G^{\chi_{p^\vee}\omega^i}_{1,\chi_{-4}\chi_{p^\vee}\omega^i}(z)],$$
$$\theta_{\mathbf{1}_p}(z)^2 \sim 2(p-\chi_{-4}(p))^{-1}\{(p-2-\chi_{-4}(p))G^{\mathbf{1}_p}_{1,\chi_{-4}}(z)$$
$$+(p+\chi_{-4}(p))G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz)-2G^{\chi_{p^\vee}}_{1,\chi_{-4}p^\vee}(z)\},$$
$$\theta_{\omega^{2i}}(z)\theta(p^2z) \sim 2(p-\chi_{-4}(p))^{-1}\{G^{\omega^i}_{1,\chi_{-4}\omega^i}(z)+G^{\chi_{p^\vee}\omega^i}_{1,\chi_{-4}\chi_{p^\vee}\omega^i}(z)\},$$

and

$$\theta_{\mathbf{1}_p}(z)\theta(p^2z) \sim 2(p-\chi_{-4}(p))^{-1}\{G^{\chi_{p^\vee}}_{1,\chi_{-4}\chi_{p^\vee}}(z)+G^{\mathbf{1}_p}_{1,\chi_{-4}}(z)-\chi_{-4}(p)G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz)\}.$$

7. Shimura lifts of products of theta series

Let a be a square-free natural number. It is necessary to express

$$\mathscr{S}_{a^*,\omega^{2(i+j)}\chi_{-4}}(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)),\ \mathscr{S}_{a^*,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\mathbf{1}_p}(z)\theta_{\omega^{2i}}(z)),$$
$$\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)\theta_{\mathbf{1}_p}(z)^2),\ \mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)\theta_{\omega-2i}(z)\theta(p^2z)),$$
$$\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)\theta_{\mathbf{1}_p}(z)\theta(p^2z)) \tag{25}$$

as linear combinations of Eisenstein series up to cusps forms. We present how to obtain the expressions. From Proposition 6.2 and from what we stated before Lemma 5.1, this argument is reduced to express $\mathscr{S}_{a^*,\omega^{2i}\chi_{-4}}(\theta(z)G^{\omega^i}_{1,\chi_{-4}\omega^i}(z))\,(1 \le i < p-1)$, $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(z))$ and $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz))$.

Let $K = \mathbf{Q}(\sqrt{a})$, and let N be as in Section 4. Put

$$\rho = \omega \circ \mathrm{N},\ \psi_{-4} = \chi_{-4} \circ \mathrm{N}. \tag{26}$$

Then by Theorem 5.2, $\mathscr{S}_{a^*,\omega^{2i}\chi_{-4}}(\theta(z)G^{\omega^i}_{1,\chi_{-4}\omega^i}(z)) = 2^{-1}U_2(\lambda^{\omega^i}_{2k,\omega^i}(z;a^*,1))$ $(1 \le i < p-1)$ for $a \equiv 1 \pmod 4$, $\mathscr{S}_{a^*,\omega^{2i}\chi_{-4}}(\theta(z)G^{\omega^i}_{1,\chi_{-4}\omega^i}(z)) = 2^{-1}\lambda^{\omega^i}_{2k,\omega^i}(z;a^*,1)$ $(1 \le i < p-1)$ for $a \not\equiv 1 \pmod 4$, $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(z)) = 2^{-1}U_2(\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,1))$ for $a \equiv 1 \pmod 4$, $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(z)) = 2^{-1}\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,1)$ for $a \not\equiv 1 \pmod 4$, $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz)) = 2^{-1}U_2(\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,p))$ for $a \equiv 1 \pmod 4$ and $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz)) = 2^{-1}\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,p)$ for $a \not\equiv 1 \pmod 4$ where $\lambda^{\omega^i}_{2k,\omega^i}(z;a^*,1) = \widetilde{\lambda}^{\rho^i}_{k,\mathcal{O}_K,\psi_{-4}\rho^i}(z,z)$ and $\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,1) = \widetilde{\lambda}^{\mathbf{1}_p}_{k,\mathcal{O}_K,\psi_{-4}}(z,z)$, and where $\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,p) = \widetilde{\lambda}^{\mathbf{1}_p}_{k,\mathfrak{P}_p,\psi_{-4}}(z,z)+\widetilde{\lambda}^{\mathbf{1}_p}_{k,\overline{\mathfrak{P}}_p,\psi_{-4}}(z,z)-\chi_{-4}(p)\widetilde{\lambda}^{\mathbf{1}_p}_{k,(p),\psi_{-4}}(z,z)$ if $(\chi_{a^*}(p) = 1)$, $\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,p) = \widetilde{\lambda}^{\mathbf{1}_p}_{k,(p),\psi_{-4}}(z,z)$ if $\chi_{a^*}(p) = -1$ and $\lambda^{\mathbf{1}_p}_{2k,\chi_{-4}}(z;a^*,p) = \widetilde{\lambda}^{\mathbf{1}_p}_{k,\mathfrak{P}_p,\psi_{-4}}(z,z)$ if $\chi_{a^*}(p) = 0$, $\mathfrak{P}_p$, $\overline{\mathfrak{P}}_p$ being as in Section 4. Proposition 4.1 gives the values at cusps of these functions. If the restrictions to $\mathcal{C}(2p^2)$, of functions of (25) are obtained, then the functions are represented as linear combinations of Eisenstein series up to cusps forms, indeed, from Proposition 3.1 there hold

$$\begin{pmatrix} \kappa_{\mathbf{1},2p^2}(\mathbf{1},2p^2) \\ \kappa_{\mathbf{1},2p^2}(\mathbf{1},p^2) \\ \kappa_{\mathbf{1},2p^2}(\mathbf{1},2p) \\ \kappa_{\mathbf{1},2p^2}(\mathbf{1},p) \\ \kappa_{\mathbf{1},2p^2}(\mathbf{1},2) \\ \kappa_{\mathbf{1},2p^2}(\mathbf{1},1) \end{pmatrix} = A \begin{pmatrix} G_2(z)|_{\mathcal{C}(2p^2)} \\ G_2(2z)|_{\mathcal{C}(2p^2)} \\ G_2(pz)|_{\mathcal{C}(2p^2)} \\ G_2(2pz)|_{\mathcal{C}(2p^2)} \\ G_2(p^2z)|_{\mathcal{C}(2p^2)} \\ G_2(2p^2z)|_{\mathcal{C}(2p^2)} \end{pmatrix},$$

$$\begin{pmatrix} \kappa_{\omega^{2i},2p^2}(\overline{\omega}^i,2p) \\ \kappa_{\omega^{2i},2p^2}(\overline{\omega}^i,p) \end{pmatrix} = B \begin{pmatrix} G^{\omega^i}_{2,\omega^i}(z)|_{\mathcal{C}(2p^2)} \\ G^{\omega^i}_{2,\omega^i}(2z)|_{\mathcal{C}(2p^2)} \end{pmatrix} \quad (1 \le i < p-1) \tag{27}$$

with

$$A = \frac{4}{p^2-1} \begin{pmatrix} 0 & 0 & -1 & 4 & p^2 & -4p^2 \\ 0 & 0 & 4 & -4 & -4p^2 & 4p^2 \\ -1 & 4 & p^2+1 & -4(p^2+1) & -p^2 & 4p^2 \\ 4 & -4 & -4(p^2+1) & 4(p^2+1) & 4p^2 & -4p^2 \\ p^2 & -4p^2 & -p^2 & 4p^2 & 0 & 0 \\ -4p^2 & 4p^2 & 4p^2 & -4p^2 & 0 & 0 \end{pmatrix},$$

$$B = \frac{4p}{(p^2-1)\tau(\omega^i)} \begin{pmatrix} \overline{\omega}(2)^{2i} & -4 \\ -4 & 4\omega(2)^{2i} \end{pmatrix}.$$

Suppose that $a \equiv 1 \pmod 4$ and $\chi_{p^\vee}(a) = 1$. Then by Proposition 4.1 and Lemma 5.1, we have

$$\begin{aligned}
&U_2(\tilde{\lambda}^{\rho^i}_{1,\mathcal{O}_K,\psi_{-4}\rho^i}(z,z))|_{\mathcal{C}(2p^2)} \\
&\quad = 2^{-2}\omega(2^2a)^i\tau(\omega^i)^2(1-\chi_{-4}(p)p^{-1})^2L(0,\chi_{-4a}) \\
&\qquad \times \{2\omega(2)^{2i}\kappa_{\omega^{4i},2p^2}(\varpi^{2i},2p) - \kappa_{\omega^{4i},2p^2}(\varpi^{2i},p)\} \quad (1 \le i < (p-1)/2), \\
&U_2(\tilde{\lambda}^{\mathbf{1}_p}_{1,\mathcal{O}_K,\psi_{-4}}(z,z))|_{\mathcal{C}(2p^2)} \\
&\quad = -2^{-2}L(0,\chi_{-4a})[(1-\chi_{-4}(p))^2\kappa_{\mathbf{1},2p^2}(\mathbf{1},p^2) + (1-\chi_{-4}(p))^2\kappa_{\mathbf{1},2p^2}(\mathbf{1},p) \\
&\qquad - \frac{2(p-\chi_{-4}(p))^2}{p^2}\kappa_{\mathbf{1},2p^2}(\mathbf{1},2) + \frac{(p-1)^2}{p^2}\kappa_{\mathbf{1},2p^2}(\mathbf{1},1)]
\end{aligned}$$

and

$$\begin{aligned}
&U_2(\tilde{\lambda}^{\mathbf{1}_p}_{1,\mathfrak{P}_p,\psi_{-4}}(z,z) + \tilde{\lambda}^{\mathbf{1}_p}_{1,\overline{\mathfrak{P}}_p,\psi_{-4}}(z,z) - \chi_{-4}(p)\tilde{\lambda}^{\mathbf{1}_p}_{1,(p),\psi_{-4}}(z,z))|_{\mathcal{C}(2p^2)} \\
&= -2^{-2}L(0,\chi_{-4a})[\chi_{-4}(p)(1-\chi_{-4}(p))^2\kappa_{\mathbf{1},2p^2}(\mathbf{1},p^2) + \tfrac{2\chi_{-4}(p)(p-\chi_{-4}(p))^2}{p^2}\kappa_{\mathbf{1},2p^2}(\mathbf{1},2p) \\
&\quad - \tfrac{\chi_{-4}(p)(p-1)\{(2\chi_{-4}(p)-1)p-1\}}{p^2}\kappa_{\mathbf{1},2p^2}(\mathbf{1},p) - \tfrac{2\chi_{-4}(p)(2p-1)(p-\chi_{-4}(p))^2}{p^4}\kappa_{\mathbf{1},2p^2}(\mathbf{1},2) \\
&\quad + \tfrac{(p-1)^2(2p-\chi_{-4}(p))}{p^4}\kappa_{\mathbf{1},2p^2}(\mathbf{1},1)],
\end{aligned}$$

the halves of which are equal to $\mathscr{S}_{a^*,\omega^{2i}\chi_{-4}}(\theta(z)G^{\omega^i}_{1,\chi_{-4}\omega^i}(z))|_{\mathcal{C}(2p^2)}$, $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z) \times G^{\mathbf{1}_p}_{1,\chi_{-4}}(z))|_{\mathcal{C}(2p^2)}$ and $\mathscr{S}_{a^*,\chi_{-4}\mathbf{1}_p}(\theta(z)G^{\mathbf{1}_p}_{1,\chi_{-4}}(pz))|_{\mathcal{C}(2p^2)}$ respectively by calculation after (26). Then $\mathscr{S}_{a,\chi_{-4}\omega^{2i+2j}}(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z))|_{\mathcal{C}(2p^2)}$, $\mathscr{S}_{a,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\mathbf{1}_p}(z)\times\theta_{\omega^{2i}}(z))|_{\mathcal{C}(2p^2)}$, $\mathscr{S}_{a,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))|_{\mathcal{C}(2p^2)}$ and $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta_{\mathbf{1}_p}(z)\theta(p^2z))|_{\mathcal{C}(2p^2)}$ are written as linear combinations of κ's by Proposition 6.2 and by what we stated before Lemma 5.1. Modular forms whose values at cusps are known, are written as linear combination of Eisenstein series up to cusp forms by using (27). Then we have

$$\begin{aligned}
&\mathscr{S}_{a,\chi_{-4}\omega^{2i+2j}}(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)) \\
&\quad \sim \frac{12\omega(2^2a)^{i+j}L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))}\{J(\omega^i,\omega^j,\omega^{i+j}) + J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j,\omega^{i+j}) \\
&\qquad + J(\chi_{p^\vee}\omega^i,\omega^j,\chi_{p^\vee}\omega^{i+j}) + J(\omega^i,\chi_{p^\vee}\omega^j,\chi_{p^\vee}\omega^{i+j})\}G^{\omega^{2(i+j)}}_{2,\mathbf{1}_2\omega^{2(i+j)}}(z) \\
&\quad (i+j \ne (p-1)/2), \\
&\mathscr{S}_{a,\chi_{-4}\omega^{2i+2j}}(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)) \\
&\quad \sim \frac{12L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))}[\{-\chi_{-4}(p)J(\omega^i,\omega^j) - \chi_{-4}(p)J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j)\} \\
&\qquad \times \{G^{\mathbf{1}_p}_{2,\mathbf{1}_{2p}}(z) - p(p-1)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(pz)\} + \omega(-1)^i(1+\chi_{-4}(p))pG^{\mathbf{1}_p}_{2,\mathbf{1}_{2p}}(z)] \\
&\quad (i+j = (p-1)/2),
\end{aligned}$$

$$\begin{aligned}
&\mathscr{S}_{a,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\mathbf{1}_p}(z)\theta_{\omega^{2i}}(z))\\
&\quad\sim \frac{6\omega(2^2a)^i(p-\chi_{-4}(p))L(0,\chi_{-4a})}{p(p^2-1)}\{2J(\omega^i,\chi_{p^\vee},\chi_{p^\vee}\omega^i)-J(\omega^i,\omega^i)\\
&\qquad -J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^i)\}G^{\omega^{2i}}_{2,\mathbf{1}_2\omega^{2i}}(z),\\
&\mathscr{S}_{a,\chi_{-4}\mathbf{1}_p}(\theta(z)\theta_{\mathbf{1}_p}(z)^2)\\
&\quad\sim \begin{cases} \frac{6L(0,\chi_{-4a})}{p(p+1)}\{(p^2-p+4)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(z)-p(3p+1)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(pz)\} & (\chi_{-4}(p)=1)\\ \frac{6L(0,\chi_{-4a})}{p-1}\{(p-3)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(z)+(3p-1)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(pz)\} & (\chi_{-4}(p)=-1)\end{cases},\\
&\mathscr{S}_{a,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))\\
&\quad\sim \frac{6\omega(2^2a)^i(p-\chi_{-4}(p))L(0,\chi_{-4a})}{p(p^2-1)}\{J(\omega^i,\omega^i)+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^i)\}G^{\omega^{2i}}_{2,\mathbf{1}_2\omega^{2i}}(z)
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta_{\mathbf{1}_p}(z)\theta(p^2z))\\
&\quad\sim \begin{cases} \frac{6L(0,\chi_{-4a})}{p(p+1)}\{(p-3)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(z)+p(p+1)G^{\mathbf{1}_p}_{2,\mathbf{1}_2}(pz)\} & (\chi_{-4}(p)=1)\\ \frac{6L(0,\chi_{-4a})}{p}G^{\mathbf{1}_p}_{2,\mathbf{1}_{2p}}(z) & (\chi_{-4}(p)=-1)\end{cases}
\end{aligned}$$

where $1\le i,j<(p-1)/2$.

We are done in the case that $a\equiv 1 \pmod 4$ and $\chi_{p^\vee}(a)=1$. The similar method works also for other cases.

8. Sums of three squares under the congruence condition

For a square free natural number a, we define an operator $\mathcal{P}_a$ on the set of Fourier series by

$$\mathcal{P}_a(\sum_{n=0}^{\infty} c_n\mathbf{e}(nz)) = \sum_{n=1}^{\infty} c_{an^2}\mathbf{e}(an^2z).$$

As stated at the beginning of Section 6, we have

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\sum_{n^2\equiv\alpha_2^2 (\mathrm{mod}\ p)}\mathbf{e}(n^2z)\sum_{n^2\equiv\alpha_3^2(\mathrm{mod}\ p)}\mathbf{e}(n^2z))\\
&\quad=\sum_{n=1}^{\infty} r_{(\alpha(an^2),\alpha_2,\alpha_3)}(an^2)\mathbf{e}(an^2) \qquad (28)
\end{aligned}$$

where $\alpha(an^2)$ denotes integer between 0 and $p-1$ satisfying $\alpha(an^2)^2+\alpha_2^2+\alpha_3^2\equiv an^2 \pmod p$.

From the previous section, we can obtain the Fourier coefficient of Shimura lifts $\mathscr{S}_{a^*,\chi_{-4}\omega^{2i+2j}}(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z))$ $(0 \le i,j < (p-1)/2)$, $\mathscr{S}_{a^*,\chi_{-4}\omega^{2i}}(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))$ $(0 \le i < (p-1)/2)$ with the error terms coming Fourier coefficients of cusp forms, and hence those of $\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)$, $\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z)$ by (21).

(i) The case $\chi_{p^\vee}(a) = 1$.

From the computation of the preceding section and from Lemma 2.1 we have for $0 \le i,j < (p-1)/2$,

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z))\\
&\quad\sim \frac{12\omega(2^2a)^{i+j}L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))}\{J(\omega^i,\omega^j,\omega^{i+j}) + J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j,\omega^{i+j})\\
&\qquad + J(\chi_{p^\vee}\omega^i,\omega^j,\chi_{p^\vee}\omega^{i+j}) + J(\omega^i,\chi_{p^\vee}\omega^j,\chi_{p^\vee}\omega^{i+j})\}\\
&\qquad \times \sum_{n=1}^{\infty}\omega(n)^{2(i+j)}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)\\
&\qquad + \delta_{\omega^{i+j},\chi_{p^\vee}}X_a(i) + \delta_{(i,j),(0,0)}Y_a
\end{aligned}$$

with

$$\begin{aligned}
X_a(i) &= \frac{12L(0,\chi_{-4a})}{p(p+1)}[\chi_{-4}(p)p(p-\chi_{-4}(p))\{J(\omega^i,\chi_{p^\vee}\varpi^i) + J(\chi_{p^\vee}\omega^i,\varpi^i)\}\\
&\quad \times \sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/(pd))\mathbf{e}(an^2z) + (1+\chi_{-4}(p))\omega(-1)^i(p-1)\\
&\quad \times \sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)],\\
Y_a &= \frac{12(p-1)^2L(0,\chi_{-4a})}{p+\chi_{-4}(p)}\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/(pd))\mathbf{e}(an^2z)
\end{aligned}$$

where $\delta_{\omega^{i+j},\chi_{p^\vee}}$ (resp. $\delta_{(i,j),(0,0)}$) denotes 1 if $\omega^{i+j} = \chi_{p^\vee}$ (resp. $(i,j) = (0,0)$) and it denotes 0 if otherwise. Further

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))\\
&\quad\sim \frac{12L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))}[\omega(2^2a)^i\{J(\omega^i,\omega^i) + J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^i)\}\sum_{n=1}^{\infty}\omega(n)^{2i}\\
&\qquad \times \sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)\\
&\qquad + \delta_{i,0}(1+\chi_{-4}(p))p(p-1)\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\{\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/(pd))\}\mathbf{e}(an^2z)]
\end{aligned}$$

for $0 \leq i < (p-1)$, where $\delta_{i,0}$ denotes the Kronecker delta. We assume that $\alpha_2 \not\equiv 0 \pmod p$. By (5) and by Proposition 2.2, we have, including the case $\alpha_3 \equiv 0 \pmod p$,

$$\mathcal{P}_a(\theta(z) \sum_{n^2\equiv\alpha_2^2 (\mathrm{mod}\, p)} \mathbf{e}(n^2 z) \sum_{n^2\equiv\alpha_3^2 (\mathrm{mod}\, p)} \mathbf{e}(n^2 z))$$
$$\sim \frac{12L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))} \sum_{n=1}^{\infty} \bar{r}^{(p)}_{(\alpha(an^2),\alpha_2,\alpha_3)}(an^2) \sum_{d|n} (\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/d)\mathbf{e}(an^2 z),$$

where we note that $\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/(pd)) = p^{-1}\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/p)$ for n with $p|n$. Hence if $|\boldsymbol{\alpha}|^2 \equiv an^2 \pmod p$ and $\boldsymbol{\alpha} \not\equiv \mathbf{0} \pmod p$, then

$$r_{\boldsymbol{\alpha}}(an^2)$$
$$= \frac{12L(0,\chi_{-4a})}{p(p+\chi_{-4}(p))} \bar{r}^{(p)}_{\boldsymbol{\alpha}}(an^2) \sum_{d|n} (\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/d) + O((an^2)^{13/28+\varepsilon}) \quad (29)$$

by (28).

(ii) The case $\chi_{p^\vee}(a) = -1$.
We have

$$\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z))$$
$$\sim \frac{12\omega(2^2a)^{i+j}L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))}\{-J(\omega^i,\omega^j,\omega^{i+j}) - J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^j,\omega^{i+j})$$
$$+ J(\chi_{p^\vee}\omega^i,\omega^j,\chi_{p^\vee}\omega^{i+j}) + J(\omega^i,\chi_{p^\vee}\omega^j,\chi_{p^\vee}\omega^{i+j})\}$$
$$\times \sum_{n=1}^{\infty} \omega(n)^{2(i+j)} \sum_{d|n} (\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_{2p}}(n/d)\mathbf{e}(an^2 z)$$
$$+ \delta_{\omega^{i+j},\chi_{p^\vee}} X_a(i) + \delta_{(i,j),(0,0)} Y_a$$

for $0 \leq, i, j < (p-1)/2$ with

$$X_a(i) = \tfrac{12(p-1)L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))}[-\omega(-1)^i(1+\chi_{-4}(p)) \sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_{2p}}(n/d)\mathbf{e}(an^2 z)$$
$$+ \chi_{-4}(p)\{J(\omega^i,\chi_{p^\vee}\bar{\omega}^i) + J(\chi_{p^\vee}\omega^i,\bar{\omega}^i)\}p$$
$$\times \sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/(pd))\mathbf{e}(an^2 z)],$$
$$Y_a = \tfrac{24(p-1)^2L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))} \sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\{\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_{2p}}(n/d) + \frac{p}{2}\sigma^{\mathbf{1}_p}_{1,\mathbf{1}_2}(n/(pd))\}\mathbf{e}(an^2 z).$$

Further for $0 \le i < (p-1)/2$,

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z)) \\
&\sim \frac{12\omega(2^2a)^i L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))}\{-J(\omega^i,\omega^i)+J(\chi_{p^\vee}\omega^i,\chi_{p^\vee}\omega^i)\} \\
&\quad \times \sum_{n=1}^{\infty}\omega(n)^{2i}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)
\end{aligned}$$

with the additional term

$$\frac{24\delta_{i,0}L(0,\chi_{-4a})}{p}\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\{\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/d)+p\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/(pd))\}\mathbf{e}(an^2z)$$

if $\chi_{-4}(p)=1$, and

$$\frac{24(p-1)\delta_{i,0}L(0,\chi_{-4a})}{p(p+1)}\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_{2p}}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)$$

if $\chi_{-4}(p)=-1$. By (5) and by Proposition 2.2,

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\sum_{n^2\equiv\alpha_2^2 (\mathrm{mod}\ p)}\mathbf{e}(n^2z)\sum_{n^2\equiv\alpha_3^2 (\mathrm{mod}\ p)}\mathbf{e}(n^2z)) \\
&\sim \frac{12L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))}\sum_{n=1}^{\infty}\bar{r}^{(p)}_{(\alpha(an^2),\alpha_2,\alpha_3)}(an^2)\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z).
\end{aligned}$$

If $|\boldsymbol{\alpha}|^2 \equiv an^2 \pmod p$ and $\boldsymbol{\alpha} \not\equiv \mathbf{0} \pmod p$, then

$$\begin{aligned}
&r_{\boldsymbol{\alpha}}(an^2) \\
&= \frac{12L(0,\chi_{-4a})}{p(p-\chi_{-4}(p))}\bar{r}^{(p)}_{\boldsymbol{\alpha}}(an^2)\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)+O((an^2)^{13/28+\varepsilon})
\end{aligned} \tag{30}$$

by (28).

(iii) The case $\chi_{p^\vee}(a)=0$.
We have

$$\begin{aligned}
&\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta_{\omega^{2j}}(z)) \\
&\sim \delta_{\omega^{i+j},\chi_{p^\vee}}\frac{12\chi_{-4}(p)L(0,\chi_{4a})}{p+1}\{J(\omega^i,\chi_{p^\vee}\bar{\omega}^i)+J(\chi_{p^\vee}\omega^i,\bar{\omega}^i)\} \\
&\quad \times \sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)
\end{aligned}$$

$$+\,\delta_{(i,j),(0,0)}\frac{12(p-1)L(0,\chi_{-4a})}{p+1}\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)$$

for $0\le,i,j<(p-1)/2$. Further

$$\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))\sim\frac{48L(0,\chi_{-4a})}{p^2-1}\sum_{n=1}^{\infty}\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z)$$

for $0\le i<(p-1)/2$ if $\chi_{-4}(p)=1$, and $\mathcal{P}_a(\theta(z)\theta_{\omega^{2i}}(z)\theta(p^2z))\sim 0$ if $\chi_{-4}(p)=-1$. By (5) and by Proposition 2.2,

$$\begin{aligned}&\mathcal{P}_a(\theta(z)\sum_{n^2\equiv\alpha_2^2(\bmod p)}\mathbf{e}(n^2z)\sum_{n^2\equiv\alpha_3^2(\bmod p)}\mathbf{e}(n^2z))\\&\sim\frac{12L(0,\chi_{-4a})}{p^2-1}\sum_{n=1}^{\infty}\bar{r}^{(p)}_{(\alpha(an^2),\alpha_2,\alpha_3)}(an^2)\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)\mathbf{e}(an^2z).\end{aligned}$$

If $|\boldsymbol{\alpha}|^2\equiv an^2\pmod p$ and $\boldsymbol{\alpha}\not\equiv\mathbf{0}\pmod p$, then

$$\begin{aligned}&r_{\boldsymbol{\alpha}}(an^2)\\&=\frac{12L(0,\chi_{-4a})}{p^2-1}\bar{r}^{(p)}_{\boldsymbol{\alpha}}(an^2)\sum_{d|n}(\mu\chi_{-4a}\mathbf{1}_p)(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)+O((an^2)^{13/28+\varepsilon})\end{aligned}\tag{31}$$

by (28).

From (29), (30) and (31), the following theorem is proved.

Theorem 8.1. *Let a, n be natural numbers where a is square-free. Let p be an odd prime. Then $r_{\mathbf{0}}^{(p)}(an^2)=r_3(an^2/p^2)$, and if $\boldsymbol{\alpha}\in\mathbf{Z}^3$ is not congruent to $(0,0,0)$ modulo p, then*

$$r_{\boldsymbol{\alpha}}^{(p)}(an^2)=\frac{12\,\bar{r}^{(p)}_{\boldsymbol{\alpha}}(an^2)L(0,\chi_{-4a})}{\bar{r}^{(p)}_3(a)}\sum_{d|n}\mu(d)\chi_{-4a}(d)\sigma_{1,\mathbf{1}_2}^{\mathbf{1}_p}(n/d)+O((an^2)^{13/28+\varepsilon})$$

for any $\varepsilon>0$.

Let $h(a)$ denote the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{-a})$. If $a>3$, then $L(0,\chi_{-4a})$ equals $h(a)$, $2h(a)$ or 0 according as $a\equiv 1,2\pmod 4$, $a\equiv 3\pmod{}$ or $a\equiv 7\pmod 8$. Siegel's celebrated theorem on the class numbers shows that $h(a)>a^{1/2-\varepsilon}$ as $a\longrightarrow\infty$. Let $a\not\equiv 7\pmod 8$. Then $L(0,\chi_{-4a})/a^{13/28+\varepsilon}\longrightarrow\infty$ as $a\longrightarrow\infty$ for a sufficiently small $\varepsilon>0$. Theorem 8.1 implies that if $\bar{r}^{(p)}_{\boldsymbol{\alpha}}(an^2)>0$ and if an^2 is sufficiently large with bounded 2-adic valuation, then $r^{(p)}_{\boldsymbol{\alpha}}(an^2)$ is positive.

Corollary 8.2. *Let n be either an odd number not congruent to 7 modulo 8, or $2\|n$. Let p be an odd prime. Then there is a constant C_p depending only on p which satisfies the following. If $n > C_p$ and if $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \equiv n \pmod{p}$ has a solution in $\mathbf{F}_p^3$ with $(\alpha_1, \alpha_2, \alpha_3) \not\equiv (0,0,0) \pmod{p}$, then*

$$x_i \equiv \alpha_i \pmod{p} \quad (i = 1,2,3), \quad x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution.

It is easy to see that we can take 0 as C_3. However C_p might be very large in general. It has been verified by using computer, that up to $2 \cdot 10^6$, the result of the corollary holds for $p = 5$ (resp. $p = 7$) if $n > 42\,211$ (resp. $n > 308\,995$). Hence $C_5 \geq 42\,211$ and $C_7 \geq 308\,995$, where there are possibilities that the equalities hold. We have checked also that $C_{11} > 2 \cdot 10^6$.

9. Sums of three squares under congruence condition modulo 8

Let $\eta(z)$ be the Dedekind eta function. We put

$$f_{3/2}(z) := \eta(8z)\eta(16z)\theta(2z) \in \mathbf{S}_{3/2}(128, \chi_{-4}),$$
$$f'_{3/2}(z) := \eta(8z)\eta(16z)\theta(8z) \in \mathbf{S}_{3/2}(128, \chi_{-4}).$$

Let $c_{f_{3/2}}(n)$, $c_{f'_{3/2}}(n)$ be the n-th Fourier coefficients of $f_{3/2}(z)$, $f'_{3/2}(z)$ respectively. We have $c_{f_{3/2}}(n) = c_{f'_{3/2}}(n) = 0$ for $n \equiv 5,6,7 \pmod{8}$, and $c_{f'_{3/2}}(n) = 0$ for $n \equiv 3 \pmod{8}$. The coefficients $c_{f_{3/2}}(n)$ and $c_{f'_{3/2}}(n)$ coincide except for $n \equiv 3 \pmod{8}$. Both of $f_{3/2}(z)$, $f'_{3/2}(z)$ are Hecke eigen forms. Tunnell [21] (see also Koblitz [11]) showed under the weak Birch–Swinnerton–Dyer conjecture, that an odd natural number n is a congruent number if and only if $c_{f_{3/2}}(n) = 0$.

Let

$$r_{\boldsymbol{\alpha}}^{(8)}(n) := \#\{\boldsymbol{x} \in \mathbf{Z}^3 \mid x_i \equiv \pm\alpha_i \pmod{8}\ (i = 1,2,3),\ |\boldsymbol{x}|^2 = n\}$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}^3$, and

$$r_{(\alpha_1,\alpha_2,4*)}^{(8)}(n) := \#\{\boldsymbol{x} \in \mathbf{Z}^3 \mid x_i \equiv \pm\alpha_i \pmod{8}\ (i = 1,2),\ 4|x_3,\ |\boldsymbol{x}|^2 = n\}.$$

We define also $r_{(\alpha_1,4*,4*)}^{(8)}(n)$ in the similar way. If all of α_1, α_2, α_3 are even, then we can reduce the argument to the case at least one of them is odd. For example, if all α_i's are odd, then $r_{2\boldsymbol{\alpha}}^{(8)}(n) = r_{\boldsymbol{\alpha}}^{(8)}(n/2^2)$ and $r_{(2\alpha_1,2\alpha_2,4*)}^{(8)}(n) = r_{(\alpha_1,\alpha_2,2)}^{(8)}(n/2^2) + r_{(\alpha_1,\alpha_2,6)}^{(8)}(n/2^2) + r_{(\alpha_1,\alpha_2,4*)}^{(8)}(n/2^2)$.

Theorem 9.1. *Let a be a square-free natural number and let n be an odd natural number. Put*

$$\xi(a,n) := L(0,\chi_{-4a}) \sum_{0<d|n} \mu(d)\chi_{-4a}(d)\sigma_1(n/d).$$

(i) *Then $r^{(8)}_{(1,4*,4*)}(n^2)$ (resp. $r^{(8)}_{(3,2,2)}(n^2)$) is $2\xi(1,n) + \chi_{-4}(n)n$ (resp. $2\xi(1,n) - \chi_{-4}(n)n$) or 0 according as $n \equiv \pm 1 \pmod 8$ or not. Further $r^{(8)}_{(3,4*,4*)}(n^2)$ (resp. $r^{(8)}_{(1,2,2)}(n^2)$) is $2\xi(1,n) + \chi_{-4}(n)n$ (resp. $2\xi(1,n) - \chi_{-4}(n)n$) or 0 according as $n \equiv \pm 3 \pmod 8$ or not.*

(ii) *Let $a \equiv 1 \pmod 8), > 1$. Then $r^{(8)}_{(1,4*,4*)}(an^2) = r^{(8)}_{(3,2,2)}(an^2)$, and this is equal to $2\xi(a,n)$ or 0 according as $an^2 \equiv 1 \pmod{16}$ or not. Further $r^{(8)}_{(3,4*,4*)}(an^2) = r^{(8)}_{(1,2,2)}(an^2)$, and this is equal to $2\xi(a,n)$ or 0 according as $an^2 \equiv 9 \pmod{16}$ or not.*

(iii) *Let $a \equiv 5 \pmod 8$. Then $r^{(8)}_{(1,2,4*)}(an^2)$ (resp. $r^{(8)}_{(3,2,4*)}(an^2)$) is $2\xi(a,n)$ or 0 according as $an^2 \equiv 5 \pmod{16}$ (resp. $an^2 \equiv 13 \pmod{16}$) or not.*

(iv) *Let $a \equiv 3 \pmod 8$. Then $r^{(8)}_{(1,1,1)}(an^2)$ (resp. $r^{(8)}_{(1,3,3)}(an^2)$) is $3\xi(a,n)+3c_{f_{3/2}}(an^2)$ (resp. $3\xi(a,n) - c_{f_{3/2}}(an^2)$) or 0 according as $an^2 \equiv 3 \pmod{16}$ or not. Further $r^{(8)}_{(1,1,3)}(an^2)$ (resp. $r^{(8)}_{(3,3,3)}(an^2)$) is $3\xi(a,n) - c_{f_{3/2}}(an^2)$ (resp. $3\xi(a,n) + 3c_{f_{3/2}}(an^2)$) or 0 according as $an^2 \equiv 11 \pmod{16}$ or not.*

(v) *Let $a \equiv 2 \pmod 8$. Then $r^{(8)}_{(1,1,4*)}(an^2)$ (resp. $r^{(8)}_{(3,3,4*)}(an^2)$) is $2\xi(a,n) + 2c_{f_{3/2}}(an^2/2)$ (resp. $2\xi(a,n) - 2c_{f_{3/2}}(an^2/2)$) or 0 according as $an^2 \equiv 2 \pmod{16}$ or not. Further $r^{(8)}_{(1,3,4*)}(an^2)$ is $2\xi(a,n)$ or 0 according as $an^2 \equiv 10 \pmod{16}$ or not.*

(vi) *Let $a \equiv 6 \pmod 8$. Then $r^{(8)}_{(1,1,2)}(an^2)$ (resp. $r^{(8)}_{(3,3,2)}(an^2)$) is $2\xi(a,n) + 2c_{f_{3/2}}(an^2/2)$ (resp. $2\xi(a,n) - 2c_{f_{3/2}}(an^2/2)$) or 0 according as $an^2 \equiv 6 \pmod{16}$ or not. Further $r^{(8)}_{(1,3,2)}(an^2)$ is $2\xi(a,n)$ or 0 according as $an^2 \equiv 14 \pmod{16}$ or not.*

We give the proof of the theorem in the next section.

From (i) of the theorem we see in particular that $r_{(3,2,2)}(p^2) = 0$ (resp. $r_{(1,2,2)}(p^2) = 0$) if p is a prime with $p \equiv 1 \pmod 8$ (resp. $p \equiv 5 \pmod 8$), while the congruence relation $3^2 + 2^2 + 2^2 \equiv p^2 \pmod 8$ (resp. $1^2 + 2^2 + 2^2 \equiv p^2 \pmod 8$) holds. Since $c_{f_{3/2}}(an^2) = O((an^2)^{13/28+\varepsilon})$ by Duke and Schulze-Pillot [4], we have the following:

Corollary 9.2. *There is a constant C which satisfies the following. Let n be either an odd non-square number not congruent to 7 modulo 8, or $2\|n$. Let α_1, α_2, α_3 be any integer with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \equiv n \pmod 8$. Then if $n > C$, then there are integers x_1, x_2, x_3 with $x_i \equiv \alpha_i \pmod 8$ for α_i not divisible by 4 and with $x_i \equiv \alpha_i \pmod 4$ for α_i divisible by 4 which satisfy $x_1^2 + x_2^2 + x_3^2 = n$.*

It is checked by computer, that up to $2 \cdot 10^6$, the result of the corollary holds if $n > 83\,227$. The constant C in the corollary is at least $83\,227$, and there is a possibility that $C = 83\,227$.

By Theorem 9.1 (iv), we see for square-free $n \equiv 1 \pmod 8, > 1$, that $c_{f_{3/2}}(n)$ vanishes if and only if $r^{(8)}_{(1,1,4*)}(2n) = r^{(8)}_{(3,3,4*)}(2n)$. By (v), for square-free $n \equiv 3 \pmod 8$, $c_{f_{3/2}}(n)$ vanishes if and only if $r^{(8)}_{(1,1,2)}(2n) = r^{(8)}_{(3,3,2)}(2n)$. By (vi), for square-free n with $n \equiv 3 \pmod{16}$ (resp. $n \equiv 3 \pmod{16}$), $c_{f_{3/2}}(n)$ vanishes if and only if $r^{(8)}_{(1,1,2)}(2n) = r^{(8)}_{(3,3,2)}(2n)$, and $c_{f_{3/2}}(n)$ vanishes if and only if $r^{(8)}_{(1,1,1)}(n) = r^{(8)}_{(1,3,3)}(n)$ (resp. $r^{(8)}_{(1,1,3)}(n) = r^{(8)}_{(3,3,3)}(n)$). As a consequence of Tunnell's result we obtain the following:

Corollary 9.3. *Let $n \geq 1$ be a square-free odd natural number. We assume that the weak Birch–Swinnerton–Dyer conjecture is true.*

(i) *Let $n \equiv 1 \pmod 8$. Then n is a congruent number if and only if $r^{(8)}_{(1,1,4*)}(2n) = r^{(8)}_{(3,3,4*)}(2n)$.*

(ii) *Let $n \equiv 3 \pmod 8$. Then n is a congruent number if and only if $r^{(8)}_{(1,1,2)}(2n) = r^{(8)}_{(3,3,2)}(2n)$.*

(iii) *Let $n \equiv 3 \pmod{16}$ (resp. $n \equiv 11 \pmod{16}$). Then n is a congruent number if and only if $r^{(8)}_{(1,1,1)}(n) = r^{(8)}_{(1,3,3)}(n)$ (resp. $r^{(8)}_{(1,1,3)}(n) = r^{(8)}_{(3,3,3)}(n)$).*

10. The proof of Theorem 9.1

The method proving Proposition 6.2 is effective also for products of theta series of level powers of 2. In the following case, even the equalities hold. We omit the proofs of the following two lemmas.

Lemma 10.1. *There hold the equalities*

$$\begin{aligned}
\theta(z)^2 &= 2G_{1,\chi_{-4}}(z), \\
\theta(z)\theta(4z) &= G_{1,\chi_{-4}}(z) - G_{1,\chi_{-4}}(2z) + 2G_{1,\chi_{-4}}(4z), \\
\theta(z)\theta(16z) &= \frac{1}{2}G_{1,\chi_{-4}}(z) - \frac{1}{2}G_{1,\chi_{-4}}(2z) + G_{1,\chi_{-4}}(4z) - G_{1,\chi_{-4}}(8z) \\
&\quad + 2G_{1,\chi_{-4}}(16z) + \frac{1}{2}G^{\chi_8}_{1,\chi_{-8}}(z),
\end{aligned}$$

and

$$\theta_{\chi_8}(z)^2 = 2G^{\chi_8}_{1,\chi_{-8}}(2z).$$

We denote by B_m, an operator on the set of Fourier series so that $B_m(\sum_{n=0}^{\infty} c_n \mathbf{e}(nz)) = \sum_{n=0}^{\infty} c_n \mathbf{e}(nmz)$.

Lemma 10.2. *There hold the equalities*

$$\theta(z)G^{\chi_8}_{1,\chi_{-8}}(2z) - B_2(U_2(\theta(z)G^{\chi_8}_{1,\chi_{-8}}(2z))) = 2(f_{3/2}(z) - f'_{3/2}(z)),$$
$$U_2(\theta(z)\theta(16z)\theta_{\chi_8}(z)) = 4f'_{3/2}(z)$$

and

$$U_2(\theta(z)\theta(4z)\theta_{\chi_8}(z)) = 4f_{3/2}(z).$$

Let a be a square-free natural number. We investigate the numbers of representations of an^2 as sum of three squares when n is odd. Let K, $\mathfrak{P}_2$, ψ_{-4} be as in Section 7.

(I) The case $a \equiv 1 \pmod 8$.

If $|\boldsymbol{x}|^2 = an^2$ for n odd, then we may assume that x_1 is odd. Then both of x_2, x_3 are just divisible by 2, or both of them are divisible by 4.

(I-i) The case $x_2 \equiv x_3 \equiv 0 \pmod 4$.

We investigate the Fourier coefficients of $\theta(z)\theta(16z)^2 \in \mathbf{M}_{3/2}(64, \chi_{-4})$. There is a cusp form $\Theta_{\chi_{-4}}(z)$ of (23) in $\mathbf{M}_{3/2}(64, \chi_{-4})$. If $a \neq 1$, then $\mathscr{S}_{a,\chi_{-4}}(\Theta_{\chi_{-4}}(z)) = 0$ since the n-th Fourier coefficients of $\Theta_{\chi_{-4}}(z)$ is 0 for n not square. At first we consider the case $a > 1$. Since $\theta(z)\theta(16z)^2 = 2\theta(z)G_{1\chi_{-4}}(16z)$ by Lemma 10.1, $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(16z)^2) = U_2(\lambda_{2,\chi_{-4}}(z; a, 16)) = \lambda_{2,\chi_{-4}}(z; a, 4)$. By Theorem 5.2, $\lambda_{2,\chi_{-4}}(z; a, 4) = \tilde{\lambda}_{1,\mathfrak{P}_2{}^2,\psi_{-4}}(z, z) + \tilde{\lambda}_{1,\overline{\mathfrak{P}}_2{}^2,\psi_{-4}}(z, z) + \tilde{\lambda}_{1,(2),\psi_{-4}}(z) - \tilde{\lambda}_{1,2\mathfrak{P}_2,\psi_{-4}}(z, z) - \tilde{\lambda}_{1,2\overline{\mathfrak{P}}_2,\psi_{-4}}(z, z) \in \mathbf{M}_2(16)$. By Proposition 3.1, we have

$$\begin{aligned}
&\tilde{\lambda}_{1,\mathfrak{P}_2{}^2,\psi_{-4}}(z, z)|_{\mathcal{C}(16)} = \tilde{\lambda}_{1,\overline{\mathfrak{P}}_2{}^2,\psi_{-4}}(z, z)|_{\mathcal{C}(16)}\\
&\quad = 2^{-5}L(0, \chi_{-4a})\{2^4\kappa_{\mathbf{1},16}(\mathbf{1}, 16) - \kappa_{\mathbf{1},16}(\mathbf{1}, 1)\},\\
&\tilde{\lambda}_{1,(2),\psi_{-4}}(z, z)|_{\mathcal{C}(16)}\\
&\quad = 2^{-5}L(0, \chi_{-4a})\{2^4\kappa_{\mathbf{1},16}(\mathbf{1}, 16) + 2^4\kappa_{\mathbf{1},16}(\mathbf{1}, 8) - 2^2\kappa_{\mathbf{1},16}(\mathbf{1}, 2) - \kappa_{\mathbf{1},16}(\mathbf{1}, 1)\},\\
&\tilde{\lambda}_{1,2\mathfrak{P}_2{}^2,\psi_{-4}}(z, z)|_{\mathcal{C}(16)} = \tilde{\lambda}_{1,2\overline{\mathfrak{P}}_2{}^2,\psi_{-4}}(z, z)|_{\mathcal{C}(16)}\\
&\quad = 2^{-6}L(0, \chi_{-4a})\{2^5\kappa_{\mathbf{1},16}(\mathbf{1}, 16) - 2^4\kappa_{\mathbf{1},16}(\mathbf{1}, 2) - \kappa_{\mathbf{1},16}(\mathbf{1}, 1)\},
\end{aligned}$$

and

$$\lambda_{2,\chi_{-4}}(z; a, 4, 1)|_{\mathcal{C}(16)} = 2^{-4}L(0, \chi_{-4a})\{2^3\kappa_{\mathbf{1},16}(\mathbf{1}, 16) + 2^3\kappa_{\mathbf{1},16}(\mathbf{1}, 8) - \kappa_{\mathbf{1},16}(\mathbf{1}, 1)\}.$$

Hence $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(16z)^2) = L(0, \chi_{-4a})\{G_2(z) - G_2(2z) + 2G_2(4z) - 8G_2(8z)\}$, and from this we obtain $r^{(8)}_{(1,4*,4*)}(an^2) = 2\xi(a, n)$ for $an^2 \equiv 1 \pmod{16}$, and $r^{(8)}_{(3,4*,4*)}(an^2) = 2\xi(a, n)$ for $an^2 \equiv 9 \pmod{16}$.

Suppose $a = 1$. Then $\mathscr{S}_{1,\chi_{-4}}(\theta(z)\theta(16z)^2 - \Theta_{\chi_{-4}}(z)) = L(0, \chi_{-4})\{G_2(z) - G_2(2z) + 2G_2(4z) - 8G_2(8z)\}$. From this we obtain $r^{(8)}_{(1,4*,4*)}(n^2) = 2\xi(1, n) + \chi_{-4}(n)n$.

(I-ii) The case $2\|x_2$, $2\|x_3$.

Let $a>1$. Since $\theta(z)\theta(4z)^2 = 2\theta(z)G_{1\chi_{-4}}(4z)$ by Lemma 10.1, $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)^2) = U_2(\lambda_{2,\chi_{-4}}(z;a,4)) = \lambda_{2,\chi_{-4}}(z;a,1) \in \mathbf{M}_2(4)$. By Theorem 5.2, $\lambda_{2,\chi_{-4}}(z;a,1) = \tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z)$. By Proposition 3.1, we have $\tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z)|_{\mathcal{C}(4)} = 2^{-3}L(0,\chi_{-4a}) \times \{2^2\kappa_{\mathbf{1},4}(\mathbf{1},4) - \kappa_{\mathbf{1},4}(\mathbf{1},1)\}$. Hence $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)^2) = 2L(0,\chi_{-4})\{G_2(z) - 4G_2(4z)\}$. Since $r^{(8)}_{(3,2,2)}(an^2) = \#\{(x_1,2x_2,2x_3) \in \mathbf{Z}^3 \mid x_1^2+4x_2^2+4x_3^2 = an^2\} - \#\{(x_1,4x_2,4x_3) \in \mathbf{Z}^3 \mid x_1^2+16x_2^2+16x_3^2 = an^2\}$ for $an^2 \equiv 1 \pmod{16}$, we can obtain $r^{(8)}_{(3,2,2)}(an^2) = 2\xi(a,n)$ from $\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)^2 - \theta(z)\theta(16z)^2) = L(0,\chi_{-4a})\{G_2(z) + G_2(2z) - 10G_2(4z) + 8G_2(8z)\}$. The equality $r^{(8)}_{(1,2,2)}(an^2) = 2\xi(a,n)$ for $an^2 \equiv 9 \pmod{16}$ is proved similarly.

Let $a=1$, From the last part of the case (I-i), we have

$$\begin{aligned}&\mathscr{S}_{1,\chi_{-4}}(\theta(z)\theta(4z)^2 - \theta(z)\theta(16z)^2 - \Theta_{\chi_{-4}}(z))\\&\quad = L(0,\chi_{-4})\{G_2(z)+G_2(2z)-10G_2(4z)+8G_2(8z)\}.\end{aligned}$$

The equalities $r^{(8)}_{(3,2,2)}(n^2) = 2\xi(1,n) - \chi_{-4}(n)n$ for $n \equiv \pm 1 \pmod 8$ and $r^{(8)}_{(1,2,2)}(n^2) = 2\xi(1,n) - \chi_{-4}(n)n$ for $n \equiv \pm 3 \pmod 8$ follow from this.

(II) The case $a \equiv 5 \pmod 8$.

We investigate the Fourier expansion of $\theta(z)\theta(4z)\theta(16z) = \theta(z)\{G_{1,\chi_{-4}}(4z) - G_{1,\chi_{-4}}(8z) + 2G_{1,\chi_{-4}}(16z)\}$. By Theorem 5.2,

$$\begin{aligned}&\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)\theta(16z))\\&\quad = 2^{-1}U_2(\lambda_{2,\chi_{-4}}(z;a,4) + \lambda_{2,\chi_{-4}}(z;a,16))\\&\quad = 2^{-1}\{\tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z) + \tilde{\lambda}_{1,(2),\psi_{-4}}(z,z)\} \in \mathbf{M}_2(8),\end{aligned}$$

and by Proposition 3.1,

$$\begin{aligned}\tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z)|_{\mathcal{C}(8)} &= 2^{-3}L(0,\chi_{-4a})\{2^2\kappa_{\mathbf{1},8}(\mathbf{1},8) + 2^2\kappa_{\mathbf{1},8}(\mathbf{1},4) - \kappa_{\mathbf{1},8}(\mathbf{1},1)\},\\ \tilde{\lambda}_{1,(2),\psi_{-4}}(z,z)|_{\mathcal{C}(8)} &= 2^{-5}L(0,\chi_{-4a})\{2^4\kappa_{\mathbf{1},8}(\mathbf{1},8) - 2^2\kappa_{\mathbf{1},8}(\mathbf{1},2) - \kappa_{\mathbf{1},8}(\mathbf{1},1)\}.\end{aligned}$$

Then

$$\begin{aligned}&\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)\theta(16z))|_{\mathcal{C}(8)}\\&\quad = 2^{-6}L(0,\chi_{-4a})\{2^5\kappa_{\mathbf{1},8}(\mathbf{1},8) + 2^4\kappa_{\mathbf{1},8}(\mathbf{1},4) - 2^2\kappa_{\mathbf{1},8}(\mathbf{1},2) - 5\kappa_{\mathbf{1},8}(\mathbf{1},1)\},\end{aligned}$$

and

$$\mathscr{S}_{a,\chi_{-4}}(\theta(z)\theta(4z)\theta(16z)) = L(0,\chi_{-4a})\{G_2(z)+G_2(2z)-4G_2(4z)-4G_2(8z)\}.$$

Then $r^{(8)}_{(1,2,4*)}(an^2) = 2\xi(a,n)$ (resp. $r^{(8)}_{(3,2,4*)}(an^2) = 2\xi(a,n)$) for $an^2 \equiv 5 \pmod{16}$ (resp. $an^2 \equiv 13 \pmod{16}$).

(III) The case $a \equiv 3 \pmod 8$.

We have $\theta(z) - \theta(4z) + s\theta_{\chi_8}(z) = 2\sum_{\substack{-\infty<n<\infty \\ \chi_8(n)=s}} \mathbf{e}(n^2z)$ for $s = \pm 1$, and hence

$$\begin{aligned}&\theta(z)\{\theta(z) - \theta(4z) + \theta_{\chi_8}(z)\}\{\theta(z) - \theta(4z) - \theta_{\chi_8}(z)\} \\ &\quad = 4\sum_{n=1}^{\infty} \#\{\boldsymbol{x} \in \mathbf{Z}^3 \mid x_2 \equiv \pm 1, x_3 \equiv \pm 3 \pmod 8, \quad |\boldsymbol{x}|^2 = n\}\, \mathbf{e}(nz). \end{aligned} \tag{32}$$

If $n \equiv 3 \pmod{16}$ (resp. $n \equiv 11 \pmod{16}$), then n-th Fourier coefficient of (32) gives $r^{(8)}_{(1,3,3)}(n)$ (resp. $r^{(8)}_{(1,1,3)}(n)$). The product (32) is equal to $2\theta(z)G_{1,\chi_{-4}}(2z) - 2\theta(z)G_{1,\chi_{-4}}(4z) - 2\theta(z)G^{\chi_8}_{1,\chi_{-8}}(2z)$ by Lemma 10.1. By Lemma 10.2 the n-th Fourier coefficient of $\theta(z)G^{\chi_8}_{1,\chi_{-8}}(2z)$ is equal to $2c_{f_{3/2}}(n)$ if n is odd and $n \equiv 3 \pmod 8$. We compute $\mathscr{S}_{4a,\chi_{-4}}(2\theta(z)G_{1,\chi_{-4}}(2z) - 2\theta(z)G_{1,\chi_{-4}}(4z)) \in \mathbf{M}_2(4)$. We have $\mathscr{S}_{4a,\chi_{-4}}(\theta(z)G_{1,\chi_{-4}}(2z)) = \lambda_{2,\chi_{-4}}(z; 4a, 2) = \tilde{\lambda}_{1,\mathfrak{p}_2,\psi_{-4}}(z, z)$ and $\mathscr{S}_{4a,\chi_{-4}}(\theta(z) \times G_{1,\chi_{-4}}(4z)) = \lambda_{2,\chi_{-4}}(z; 4a, 4) = \tilde{\lambda}_{1,(2),\psi_{-4}}(z, z)$ by Theorem 5.2, and

$$\begin{aligned}&\mathscr{S}_{4a,\chi_{-4}}(2\theta(z)G_{1,\chi_{-4}}(2z) - 2\theta(z)G_{1,\chi_{-4}}(4z))|_{\mathcal{C}(4)} \\ &\quad = 2^{-3}L(0, \chi_{-a})\{2^4\kappa_{\mathbf{1},4}(\mathbf{1}, 4) + 2^2\kappa_{\mathbf{1},4}(\mathbf{1}, 2) - 5\kappa_{\mathbf{1},4}(\mathbf{1}, 1)\}\end{aligned}$$

by Proposition 3.1. Hence

$$\mathscr{S}_{4a,\chi_{-4}}(2\theta(z)G_{1,\chi_{-4}}(2z) - 2\theta(z)G_{1,\chi_{-4}}(4z)) = 12L(0, \chi_{-a})\{G_2(z) - G_2(2z) - 2G_2(4z)\}$$

and the an^2-th Fourier coefficient of $2\theta(z)G_{1,\chi_{-4}}(2z) - 2\theta(z)G_{1,\chi_{-4}}(4z)$ is $12\xi(a, n)$ for n odd. Since this is equal to an^2-th Fourier coefficient of $\theta(z)\{\theta(z) - \theta(4z) + \theta_{\chi_8}(z)\}\{\theta(z) - \theta(4z) - \theta_{\chi_8}(z)\} + 4f_{3/2}(z)$, we have $r^{(8)}_{(1,3,3)}(an^2) = 3\xi(a, n) - c_{f_{3/2}}(an^2)$ (resp. $r^{(8)}_{(1,1,3)}(an^2) = 3\xi(a, n) - c_{f_{3/2}}(an^2)$) for $an^2 \equiv 3 \pmod{16}$ (resp. $an^2 \equiv 11 \pmod{16}$). Because $r_{(1,1,1)}(an^2) + 3r_{(1,3,3)}(an^2) = r_3(an^2) = 12\xi(a, n)$ for $an^2 \equiv 3 \pmod{16}$ and $r_{(3,3,3)}(an^2) + 3r_{(1,1,3)}(an^2) = 12\xi(a, n)$ for $an^2 \equiv 11 \pmod{16}$, we obtain also the formulas for $r_{(1,1,1)}(an^2)$ and $r_{(3,3,3)}(an^2)$.

(IV) The case $a \equiv 2 \pmod 8$.

The number $r_{(1,1,4*)}(an^2)$ (resp. $r_{(1,3,4*)}(an^2)$) equals the an^2-th Fourier coefficient of $2^{-1}\theta(z)\theta(16z)\{\theta(z) - \theta(4z) + \theta_{\chi_8}(z)\}$ for $an^2 \equiv 2 \pmod{16}$ (resp. $an^2 \equiv 10 \pmod{16}$). By Lemma 10.2, the an^2-th Fourier coefficient of $\theta(z)\theta(16z)\theta_{\chi_8}(z)$ is equal to $4c_{f'_{3/2}}(an^2/2) = 4c_{f_{3/2}}(an^2/2)$. We consider the Fourier coefficients of $2^{-1}\theta(z)\theta(16z)\{\theta(z) - \theta(4z)\}$.

By Lemma 10.1, we have

$$\begin{aligned}\theta(z)^2\theta(16z) = 2^{-1}\theta(z)\{&G_{1,\chi_{-4}}(z) - G_{1,\chi_{-4}}(2z) + 2G_{1,\chi_{-4}}(4z) \\ &- 2G_{1,\chi_{-4}}(8z) + 4G_{1,\chi_{-4}}(16z) + G^{\chi_8}_{1,\chi_{-8}}(z)\},\end{aligned}$$

and hence

$$\begin{aligned}
&\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^2\theta(16z)) \\
&\quad = 2^{-2}\{\lambda_{2,\chi_{-4}}(z;a,1) - \lambda_{2,\chi_{-4}}(z;a,2) + 2\lambda_{2,\chi_{-4}}(z;a,4) - 2\lambda_{2,\chi_{-4}}(z;a,8) \\
&\qquad + 4\lambda_{2,\chi_{-4}}(z;a,16) + \lambda^{\chi_8}_{2,\chi_{-8}}(z;a,1)\} \\
&\quad = 2^{-2}\{\tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z) - \tilde{\lambda}_{1,\mathfrak{P}_2,\psi_{-4}}(z,z) + 2\tilde{\lambda}_{1,(2),\psi_{-4}}(z,z) - 2\tilde{\lambda}_{1,2\mathfrak{P}_2,\psi_{-4}}(z,z) \\
&\qquad + 4\tilde{\lambda}_{1,(4),\psi_{-4}}(z,z) + \tilde{\lambda}^{\psi_8}_{1,\mathcal{O}_K,\psi_{-8}}(z,z)\} \in \mathbf{M}(8).
\end{aligned}$$

Similarly we have $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)\theta(4z)\theta(16z)) = 2^{-1}\{\lambda_{2,\chi_{-4}}(z;a,4) - \lambda_{2,\chi_{-4}}(z;a,8) + 2\lambda_{2,\chi_{-4}}(z;a,16)\}$. By Proposition 3.1,

$$\begin{pmatrix} \tilde{\lambda}_{1,\mathcal{O}_K,\psi_{-4}}(z,z)|_{\mathcal{C}(8)} \\ \tilde{\lambda}_{1,\mathfrak{P}_2,\psi_{-4}}(z,z)|_{\mathcal{C}(8)} \\ \tilde{\lambda}_{1,(2),\psi_{-4}}(z,z)|_{\mathcal{C}(8)} \\ \tilde{\lambda}_{1,2\mathfrak{P}_2,\psi_{-4}}(z,z)|_{\mathcal{C}(8)} \\ \tilde{\lambda}_{1,(4),\psi_{-4}}(z,z)|_{\mathcal{C}(8)} \\ \tilde{\lambda}^{\psi_8}_{1,\mathcal{O}_K,\psi_{-8}}(z,z)|_{\mathcal{C}(8)} \end{pmatrix} = L(0,\chi_{-4a}) \begin{pmatrix} 2^{-1} & 2^{-1} & 2^{-1} & -2^{-2} \\ 2^{-1} & 2^{-1} & 0 & -2^{-3} \\ 2^{-1} & 2^{-1} & -2^{-2} & -2^{-4} \\ 2^{-1} & 0 & -2^{-3} & -2^{-5} \\ 2^{-1} & -2^{-2} & -2^{-4} & -2^{-6} \\ 0 & 0 & 2^{-1} & -2^{-3} \end{pmatrix} \begin{pmatrix} \kappa_{\mathbf{1},8}(\mathbf{1},8) \\ \kappa_{\mathbf{1},8}(\mathbf{1},4) \\ \kappa_{\mathbf{1},8}(\mathbf{1},2) \\ \kappa_{\mathbf{1},8}(\mathbf{1},1) \end{pmatrix},$$

and hence $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)^2\theta(16z))|_{\mathcal{C}(8)} = 2^{-5}L(0,\chi_{-4a})\{2^4\kappa_{\mathbf{1},8}(\mathbf{1},8) + 2^2\kappa_{\mathbf{1},8}(\mathbf{1},2) - 3\kappa_{\mathbf{1},8}(\mathbf{1},1)\}$ and $\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)\theta(4z)\theta(16z))|_{\mathcal{C}(8)} = 2^{-5}L(0,\chi_{-4a})\{2^4\kappa_{\mathbf{1},8}(\mathbf{1},8) - 2^2\kappa_{\mathbf{1},8}(\mathbf{1},2) - \kappa_{\mathbf{1},8}(\mathbf{1},1)\}$. Then

$$\mathscr{S}_{4a,\chi_{-4}}(2^{-1}\theta(z)\theta(16z)\{\theta(z) - \theta(4z)\}) = L(0,\chi_{-4a})\{G_2(z) - 3G_2(2z) + 2G_2(4z)\}.$$

From this we obtain $r_{(1,1,4*)}(an^2) - 2c_{f_{3/2}}(an^2/2) = 2\xi(a,n)$ for $an^2 \equiv 2 \pmod{16}$ and $r_{(1,3,4*)}(an^2) = 2\xi(a,n)$ for $an^2 \equiv 10 \pmod{16}$, where we note that $c_{f_{3/2}}(n) = 0$ if $n \equiv 5 \pmod{16}$. Since $3r_{(1,1,4*)}(an^2) + 3r_{(3,3,4*)}(an^2) = r_3(an^2)$ for $an^2 \equiv 2 \pmod{16}$, the formula for $r_{(3,3,4*)}(an^2)$ is also obtained.

(V) The case $a \equiv 6 \pmod 8$.

The number $r_{(1,1,2)}(an^2)$ (resp. $r_{(1,3,2)}(an^2)$) equals the an^2-th Fourier coefficient of $2^{-1}\theta(z)\theta(4z)\{\theta(z) - \theta(4z) + \theta_{\chi_8}(z)\}$ for $an^2 \equiv 6 \pmod{16}$ (resp. $an^2 \equiv 14 \pmod{16}$). By Lemma 10.2, the an^2-th Fourier coefficient of $\theta(z)\theta(4z)\theta_{\chi_8}(z)$ is equal to $4c_{f_{3/2}}(an^2/2)$. We consider the Fourier coefficients of $2^{-1}\theta(z)\theta(4z)\{\theta(z) - \theta(4z)\}$. We already know that $\mathscr{S}_{4a,\chi_{-4}}(\theta(z)^2\theta(4z)) = 2L(0,\chi_{-4a})\{G_2(z) - 4G_2(4z)\}$, $\mathscr{S}_{4a,\chi_{-4}}(\theta(z)\theta(4z)^2) = 6L(0,\chi_{-4a})\{G_2(2z) - 2G_2(4z)\}$ and hence

$$\mathscr{S}_{4a,\chi_{-4}}(2^{-1}\theta(z)\theta(4z)\{\theta(z) - \theta(4z)\}) = L(0,\chi_{-4a})\{G_2(z) - 3G_2(2z) + 2G_2(4z)\}.$$

Then we have $r^{(8)}_{(1,1,2)}(an^2) - 2c_{f_{3/2}}(an^2/2) = 2\xi(a,n)$ for $an^2 \equiv 6 \pmod 8$ and $r^{(8)}_{(3,3,2)}(an^2) = 2\xi(a,n)$ for $an^2 \equiv 14 \pmod 8$ where we note that $c_{f_{3/2}}(n) = 0$ for $n \equiv 7 \pmod 8$. Since $3r^{(8)}_{(1,1,2)}(an^2) + 3r^{(8)}_{(3,3,2)}(an^2) = r_3(an^2)$ for $an^2 \equiv 6 \pmod 8$, we obtain also the formula for $r^{(8)}_{(3,3,2)}(an^2)$.

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