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Asymptotic expansions for the gamma function

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A B S T R A C T

Mortici (2015) [31] proposed a new formula for approximating the gamma function and the convergence of the corresponding asymptotic series is very fast in comparison with other classical or recently discovered asymptotic series. In this paper, by the Lagrange–Bürmann formula we give an explicit formula for determining the coefficients a_k $(k = 1, 2, \ldots)$ in Mortici's formula such that

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \exp\left\{\sum_{k=1}^{\infty} a_k \left(\frac{x}{12x^2+\frac{2}{5}}\right)^k\right\}, \quad x \to \infty.$$

Moreover, by the cycle indicator polynomial of symmetric group, we give an explicit expression for the coefficients b_k $(k = 0, 1, \ldots)$ of the following expansion:

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \left(\sum_{k=0}^{\infty} b_k \left(\frac{x}{12x^2+\frac{2}{5}}\right)^k\right)^{1/r}, \quad x \to \infty.$$

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A recursive formula for calculating the coefficients b_k ($k = 0, 1, \ldots$) is also given.

1. Introduction

In the theory of mathematical constants, Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{1.1}$$

named after James Stirling, though it was first stated by Abraham de Moivre, is an approximation for factorials. It is not only a very powerful tool of approximation but also has many applications in statistical physics, probability theory and number theory.

The Euler gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0$$

is one of the most important functions in mathematical analysis, especially in the area of special functions. It has a lot of applications in various diverse areas and it has been staying in the middle of attention of many authors in years. A formula for approximation of $\Gamma(x)$ for large value of x, a more general one than (1.1), is of special attraction. It is stated as follows

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x \to \infty.$$

This formula was improved by an asymptotic series which is often called the Stirling series

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left\{\sum_{i=1}^\infty \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right\}, \quad x \to \infty, \tag{1.2}$$

where B_i denotes the ith Bernoulli number defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{i=1}^\infty \frac{B_i}{i!}x^i.$$

It is well known that the first Bernoulli numbers are $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = -1/30$ with $B_{2i+1} = 0$, for each integer $i \geq 1$. Laplace [1] obtained the following asymptotic expansion for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right\}, \quad x \to \infty.$$

In [8], Chen and Lin proved that the gamma function has the following asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}\right)^{x^l/r}, \quad x \to \infty,$$

where the coefficients b_j are given by

$$b_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{j+1}}{j\cdot(j+1)}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+l)k_1 + (2+l)k_2 + \cdots + (j+l)k_j = j.$$

When $l = 0$, this result reduces to the main theorem in [6]. Recently, Chen and Tong [9] gave a recurrence relation for determining the coefficients a_j such that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x+\sum_{j=0}^{\infty} a_j x^{-j}}\right), \quad x \to \infty.$$

They also provided a pair of recurrence relations for determining the constants α_l and β_l such that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\sum_{l=1}^{\infty}\frac{\alpha_l}{(x+\beta_l)^{2l-1}}\right), \quad x \to \infty.$$

In the recent paper [31], based on the first approximation in (1.2)

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \exp\left(\frac{1}{12x}\right),$$

Mortici proposed a new formula

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \exp\left(\frac{x}{12x^2+2/5}\right),$$

which is the best in the following class of approximations for every real a:

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \exp\left(\frac{x}{12x^2+a}\right).$$

By numerical comparison [31], the convergence of this formula is faster than classical formulas due to Burnside [5]:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}},$$

Gosper [11]:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\sqrt{x+\frac{1}{6}},$$

and Ramanujan [35]:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\sqrt[6]{x^3+\frac{1}{2}x^2+\frac{1}{8}x+\frac{1}{240}}.$$

Furthermore, Mortici [31] presented a new asymptotic series in terms of $y = x/(12x^2 + 2/5)$ as x approaches infinity:

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}} \sim \exp\left\{\sum_{k=1}^{\infty} a_k y^k\right\}, \tag{1.3}$$

where the coefficients a_k are given by the following linear system:

$$\sum_{0\le j<m/2} 12^{j-m}\left(\frac{2}{5}\right)^{j}\binom{-m+2j}{j}a_{m-2j} = \frac{B_m}{m(m+1)}, \quad m \ge 1.$$

For more works on extensions of Stirling's formula and approximations and asymptotic expansions of the gamma function, one is referred to [13–34] and references therein.

Motivated by these interesting works, in this paper we provide an explicit expression for the coefficients a_k $(k = 1, 2, \ldots)$ in Mortici's expansion (1.3). Our method is based on the Lagrange–Bürmann formula. Moreover, by the cycle indicator polynomial of symmetric group, which is an important tool in enumerative combinatorics, we give an explicit expression for the coefficients b_k $(k = 0, 1, \ldots)$ of the following expansion in terms of $y = x/(12x^2 + 2/5)$:

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}} \sim \left(\sum_{k=0}^{\infty} b_k y^k\right)^{1/r}, \quad x \to \infty.$$

A recursive formula for calculating the coefficients b_k $(k = 0, 1, \ldots)$ will be also given.

2. Main results

Recall that the classical Lagrange–Bürmann expansion formula [10] is a landmark discovery in the history of analysis, see also [36]. In this section, we will make use of this well-known formula to given an explicit expression for the coefficients a_k in (1.3).

Let $[t^i]$ be the operator which gives the ith coefficient in the series development of a generating function. Now, we state the Lagrange–Bürmann formula as a lemma.

Lemma 2.1. *Let $F(t)$ and $\phi(t)$ be analytic around $t = 0$ and $\phi(0) \neq 0$. We have*

$$F(t) = F(0) + \sum_{k=1}^{\infty} a_k \left(\frac{t}{\phi(t)}\right)^k,$$

where

$$a_k = \frac{1}{k}[t^{k-1}]\left\{F'(t)\phi(t)^k\right\}, \quad k \geq 1.$$

Applying the above Lagrange–Bürmann formula, we have the following theorem.

Theorem 2.1. *Let $y = \frac{x}{12x^2+\frac{2}{5}}$ and $k \geq 1$ be an integer. For $x \to \infty$ we have the following asymptotic series*

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \exp\left\{\sum_{k=1}^{\infty} a_k y^k\right\}, \tag{2.1}$$

where the coefficients a_k are explicitly given by

$$a_k = \begin{cases} \frac{1}{k}\sum_{i=0}^{\frac{k-1}{2}} \frac{B_{2i+2}}{2i+2}\binom{k}{\frac{k-1}{2}-i}\left(\frac{2}{5}\right)^{\frac{k-1}{2}-i} 12^{\frac{k+1}{2}+i}, & k \text{ is odd}, \\ 0, & k \text{ is even}. \end{cases} \tag{2.2}$$

Proof. According to (1.2) we have

$$\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \sum_{k=1}^{\infty} a_k \left(\frac{x}{12x^2+\frac{2}{5}}\right)^k. \tag{2.3}$$

Let $t = 1/x$, $F(t) = \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} t^{2i-1}$ and $\phi(t) = \frac{2}{5}t^2 + 12$. Thus, Eq. (2.3) can be rewritten as

$$F(t) = \sum_{k=1}^{\infty} a_k \left(\frac{t}{\phi(t)}\right)^k.$$

In view of Lemma 2.1 we obtain

$$\begin{aligned} a_k &= \frac{1}{k}[t^{k-1}]\left\{\sum_{i=1}^{\infty} \frac{B_{2i}}{2i} t^{2i-2}\left(\frac{2}{5}t^2+12\right)^k\right\} \\ &= \frac{1}{k}[t^{k-1}]\left\{\sum_{i=0}^{\infty} \frac{B_{2i+2}}{2i+2} t^{2i} \sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{5}\right)^j 12^{k-j} t^{2j}\right\} \\ &= \frac{1}{k}[t^{k-1}]\left\{\sum_{m=0}^{\infty} t^{2m} \sum_{i=\max\{0,m-k\}}^{m} \frac{B_{2i+2}}{2i+2}\binom{k}{m-i}\left(\frac{2}{5}\right)^{m-i} 12^{k+i-m}\right\}, \end{aligned}$$

which implies (2.2) is true. □

Remark 2.1. From this theorem, we can easily obtain the first few cases of the a_k's:

$$a_1 = 1, \quad a_3 = 0, \quad a_5 = \frac{30528}{175},$$
$$a_7 = -\frac{2128896}{125}, \quad a_9 = \frac{178552553472}{48125}, \ldots.$$

Let us recall the notions of groups of permutations and cycle indicator polynomial. For more details one can refer to [10]. A group $\mathfrak{G}$ of permutation of a finite set N be a subgroup of the group $\mathfrak{S}(N)$ of all permutations of N, and we denote $\mathfrak{G} \leq \mathfrak{S}(N)$. $|\mathfrak{G}|$ is called the order of $\mathfrak{G}$, and $|N|$ its degree.

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{N}$ be a set of non-negative integers. For every permutation $\sigma \in \mathfrak{G}(N)$, $|N| = n$, denote $c_i(\sigma)$ the number of orbits of length i of σ, $i \in [n]$. We define the cycle indicator polynomial $Z(x_1, x_2, \ldots, x_n) := Z(\mathfrak{G}; x_1, x_2, \ldots, x_n)$ of a group of permutations $\mathfrak{G}$ of N:

$$Z(x_1, x_2, \ldots, x_n) = \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \cdots x_n^{c_n(\sigma)}.$$

If $\mathfrak{G} = \mathfrak{S}(N)$ (the symmetric group of degree n), the cycle indicator polynomial denoted by $Z_n(x_1, x_2, \ldots, x_n) := Z_n(\mathfrak{G}; x_1, x_2, \ldots, x_n)$ is explicitly expressed by

$$Z_n(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{c \in \varpi_n} (c; n)^* x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n},$$

where

$$\varpi_n = \left\{ c := (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n \,\middle|\, \sum_{k=1}^{n} k c_k = n \right\}$$

is a set in which an element corresponds to a way of partition of n, and

$$(c; n)^* = \frac{n!}{c_1! c_2! \cdots c_n! 1^{c_1} 2^{c_2} \cdots n^{c_n}}$$

is the number of permutations of type $[\![c]\!] = [\![c_1, c_2, \ldots, c_n]\!]$. The first few cases are

$$Z_0 = 1,$$

$$Z_1(x_1) = x_1,$$

$$Z_2(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2),$$

$$Z_3(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1 x_2 + 2x_3),$$

$$Z_4(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^4 + 6x_1^2 x_2 + 3x_2^2 + 8x_1 x_3 + 6x_4),$$

$$Z_5(x_1,x_2,x_3,x_4,x_5)=\frac{1}{120}(x_1^5+10x_1^3x_2+15x_1x_2^2+20x_1^2x_3+20x_2x_3+30x_1x_4+24x_5),$$

$$Z_6(x_1,x_2,x_3,x_4,x_5,x_6)=\frac{1}{720}(x_1^6+15x_1^4x_2+45x_1^2x_2^2+40x_1^3x_2+40x_1^3x_3 \\ +15x_2^3+120x_1x_2x_3+90x_1^2x_4+40x_3^2+90x_2x_4+144x_1x_5+120x_6).$$

The cycle indicator polynomial can also be defined by its ordinary generating function

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m}\right)=1+\sum_{n=1}^{\infty} Z_n(x_1,x_2,\dots,x_n)t^n. \tag{2.4}$$

From (2.4) we have the following recurrence relation:

$$Z_0=1, \quad nZ_n(x_1,x_2,\dots,x_n)=\sum_{k=1}^{n} x_k Z_{n-k}(x_1,x_2,\dots,x_{n-k}),\ n\ge 1. \tag{2.5}$$

It is worth noticing that the cycle indicator polynomials are well connected with the well-known Bell polynomials [2,10] by

$$Z_n(x_1,x_2,\dots,x_n)=\frac{1}{n!}Y_n(0!x_1,1!x_2,\dots,(n-1)!x_n),\ n=1,2,\dots.$$

Using the cycle indicator polynomials, we present the following expansion in terms of $x/(12x^2+2/5)$:

Theorem 2.2. *Let r be a given nonzero real number and $k\ge 1$ be an integer. For $y=\frac{x}{12x^2+\frac{2}{5}}$ and $x\to\infty$ we have the following asymptotic series*

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}\sim\left(1+\sum_{k=1}^{\infty} b_k y^k\right)^{1/r},$$

where the coefficients b_k are explicitly given by

$$b_k=Z_k(ra_1,2ra_2,\dots,kra_k), \tag{2.6}$$

and the a_k's are given as in (2.2).

Proof. From (2.1) it is obvious that

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}\right)^r\sim\exp\left\{\sum_{k=1}^{\infty} ra_k y^k\right\}, \tag{2.7}$$

where the a_k's are as given in (2.2). Let $t=y$, $x_m=mra_m$ in (2.4). Combining with (2.7) and (2.4) yields (2.6). □

Formula (2.6) seems very complicated, even though it is a real explicit formula for the coefficients b_k $(k = 1, 2, \ldots)$. The concrete computation is almost practical impossible. Having in mind this fact, we propose a new approach using recurrences, where only simple computations are required to find the coefficients b_k $(k = 1, 2, \ldots)$.

Now, we need the following lemma about functional transformations of asymptotic series, which is equivalent to (2.5). This lemma has his origin in Euler's work. See [12] for the explanation in the case of Taylor series, and [3,4,7,37] for its use in the case of asymptotic series.

Lemma 2.2. *Let $G(x) = \sum_{n=1}^{\infty} g_n x^n$ be a formal power series. Then the composition*

$$\exp(G(x)) = \sum_{n=0}^{\infty} f_n x^n,$$

where the coefficients f_n $(n = 0, 1, \ldots)$ satisfy

$$f_0 = 1, \quad f_n = \frac{1}{n} \sum_{k=1}^{n} k g_k f_{n-k}, \; n \geq 1.$$

Applying this lemma we have

Theorem 2.3. *Let r be a given nonzero real number and $k \geq 1$ be an integer. For $y = \frac{x}{12x^2+\frac{2}{5}}$ and $x \to \infty$ we have the following asymptotic series*

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \sim \left(\sum_{k=0}^{\infty} b_k y^k\right)^{1/r},$$

where the coefficients b_k are recursively given by

$$b_0 = 1, \quad b_k = \frac{r}{k} \sum_{i=1}^{k} i a_i b_{k-i}, \; k \geq 1, \tag{2.8}$$

and the a_k's are given as in (2.2).

Remark 2.2. Because $a_{2i} = 0$ $(i = 1, 2, \ldots)$, (2.8) can be rewritten as

$$b_{2k-1} = \frac{r}{2k-1} \sum_{i=1}^{k} (2i-1) a_{2i-1} b_{2k-2i},$$

$$b_{2k} = \frac{r}{2k} \sum_{i=1}^{k} (2i-1) a_{2i-1} b_{2k-2i+1}, \quad k \geq 1.$$

Remark 2.3. Using the recurrence relation (2.8), we here present the first few coefficients:

$$\begin{aligned}
b_1 &= a_1 b_0 = r, \\
b_2 &= \frac{r}{2} a_1 b_1 = \frac{r}{2} \times r = \frac{r^2}{2}, \\
b_3 &= \frac{r}{3}(a_1 b_2 + 3a_3 b_0) = \frac{r^3}{6}, \\
b_4 &= \frac{r}{4}(a_1 b_3 + 3a_3 b_1) = \frac{r}{4} \times \frac{r^3}{6} = \frac{r^4}{24}, \\
b_5 &= \frac{r}{5}(a_1 b_4 + 3a_3 b_2 + 5a_5 b_0) = \frac{r}{5}\left(\frac{r^4}{24} + 5 \times \frac{30528}{175}\right) = \frac{r^5}{120} + \frac{30528r}{175}, \\
b_6 &= \frac{r}{6}(a_1 b_5 + 3a_3 b_3 + 5a_5 b_1) = \frac{r}{6}\left(\frac{r^5}{120} + \frac{30528r}{175} + 5 \times \frac{30528r}{175}\right) = \frac{r^6}{720} + \frac{30528r^2}{175}.
\end{aligned}$$

In particular, for $r = 1$ we have

$$\begin{aligned}
&b_1 = 1, \quad b_2 = \frac{1}{2}, \quad b_3 = \frac{1}{6}, \quad b_4 = \frac{1}{24}, \\
&b_5 = \frac{732707}{4200}, \quad b_6 = \frac{4396067}{25200}.
\end{aligned}$$

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