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Conditional expanding bounds for two-variable functions over arbitrary fields

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ABSTRACT

In this paper, we prove some results on the sum-product problem over arbitrary fields which improve and generalize results given by Hegyvári and Hennecart [5]. More precisely, we prove that, for related pairs of two-variable functions $f(x, y)$ and $g(x, y)$, if A and B are two sets in an arbitrary field $\mathbb{F}$ with $|A| = |B|$, then

$$\max\{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+c},$$

for some $c > 0$.

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1. Introduction

Throughout this paper, by $\mathbb{F}$ we refer to any arbitrary field, while by $\mathbb{F}_p$, we only refer to the fields of prime order p. We denote the set of non-zero elements by $\mathbb{F}^*$ and $\mathbb{F}_p^*$, respectively. Furthermore, we use the following convention: if the characteristic of $\mathbb{F}$ is positive, then we denote its characteristic by p; if the characteristic of $\mathbb{F}$ is zero, then we

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set $p = \infty$. So a term like $N < p^{5/8}$ is restrictive in positive characteristic, but vacuous for zero one.

For $A \subset \mathbb{F}$, the sum and the product sets are defined as follows:

$$A + A = \{a + a' : a, a' \in A\}, \ A \cdot A = \{a \cdot a' : a, a' \in A\}.$$

For $A \subset \mathbb{F}_p$, Bourgain, Katz and Tao ([2]) proved that if $p^\delta < |A| < p^{1-\delta}$ for some $\delta > 0$, then we have

$$\max\{|A + A|, |A \cdot A|\} \gg |A|^{1+\epsilon},$$

for some $\epsilon = \epsilon(\delta) > 0$. Here, and throughout, by $X \ll Y$ we mean that there exists the constant $C > 0$ such that $X \leq CY$.

In a recent paper [8], Roche-Newton, Rudnev, and Shkredov improved and generalized this result to the setting of arbitrary fields. More precisely, they showed that for $A \subset \mathbb{F}$, the sum set, product set, and quotient set satisfy

$$\max\{|A \pm A|, |A \cdot A|\} \gg |A|^{6/5}, \ \max\{|A \pm A|, |A : A|\} \gg |A|^{6/5}.$$

We note that the same bound also holds for $|A(1 + A)|$ [10], and $|A + A^2|$, $\max\{|A + A|, |A^2 + A^2|\}$ [7]. We refer the reader to [1,3,8,6] and references therein for recent results on the sum-product topic.

Let G be a subgroup of $\mathbb{F}^*$, and $g\colon G \to \mathbb{F}^*$ be an arbitrary function. We define

$$\mu(g) = \max_{t \in \mathbb{F}^*} |\{x \in G\colon g(x) = t\}|.$$

For $A, B \subset \mathbb{F}_p$ and a related pair of two-variable functions $f(x, y)$ and $g(x, y)$ in $\mathbb{F}_p[x, y]$, Hegyvári and Hennecart [5], using graph theoretic techniques, proved that there exists $\Delta(\alpha) > 0$ such that for $|A| = |B| = p^\alpha$, we have

$$\max\{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+\Delta(\alpha)},$$

where $f(A, B) = \{f(a, b)\colon a \in A, b \in B\}$. More precisely, they established the following results.

Theorem 1.1 *(**Hegyvári and Hennecart**, [5]). Let G be a subgroup of $\mathbb{F}_p^*$. Consider the function $f(x, y) = g(x)(h(x)+y)$ on $G \times \mathbb{F}_p^*$, where $g, h\colon G \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A \subset G$ and $B, C \subset \mathbb{F}_p^*$, we have*

$$|f(A, B)|\,|B \cdot C| \gg \min\left\{\frac{|A||B|^2|C|}{pm^2}, \frac{p|B|}{m}\right\}.$$

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Theorem 1.2 (***Hegyvári and Hennecart***, *[5]*). *Let G be a subgroup of $\mathbb{F}_p^*$. Consider the function $f(x,y)=g(x)(h(x)+y)$ on $G\times\mathbb{F}_p^*$, where $g,h\colon G\to\mathbb{F}_p^*$ are arbitrary functions. Define $m=\mu(g)$. For any subsets $A\subset G$, $B,C\subset\mathbb{F}_p^*$, we have*

$$|f(A,B)||B+C|\gg\min\left\{\frac{|A||B|^2|C|}{pm^2},\frac{p|B|}{m}\right\}.$$

It is worth noting that the result established by Bukh and Tsimerman in [[3], Theorem 6] does not cover such a function defined in Theorem 1.2. The reader can also find the generalizations of Theorems 1.1 and 1.2 in the setting of finite valuation rings in [4].

Suppose $f(x,y)=g(x)(h(x)+y)$ with $\mu(g),\mu(h)=O(1)$ and $A=B=C$. Then, it follows from Theorems 1.1 and 1.2 that

1. If $|A|\gg p^{2/3}$, then we have
$$|f(A,A)||A\cdot A|,|f(A,A)||A+A|\gg p|A|.$$
2. If $|A|\ll p^{2/3}$, then we have
$$|f(A,A)||A\cdot A|,|f(A,A)||A+A|\gg |A|^4/p. \tag{1}$$

The main goal of this paper is to improve and generalize Theorems 1.1 and 1.2 to arbitrary fields for small sets. Our first result is an improvement of Theorem 1.1 for small sets.

Theorem 1.3. *Let $f(x,y)=g(x)(h(x)+y)$ be a function defined on $\mathbb{F}^*\times\mathbb{F}^*$, where $g,h\colon\mathbb{F}^*\to\mathbb{F}^*$ are arbitrary functions. Define $m=\mu(g\cdot h)$. For any subsets $A,B,C\subset\mathbb{F}^*$ with $|A|,|B|,|C|\leq p^{5/8}$, we have*

$$\max\{|f(A,B)|,|B\cdot C|\}\gg\min\left\{\frac{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}}{m^{\frac{4}{5}}},\frac{|B||C|^{\frac{1}{2}}}{m},\frac{|B||A|^{\frac{1}{2}}}{m},\frac{|B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}}{m^{\frac{2}{3}}}\right\}.$$

Remark 1.1. Consider the function $f(x,y)=g(x)(h(x)+y)$ with $\mu(g),\mu(h)=O(1)$, and the subsets $A,B,C\subset\mathbb{F}_p^*$ with $|A|,|B|,|C|\leq p^{5/8}$. By Theorem 1.1 we have

$$|f(A,B)|\,|B\cdot C|\gg\frac{|A||B|^2|C|}{p}.$$

Therefore, Theorem 1.3 improves this bound in the case

$$\min\left\{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}},|B||C|^{\frac{1}{2}},|B||A|^{\frac{1}{2}},|B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}\right\}\gg\frac{|A|^{\frac{1}{2}}|B||C|^{\frac{1}{2}}}{p^{\frac{1}{2}}},$$

which is equivalent to

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$$\max\left\{|A|^{\frac{3}{5}}|B|^{\frac{2}{5}}|C|^{\frac{3}{5}}, |A|, |C|, |A|^{\frac{1}{3}}|B|^{\frac{2}{3}}|C|^{\frac{1}{3}}\right\} \ll p. \tag{2}$$

As a special case, when $|A| = |B| = |C| \leq p^{5/8}$, one can check that inequality (2) holds, thus Theorem 1.3 improves the bound in Theorem 1.1 in this case.

The following are consequences of Theorem 1.3. The first one improves the multiplicative bound in (1).

Corollary 1.4. *Let $f(x,y) = g(x)(h(x)+y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h\colon \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions with $\mu(g \cdot h) = O(1)$. For any subset $A \subset \mathbb{F}^*$ with $|A| \leq p^{5/8}$, we have*

$$\max\{|f(A,A)|, |A \cdot A|\} \gg |A|^{\frac{6}{5}}.$$

Corollary 1.5. *Consider the subsets $A \subset \mathbb{F}^*$, and $B, C \subset \mathbb{F}$ with $|A|, |B|, |C| \leq p^{5/8}$.*

1. *By fixing $g(x) = 1$ and $h(x) = x^{-1}$, we get*

$$\max\left\{|A^{-1}+B|, |B \cdot C|\right\} \gg \min\left\{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}, |B||C|^{\frac{1}{2}}, |B||A|^{\frac{1}{2}}, |B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}\right\}.$$

2. *By fixing $g(x) = x$ and $h(x) = 1$, we have*

$$\max\{|A(B+1)|, |B \cdot C|\} \gg \min\left\{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}, |B||C|^{\frac{1}{2}}, |B||A|^{\frac{1}{2}}, |B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}\right\}.$$

It follows from Corollary 1.5, 2. that if $B = A$ and $C = A+1$ then we have $|A(A+1)| \gg |A|^{6/5}$, which recovers the result of Stevens and de Zeeuw [10]. Corollary 1.5, 2. also gives an improvement of Theorem 1 in [12].

Our next result is the additive version of Theorem 1.3, which improves Theorem 1.2 in the case inequality (2) is satisfied.

Theorem 1.6. *Let $f(x,y) = g(x)(h(x)+y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h\colon \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A, B, C \subset \mathbb{F}^*$ with $|A|, |B|, |C| \leq p^{5/8}$, we have*

$$\max\{|f(A,B)|, |B+C|\} \gg \min\left\{\frac{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}}{m^{\frac{4}{5}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}}{m^{\frac{2}{3}}}\right\}.$$

The following result gives an improvement of (1) in the additive case.

Corollary 1.7. *Let $f(x,y) = g(x)(h(x)+y)$ be a function defined on $\mathbb{F}^* \times \mathbb{F}^*$, where $g, h\colon \mathbb{F}^* \to \mathbb{F}^*$ are arbitrary functions with $\mu(g) = O(1)$. For any subset $A \subset \mathbb{F}^*$ with $|A| \leq p^{5/8}$, we have*

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$$\max\{|f(A,A)|,|A+A|\} \gg |A|^{\frac{6}{5}}.$$

Let $g(x)=x$ and $h(x)=1$, we have the following corollary.

Corollary 1.8. *For $A,B,C\subset\mathbb{F}$ with $|A|,|B|,|C|\leq p^{5/8}$, we have*

$$\max\{|A(B+1)|,|B+C|\} \gg \min\left\{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}, |B||C|^{\frac{1}{2}}, |B||A|^{\frac{1}{2}}, |B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}\right\}.$$

By fixing $g(x)=x$ and $h(x)=0$, we have the following result.

Corollary 1.9. *For $A,B,C\subset\mathbb{F}$ with $|A|,|B|,|C|\ll p^{5/8}$, we have*

$$\max\{|A\cdot B|,|B+C|\} \gg \min\left\{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}, |B||C|^{\frac{1}{2}}, |B||A|^{\frac{1}{2}}, |B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}\right\}.$$

In the case $A=B=C$, we recover the following result due to Roche-Newton, Rudnev, and Shkredov [8], which says that $\max\{|A+A|,|A\cdot A|\}\gg|A|^{6/5}$.

It has been shown in [10] that if $f(x,y)=x(x+y)$, then $|f(A,A)|\gg|A|^{5/4}$ under the condition $|A|\leq p^{2/3}$. In the following theorem, we show that if either $|A+A|$ or $|A\cdot A|$ is sufficiently small, then the exponent $5/4$ can be improved for a larger family of functions over $\mathbb{F}^*\times\mathbb{F}^*$.

Theorem 1.10. *Let $f(x,y)=g(x)(h(x)+y)$ be a function defined on $\mathbb{F}^*\times\mathbb{F}^*$, where $g,h\colon\mathbb{F}^*\to\mathbb{F}^*$ are arbitrary functions with $\mu(f),\mu(g)=O(1)$. Consider the subset $A\subset\mathbb{F}^*$ with $|A|\leq p^{5/8}$, satisfying*

$$\min\{|A+A|,|A\cdot A|\}\leq|A|^{\frac{9}{8}-\epsilon},$$

for some $\epsilon>0$. Then, we have

$$|f(A,A)|\gg|A|^{\frac{5}{4}+\frac{2\epsilon}{3}}.$$

Consider the set $A\subset\mathbb{F}_p^*$ with $|A|\leq p^{\frac{1}{2}-\frac{1}{500}}$. Let $f(x,y)=xy(x+y)$. Using Plünnecke–Ruzsa's inequality and the Balog–Szemer´edi–Gowers Theorem, Hegyvári and Hennecart [5] proved that

$$\max\{|f(A,A)|,|A\cdot A|\}\gg|A|^{1+\frac{1}{800}}.$$

In this note, we use a recent point-line incidence bound due to Stenvens and de Zeeuw [10] to give an improvement of this result over arbitrary fields.

Theorem 1.11. *Consider the set $A\subset\mathbb{F}^*$ with $|A|\leq p^{\frac{11}{17}}$. For the polynomial $f(x,y)=xy(x+y)\in\mathbb{F}[x,y]$ we have*

$$\max\{|f(A,A)|,|A\cdot A|\}\gg|A|^{1+\frac{1}{22}}.$$

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2. Proofs of Theorems 1.3, 1.6, and 1.10

Let $\mathcal{R}$ be a set of points in $\mathbb{F}^3$ and $\mathcal{S}$ be a set of planes in $\mathbb{F}^3$. We write $\mathcal{I}(\mathcal{R},\mathcal{S}) = |\{(r,s) \in \mathcal{R}\times\mathcal{S} : r \in s\}|$ for the number of *incidences* between $\mathcal{R}$ and $\mathcal{S}$. To prove Theorems 1.3 and 1.6, we make use of the following point-plane incidence bound due to Rudnev [9]. A short proof can be found in [11].

Theorem 2.1 *(**Rudnev**, [9]). Let $\mathcal{R}$ be a set of points in $\mathbb{F}^3$ and let $\mathcal{S}$ be a set of planes in $\mathbb{F}^3$, with $|\mathcal{R}| \ll |\mathcal{S}|$ and $|\mathcal{R}| \ll p^2$. Assume that there is no line containing k points of $\mathcal{R}$. Then*

$$\mathcal{I}(\mathcal{R},\mathcal{S}) \ll |\mathcal{R}|^{1/2}|\mathcal{S}| + k|\mathcal{S}|.$$

Proof of Theorem 1.3. For $\lambda \in B\cdot C$, let

$$E_\lambda = \big|\big\{\big(f(a,b), c\cdot g(a)^{-1}, c\cdot h(a)\big) : (a,b,c) \in A\times B\times C,\\ f(a,b)\cdot c\cdot g(a)^{-1} - c\cdot h(a) = \lambda\big\}\big|,$$

where by $g(a)^{-1}$ we mean the multiplicative inverse of $g(a)$ in $\mathbb{F}^*$. For a given triple $(x,y,z) \in (\mathbb{F}^*)^3$, we count the number of solutions $(a,b,c) \in A\times B\times C$ to the following system

$$g(a)(h(a)+b) = x,\ c\cdot g(a)^{-1} = y,\ c\cdot h(a) = z.$$

This implies that

$$g(a)h(a) = zy^{-1}.$$

Since $\mu(g\cdot h) = m$, there are at most m different values of a satisfying the equation $g(a)h(a) = zy^{-1}$, and b, c are uniquely determined in term of a by the first and second equations of the system. This implies that

$$|A||B||C|/m \le \sum_{\lambda\in B\cdot C} E_\lambda.$$

By the Cauchy–Schwarz inequality, we get

$$(|A||B||C|/m)^2 \le \left(\sum_{\lambda\in B\cdot C} E_\lambda\right)^2 \le E\cdot|B\cdot C|, \tag{3}$$

where $E = \sum_{\lambda\in B\cdot C} E_\lambda^2$.

Define the point set $\mathcal{R}$ as

$$\mathcal{R} = \big\{\big(c\cdot g(a)^{-1}, c\cdot h(a), g(a')(h(a')+b')\big) : a,a' \in A, b' \in B, c \in C\big\}$$

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and the set of planes $\mathcal{S}$ as

$$\mathcal{S} = \left\{ g(a)(h(a)+b)X - Y - c'g(a')^{-1}Z = -c' \cdot h(a') \colon a, a' \in A, b \in B, c' \in C \right\}.$$

We have $E \leq I(\mathcal{R}, \mathcal{S})$, and $|\mathcal{R}| = |\mathcal{S}| \leq |f(A,B)||A||C|$. To apply Theorem 2.1, we need to find an upper bound on k which is the maximum number of collinear points in $\mathcal{R}$. The projection of $\mathcal{R}$ into the first two coordinates is the set $\mathcal{T} = \{(c \cdot g(a)^{-1}, c \cdot h(a)) \colon a \in A, c \in C\}$. The set $\mathcal{T}$ can be covered by the lines of the form $y = g(a)h(a)x$ with $a \in A$. This implies that $\mathcal{T}$ can be covered by at most $|A|$ lines passing through the origin, with each line containing $|C|$ points of $\mathcal{T}$. Therefore, a line in $\mathbb{F}^3$ contains at most $\max\{|A|, |C|\}$ points of $\mathcal{R}$, unless it is vertical, in which case it contains at most $|f(A,B)|$ points. In other words, we get

$$k \leq \max\{|A|, |C|, |f(A,B)|\}.$$

If $|\mathcal{R}| \gg p^2$, then we get $|f(A,B)||A||C| \gg p^2$. Since $|A|, |C| \leq p^{5/8}$, we have $|f(A,B)| \gg p^{3/4} \gg |A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}$, and we are done in this case. Thus, we can assume that $|\mathcal{R}| \ll p^2$. Applying Theorem 2.1, we obtain

$$I(\mathcal{R}, \mathcal{S}) \leq |f(A,B)|^{3/2}|A|^{3/2}|C|^{3/2} + k|f(A,B)|A||C|. \tag{4}$$

Putting (3) and (4) together gives us

$$\max\{|f(A,B)|, |B \cdot C|\} \gg \min\left\{ \frac{|A|^{\frac{1}{5}}|B|^{\frac{4}{5}}|C|^{\frac{1}{5}}}{m^{\frac{4}{5}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{2}{3}}|C|^{\frac{1}{3}}|A|^{\frac{1}{3}}}{m^{\frac{2}{3}}} \right\}.$$

This completes the proof of the theorem. □

Proof of Theorem 1.6. The proof goes in the same direction as Theorem 1.3, but for the sake of completeness, we include the detailed proof. For $\lambda \in B + C$, let

$$E_\lambda = \big|\big\{ \big(f(a,b), g(a)^{-1}, c - h(a)\big) : (a,b,c) \in A \times B \times C, \\ f(a,b) \cdot g(a)^{-1} + (c - h(a)) = \lambda \big\}\big|.$$

For a given triple $(x,y,z) \in (\mathbb{F}^*)^3$, we count the number of solutions $(a,b,c) \in A \times B \times C$ to the following system

$$g(a)(h(a)+b) = x, \ g(a)^{-1} = y, \ c - h(a) = z.$$

Since $\mu(g) = m$, there are at most m different values of a satisfying the equation $g(a) = y^{-1}$, and b, c are uniquely determined in term of a by the first and third equations of the system. This implies that

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$$|A||B||C|/m \leq \sum_{\lambda \in B+C} E_\lambda.$$

By the Cauchy–Schwarz inequality, we have

$$(|A||B||C|/m)^2 \leq \left(\sum_{\lambda \in B+C} E_\lambda \right)^2 \leq E \cdot |B+C|, \tag{5}$$

where $E = \sum_{\lambda \in B+C} E_\lambda^2$. Define the point set $\mathcal{R}$ as

$$\mathcal{R} = \left\{ \left(g(a)^{-1}, c - h(a), g(a')(h(a') + b') \right) : a, a' \in A, b' \in B, c \in C \right\},$$

and the collection of planes $\mathcal{S}$ as

$$\mathcal{S} = \left\{ g(a)(h(a) + b)X + Y - g(a')^{-1}Z = c' - h(a') \colon a, a' \in A, b \in B, c' \in C \right\}.$$

It is clear that $|\mathcal{R}| = |\mathcal{S}| \leq |f(A,B)||A||C|$, and $E \leq I(\mathcal{R}, \mathcal{S})$. To apply Theorem 2.1, we need to find an upper bound on k which is the maximum number of collinear points in $\mathcal{R}$. The projection of $\mathcal{R}$ into the first two coordinates is the set $\mathcal{T} = \{ \left(g(a)^{-1}, c - h(a) \right) : a \in A, c \in C \}$. The set $\mathcal{T}$ can be covered by at most $|A|$ lines of the form $x = g(a)^{-1}$ with $a \in A$, where each line contains $|C|$ points of $\mathcal{T}$. Therefore, a line in $\mathbb{F}^3$ contains at most $\max\{|A|, |C|\}$ points of $\mathcal{R}$, unless it is vertical, in which case it contains at most $|f(A,B)|$ points. So we get

$$k \leq \max\{|A|, |C|, |f(A,B)|\}.$$

If $|\mathcal{R}| \gg p^2$, this implies that $|f(A,B)||A||C| \gg p^2$. Since $|A|, |C| \leq p^{5/8}$, we have $|f(A,B)| \gg p^{3/4} \gg |A|^{\frac{1}{5}} |B|^{\frac{4}{5}} |C|^{\frac{1}{5}}$, and we are done. Thus, we can assume that $|\mathcal{R}| \ll p^2$. Applying Theorem 2.1, we obtain

$$I(\mathcal{R}, \mathcal{S}) \leq |f(A,B)|^{3/2} |A|^{3/2} |C|^{3/2} + k|f(A,B)||A||C|. \tag{6}$$

Putting (5) and (6) together gives us

$$\max\{|f(A,B)|, |B+C|\} \gg \min\left\{ \frac{|A|^{\frac{1}{5}} |B|^{\frac{4}{5}} |C|^{\frac{1}{5}}}{m^{\frac{4}{5}}}, \frac{|B||C|^{\frac{1}{2}}}{m}, \frac{|B||A|^{\frac{1}{2}}}{m}, \frac{|B|^{\frac{2}{3}} |C|^{\frac{1}{3}} |A|^{\frac{1}{3}}}{m^{\frac{2}{3}}} \right\}.$$

This completes the proof. □

Proof of Theorem 1.10. One can assume that $|f(A,A)| \leq |A|^2$, since otherwise we are done. Now by the proofs of Theorems 1.3 and 1.6 for $A \subset \mathbb{F}^*$ with $|A| \leq p^{5/8}$, we have

$$|f(A,A)|^{3/2} |A \cdot A| \gg |A|^3, \ |f(A,A)|^{3/2} |A + A| \gg |A|^3.$$

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Since $\min\{|A+A|, |A\cdot A|\} \leq |A|^{\frac{9}{8}-\epsilon}$, we get $|f(A,A)|^{3/2} \gg |A|^{3-\frac{9}{8}+\epsilon}$, which concludes the proof of the theorem. □

3. Proof of Theorem 1.11

To prove Theorem 1.11, we use the following theorem on the point-line incidence bound due to Stevens and de Zeeuw [10].

Theorem 3.1 *(**Stevens–de Zeeuw**, [10]). Let $\mathcal{P}$ be a set of m points in $\mathbb{F}^2$ and $\mathcal{L}$ a set of n lines in $\mathbb{F}^2$, with $m^{7/8} < n < m^{8/7}$. If $m^{-2}n^{13} \ll p^{15}$, then*

$$\mathcal{I}(\mathcal{P},\mathcal{L}) \ll m^{11/15}n^{11/15}.$$

Proof of Theorem 1.11. Let $\mathcal{P}$ be a set of points in $\mathbb{F}^2$ defined as follows

$$\mathcal{P} := \left\{\left(\frac{c^2}{a}, ca\right) : a,c \in A\right\},$$

and $\mathcal{L}$ be a set of lines in $\mathbb{F}^2$ defined as

$$\mathcal{L} := \left\{y = \frac{f(a,b)}{bc}x - bc\colon a,b,c \in A\right\}.$$

We have $m := |\mathcal{P}| = \Theta(|A|^2)$. Indeed, for each pair $(u,v) \in \mathcal{P}$, the number of pairs $(a,c) \in A\times A$ such that $ca = v, c^2/a = u$ is at most 3, since we have $c^3 = uv$ which implies that there are at most three possible solutions for c. Moreover, a is uniquely determined in terms of c and v. So we get $m \geq |A|^2/3$. The upper bound $m \leq |A|^2$ is trivial. Similarly, we have $|A|^2 \ll n := |\mathcal{L}| \leq |f(A,A)|\cdot|A\cdot A|$.

Given a quadruple $(u,v,w,t) \in (\mathbb{F}^*)^4$, we count the number of solutions (a,b,c) to the following system

$$\frac{ab(a+b)}{bc} = u,\ bc = v,\ \frac{c^2}{a} = t,\ ca = w.$$

This implies that $a^3 = w^2/t$. Thus, there are at most 3 values of a satisfying the equation $a^3 = w^2/t$, and b,c are uniquely determined in terms of a by the second and fourth equations. Therefore, we get $|A|^3 \ll \mathcal{I}(\mathcal{P},\mathcal{L})$.

To apply Theorem 3.1, we need to check the conditions. The condition $m^{7/8} < n$ is satisfied. If $|f(A,A)||A\cdot A| > (|A|^2)^{8/7}$, then we have

$$\max\{|f(A,A)|, |A\cdot A|\} \gg |A|^{1+\frac{1}{7}},$$

and we are done. So we can assume that $n \leq |f(A,A)||A\cdot A| < (|A|^2)^{8/7}$.

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If $(|f(A,A)||A\cdot A|)^{13} \gg p^{15}m^2 \gg p^{15}|A|^4$, then we have

$$\max\{|f(A,A)|, |A\cdot A|\} \gg |A|^{1+\frac{1}{22}},$$

under the condition $|A| \leq p^{\frac{11}{17}}$, and we are done. Thus, we can assume that $(|f(A,A)||A\cdot A|)^{13} \ll p^{15}m^2$, which implies that the condition $m^{-2}n^{13} \ll p^{15}$ is satisfied. It follows from Theorem 3.1 that

$$|A|^3 \ll I(\mathcal{P},\mathcal{L}) \ll |A|^{22/15}\left(|f(A,A)|\cdot|A\cdot A|\right)^{11/15},$$

implying

$$\max\{|f(A,A)|, |A\cdot A|\} \gg |A|^{1+\frac{1}{22}},$$

which concludes the proof of the theorem. □

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