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POWER MOMENTS OF HECKE EIGENVALUES FOR CONGRUENCE GROUP

PING SONG, WENGUANG ZHAI, AND DEYU ZHANG

ABSTRACT. Let $H_\ell(N)$ be the set of normalized primitive holomorphic cusp forms of even integral weight ℓ for the congruence group $\Gamma_0(N)$. For any $f \in H_\ell(N)$, we study the higher power moments of $\mathcal{S}_f(x;N) := \sum_{n\le x} \lambda_f(n)$ and derive the asymptotic formulas for

$$\int_1^T \mathcal{S}_f^k(x;N)dx, \ k = 2, 3, \ldots 7,$$

by using Ivić's large value arguments, where $\lambda_f(n)$ is the n-th Hecke eigenvalue of f.

1. Introduction

Let $N \ge 1$ be positive integer and $H_\ell(N)$ be the set of normalized primitive holomorphic cusp forms of even integral weight ℓ for the congruence group $\Gamma_0(N)$. For any $f \in H_\ell(N)$, we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(\ell-1)/2} e^{2\pi i n z},$$

where $\lambda_f(n)$ are Hecke eigenvalues with $\lambda_f(1) = 1$. It is well-known that the arithmetical function $\lambda_f(n)$ is real multiplicative and verifies Deligne's inequality

$$|\lambda_f(n)| \le \tau(n), \tag{1.1}$$

for all $n \ge 1$, where $\tau(n)$ is the divisor function. In order to detect sign changes or cancellations among $\lambda_f(n)$, it is natural to study the summatory function

$$\mathcal{S}_f(x;N) := \sum_{n\le x} \lambda_f(n). \tag{1.2}$$

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In 1927, Hecke [8] proved that

$$\mathcal{S}_f(x;N) \ll_f x^{1/2},$$

for all $f \in H_\ell(N)$ and $x \geq 1$. Many mathematicians studied the upper bound of $\mathcal{S}_f(x;N)$ later. For example, see [4, 5, 7, 8, 14, 18, 19, 21, 22, 23]. The best result to date is

$$\mathcal{S}_f(x;N) \ll x^{1/3}(\log x)^{-0.118\ldots} \tag{1.3}$$

proved by Wu [24]. In the opposite direction, Ivić and Hafner [7] also showed that there is a positive constant D such that

$$\mathcal{S}_f(x;N) = \Omega_\pm \left(x^{1/4} \exp\left\{ \frac{D(\log_2 x)^{1/4}}{(\log_2 x)^{3/4}} \right\} \right).$$

In fact, it is conjectured that $\mathcal{S}_f(x;N) \ll x^{1/4+\varepsilon}$. In the absence a proof of conjecture, it is natural to consider the mean value of $\mathcal{S}_f(x;N)$. For $N=1$, A. Walfisz [21] showed the mean square estimate

$$\int_1^T \mathcal{S}_f^2(x;1)dx = B_2 T^{3/2} + O(T\log^2 T),$$

where $B_2 = \frac{1}{6\pi^2}\sum_{n=1}^\infty \lambda_f^2(n) n^{-3/2}$. Cai [2] studied the third and fourth power moments of $\mathcal{S}_f(x;1)$. He proved that

$$\int_1^T \mathcal{S}_f^3(x;1)dx = B_3 T^{7/4} + O(T^{7/4-1/14+\varepsilon}),$$

$$\int_1^T \mathcal{S}_f^4(x;1)dx = B_4 T^{2} + O(T^{2-1/23+\varepsilon}),$$

where B_3, B_4 are computable constants. Later Ivić[10], Zhai[26] improved the two results by large value arguments.

In this paper, we investigate the higher power moments of Hecke eigenvalue for the congruence group $\Gamma_0(N)$ and determine the explicit dependence on the level. We first study the mean square estimates of $\mathcal{S}_f(x;N)$ and obtain the following theorem.

Theorem 1. For any $f \in H_\ell(N)$, $1 \leq N \ll T^{1-\varepsilon}$, we have

$$\int_1^T \mathcal{S}_f^2(x;N)dx = \frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\sum_{n=1}^\infty \frac{\lambda_f^2(n)}{n^{3/2}} T^{3/2} + O_\ell\left(N^{3/4}T^{5/4+\varepsilon}\right),$$

where $\epsilon_f = \pm 1$.

For higher moments, we start with the large value of $\mathcal{S}_f(x;N)$ (see Theorem 4) by Halasz-Montgomery inequality and the estimates of exponential sums. Then we divide the interval $[T/2,T]$ into subintervals $[T/2+j-1,T/2+j],(j=1,2,\ldots)$, and pick the maximal $|\mathcal{S}_f(\tau_j;N)|$ in j intervals of length V. Finally, we can get the higher power moments of $\mathcal{S}_f(x;N)$ by Theorem 4. More precisely, we have the following theorem.

Theorem 2. Suppose $A>2$ is a fixed constant and $1\le N\ll T^{1-\varepsilon}$. Then for any $f\in H_\ell(N)$, we have

$$\int_1^T |\mathcal{S}_f(x;N)|^A dx \ll_\ell N^{\frac{A}{4}} T^{1+\frac{A}{4}} \log^{29} T, \tag{1.4}$$

if $2<A\le 8$.

Before stating the asymptotic results, we introduce some notations. Define

$$s_{k;l}(\lambda_f) := \sum_{\sqrt{n_1}+\cdots+\sqrt{n_l}=\sqrt{n_{l+1}}+\cdots+\sqrt{n_k}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \quad (1\le l<k), \tag{1.5}$$

$$B_k(\lambda_f) := \sum_{l=1}^{k-1} \begin{pmatrix} k-1 \\ l \end{pmatrix} s_{k;l}(\lambda_f)\cos\left(\frac{(k-2l)\pi}{4}\right). \tag{1.6}$$

Suppose $A_0>3$ is a real number. Define

$$\begin{aligned} &K_0=\min\{n\in\mathbb{N}: n\ge A_0, 2|n\}, \qquad b(k):=2^{k-2}-\frac{1}{2}+\frac{k}{4},\\ &\sigma(k,A_0):=\frac{A_0-k}{2(A_0-2)}, \quad \omega(k,A_0)=\frac{k-2}{2(A_0-2)},\ 3\le k<A_0,\\ &\delta(k,A_0):=\frac{\sigma(k,A_0)}{2b(k)+2\sigma(k,A_0)}. \end{aligned}$$

Theorem 3. Let $A_0\ge 8$ be a real number such that (1.4) holds. Then for any integer $3\le k<A_0$, and $1\le N\ll T^{1-\varepsilon}$, we have the asymptotic formula

$$\begin{aligned} \int_1^T \mathcal{S}_f^k(x;N)dx = &\frac{\varepsilon_f^k N^{\frac{k}{4}} B_k(\lambda_f)}{(\sqrt{2}\pi)^k 2^{k-3}(k+4)} T^{1+\frac{k}{4}} \\ &+ O_\ell\left(N^{\frac{k}{4}}T^{1+\frac{k}{4}-\delta(k,A_0)+\varepsilon} + N^{\frac{k}{4}+\sigma(k,A_0)}T^{\frac{k}{4}+\frac{1}{2}+\omega(k,A_0)}\right). \end{aligned} \tag{1.7}$$

Remark 1.1. For $f\in H_\ell(N)$, the mean value of $\lambda_f(n)$ in arithmetic progressions also attracts the attention of mathematicians, see [1, 6, 16, 17, 25, 33]. Let a and r be two positive integers. It is well-known that $\{\lambda_f(n)|n \equiv a \bmod r\}$ determines a cusp form of higher level. More precisely, (e.g., see [31, Lemma 3.1])

$$g(z) := \sum_{\substack{n\geq 1 \\ n\equiv a \bmod r}} \lambda_f(n) n^{(\ell-1)/2} e^{2\pi i n z}$$

is a cusp form on $\Gamma_0(Nr^2)$. If g is also a Hecke eigenform on $\Gamma_0(Nr^2)$, then we could get the higher power moments of Hecke eigenvalues in arithmetic progressions by Theorem 1-Theorem 3.

Notation. Throughout the paper ε always denotes a fixed but sufficiently small positive constant. We write $f(x) \ll g(x)$, or $f(x) = O(g(x))$, to mean that $|f(x)| \leq Cg(x)$. $\sum_{n\sim N}$ denote that the sum over $N < n \leq 2N$. $f(x) = \Omega(g(x))$ means that there exists a suitable constant $C > 0$ such that $|f(x)| > Cg(x)$ holds for a sequence $x = x_n$ such that $\lim\limits_{n\to\infty} x_n = \infty$.

2. Preliminary lemmas

In this section, we introduce some tools used in this paper. These lemmas are well-known and we introduce them for self-contained.

Lemma 2.1. Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic and $F'(x) \geq m > 0$ or $F'(x) \leq -m < 0$ for $a \leq x \leq b$. Then

$$\left|\int_a^b \cos F(x) dx\right| \leq 2m^{-1}, \quad \left|\int_a^b \sin F(x) dx\right| \leq 2m^{-1}.$$

Proof. See Lemma 2.1 of [9]. □

Lemma 2.2. Let S be an inner-product vector space over $\mathbb{C}$, (a, b) denote the inner product in S and $\|a\|^2 = (a, a)$. Suppose that $\xi, \varphi_1, \ldots, \varphi_R$ are arbitrary vectors in S. Then

$$\sum_{l\leq R} |(\xi, \varphi_l)|^2 \leq \|\xi\|^2 \max_{l_1\leq R} \sum_{l_2\leq R} |(\varphi_{l_1}, \varphi_{l_2})|.$$

Proof. This is the well-known Halasz-Montgomery inequality. See (A.40) of [9]. □

Lemma 2.3. Suppose $y > 1$ is a large parameter, $f(n) \ll n^{\varepsilon}$ and define

$$s_{k;l}(f; y) = \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_l}=\sqrt{n_{l+1}}+\cdots+\sqrt{n_k} \\ n_1,\cdots,n_k\leq y}} \frac{f(n_1)\cdots f(n_k)}{(n_1\cdots n_k)^{3/4}}, \quad 1 \leq l < k.$$

Then

$$|s_{k;l}(f) - s_{k;l}(f;y)| \ll y^{-1/2+\varepsilon}, \quad 1 \le l < k.$$

Proof. See Lemma 3.1 of [30]. □

Lemma 2.4. Suppose $k \ge 3$ and $(i_1, \cdots, i_{k-1}) \in \{0,1\}^{k-1}$ are such that

$$\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k} \ne 0.$$

Then

$$|\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k}| \gg \max(n_1, \cdots, n_k)^{-(2^{k-2}-2^{-1})}.$$

Proof. See Lemma 2.2 of [30]. □

Lemma 2.5. Let $f \in H_\ell(N)$. For any $\varepsilon > 0$ and $C > 0$, we have

$$\begin{aligned}\mathcal{S}_f(x;N) =& \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{n \le M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) \\ &+ O_\ell\left(N^{1/2}\left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\}(Nx)^\varepsilon\right)\end{aligned} \tag{2.1}$$

uniformly for $2 \le M \le x^C$, where $\epsilon_f = \pm 1$.

Proof. See (3.2) of Lau and Wu [15]. □

3. Proof of Theorem 1

Suppose $1 \le N^{1+\varepsilon} \ll T$. For $1 \le x \le N^{1+\varepsilon} \ll T$, taking $M = N$, we use formula (2.1) to get

$$\mathcal{S}_f(x;N) = \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) + O_\ell\left(N^{3/4+\varepsilon}x^{-1/4+\varepsilon}\right).$$

Squaring and integrating, we have

$$\begin{aligned}\int_1^{N^{1+\varepsilon}} \mathcal{S}_f^2(x;N)dx \ll& N^{1/2}\int_1^{N^{1+\varepsilon}} x^{1/2} \sum_{m \le N}\sum_{n \le N} \lambda_f(m)\lambda_f(n)(mn)^{-3/4} \\ &\times \cos\left(4\pi\sqrt{\frac{mx}{N}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right)dx + N^{2+\varepsilon}.\end{aligned} \tag{3.1}$$

The first term with $m = n$ contribute

$$N^{1/2} \sum_{n \leq N} \frac{\lambda_f^2(n)}{n^{3/2}} \int_1^{N^{1+\varepsilon}} x^{1/2} \cos^2\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) dx$$

$$\ll N^{2+\varepsilon} \sum_{n \leq N} \frac{\lambda_f^2(n)}{n^{3/2}} \ll N^{2+\varepsilon}. \tag{3.2}$$

In view of $2\cos X \cos Y = \cos(X+Y) + \cos(X-Y)$, the term in (3.1) with $m \neq n$ contribute

$$\ll N^{1/2} \sum_{m \neq n \leq N} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}} \int_1^{N^{1+\varepsilon}} x^{1/2} \cos\left(4\pi\sqrt{\frac{mx}{N}} - 4\pi\sqrt{\frac{nx}{N}}\right) dx$$

$$+ N^{1/2} \sum_{m \neq n \leq N} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}} \int_1^{N^{1+\varepsilon}} x^{1/2} \sin\left(4\pi\sqrt{\frac{mx}{N}} + 4\pi\sqrt{\frac{nx}{N}}\right) dx,$$

$$:= S_1 + S_2.$$

Estimating S_2 by Lemma 2.1, we have

$$S_2 \ll N^{3/2} \sum_{m < n \leq N} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}} (\sqrt{m} + \sqrt{n})^{-1}$$

$$\ll N^{3/2} \sum_{m < n \leq N} \frac{\lambda_f(m)\lambda_f(n)}{mn}$$

$$\ll N^{2+\varepsilon}, \tag{3.3}$$

if we notice that $\sqrt{m} + \sqrt{n} \geq 2(mn)^{1/4}$. Analogously, we obtain

$$S_1 \ll N^{3/2} \sum_{n < m \leq N} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}} (\sqrt{m} - \sqrt{n})^{-1}$$

$$= N^{3/2} \left(\sum_{n \leq \frac{m}{2}} + \sum_{n > \frac{m}{2}} \right) := N^{3/2}(S_1' + S_1''), \tag{3.4}$$

say. We have

$$S_1' \ll \sum_{m \leq N} \frac{\lambda_f(m)}{m^{1/4}} \sum_{n \leq \frac{m}{2}} \frac{\lambda_f(n)}{n^{3/4}} (m-n)^{-1}$$

$$\ll \sum_{m \leq N} \frac{\lambda_f(m)}{m^{5/4}} m^{1/4+\varepsilon} \ll N^{\varepsilon}, \tag{3.5}$$

and

$$S_1'' \ll \sum_{m\le N} \frac{\lambda_f(m)}{m} \sum_{\frac{m}{2}<n<m} \lambda_f(n)(m-n)^{-1}$$

$$\ll N^{\varepsilon} \sum_{m\le N} \frac{\lambda_f(m)}{m} \ll N^{\varepsilon}. \tag{3.6}$$

So we have

$$\int_1^{N^{1+\varepsilon}} \mathcal{S}_f^2(x;N)dx \ll N^{2+\varepsilon}. \tag{3.7}$$

For $N^{1+\varepsilon} \le x \le T$, we use the method of bisection. It suffices to prove the corresponding formula for the interval $[T/2, T]$. Taking $M = T$ in the truncated Voronoi formula(2.1), we have

$$\mathcal{S}_f(x;N) = \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{n\le T} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) + O_\ell\left(N^{1/2}T^{\varepsilon}\right).$$

Integrating term by term, we get that

$$\begin{aligned}\int_{T/2}^{T} \mathcal{S}_f^2(x;N)dx = {}& \frac{\epsilon_f^2 N^{1/2}}{2\pi^2} \int_{T/2}^{T} x^{1/2} \sum_{m\le T}\sum_{n\le T} \lambda_f(m)\lambda_f(n)(mn)^{-3/4} \\ & \times \cos\left(4\pi\sqrt{\frac{mx}{N}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right)dx \\ & + O_\ell\left(N^{3/4}T^{1/4+\varepsilon}\int_{T/2}^{T}\left|\sum_{n\le T}\frac{\lambda_f(n)}{n^{3/4}}\cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right)\right|dx\right) \\ & + O_\ell\left(NT^{1+\varepsilon}\right).\end{aligned} \tag{3.8}$$

For the first sum we distinguish the cases $m = n$ and $m \ne n$. The terms with $m = n$ contribute

$$\begin{aligned}&\frac{\epsilon_f^2 N^{1/2}}{2\pi^2} \sum_{n\le T} \frac{\lambda_f^2(n)}{n^{3/2}} \int_{T/2}^{T} x^{1/2}\cos^2\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right)dx \\ &= \frac{\epsilon_f^2 N^{1/2}}{4\pi^2} \sum_{n\le T} \frac{\lambda_f^2(n)}{n^{3/2}} \int_{T/2}^{T} x^{1/2}\left(1+\cos\left(8\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{2}\right)\right)dx \\ &= \frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\left(T^{3/2} - (T/2)^{3/2}\right)\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^{3/2}} + O_\ell\left(TN + N^{1/2}T^{1+\varepsilon}\right).\end{aligned} \tag{3.9}$$

Here we have used Lemma 2.1 in the last step to estimate

$$\begin{aligned}&\frac{\epsilon_f^2 N^{1/2}}{4\pi^2}\sum_{n\leq T}\frac{\lambda_f^2(n)}{n^{3/2}}\int_{T/2}^{T} x^{1/2}\cos\left(8\pi\sqrt{\frac{nx}{N}}-\frac{\pi}{2}\right)dx\\ &\ll TN\sum_{n\leq T}\frac{\lambda_f^2(n)}{n^2}\ll TN,\end{aligned}$$

and we also used partial summation and Deligne's bound (1.1) to estimate

$$\sum_{n\geq T}\frac{\lambda_f^2(n)}{n^{3/2}}\ll T^{-1/2+\varepsilon}.$$

By $2\cos X\cos Y=\cos(X+Y)+\cos(X-Y)$, the terms in (3.8) with $m\neq n$ are a multiple of

$$\begin{aligned}&N^{1/2}\sum_{m\neq n\leq T}\frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}}\int\limits_{T/2}^{T} x^{1/2}\cos\left(4\pi\sqrt{\frac{mx}{N}}-4\pi\sqrt{\frac{nx}{N}}\right)dx\\ &+N^{1/2}\sum_{m\neq n\leq T}\frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}}\int\limits_{T/2}^{T} x^{1/2}\sin\left(4\pi\sqrt{\frac{mx}{N}}+4\pi\sqrt{\frac{nx}{N}}\right)dx\\ &\ll NT^{1+\varepsilon},\end{aligned}$$

where we use the same method as (3.3) and (3.4). Therefore by the preceding estimates the first sum in (3.8) is equal to

$$\frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\left(T^{3/2}-(T/2)^{3/2}\right)\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^{3/2}}+O_\ell(NT^{1+\varepsilon}).$$

The first O-term in (3.8) is estimated by the Cauchy -Schwarz inequality as

$$(TN)^{3/4+\varepsilon}\left(\int\limits_{T/2}^{T}\left|\sum_{n\leq T}\frac{\lambda_f(n)}{n^{3/4}}\cos\left(4\pi\sqrt{\frac{nx}{N}}-\frac{\pi}{4}\right)\right|^2 dx\right)^{1/2}\ll N^{3/4}T^{5/4+\varepsilon},$$

where we square out the modulus under the integral sign and treat the terms $m=n$ and $m\neq n$ similarly as before. Then we have

$$\int_{T/2}^{T}\mathcal{S}_f^2(x;N)dx=\frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\left(T^{3/2}-(T/2)^{3/2}\right)\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^{3/2}}+O_\ell(N^{3/4}T^{5/4+\varepsilon}),$$

if $1 \le N \ll T^{1-\varepsilon}$. Summing over intervals of the form $[2^{-j}T, 2^{1-j}T], j \ge 1$, to get

$$\begin{aligned}\int_{N^{1+\varepsilon}}^{T} \mathcal{S}_f^2(x;N)dx &= \frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^{3/2}}T^{3/2} + O_\ell(N^{2+\varepsilon} + N^{3/4}T^{5/4+\varepsilon}) \\ &= \frac{\epsilon_f^2 N^{1/2}}{6\pi^2}\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^{3/2}}T^{3/2} + O_\ell(N^{3/4}T^{5/4+\varepsilon}), \end{aligned} \tag{3.10}$$

if $1 \le N \ll T^{1-\varepsilon}$. Theorem 1 follows from (3.10) and (3.7).

4. A large value estimate of $\mathcal{S}_f(x;N)$

Theorem 4. Suppose $N^{1+\varepsilon} \le H \le x_1 < x_2 < \cdots < x_R \le 2H$ satisfy $|\mathcal{S}_f(x_r;N)| \gg VN^{1/4}$ $(r = 1, 2, \ldots, R)$ and $|x_j - x_i| \gg V \gg \max\{H^{7/32}N^{-1/16}(\log H)^{45/16}, N^{1/4}H^{\varepsilon}\}$ $(i \ne j)$. Then we have

$$R \ll HN^{1/2}V^{-3}\log^6 H + H^{15/4}V^{-12}\log^{29} H.$$

Proof. Suppose that $H_0 > V$ is a parameter to be determined later. Let I be any subinterval of $[H, 2H]$ of length not exceeding H_0 and let $G = I \cap \{x_1, x_2, \ldots, x_R\}$. Without loss of generality, we may assume that $G = \{x_1, x_2, \ldots, x_{R_0}\}$.

Suppose $J = \left[\frac{(1+4\varepsilon)\log H - 2\log V}{\log 2}\right]$. For any $x_r \in \{x_1, x_2, \ldots, x_R\}$, we apply formula (2.1) with $x = x_r$ and $M = 2^{J+1} \asymp H^{1+4\varepsilon}V^{-2}N^{1/2}$ to get

$$\begin{aligned}\mathcal{S}_f(x_r;N) &\ll (HN)^{\frac14}\left|\sum_{n\le M}\frac{\lambda_f(n)}{n^{3/4}}e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right| + H^{-\varepsilon}VN^{1/4} + N^{1/2}H^{\varepsilon} \\ &\ll (HN)^{\frac14}\left|\sum_{j=0}^{J}\sum_{n\sim 2^j}\frac{\lambda_f(n)}{n^{3/4}}e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right|,\end{aligned}$$

if we notice that $|\mathcal{S}_f(x_r;N)| \gg VN^{1/4}$ and $V \gg N^{1/4}H^{\varepsilon}$. Squaring, summing over the set G and then using the Cauchy inequality, we get that

$$\begin{aligned}\sum_{r\le R_0}\mathcal{S}_f^2(x_r;N) &\ll (HN)^{\frac12}\sum_{r\le R_0}\left|\sum_{j=0}^{J}\sum_{n\sim 2^j}\frac{\lambda_f(n)}{n^{3/4}}e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right|^2 \\ &\ll (HN)^{\frac12}\log H\sum_{r\le R_0}\sum_{j=0}^{J}\left|\sum_{n\sim 2^j}\frac{\lambda_f(n)}{n^{3/4}}e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right|^2 \\ &\ll (HN)^{\frac12}\log^2 H\sum_{r\le R_0}\left|\sum_{n\sim 2^{j_0}}\frac{\lambda_f(n)}{n^{3/4}}e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right|^2 \end{aligned} \tag{4.1}$$

for some $0 \le j_0 \le J$. Let $M_0 = 2^{j_0}$. Take $\xi(d) = \{\xi_n(d)\}_{n=1}^{\infty}$ with $\xi_n(d) = \lambda_f(n)n^{-3/4}$ for $n \sim M_0$ and zero otherwise. Take $\varphi_r^{\pm}(d) = \left\{\varphi_{r,n}^{\pm}(d)\right\}_{n=1}^{\infty}$ with $\varphi_{r,n}^{\pm}(d) = e\left(\mp 2\sqrt{\frac{nx_r}{N}}\right)$ for $n \sim M_0$ and zero otherwise. Then

$$
\begin{aligned}
(\xi(d), \varphi_r^{\pm}(d)) &= \sum_{n\sim M_0} \frac{\lambda_f(n)}{n^{3/4}} e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right), \\
(\varphi_{r_1}^{\pm}(d), \varphi_{r_2}^{\pm}(d)) &= \sum_{n\sim M_0} e\left(\mp 2\sqrt{\frac{n}{N}}(\sqrt{x_{r_1}} - \sqrt{x_{r_2}})\right), \\
\|\xi(d)\|^2 &= \sum_{n\sim M_0} \frac{\lambda_f^2(n)}{n^{3/2}} \ll M_0^{-1/2} \log^3 M_0,
\end{aligned}
$$

where we have used Deligne's bound, and the well-known estimate

$$
\sum_{n\sim M_0} \lambda_f^2(n) \ll \sum_{n\sim M_0} \tau^2(n) \ll M_0 \log^3 M_0.
$$

By (4.1) and Lemma 2.2, we get

$$
\begin{aligned}
R_0 V^2 N^{1/2} &\ll (HN)^{\frac{1}{2}} \log^2 H \sum_{r\le R_0} \left| \sum_{n\sim M_0} \frac{\lambda_f(n)}{n^{3/4}} e\left(\pm 2\sqrt{\frac{nx_r}{N}}\right)\right|^2 \\
&\ll \frac{(HN)^{\frac{1}{2}} \log^5 H}{M_0^{1/2}} \max_{r_1\le R_0} \sum_{r_2\le R_0} \left| \sum_{n\sim M_0} e\left(\mp 2\sqrt{\frac{n}{N}}(\sqrt{x_{r_1}} - \sqrt{x_{r_2}})\right)\right| \\
&\ll \frac{(HN)^{\frac{1}{2}} \log^5 H}{M_0^{1/2}} \left(M_0 + \max_{r_1\le R_0} \sum_{\substack{r_2\le R_0\\ r_2\ne r_1}} \left| \sum_{n\sim M_0} e\left(\mp 2\sqrt{\frac{n}{N}}(\sqrt{x_{r_1}} - \sqrt{x_{r_2}})\right)\right| \right). \qquad (4.2)
\end{aligned}
$$

Following the calculation on [28, page 236], we get, by the Kuzmin-Landau inequality and the exponent pair $(4/18, 11/18)$,

$$
\begin{aligned}
\sum_{n\sim M_0} e\left(2\sqrt{\frac{n}{N}}(\sqrt{x_{r_1}} - \sqrt{x_{r_2}})\right) &\ll \frac{(NM_0)^{\frac{1}{2}}}{|\sqrt{x_{r_1}} - \sqrt{x_{r_2}}|} + \left(\frac{|\sqrt{x_{r_1}} - \sqrt{x_{r_2}}|}{(NM_0)^{1/2}}\right)^{\frac{4}{18}} M_0^{\frac{11}{18}} \\
&\ll \frac{(HNM_0)^{\frac{1}{2}}}{|x_{r_1} - x_{r_2}|} + \left(\frac{|x_{r_1} - x_{r_2}|}{(HNM_0)^{1/2}}\right)^{\frac{4}{18}} M_0^{\frac{11}{18}} \\
&\ll \frac{(HNM_0)^{\frac{1}{2}}}{|x_{r_1} - x_{r_2}|} + H^{-\frac{1}{9}} H_0^{\frac{2}{9}} M_0^{\frac{1}{2}} N^{-\frac{1}{9}},
\end{aligned}
$$

where we have used the mean value theorem and the estimate $|x_{r_1} - x_{r_2}| \leq H_0$. Substituting this estimate into (4.2), we get

$$\begin{aligned} R_0V^2 &\ll \frac{H^{\frac{1}{2}}\log^5 H}{M_0^{1/2}}\left(M_0 + \max_{r_1\leq R_0}\sum_{\substack{r_2\leq R_0\\ r_2\neq r_1}}\left(\frac{(HNM_0)^{\frac{1}{2}}}{|x_{r_1}-x_{r_2}|} + \frac{H_0^{\frac{2}{9}}M_0^{\frac{1}{2}}}{H^{\frac{1}{9}}N^{\frac{1}{9}}}\right)\right)\\ &\ll \frac{H^{\frac{1}{2}}\log^5 H}{M_0^{1/2}}\left(M_0 + \frac{(HNM_0)^{1/2}\log H}{V} + \frac{H_0^{\frac{2}{9}}M_0^{\frac{1}{2}}}{H^{\frac{1}{9}}N^{\frac{1}{9}}}R_0\right)\\ &\ll HN^{1/2}V^{-1}\log^6 H + H^{\frac{7}{18}}N^{-\frac{1}{9}}H_0^{\frac{2}{9}}R_0\log^5 H, \end{aligned}$$

where we use the facts that $\{x_r\}$ is V-spaced and $M_0 \ll H^{1+4\varepsilon}V^{-2}N^{1/2}$. Take

$$H_0 = cV^9H^{-7/4}N^{1/2}(\log H)^{-45/2},$$

for some sufficiently small constant c such that we can get for this H_0 that

$$R_0V^2 \ll HN^{1/2}V^{-1}\log^6 H,$$

i.e.

$$R_0 \ll HN^{1/2}V^{-3}\log^6 H.$$

It is easy to check that $H_0 \gg V$ if $V \gg H^{7/32}N^{-1/16}(\log H)^{45/16}$.

Now we divide the interval $[H, 2H]$ into $O(1 + H/H_0)$ subintervals of length not exceeding H_0. In each interval of this type, the number of x_r's is at most $O(HN^{1/2}V^{-3}\log^6 H)$. So we have

$$\begin{aligned} R &\ll HN^{1/2}V^{-3}\log^6 H(1 + H/H_0)\\ &\ll HN^{1/2}V^{-3}\log^6 H + H^{15/4}V^{-12}\log^{29} H. \end{aligned}$$

□

5. Proof of Theorem 2

Suppose $1 \leq N^{1+\varepsilon} \ll T$. For $1 \leq x \leq N^{1+\varepsilon} \ll T$, we use the same method as the proof of Theorem 1 to deal with it. For $N^{1+\varepsilon} \leq x \leq T$, it is sufficient to prove the corresponding formula for the interval $[T/2, T]$ and then to sum over intervals of the form $[2^{-i}T, 2^{1-i}T], 1 \leq i \leq [\log_2 T] + 1$. Put $H = 2^{-i}T \geq N^{1+\varepsilon}$ such that

$$H^{1/4} \leq 2^m = V \ll H^{\frac{1}{3}}N^{-1/4}.$$

Obviously, there are $O(\log H) = O(\log T)$ choices for V. We denote by $\tau_j(V)$ the point with

$$|\mathcal{S}_f(\tau_j(V); N)| = \sup_{x\in[H+(j-1)V, H+jV]} |\mathcal{S}_f(x; N)|, \ (j = 1, 2, \ldots, [H/V] + 1).$$

We consider those $\tau_{j_r}(V)$ with

$$VN^{1/4} \le |\mathcal{S}_f(\tau_{j_r}(V); N)| < 2VN^{1/4},$$

where $V \ll H^{1/3}N^{-1/4}$. According to the parity of subindex r, we divide these points $\tau_{j_r}(V)$ into two groups of points $x_1^{\pm}(V), x_2^{\pm}(V), \ldots, x_{R^{\pm}}^{\pm}(V)$, which satisfy

$$VN^{1/4} \le |\mathcal{S}_f(x_j^{\pm}(V); N)| < 2VN^{1/4}, \ \ |x_i^{\pm}(V) - x_j^{\pm}(V)| \ge V$$

for $i \ne j \le R^{\pm} = R^{\pm}(V)$. Then we have

$$\int_H^{2H} |\mathcal{S}_f(x; N)|^A dx \ll H^{\frac{4+A}{4}} N^{\frac{A}{4}} + \sum_V V \sum_{j\le R^{\pm}} |\mathcal{S}_f(x_j^{\pm}(V); N)|^A. \tag{5.1}$$

Recall that

$$V \ge H^{1/4} \gg \max\{H^{7/32}N^{-1/16}(\log H)^{45/16}, N^{1/4}H^{\varepsilon}\},$$

if $1 \le N \ll T^{1-\varepsilon}$. Therefore, by Theorem 4, we have

$$\begin{aligned} & V \sum_{j\le R^{\pm}} |\mathcal{S}_f(x_j^{\pm}(V); N)|^A \\ & \ll RV^{A+1}N^{\frac{A}{4}} \ll HV^{A-2}N^{\frac{A}{4}+\frac{1}{2}} \log^6 H + H^{15/4}V^{A-11}N^{\frac{A}{4}} \log^{29} H \\ & \ll T^{1+\frac{A}{4}} N^{\frac{A}{4}} \log^{29} T, \end{aligned}$$

if $4 \le A \le 8$ and $H^{1/4} \le V \ll H^{\frac{1}{3}}N^{-1/4}$. Let $A = 8$. Then we can get

$$\int_1^T |\mathcal{S}_f(x; N)|^8 dx \ll N^2T^3 \log^{29} T.$$

Also Theorem 1 yields

$$\int_1^T |\mathcal{S}_f(x; N)|^2 dx \ll N^{1/2}T^{3/2}.$$

Then for $2 < A \leq 8$, by Hölder's inequality, we have

$$\begin{aligned}&\int_1^T |\mathcal{S}_f(x;N)|^A dx\\ &\ll \left(\int_1^T |\mathcal{S}_f(x;N)|^8 dx\right)^{\frac{A-2}{6}} \left(\int_1^T |\mathcal{S}_f(x;N)|^2 dx\right)^{\frac{8-A}{6}}\\ &\ll T^{1+\frac{A}{4}} N^{\frac{A}{4}} \log^{29} T.\end{aligned}$$

6. Proof of Theorem 3

Let $N^{1+\varepsilon} \ll T$ be a real number. For $1 \leq x \leq N^{1+\varepsilon} \ll T$, we use the same method as the proof of Theorem 1 to deal with it. For $N^{1+\varepsilon} \leq x \leq T$, it suffices to evaluate the integral $\int_T^{2T} S_f^k(x;N)dx$. Suppose y is a parameter to be determined. For any $T \leq x \leq 2T$, define

$$\begin{aligned}\mathcal{R}_1 &:= \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{n\leq y} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right),\\ \mathcal{R}_2 &:= S_f^k(x;N) - \mathcal{R}_1.\end{aligned}$$

6.1. **Evaluation of the integral $\int_T^{2T} \mathcal{R}_1^k dx$.** Let $\mathbb{I} = \{0,1\}$ and $\mathbb{N}^k = \{\mathbf{n} : \mathbf{n} = (n_1,\cdots,n_k), n_j \in \mathbb{N}, 1 \leq j \leq k\}$. For each element $\mathbf{i} = (i_1,\cdots,i_{k-1}) \in \mathbb{I}^{k-1}$, put $|\mathbf{i}| = i_1 + \cdots + i_{k-1}$. By the elementary formula

$$\cos a_1 \cdots \cos a_k = \frac{1}{2^{k-1}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\left(a_1 + (-1)^{i_1} a_2 + (-1)^{i_2} a_3 + \cdots + (-1)^{i_{k-1}} a_k\right),$$

we have

$$\begin{aligned}\mathcal{R}_1^k &= \frac{\epsilon_f^k (xN)^{k/4}}{(\pi\sqrt{2})^k} \sum_{n_1\leq y} \cdots \sum_{n_k\leq y} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \prod_{j=1}^k \cos\left(4\pi\sqrt{\frac{n_j x}{N}} - \frac{\pi}{4}\right)\\ &= \frac{\epsilon_f^k (xN)^{k/4}}{(\pi\sqrt{2})^k 2^{k-1}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{n_1\leq y} \cdots \sum_{n_k\leq y} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \cos\left(4\pi\sqrt{\frac{x}{N}}\alpha(\mathbf{n};\mathbf{i}) - \frac{\pi}{4}\beta(\mathbf{i})\right),\end{aligned}$$

where

$$\begin{aligned}\alpha(\mathbf{n};\mathbf{i}) &:= \sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k},\\ \beta(\mathbf{i}) &:= 1 + (-1)^{i_1} + (-1)^{i_2} + \cdots + (-1)^{i_{k-1}}.\end{aligned}$$

Suppose that

$$\mathcal{R}_1^k = \frac{\epsilon_f^k N^{k/4}}{(\pi\sqrt{2})^k 2^{k-1}} \left(S_1(x) + S_2(x)\right), \tag{6.1}$$

where

$$S_1(x) := x^{k/4} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\left(-\frac{\pi}{4}\beta(\mathbf{i})\right) \sum_{\substack{n_j\le y, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}},$$

$$S_2(x) := x^{k/4} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\le y, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \cos\left(4\pi\sqrt{\frac{x}{N}}\alpha(\mathbf{n};\mathbf{i}) - \frac{\pi}{4}\beta(\mathbf{i})\right).$$

Firstly we consider the contribution of $S_1(x)$ to get

$$\int_T^{2T} S_1(x)dx = \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\left(-\frac{\pi}{4}\beta(\mathbf{i})\right) \sum_{\substack{n_j\le y, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \int_T^{2T} x^{k/4}dx. \tag{6.2}$$

It is easy to see that if $\alpha(\mathbf{n};\mathbf{i}) = 0$, then $1 \le |\mathbf{i}| \le k-1$. Let $|\mathbf{i}| = l$, then

$$\sum_{\substack{n_j\le y, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} = s_{k;l}(\lambda_f; y),$$

where $s_{k;l}(f;y)$ has been defined in (1.5). By Lemma 2.3, we have

$$\int_T^{2T} S_1(x)dx = B_k^*(\lambda_f)\int_T^{2T} x^{\frac{k}{4}}dx + O(T^{1+k/4+\varepsilon}y^{-1/2}), \tag{6.3}$$

where

$$B_k^*(\lambda_f) = \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\left(-\frac{\pi}{4}\beta(\mathbf{i})\right) \sum_{\substack{(n_1,\cdots,n_k)\in\mathbb{N}^k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}}.$$

For any $\mathbf{i} \in \mathbb{I}^{k-1}\backslash\mathbf{0}$, let

$$S(\lambda_f;\mathbf{i}) := \sum_{\substack{(n_1,\cdots,n_k)\in\mathbb{N}^k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}}.$$

It is easy to see that if $|\mathbf{i}| = |\mathbf{i}'|$ or $|\mathbf{i}| + |\mathbf{i}'| = k$, then

$$S(\lambda_f;\mathbf{i}) = S(\lambda_f;\mathbf{i}') = s_{k;|\mathbf{i}|}(\lambda_f).$$

We also have $\beta(\mathbf{i}) = k - 2|\mathbf{i}|$ if we notice that $(-1)^j = 1 - 2j,\ (j = 0, 1)$. So we have

$$\begin{aligned} B_k^*(\lambda_f) &= \sum_{l=1}^{k-1} \sum_{|\mathbf{i}|=l} \cos\left(-\frac{\pi}{4}\beta(\mathbf{i})\right) S(\lambda_f; \mathbf{i}) \\ &= \sum_{l=1}^{k-1} s_{k;l}(\lambda_f) \cos\left(\frac{\pi(k-2l)}{4}\right) \sum_{|\mathbf{i}|=l} 1 \\ &= \sum_{l=1}^{k-1} \begin{pmatrix} k-1 \\ l \end{pmatrix} s_{k;l}(\lambda_f) \cos\left(\frac{\pi(k-2l)}{4}\right) = B_k(\lambda_f). \end{aligned} \tag{6.4}$$

By Lemma 2.4 and the first derivative test we have

$$\begin{aligned} \int_T^{2T} S_2(x)dx &\ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j \le y, 1\le j\le k \\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|} \\ &\ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{2^{k-2}-\frac{1}{2}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j \le y, 1\le j\le k \\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \\ &\ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{2^{k-2}-\frac{1}{2}+\frac{k}{4}}. \end{aligned} \tag{6.5}$$

From (6.1) to (6.5) we get

Lemma 6.1. For any fixed $k \ge 3$ and $y^{2b(k)} \ll TN^{-1}$, we have

$$\int_T^{2T} \mathcal{R}_1^k dx = \frac{\epsilon_f^k N^{k/4} B_k(\lambda_f)}{(\pi\sqrt{2})^k 2^{k-1}} \int_T^{2T} x^{\frac{k}{4}} dx + O\left(N^{\frac{k}{4}}\left(T^{1+k/4+\varepsilon} y^{-1/2} + N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{b(k)}\right)\right), \tag{6.6}$$

where $b(k) = 2^{k-2} - \frac{1}{2} + \frac{k}{4}$.

6.2. **Upper bound of the integral $\int_T^{2T} \mathcal{R}_2^A dx$.** Taking $M = T$ in the formula (2.1), we have

$$\mathcal{R}_2 = \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{y<n\le T} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) + O_k\left(N^{1/2+\varepsilon} T^{\varepsilon}\right). \tag{6.7}$$

Then

$$
\begin{aligned}
\int_T^{2T} \mathcal{R}_2^2 dx &\ll NT^{1+\varepsilon} + N^{1/2}\int_T^{2T} \left| x^{1/4} \sum_{y<n\le T} \frac{\lambda_f(n)}{n^{3/4}} e\left(2\sqrt{\frac{nx}{N}}\right)\right|^2 dx \\
&\ll NT^{1+\varepsilon} + N^{1/2}T^{3/2} \sum_{y<n\le T} \frac{\lambda_f^2(n)}{n^{3/2}} + NT \sum_{y<m<n\le T} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})} \\
&\ll N^{1/2}T^{3/2+\varepsilon}y^{-1/2} + NT^{1+\varepsilon}, \qquad (6.8)
\end{aligned}
$$

where we used the bound $\sum_{n\le x} \lambda_f^2(n) \ll x^{1+\varepsilon}$ and the estimate

$$
\sum_{y<m<n\le T} \frac{\lambda_f(m)\lambda_f(n)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})} \ll T^{\varepsilon}.
$$

Now we suppose y satisfies $y^{2b(K_0)} \le TN^{-1}$. From Lemma 6.1 we get

$$
\int_T^{2T} |\mathcal{R}_1|^{K_0} dx \ll N^{K_0/4}T^{1+K_0/4+\varepsilon},
$$

which implies

$$
\int_T^{2T} |\mathcal{R}_1|^{A_0} dx \ll N^{A_0/4}T^{1+A_0/4+\varepsilon}, \qquad (6.9)
$$

if we notice that $A_0 \le K_0$. From (6.6) and (6.9) we have

$$
\int_T^{2T} |\mathcal{R}_2|^{A_0} dx \ll \int_T^{2T} \left(|S_f(x;N)|^{A_0} + |\mathcal{R}_1^{A_0}|\right) dx \ll N^{A_0/4}T^{1+A_0/4+\varepsilon}. \qquad (6.10)
$$

For any $2 < A < A_0$, from (6.8), (6.10) and Hölder's inequality we get

$$
\begin{aligned}
\int_T^{2T} |\mathcal{R}_2|^{A} dx &= \int_T^{2T} |\mathcal{R}_2|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}} dx \qquad (6.11)\\
&\ll \left(\int_T^{2T} \mathcal{R}_2^2 dx\right)^{\frac{A_0-A}{A_0-2}} \left(\int_T^{2T} |\mathcal{R}_2|^{A_0} dx\right)^{\frac{A-2}{A_0-2}} \\
&\ll N^{\frac{A}{4}}T^{1+\frac{A}{4}+\varepsilon}y^{-\frac{A_0-A}{2(A_0-2)}} + N^{\frac{A}{4}+\frac{A_0-A}{2(A_0-2)}}T^{\frac{A}{4}+\frac{1}{2}+\frac{A-2}{2(A_0-2)}}.
\end{aligned}
$$

Lemma 6.2. Suppose $T^{\varepsilon} \le y \le (TN^{-1})^{1/2b(K_0)},\ 2 < A < A_0$, then

$$
\int_T^{2T} |\mathcal{R}_2|^{A} dx \ll N^{\frac{A}{4}}T^{1+\frac{A}{4}+\varepsilon}y^{-\frac{A_0-A}{2(A_0-2)}} + N^{\frac{A}{4}+\frac{A_0-A}{2(A_0-2)}}T^{\frac{A}{4}+\frac{1}{2}+\frac{A-2}{2(A_0-2)}}. \qquad (6.12)
$$

6.3. **Evaluation of the integral $\int_T^{2T} \mathcal{R}_1^{k-1}\mathcal{R}_2 dx$.** Let $T^\varepsilon \le y \le (TN^{-1})^{1/2b(K_0)}$. Then by Lemma 6.1 we get

$$\int_T^{2T} |\mathcal{R}_1|^{k-1} dx \ll N^{\frac{k-1}{4}} T^{1+\frac{k-1}{4}}.$$

From (6.7) we have

$$\int_T^{2T} \mathcal{R}_1^{k-1}\mathcal{R}_2 dx = \int_T^{2T} \mathcal{R}_1^{k-1}\mathcal{R}_2^* dx + O(N^{\frac{k+1}{4}+\varepsilon} T^{1+\frac{k-1}{4}+\varepsilon}), \tag{6.13}$$

where

$$\mathcal{R}_2^* = \frac{\epsilon_f (xN)^{1/4}}{\pi\sqrt{2}} \sum_{y<n\le T} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right).$$

We can write

$$\mathcal{R}_1^{k-1}\mathcal{R}_2^* = \frac{\epsilon_f^k N^{k/4}}{(\pi\sqrt{2})^k 2^{k-1}} \left(S_3(x) + S_4(x)\right),$$

where

$$S_3(x) := x^{k/4} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y, 2\le j\le k \\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \cos\left(-\frac{\pi}{4}\beta(\mathbf{i})\right),$$

$$S_4(x) := x^{k/4} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y, 2\le j\le k \\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \cos\left(4\pi\sqrt{\frac{x}{N}}\alpha(\mathbf{n};\mathbf{i}) - \frac{\pi}{4}\beta(\mathbf{i})\right).$$

By Lemma 2.3 the contribution of $S_3(x)$ is

$$\begin{aligned} \int_T^{2T} S_3(x) dx &\ll \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y, 2\le j\le k \\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}} \int_T^{2T} x^{k/4} dx \\ &\ll \sum_{l=1}^{k-1} |s_{k;l}(\lambda_f; y) - s_{k;l}(\lambda_f)| \int_T^{2T} x^{k/4} dx \\ &\ll T^{1+\frac{k}{4}+\varepsilon} y^{-1/2}. \end{aligned} \tag{6.14}$$

By Lemma 2.4 and the first derivative test we have

$$\int_T^{2T} S_4(x)dx \ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y, 2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|}$$

$$\ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le k^2 y} \sum_{\substack{n_j\le y, 2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|}$$

$$+ N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{n_1> k^2 y} \sum_{\substack{n_j\le y, 2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_k)}{(n_1\cdots n_k)^{3/4} n_1^{1/2}}$$

$$\ll N^{\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{2^{k-2}-\frac{1}{2}+\frac{k}{4}}. \tag{6.15}$$

Thus from (6.13)-(6.15) we get

$$\int_T^{2T} \mathcal{R}_1^{k-1}\mathcal{R}_2 dx \ll N^{\frac{k+1}{4}+\varepsilon} T^{1+\frac{k-1}{4}+\varepsilon} + N^{\frac{k}{4}} T^{1+\frac{k}{4}+\varepsilon} y^{-1/2} + N^{\frac{k}{4}+\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{b(k)}, \tag{6.16}$$

where $b(k) = 2^{k-2} - \frac{1}{2} + \frac{k}{4}$.

6.4. **Proof of Theorem 3.** Let $3 \le k < A_0$ and $T^\varepsilon \le y \le (TN^{-1})^{1/2b(K_0)}$. By the elementary formula $(a+b)^k = a^k + ka^{k-1}b + O(|a^{k-2}b^2| + |b|^k)$, we have

$$\int_T^{2T} S_f^k(x;N)dx = \int_T^{2T} \mathcal{R}_1^k dx + k\int_T^{2T} \mathcal{R}_1^{k-1}\mathcal{R}_2 dx + O\left(\int_T^{2T} |\mathcal{R}_1^{k-2}\mathcal{R}_2^2| dx\right) + O\left(\int_T^{2T} |\mathcal{R}_2^k| dx\right). \tag{6.17}$$

By Hölder's inequality, Lemma 6.2 and (6.9) we have

$$\int_T^{2T} |\mathcal{R}_1^{k-2}\mathcal{R}_2^2| dx \ll \left(\int_T^{2T} |\mathcal{R}_1|^{A_0} dx\right)^{\frac{k-2}{A_0}} \left(\int_T^{2T} |\mathcal{R}_2|^{\frac{2A_0}{A_0-k+2}} dx\right)^{\frac{A_0-k+2}{A_0}}$$

$$\ll N^{\frac{k}{4}} T^{1+\frac{k}{4}+\varepsilon} y^{-\frac{A_0-k}{2(A_0-2)}} + N^{\frac{k}{4}+\frac{A_0-k}{2(A_0-2)}} T^{\frac{k}{4}+\frac{1}{2}+\frac{k-2}{2(A_0-2)}}. \tag{6.18}$$

From (6.16)-(6.18)and Lemma 6.2 with $A = k$ we get

$$\int_T^{2T} S_f^k(x;N)dx = \int_T^{2T} \mathcal{R}_1^k dx + O\left(N^{\frac{k}{4}} T^{1+\frac{k}{4}+\varepsilon} y^{-\sigma(k,A_0)} + N^{\frac{k}{4}+\frac{1}{2}} T^{\frac{k}{4}+\frac{1}{2}} y^{b(k)} + N^{\frac{k}{4}+\sigma(k,A_0)} T^{\frac{k}{4}+\frac{1}{2}+\omega(k,A_0)}\right). \tag{6.19}$$

where $\sigma(k,A_0)=\frac{A_0-k}{2(A_0-2)}, \omega(k,A_0)=\frac{k-2}{2(A_0-2)}$. Take $y=(TN^{-1})^{\frac{1}{2b(k)+2\sigma(k,A_0)}}$. We have

$$\begin{aligned}\int_T^{2T} S_f^k(x;N)dx =& \frac{\epsilon_f^k N^{k/4} B_k(\lambda_f)}{(\pi\sqrt{2})^k 2^{k-1}} \int_T^{2T} x^{\frac{k}{4}} dx \\ &+ O\left(N^{\frac{k}{4}} T^{1+\frac{k}{4}-\delta(k,A_0)+\varepsilon} + N^{\frac{k}{4}+\sigma(k,A_0)} T^{\frac{k}{4}+\frac{1}{2}+\omega(k,A_0)}\right),\end{aligned} \tag{6.20}$$

where $\delta(k,A_0)=\frac{\sigma(k,A_0)}{2b(k)+2\sigma(k,A_0)}$.

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