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Satake compactification of analytic Drinfeld modular varieties

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We construct a normal projective rigid analytic compactification of an arbitrary Drinfeld modular variety whose boundary is stratified by modular varieties of smaller dimensions. This generalizes work of Kapranov. Using an algebraic modular compactification that generalizes Pink and Schieder's, we show that the analytic compactification is naturally isomorphic to the analytification of Pink's normal algebraic compactification. We interpret analytic Drinfeld modular forms as the global sections of natural ample invertible sheaves on the analytic compactification and deduce finiteness results for spaces of such modular forms.

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1. Introduction

Consider any global function field F of characteristic $p > 0$ and any place ∞ of F. Let $A \subset F$ be the subring of elements that are regular outside of ∞. The basic example for A is the polynomial ring over a finite field. Denote by E the completion of F with respect to ∞. Let C be any non-Archimedean complete algebraically closed valued field containing E as a valued subfield.

For any module M over any ring R and any ring extension $R \subset R'$ let

$$M_{R'} := M \otimes_R R'$$

denote the module over R' obtained by extension of scalars.

Drinfeld modular varieties and modular forms

Drinfeld A-modules with level structure, introduced by Drinfeld [12] in 1974, are a function field analogue to elliptic curves with level structure.

Let $d \geq 1$ be any positive integer. Consider any ring R over F and denote by $\iota\colon A \to R$ the structure morphism. Denote by $R\{\tau\} \subset R[T]$, with $\tau := T^p$, the subgroup of additive polynomials and equip it with the ring structure for which multiplication is given by composition. A *Drinfeld A-module* of rank d over R is a ring homomorphism

$$\varphi\colon A \to R\{\tau\}, 0 \neq a \mapsto \varphi_a = \sum_{0 \leq i \leq d \cdot \deg(a)} \varphi_{a,i}\tau^i \tag{1}$$

with $\varphi_{a,0} = \iota(a)$ and $\varphi_{a,d\cdot\deg(a)} \in R^\times$, where $\deg(a) := \dim_{\mathbb{F}_p}(A/(a))$. Consider any non-zero non-unital $t \in A$. Let V be a free (A/t)-module of rank d. A *level* (t) *structure* for such a φ is a map $\lambda\colon V \to R$ with $\lambda(V \setminus \{0\}) \subset R^\times$ and

$$\varphi_t(T) = t \cdot T \prod_{0 \neq v \in V} \left(1 - \frac{T}{\lambda(v)}\right)$$

for which the induced map $V \to \mathrm{Ker}(R \xrightarrow{\varphi_t} R)$ is an A-linear isomorphism.

Consider any ideal $0 \neq I \subsetneq A$. More generally, one defines (see Section 7.1) Drinfeld A-modules with level I structures over arbitrary schemes S over F. The functor which associates to such an S the set of isomorphism classes of Drinfeld A-modules of rank d over S with level I structure is then represented by an irreducible smooth affine variety X_I^d of dimension $d-1$ over F (see [12, Section 5]). Consequently, X_I^d is non-compact if $d \geq 2$.

Analytically, Drinfeld [12, Section 6] described $X_I^d(C)$ as follows. Consider any non-zero finite dimensional E-vector space $\mathcal{V}$. Let $\mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$ be

$$(\mathrm{Hom}_C(\mathcal{V}_C, C) \setminus \{0\})/C^\times$$

with its natural structure of projective rigid analytic variety over C (see Example 2.21). Drinfeld's *period domain* is the $\mathrm{PGL}(\mathcal{V})$-invariant subset

$$\Omega_{\mathcal{V}} \subset \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$$

of those $C^\times$-classes $[l]$ of C-linear maps $l\colon \mathcal{V}_C \to C$ with $\mathrm{Ker}(l) \cap \mathcal{V} = 0$. For any non-zero finitely generated projective A-module Λ set $\Omega_\Lambda := \Omega_{\Lambda_E}$. Let Λ be such an A-module and consider a *congruence subgroup* $\Gamma \subset \mathrm{Aut}_A(\Lambda)$, i.e., a subgroup that contains

$$\Gamma(J) := \mathrm{Ker}(\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(\Lambda/J\Lambda))$$

for some ideal $0 \neq J \subset A$. Then the quotient

$$\Omega_\Gamma := \Gamma \backslash \Omega_\Lambda$$

is a rigid analytic variety over C and, if $\mathrm{rank}_A(\Lambda) = d$ and $\Gamma = \Gamma(I)$, it is naturally isomorphic to an irreducible component of the rigid analytic variety associated with $X_I^d(C)$. Conversely, any irreducible component arises in this way.

In Section 4 we provide comprehensive proofs of widely known or accepted facts about the rigid analytic structure of quotients of $\Omega_{\mathcal{V}}$ by discrete subgroups of $\mathrm{PGL}(\mathcal{V})$ which are fundamental for the construction of Ω_Γ and its compactification.

In analogy to the classical weak modular forms on the complex upper half space, the quotient Ω_Γ is naturally equipped with an invertible sheaf $\mathcal{O}_\Gamma(k)$ of weak modular forms with respect to Γ of any integer weight k. By means of Fourier expansions of weak modular forms at various *cusps*, i.e., at various irreducible components of modular varieties of codimension 1, modular forms may be distinguished. They form a C-subspace

$$\mathcal{M}_\Gamma(k) \subset \mathcal{O}_\Gamma(k)(\Omega_\Gamma).$$

Fourier expansions were defined and studied by Gekeler [14] and Goss [17,18] mainly in the case of Drinfeld A-modules of rank 2, implicitly by Kapranov [25, Proof of Prop. 1.19] when A is a polynomial ring and in general by Basson, Breuer and Pink [3–6].

We also provide everything required for the definition of Fourier expansions and believe to have taken rigorous care of the rigid analysis involved.

Analytic compactification

We construct the Satake compactification Ω_Γ^* of Ω_Γ as a Grothendieck ringed space roughly as follows. Let Ω_Λ^* denote the disjoint union of the sets Ω_L for all direct summands $0 \neq L \subset \Lambda$ and equip it with the left Γ-action for which any $\gamma \in \Gamma$ maps any $[l] \in \Omega_L$ to $[\gamma l] \in \Omega_{\gamma(L)}$, where $(\gamma l)(v) := l(\gamma^{-1}v)$ for all $v \in \gamma(L)_C$. We endow Ω_Λ^* with a certain Grothendieck topology which induces the rigid analytic Grothendieck topology on any stratum Ω_L and which contains Ω_Λ as a dense admissible subset. We set

$$\Omega_\Gamma^* := \Gamma \backslash \Omega_\Lambda^*$$

and endow this quotient with the quotient Grothendieck topology. For any orbit $\mathfrak{O}$ of the action $(\gamma, L) \mapsto \gamma(L)$ of Γ on the set of direct summands $0 \neq L \subset \Lambda$ denote by

$$\Omega_{\mathfrak{O}} \subset \Omega_\Gamma^*$$

the quotient by Γ of the union of the Ω_L for all $L \in \mathfrak{O}$. We further define a sheaf of rings $\mathcal{O}_\Gamma^*$ on Ω_Γ^* and a sheaf of graded $\mathcal{O}_\Gamma^*$-algebras

$$\mathcal{R}_\Gamma^* = \sum_{k \geq 0} \mathcal{O}_\Gamma^*(k)$$

on Ω_Γ^* with $\mathcal{O}_\Gamma^*(0) = \mathcal{O}_\Gamma^*$ and such that the $\mathcal{O}_\Gamma^*$-module $\mathcal{O}_\Gamma^*(k)$ of the homogeneous sections of weight k extends the sheaf $\mathcal{O}_\Gamma(k)$ of weak modular forms for any $k \geq 0$.

If $\operatorname{rank}_A(\Lambda) = d$ and $\Gamma = \Gamma(I)$, the basic examples of modular forms with respect to Γ are the level I Eisenstein series of weight 1 indexed by $(A/I)^d \setminus \{0\}$. In that case, we directly write down global sections E_α of $\mathcal{O}_\Gamma^*(1)$ for all $\alpha \in (I^{-1}\Lambda/\Lambda) \setminus \{0\}$ which, by Theorem 1.1, iii), a posteriori uniquely restrict to these series up to an isomorphism $I^{-1}\Lambda/\Lambda \cong (A/I)^d$.

The following results on Ω_Γ^* are in Theorem 9.8.

Theorem 1.1. *(Analytic Satake compactification)*

i) The Grothendieck ringed space $(\Omega_\Gamma^*, \mathcal{O}_\Gamma^*)$ is an integral normal projective rigid analytic variety over C containing Ω_Γ as a dense admissible subvariety.

ii) *If for some maximal ideal $\mathfrak{p} \subset A$ the image of Γ in $\mathrm{Aut}_A(\Lambda/\mathfrak{p}\Lambda)$ is unipotent, $\mathcal{O}^*_\Gamma(k)$ is ample invertible for any $k \geq 1$.*

iii) *For any $k \geq 0$ the restriction morphism $\mathcal{O}^*_\Gamma(k)(\Omega^*_\Gamma) \to \mathcal{O}_\Gamma(k)(\Omega_\Gamma)$ is injective and its image is the space of modular forms $\mathcal{M}_\Gamma(k)$.*

iv) *The graded C-algebra $\mathcal{M}_\Gamma := \sum_{k \geq 0} \mathcal{M}_\Gamma(k)$ is finitely generated with $\mathcal{M}_\Gamma(0) = C$ and Ω^*_Γ is the analytification of $\mathrm{Proj}(\mathcal{M}_\Gamma)$. Moreover, $\mathcal{M}_\Gamma(k)$ is a finite dimensional vector space over C for any $k \geq 0$.*

v) *Consider any Γ-orbit $\mathfrak{O} = \Gamma \cdot L \neq \{0\}$. With respect to the Zariski topology, the subset $\Omega_\mathfrak{O} \subset \Omega^*_\Gamma$ is irreducible, locally closed and its closure is the union of all $\Omega_{\Gamma \cdot L'}$ for all direct summands $0 \neq L' \subset L$.*

vi) *Consider any direct summand $0 \neq L \subset \Lambda$ and set $\mathfrak{O} := \Gamma \cdot L$ and $\overline{\Gamma}_L := \{\gamma' \in \mathrm{Aut}_A(L) \mid \exists \gamma \in \Gamma \colon \gamma|_L = \gamma'\}$. The composition of the canonical bijection $\overline{\Gamma}_L \backslash \Omega_L \to \Omega_\mathfrak{O}$ with the inclusion $\Omega_\mathfrak{O} \subset \Omega^*_\Gamma$ is a locally closed immersion (in the sense of Definition 2.25) of rigid analytic varieties.*

Observe that under the map $L \mapsto L_F$, direct summands of Λ correspond bijectively to vector subspaces of Λ_F. The locally closed boundary strata in part v) are thus parametrized by the Γ-conjugacy classes of the maximal parabolic subgroups of $\mathrm{Aut}_F(\Lambda_F)$.

We briefly outline the proof of Theorem 1.1. If $\Gamma = \Gamma(I)$ and $I = (t)$ for some $t \in A$ whose divisors in A generate A, parts i), ii) and v) follow by comparison with an algebraic compactification (see Theorem 1.6 below); underlying this comparison is a projective embedding defined by the E_α. In general, parts i), ii) and v) are then obtained by choosing $t \in A$ as before small enough such that $\Gamma' := \Gamma((t)) \subset \Gamma$ and using that, by construction, Ω^*_Γ is the quotient of $\Omega^*_{\Gamma'}$ by the finite group Γ/Γ', that $\Omega_{\Gamma \cdot L}$ is the image under the quotient map of $\Omega_{\Gamma' \cdot L}$ for any $0 \neq L \subset \Lambda$ and that $\mathcal{O}^*_\Gamma(k)$ is the subsheaf of $\mathcal{O}^*_{\Gamma'}(k)$ of (Γ/Γ')-invariants. For part iii) we use that the restriction morphism factors through the restriction morphisms

$$\mathcal{O}^*_\Gamma(k)(\Omega^*_\Gamma) \to \mathcal{O}^*_\Gamma(k)(\Omega^{<2}_\Gamma) \to \mathcal{O}_\Gamma(k)(\Omega_\Gamma),$$

where $\Omega^{<2}_\Gamma$ is the union of the $\Omega_\mathfrak{O}$ for all Γ-orbits $\mathfrak{O}$ of direct summands $0 \neq L \subset \Lambda$ of co-rank < 2; this is a Zariski open admissible subset of Ω^*_Γ whose complement has codimension 2. The first restriction morphism is thus bijective by Riemann's extension theorem using that Ω^*_Γ is normal. Moreover, the theory of Fourier expansions shows that the second restriction morphism is injective with image $\mathcal{M}_\Gamma(k)$. Part iv) follows by standard arguments from the previous parts. For part vi), it remains to show that the composition induces surjective maps on stalks. In fact, by means of explicit admissible neighborhoods in Ω^*_Γ of points in $\Omega_\mathfrak{O}$, we show that locally on the target the composition has a left-inverse.

The compactification of all of $X^d_I(C)$ is constructed using adelic language such that any of its connected components is isomorphic to some Ω^*_Γ as above. Varying A, d, I

suitably in fact yields morphisms between the compactifications of the varying modular varieties (see [20, Section 7.2]).

When A is the polynomial ring, the topology on Ω_Γ^* induced by the Grothendieck topology was already defined by Kapranov in [25], where he further carried out a projective embedding of Ω_Γ^* using Eisenstein series of high weight and then defined the Satake compactification of Ω_Γ as the normalization of the image of Ω_Γ^*. Although he did not specify the Grothendieck topology itself nor the sheaves $\mathcal{O}_\Gamma^*(k)$, he already argued towards parts iii) along with Goss [16]. In the polynomial case, Gekeler [15] has recently, and independently of the work presented here, improved on Kapranov's approach by carrying out an embedding defined by the Eisenstein series of weight 1.

Algebraic compactifications

Using that A is finitely generated, let $t \in A$ be such that its divisors

$$\mathrm{Div}_A(t) := \{a \in A |\ t \in (a)\}$$

generate A. Let V be a free $A/(t)$-module of rank d and set $\mathring{V} := V \setminus \{0\}$.

We obtain a compactification of $X_{(t)}^d$ which has a modular interpretation and whose boundary is stratified by modular varieties of smaller dimensions and whose normalization is Pink's normal compactification. When A is the polynomial ring $\mathbb{F}_q[t]$ over a finite field $\mathbb{F}_q$, this modular compactification specializes to the one due to Pink and Schieder and, in general, it is inspired by their compactification.

Before discussing the modular compactification, we recall Pink's normal compactification. In [29], he introduced the notion of *generalized Drinfeld A-module* of rank $\leq d$ over any scheme S over F. It generalizes the notion of Drinfeld A-module over S in that its fibers over the points of S – which are Drinfeld A-modules of the form (1) – are allowed to have rank $\leq d$ rather than only $= d$. A generalized Drinfeld A-module over S is *weakly separating* if for any Drinfeld A-module φ over any field extension $F' \supset F$ at most finitely many fibers of the generalized Drinfeld A-module over F'-valued points of S are isomorphic to φ.

Theorem 1.2. *(Pink [29]) Uniquely up to unique isomorphism, there exist an integral normal projective algebraic variety $\overline{X}_I^d$ over F together with an embedding of X_I^d and a weakly separating generalized Drinfeld module on $\overline{X}_I^d$ extending the universal family on X_I^d.*

The notion of level structure does not directly generalize to generalized Drinfeld modules in a satisfying way so as to turn $\overline{X}_I^d$ in a fine moduli space. For $A = \mathbb{F}_q[t]$ and $I = (t)$ and thus $V \cong \mathbb{F}_q^d$, Pink and Schieder [31] instead introduced and studied the notion of a *reciprocal map*. Over any ring R over $\mathbb{F}_q$, the injective reciprocal maps are precisely the ones that arise from the injective $\mathbb{F}_q$-linear morphisms $\lambda\colon V \to R$ with $\lambda(\mathring{V}) \subset R^\times$ by the rule

$$\rho_\lambda \colon \mathring{V} \to R^\times, v \mapsto \frac{1}{\lambda(v)}.$$

The maps ρ_λ thus obtained are the injective maps $\rho\colon \mathring{V} \to R^\times$ such that

- $\forall \alpha \in \mathbb{F}_q^\times, v \in \mathring{V} \colon \rho(\alpha \cdot v) = \alpha^{-1} \cdot \rho(v)$,
- $\forall v, v' \in \mathring{V} \colon [v + v' \in \mathring{V} \Rightarrow \rho(v) \cdot \rho(v') = \rho(v + v') \cdot (\rho(v) + \rho(v'))]$.

A general reciprocal map over R is then defined to be any map $\rho\colon \mathring{V} \to R$ satisfying these polynomial conditions. Globally, reciprocal maps are defined more generally as certain maps from $\mathring{V}$ to the set of global sections $\Gamma(S, \mathcal{L})$ of invertible sheaves $\mathcal{L}$ on schemes S over $\mathbb{F}_q$.

Theorem 1.3. *(Pink, Schieder [31, Theorems 1.7 and 7.10]) Consider the functor that associates with any scheme S over $\mathbb{F}_q$ the set of isomorphism classes of reciprocal maps $\mathring{V} \to \Gamma(S, \mathcal{L})$ whose induced morphism $\mathring{V} \to \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ is non-zero for every point $s \in S$. It is represented by a normal projective scheme Q_V over $\mathbb{F}_q$.*

Using that a Drinfeld $\mathbb{F}_q[t]$-module over a scheme over F is uniquely determined by a level (t) structure, Pink deduced from Theorem 1.3:

Theorem 1.4. *([29, Section 7]) If $A = \mathbb{F}_q[t]$, then $\overline{X}^d_{(t)}$ equals the pullback of Q_V to* $\mathrm{Spec}(F)$ *and is stratified by copies of $X^{d'}_{(t)}$ for all $1 \leq d' \leq d$ indexed by the non-zero $\mathbb{F}_q$-subspaces of V.*

In fact, Pink proved Theorem 1.2 by reduction to the case $A = \mathbb{F}_q[t]$ and $I = (t)$ which he proved jointly with Theorem 1.4 using Theorem 1.3.

However, as long as $\mathrm{Div}_A(t)$ generates A, it remains true for general A that a Drinfeld A-module over any ring R over F with level (t) structure λ is uniquely determined by λ and hence by its reciprocal map

$$\rho\colon \mathring{V} \to R^\times, v \mapsto \frac{1}{\lambda(v)}.$$

We find a set of necessary and sufficient polynomial conditions for an injective map $\rho\colon \mathring{V} \to R^\times$ to arise from such a λ and define an A-*reciprocal map* over R to be any map $\rho\colon \mathring{V} \to R$ satisfying these conditions. Globally, A-reciprocal maps will be defined (see Definition 7.14) more generally to be certain maps $\mathring{V} \to \Gamma(S, \mathcal{L})$ for invertible sheaves $\mathcal{L}$ on schemes S over $\mathrm{Spec}(A)$.

Theorem 1.5. *(See Theorem 7.16 and Corollary 7.25) Suppose that* $\mathrm{Div}_A(t)$ *generates A. Consider the functor which assigns to a scheme S over* $\mathrm{Spec}(A)$ *the set of isomorphism classes of A-reciprocal maps $\mathring{V} \to \Gamma(S, \mathcal{L})$ whose induced morphism $V \to \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ is non-zero for every $s \in S$. Then the following hold:*

i) This functor is represented by $Q_V = \mathrm{Proj}(R)$, where R is the quotient of the polynomial ring $A[\{Y_v\}_{v\in\mathring{V}}]$ over A in the variables Y_v by a certain graded ideal. The invertible sheaf on Q_V underlying the universal family is the ample invertible sheaf $\mathcal{O}_{Q_V}(1)$.

ii) The pullback $Q_{V,F}$ of Q_V to $\mathrm{Spec}(F)$ contains $X^d_{(t)}$ as an open subscheme and is stratified by locally closed subschemes Ω_W for all free $A/(t)$-submodules $0 \neq W \subset V$ each of which is isomorphic to $X^{d'}_{(t)}$, where $d' := \mathrm{rank}_{A/(t)}(W)$.

iii) $\overline{X}^d_{(t)}$ is the normalization of $Q_{V,F}$ and the universal family on $Q_{V,F}$ induces a generalized Drinfeld module on $Q_{V,F}$ whose pullback to $\overline{X}^d_{(t)}$ is the weakly separating generalized Drinfeld module from Theorem 1.2.

After the work presented here was done, Pink [30] modified the notion of A-reciprocal maps by using defining conditions [30, Def. 2.3.1] that are homogeneous equations solely of weight 1. His conditions are stronger (see [30, Prop. 1.3.4 (b) and 2.4.4]) and more explicit than the ones here and enabled him to generalize computations from his and Schieder's article [31]. However, the reduced scheme underlying the modular compactification that he obtains coincides with the one underlying Q_V and this is the scheme that we use in the comparison with the analytic compactification.

Comparison of the analytic and algebraic compactifications

As before, let $t \in A$ be such that $\mathrm{Div}_A(t)$ generates A. Set $V := t^{-1}\Lambda/\Lambda$ and $\mathring{V} := V \setminus \{0\}$. Suppose that Γ is the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(\Lambda/t\Lambda)$.

Consider the reduced rigid analytic variety $Q_V(C)$ over C whose underlying set are the C-valued points of Q_V; it is the rigid analytification of $Q_V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C)$. Let $\mathcal{O}_{Q_V(C)}(1)$ be the analytification of the pullback of $\mathcal{O}_{Q_V}(1)$ under $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Evaluating the weight 1 Eisenstein series $(E_\alpha)_{\alpha\in\mathring{V}}$ at any point in Ω^*_Γ yields an isomorphism class of an A-reciprocal map $\mathring{V} \to C$ and thus a map $E\colon \Omega^*_\Gamma \to Q_V(C)$.

Theorem 1.6. *(See Theorem 8.1 and Corollary 9.4) The map E is an injective morphism of Grothendieck ringed spaces onto an irreducible component X of $Q_V(C)$. In fact, it is the normalization morphism of X in the sense of Conrad [10]. Moreover, $\mathcal{O}^*_\Gamma(k)$ is isomorphic to the pullback under E of $\mathcal{O}_{Q_V(C)}(1)^{\otimes k}$ for any $k \geq 0$.*

We finally outline our proof of Theorem 1.6. Consider any free $A/(t)$-submodule $0 \neq W \subset V$ and the restriction $E^{-1}(\Omega_W(C)) \to \Omega_W(C)$ of E via Theorem 1.5, ii). Via the isomorphism $\Omega_W \cong X^{d'}_{(t)}$, where $d' = \mathrm{rank}_{A/(t)}(W)$, this restriction is Drinfeld's isomorphism from the analytically defined modular variety to $\Omega_W(C)$ if $W \subsetneq V$. If $W = V$, it is the restriction of this isomorphism to the irreducible component Ω_Γ. Using these isomorphisms and elementary inequalities of Drinfeld's exponential functions, we prove the following result as a step towards Theorem 1.6:

Proposition 1.7. *(See Proposition 8.7 and Corollary 8.11) The morphism between Grothendieck topological spaces underlying E is an isomorphism onto an irreducible component X of $Q_V(C)$. Moreover, $X \cap \Omega_V(C)$ is an irreducible component of $\Omega_V(C)$ and*

$$X = (X \cap \Omega_V(C)) \cup (Q_V(C) \setminus \Omega_V(C)).$$

We further define a sheaf of rings $\tilde{\mathcal{O}}_X$ on X in terms of the stratification by the $\Omega_W(C)$ provided by Theorem 1.5, ii) for which the following holds:

Proposition 1.8. *(See Corollary 8.13) The isomorphism between Grothendieck topological spaces underlying E induces an isomorphism of Grothendieck ringed spaces*

$$(\Omega_\Gamma^*, \mathcal{O}_\Gamma^*) \xrightarrow{\sim} (X, \tilde{\mathcal{O}}_X). \tag{2}$$

The stratification of X may be described in terms of vanishing and non-vanishing loci of subsets of some finite set of global sections of the first twisting sheaf on X. More generally, with any finite set of global sections of an invertible sheaf on a rigid analytic variety Z may be associated (see Section 3) a stratification of Z by locally closed subvarieties and a natural sheaf of rings $\tilde{\mathcal{O}}_Z$ in terms of the stratification together with a morphism of Grothendieck ringed spaces $n_Z\colon (Z, \tilde{\mathcal{O}}_Z) \to (Z, \mathcal{O}_Z)$. In Section 3.2 we specify conditions under which n_Z is the normalization morphism. We show these conditions in the case $Z = X$ using the isomorphism in (2). The hardest condition to show is that any point in Ω_Γ^* admits a fundamental set of neighborhoods whose intersections with Ω_Γ are irreducible; this is essentially done in Section 4.3. Via Proposition 1.8, this will yield Theorem 1.6.

Analogy with the classical Satake-Baily-Borel compactification

The Drinfeld modular variety X_I^d is a function field analogue of Shimura varieties. For instance, the analytification of $X_I^d(C)$ may be described as double coset space

$$\mathrm{GL}_d(F)\backslash \left(\Omega_{\mathcal{V}} \times (\mathrm{GL}_d(\mathbb{A}_F^f)/\mathcal{K})\right),$$

where $\dim_E(\mathcal{V}) = d$ and $\mathbb{A}_F^f$ is the ring of adeles of F outside ∞ and $\mathcal{K} \subset \mathrm{GL}_d(\mathbb{A}_F^f)$ is some compact open subgroup (see for instance [20, Remark 7.14 and Theorem 9.1]). In fact, $\Omega_{\mathcal{V}}$ is an analogue of Siegel's upper half space $\mathbb{H}_g$ of genus $g \geq 1$; quotients of $\mathbb{H}_g$ by arithmetic subgroups of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ parametrize abelian varieties over $\mathbb{C}$ of dimension g with extra structures. However, $\mathbb{H}_g$ is a hermitian symmetric space isomorphic to $\mathrm{Sp}_{2g}(\mathbb{R})/K$, where $K \subset \mathrm{Sp}_{2g}(\mathbb{R})$ is a maximal compact subgroup, while GL_d has no hermitian symmetric space nor a Shimura variety when $d > 2$. On the other hand, $\Omega_{\mathcal{V}}$ has no such interpretation as coset space, but may be viewed (see for instance [20, Sections 4 and 5]) as a rigid analytic neighborhood of the Bruhat-Tits building for $\mathrm{PGL}(\mathcal{V})$ whose set of vertices is $\mathrm{PGL}(\mathcal{V})/K$ for some maximal compact subgroup K, whereas neither this coset space nor the building have a rigid analytic structure.

The quotient $\Omega_\Gamma = \Gamma\backslash\Omega_\Lambda$ is an irreducible component of a Drinfeld modular variety. The construction of its compactification Ω_Γ^* is largely analogous to Satake's [33] construction of his normal projective compactification $X_{g,N}^*$ of the quotient $X_{g,N}$ of $\mathbb{H}_g$ by the Siegel modular group of some level N. As a set $X_{g,N}^*$ is the disjoint union of $X_{g,N}$ and finitely many copies of $X_{g',N}$ for all $0 \leq g' < g$, where $X_{0,N}$ contains exactly one point ∞. If $N = 1$, there appears exactly one copy for every $0 \leq g' \leq g$. For instance, $X_{1,1}$ is the quotient of the complex upper half space $\mathbb{H}$ by $\mathrm{SL}_2(\mathbb{Z})$ and $X_{1,1}^*$ is the compact Riemann surface containing both $X_{1,1}$ and

$$U_n := \{\infty\} \cup \mathrm{SL}_2(\mathbb{Z})\backslash \bigcup_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \gamma(\{\tau \in \mathbb{H}\colon\ \mathrm{Im}(\tau) > n\})$$

for any $n > 1$ as open subspaces, where the structure of Riemann surface on U_n is such that mapping $\tau \in \mathbb{H}$ to $\exp(2\pi i\tau)$ and ∞ to 0 induces an open embedding $U_n \to \mathbb{C}$.

If $\mathrm{rank}_A(\Lambda) = 2$, the construction and behavior of Ω_Γ^* are analogous to the ones of $X_{1,N}^*$ except that Ω_Γ^* needs to be endowed with a suitable Grothendieck topology rather than just a topology in order to become a rigid analytic variety. If $g > 1$, formally the construction of $X_{g,N}^*$ is still largely analogous to the one of Ω_Γ^*; however, in this case, the boundary of $X_{g,N}^*$ has codimension $g > 1$ so that by normality all – a priori – weak Siegel modular forms on $X_{g,N}$ extend to global sections of $X_{g,N}^*$, i.e., so that the *Köcher principle* holds. By contrast, as long as $\mathrm{rank}_A(\Lambda) > 1$, the boundary of Ω_Γ^* has codimension 1 and not all weak modular forms on Ω_Γ extend to global sections.

The boundary strata of Ω_Γ^*, resp. $X_{g,N}^*$, are parametrized by conjugacy-classes of maximal F-rational, resp. $\mathbb{Q}$-rational, parabolic subgroups of GL_d, resp. Sp_{2g}, where conjugation is by Γ, resp. by the Siegel modular group of level N. Each boundary stratum is the quotient of a Drinfeld upper half space, resp. of a hermitian symmetric space, by an arithmetic subgroup of a factor of the Levi subgroup of a representing maximal parabolic subgroup. In the case of Ω_Γ^*, any boundary stratum is the quotient of Drinfeld's upper half space of dimension $d' - 1$ by an arithmetic subgroup of $\mathrm{GL}_{d'}(F)$ for some $0 < d' < d$; the corresponding Levi subgroup is isomorphic to $\mathrm{GL}_{d'} \times \mathrm{GL}_{d-d'}$. On the other hand, the Levi subgroup of any maximal parabolic subgroup of Sp_{2g} is isomorphic to $\mathrm{Sp}_{2g'} \times \mathrm{GL}_{g-g'}$ for some $0 \leq g' < g$, where $\mathrm{Sp}_{2g'}$ is the hermitian factor and, if $g' = 0$, is the trivial group; if $g' > 0$, this factor contributes an arithmetical quotient of $\mathbb{H}_{g'}$ to the boundary, and if $g' = 0$, it contributes a point.

Baily and Borel [1] later generalized Satake's work by constructing a normal projective compactification X^* of any arithmetic quotient X of the bounded symmetric domain $G(\mathbb{R})/K$, where G is any connected semi-simple algebraic group over $\mathbb{Q}$. The strata of the boundary of X^* are smaller dimensional arithmetical quotients of bounded symmetric domains and are parametrized by conjugacy-classes of maximal rational parabolic subgroups of G. Compactness of the topological space underlying X^* follows rather directly using that bounded symmetric domains admit fundamental sets with respect to arithmetic groups. Using their analyticity criterion, which requires local compactness, Baily and Borel then prove their constructed ringed space X^* to be an irreducible normal analytic space. They further show that any two points in X^* may be separated by Poincaré-Eisenstein series. By compactness, then there exist finitely many such series which yield a projective embedding of X^*. Finally, they deduce that X^* is the normalization of the image of such an embedding.

Our approach to show that Ω_Γ^* is a normal projective rigid analytic variety differs from Baily and Borel's approach as follows: First, we may not deduce any suitable variant of compactness (or even local compactness) of Ω_Γ^* directly from the construction. Although Ω_Γ^* admits Poincaré-Eisenstein series which separate points (see [20, Section 6.5]), we may thus not directly conclude a projective embedding by finitely many of those series. We may neither directly show that Ω_Γ^* is a rigid analytic variety. Instead, we define the explicit injective map E from Ω_Γ^* onto an irreducible component of the projective variety $Q_V(C)$ (see before Theorem 1.6) and show in one that E is the normalization morphism of its image, and hence that Ω_Γ^* is rigid analytic, projective and normal. However, we still use an analogue (see Theorem 3.7) of Baily and Borel's analyticity criterion in order to provide a description, not involving E, of the normalization of any irreducible component of $Q_V(C)$.

Outline

Section 2. In Section 2.1 we recall the notion of Grothendieck ringed space.

In Section 2.2 we define rigid analytic varieties over C as did Bosch, Güntzer and Remmert [8] and recall some repeatedly used facts about them.

In Section 2.3 we determine necessary conditions for the quotient of a rigid analytic variety by a group to be again a rigid analytic variety.

In Section 2.4 we recall some basic facts about A-lattices in C.

Section 3. Let S be a finite set of global sections of an invertible sheaf on a rigid analytic variety Z over C. For any $T \subset S$ denote by $\Omega(T) \subset Z$ the intersection of the non-vanishing locus of T with the vanishing locus of $S \setminus T$. These $\Omega(T)$ for all $T \subset S$ form a *stratification* of Z, i.e., a covering of Z by pairwise disjoint, locally closed subvarieties. This section will be applied in the proof of Theorem 1.6 to the case where S consists of global sections of the analytification of the pullback of $\mathcal{O}_{Q_V}(1)$ to $Q_V(C)$.

In Section 3.1 we characterize the Grothendieck topology on Z in terms of this stratification. In Section 3.2 we describe, under some conditions, the normalization (in the sense of Conrad's [10]) of Z in terms of the stratification. The criterion obtained is analogous to a special case of [1, Theorem 9.2] by Baily and Borel in the complex analytic setting.

Section 4. In Sections 4.1 and 4.2 we provide a proof of Drinfeld's results [12, Prop. 6.1, 6.2] that $\Omega_\mathcal{V} \subset \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$ is admissible and that the quotient of $\Omega_\mathcal{V}$ by any discrete subgroup of $\mathrm{PGL}(\mathcal{V})$ is a normal rigid analytic variety.

In Section 4.3 we prove, inspired by van der Put's [38], a result on the connectedness of certain subsets of $\Omega_\mathcal{V}$. It implies that $\Omega_\mathcal{V}$ itself and hence its quotient by any discrete subgroup of $\mathrm{PGL}(\mathcal{V})$ is irreducible. It further provides for any point in Ω_Γ^* a fundamental set of irreducible admissible neighborhoods. When A is the polynomial ring, Kapranov [25, Proof of Prop. 1.18] already claimed the irreducibility of similar neighborhoods.

In Section 4.4 we suppose that $\dim_E(\mathcal{V}) > 1$ and consider a natural action on $\Omega_{\mathcal{V}}$ by any discrete subgroup of any codimension 1 subspace $\mathcal{W} \subset \mathcal{V}$. We prove that a certain map, defined using exponential functions, from its quotient to $\Omega_{\mathcal{W}} \times C$ is an open embedding of rigid analytic varieties.

Section 5. In Section 5.1 we endow Ω^*_Λ with a Grothendieck topology.

In Section 5.2 we study the induced Grothendieck topology on Ω^*_Γ and define the sheaves $\mathcal{O}^*_\Gamma$ and $\mathcal{R}^*_\Gamma$.

In Section 5.3 we define Eisenstein series as global sections of $\mathcal{O}^*_\Gamma(k)$.

In Section 5.4, we provide, building on Section 4.4, a comprehensive proof that any weak modular form has a Fourier expansion at any cusp.

In **Section 6** we construct the Satake compactification of any full analytic modular variety.

Section 7. In Section 7.1 we recall the notion of (generalized) Drinfeld module and Pink's compactifications of the algebraic moduli spaces.

In Section 7.2 we define A-reciprocal maps and prove Theorem 1.5.

In **Sections 8** and **9** we prove most of the above mentioned results.

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2. Preliminaries

Let C be an algebraically closed complete non-Archimedean valued field.

2.1. Grothendieck ringed spaces

Definition 2.1.

i) A family $\{U_i\}_{i\in I}$ of subsets U_i of a set U is called a *covering* of U if $U = \bigcup_{i\in I} U_i$.
ii) A covering $\{U'_j\}_{j\in J}$ of a set U is called a *refinement* of a covering $\{U_i\}_{i\in I}$ of U if there exists a map $\tau\colon J \to I$ with $U'_j \subset U_{\tau(j)}$ for any $j \in J$.
iii) The *intersection* of a covering $\{U_i\}_{i\in I}$ of a set U with a subset $U' \subset U$ is the covering $\{U_i \cap U'\}_{i\in I}$ of U'.

iv) The *intersection* of a covering $\{U_i\}_{i\in I}$ of a subset $U \subset X$ with a covering $\{U'_j\}_{j\in J}$ of a subset $U' \subset X$ is the covering $\{U_i \cap U'_j\}_{i\in I, j\in J}$ of $U \cap U'$.

v) The *preimage* of a covering $\{U_i\}_{i\in I}$ of a subset $U \subset X$ under a map $f\colon Y \to X$ is the covering $\{f^{-1}(U_i)\}_{i\in I}$ of $f^{-1}(U)$.

Definition 2.2. A *Grothendieck topology* on a set X consists of

- a system $\mathcal{S}$ of subsets of X and
- a family $\mathcal{C} = \{\mathrm{Cov}(U)\}_{U\in S}$ of systems of coverings, where $\mathrm{Cov}(U)$ contains coverings of U by elements in S for any $U \in S$,

subject to the following conditions:

i) $U, U' \in \mathcal{S} \Rightarrow U \cap U' \in \mathcal{S}$.
ii) $U \in \mathcal{S} \Rightarrow \{U\} \in \mathrm{Cov}(U)$.
iii) If $U \in \mathcal{S}, \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $\{U_{ij}\}_{j\in J_i} \in \mathrm{Cov}(U_i)$ for any $i \in I$, then $\{U_{ij}\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.
iv) If $U, U' \in \mathcal{S}$ with $U' \subset U$ and if $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, then $\{U_i \cap U'\}_{i\in I} \in \mathrm{Cov}(U')$.
v) $\emptyset, X \in \mathcal{S}$.
vi) If $U \in \mathcal{S}$ and $U' \subset U$ such that there exists $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ with $U_i \cap U' \in \mathcal{S}$ for any $i \in I$, then $U' \in \mathcal{S}$.
vii) Consider any $U \in \mathcal{S}$ and any covering $\{U_i\}_{i\in I}$ of U with $U_i \in \mathcal{S}$ for any $i \in I$. If $\{U_i\}_{i\in I}$ has a refinement in $\mathrm{Cov}(U)$, then it is itself in $\mathrm{Cov}(U)$.

If a Grothendieck topology $(\mathcal{S}, \mathcal{C})$ on X is understood, then the elements of $\mathcal{S}$ are called the *admissible* subsets of X and the elements of any $\mathrm{Cov}(U)$ are called the *admissible* coverings of U. In this case, the topology (in the usual sense) of X whose open sets are the unions of admissible sets, is called the *canonical topology* of X.

Definition 2.3. A morphism of Grothendieck topological spaces is a map under which the preimage of any admissible subset and of any admissible covering is admissible.

Definition 2.4. Consider any Grothendieck topological space X and any ring R.

i) A *presheaf* of (graded) R-algebras on X is a contravariant functor from the category of all admissible subsets of X with inclusions as morphisms into the category of (graded) R-algebras.
ii) Given any presheaf $\mathcal{F}$ on X, we denote by

$$\mathcal{F}(U) \to \mathcal{F}(U'),\ f \mapsto f|_{U'}$$

the morphism associated with any admissible subsets $U' \subset U \subset X$.

iii) A presheaf $\mathcal{F}$ on X is called a *sheaf* if any admissible subset U of X and any admissible covering $\mathcal{C}$ of U satisfy:

- If $f, g \in \mathcal{F}(U)$ are such that $f|_{U'} = g|_{U'}$ for any $U' \in \mathcal{C}$, then $f = g$.
- For any family $(f_{U'})_{U' \in \mathcal{C}} \in (\mathcal{F}(U'))_{U' \in C}$ with

$$\forall\, U', U'' \in C\colon\; f_{U'}|_{U' \cap U''} = f_{U''}|_{U' \cap U''}$$

there exists an $f \in \mathcal{F}(U)$ such that $f|_{U'} = f_{U'}$ for any $U' \in \mathcal{C}$.

iv) The morphisms between sheaves on X are the morphisms between the underlying presheaves.

Definition-Proposition 2.5. *[8, Proposition 9.2.2.4] Any presheaf $\mathcal{F}$ on a Grothendieck topological space admits a* sheafification*, i.e., a homomorphism $i\colon \mathcal{F} \to \mathcal{F}'$ into a sheaf $\mathcal{F}'$ such that any homomorphism $\mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$ equals $\varphi \circ i$ for a unique morphism $\varphi\colon \mathcal{F}' \to \mathcal{G}$.*

Definition 2.6. A *Grothendieck (graded) ringed space* over a ring R is a pair $(X, \mathcal{F})$, where X is a Grothendieck topological space and $\mathcal{F}$ is a sheaf of (graded) R-algebras on X.

Definition 2.7. A morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ of Grothendieck (graded) ringed spaces over a ring R is a pair $(f, f^{\#})$, where $f\colon X \to Y$ is a morphism of Grothendieck topological spaces and where $f^{\#}$ is a collection of (graded) R-algebra homomorphisms

$$f_U^{\#} : \mathcal{G}(U) \to \mathcal{F}(f^{-1}(U))$$

compatible with restriction homomorphisms, where U ranges over all admissible subsets of Y.

2.2. Rigid analytic varieties

Definition 2.8. A *C-algebra norm* on a C-algebra R is a map $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ which restricts to the norm on C such that every $r, s \in R$ satisfy

- $|r| = 0 \Leftrightarrow r = 0$,
- $|r \cdot s| \leq |r| \cdot |s|$,
- $|r - s| \leq \max\{|r|, |s|\}$.

Definition 2.9. A *C-Banach algebra* is a C-algebra R together with a C-algebra norm whose induced topology on R is complete.

Definition-Proposition 2.10. *[8, Proposition 5.1.1.1] For any integer $n \geq 0$ the* Tate algebra *over C in n variables is the subalgebra T_n of $C[[X_1, \ldots, X_n]]$ of elements*

$$f = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} \cdot X_1^{i_1} \cdot \cdots \cdot X_n^{i_n}$$

for which $|a_{i_1,\ldots,i_n}| \to 0$ as $i_1 + \cdots + i_n \to \infty$. The Gauss norm

$$|f| := \max_{i_1,\ldots,i_n \geq 0} |a_{i_1,\ldots,i_n}|$$

is a C-algebra norm on T_n by means of which T_n is a C-Banach algebra.

Definition 2.11. A C-Banach algebra R is called *C-affinoid* if there exists an integer $n \geq 0$ and a continuous epimorphism $T_n \to R$.

Definition 2.12.

i) A *C-affinoid variety* is a pair $\mathrm{Sp}(R) = (\mathrm{Max}(R), R)$, where R is any C-affinoid algebra and $\mathrm{Max}(R)$ is the *maximal spectrum* of R, i.e., the set of maximal ideals of R equipped with the Zariski topology.

ii) A *morphism* $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ of C-affinoid varieties is a pair $(\sigma, \sigma^{\#})$, where $\sigma^{\#} \colon R \to S$ is any C-algebra homomorphism and

$$\sigma \colon \mathrm{Max}(S) \to \mathrm{Max}(R), \mathfrak{m} \mapsto (\sigma^{\#})^{-1}(\mathfrak{m})$$

is the induced continuous map.

Definition-Proposition 2.13. *[8, Proposition 7.2.2.1]*

i) A morphism $(i, i^{\#}) \colon \mathrm{Sp}(R') \to \mathrm{Sp}(R)$ is called an open immersion *if for any morphism*

$$(\sigma, \sigma^{\#}) \colon \mathrm{Sp}(S) \to \mathrm{Sp}(R) \quad \text{with} \quad \sigma(\mathrm{Max}(S)) \subset i(\mathrm{Sp}(R'))$$

there exists a unique morphism $(\psi, \psi^{\#}) \colon \mathrm{Sp}(S) \to \mathrm{Sp}(R')$ with

$$(\sigma, \sigma^{\#}) = (i, i^{\#}) \circ (\psi, \psi^{\#}).$$

In this case, i is injective.

ii) Any composition of open immersions is an open immersion.

iii) A subset $U \subset \mathrm{Max}(R)$ is called affinoid *if it is the image of i of an open immersion $(i, i^{\#}) \colon \mathrm{Sp}(R') \to \mathrm{Sp}(R)$. In this case, U is (uniquely up to unique isomorphism) endowed with the structure of C-affinoid variety and we identify U with* $\mathrm{Sp}(R')$.

iv) The preimage of any affinoid subset under any morphism between C-affinoid varieties is an affinoid subset.

Definition-Proposition 2.14. *[8, Proposition 9.1.4.2] The following specifies a structure of Grothendieck topology on any C-affinoid variety* $\mathrm{Sp}(R)$*:*

i) A subset $X \subset \mathrm{Max}(R)$ is admissible *if it admits a covering C by affinoid subsets of $\mathrm{Max}(R)$ whose preimage under any morphism $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ has a finite refinement by affinoid subsets of $\mathrm{Max}(S)$. In particular, the union of any finitely many affinoid subsets of $\mathrm{Max}(R)$ is admissible.*

ii) A covering $\mathcal{C}$ of an admissible subset $X \subset \mathrm{Max}(R)$ by admissible subsets is admissible *if its preimage under any morphism $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$ has a finite refinement by affinoid subsets of $\mathrm{Max}(S)$.*

Definition-Proposition 2.15. *[8, Proposition 9.2.3.1] Consider any C-affinoid variety $Y = \mathrm{Sp}(R)$. Then there exists a unique sheaf $\mathcal{O}_Y$ of C-algebras on Y with $\mathcal{O}_Y(\mathrm{Sp}(R')) = R'$ for any affinoid subset $\mathrm{Sp}(R') \subset Y$ and such that for any composition of open immersions*

$$\mathrm{Sp}(R'') \xrightarrow{(j,j^{\#})} \mathrm{Sp}(R') \subset X$$

the restriction homomorphism $\mathcal{O}_Y(\mathrm{Sp}(R')) \to \mathcal{O}_Y(\mathrm{Sp}(R''))$ equals $j^{\#}$. In particular, the pair $(Y, \mathcal{O}_Y)$ is a Grothendieck ringed space over C.

Definition 2.16. A Grothendieck ringed space $(X, \mathcal{O})$ over C is a *rigid analytic variety* over C if X admits an admissible covering $\mathcal{C}$ and any $U \in \mathcal{C}$ possesses an isomorphism $(U, \mathcal{O}|_U) \cong (Y, \mathcal{O}_Y)$ of Grothendieck ringed spaces for some C-affinoid variety Y.

As C is algebraically closed, the elements of any affinoid C-algebra A uniquely give rise to functions $\mathrm{Sp}(A) \to C$ (see [8, Section 7.1]). The global sections of any rigid analytic variety $(X, \mathcal{O})$ over C may thus be viewed as the functions $f\colon X \to C$ whose restriction to any admissible affinoid subset $\mathrm{Sp}(A)$ are induced by elements of A.

Definition 2.17. Any such $f\colon X \to C$ is called *regular*.

Definition-Proposition 2.18. *For any affinoid varieties X, Y*

$$\mathrm{Mor}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Mor}(X, Y), \ (f, f^{\#}) \mapsto (f, f^{\#}_Y)$$

constitutes a bijection by means of which we view the category of C-affinoid varieties as a full subcategory of the category of rigid analytic varieties over C.

Proposition 2.19. *[8, Theorem 6.2.4.1] For any affinoid algebra A over C the map $A \to |C|$, $f \mapsto \sup_{x \in \mathrm{Sp}(A)} |f(x)|$ is a complete norm on A.*

Example 2.20. [8, Example 9.3.4.1] For any finite set S let $\mathbb{A}_C^{S,\mathrm{rig}}$ be the affine rigid analytic variety over C whose underlying set is C^S and for which the covering by all closed balls with radius in $|C|$ is admissible affinoid, where any such ball is naturally isomorphic to $\mathrm{Sp}(T_n)$, where $n = |S|$. In the case, where $S = \{1, \dots, n\}$ for some integer $n \geq 0$, we set $\mathbb{A}_C^{n,\mathrm{rig}} := \mathbb{A}_C^{S,\mathrm{rig}}$; the natural $\mathrm{GL}_n(C)$-action on it is through rigid analytic isomorphisms. If V is a C-vector space of finite dimension n, denote by $\mathbb{A}_V^{\mathrm{rig}}$ the rigid analytic variety over C with underlying set V such that one, and hence any, C-linear isomorphism $V \to C^n$ underlies an isomorphism $\mathbb{A}_V^{\mathrm{rig}} \to \mathbb{A}_C^{n,\mathrm{rig}}$.

Example 2.21. [8, Example 9.3.4.3] For any $n \geq 1$ the standard projective variety $\mathbb{P}_C^{n,\mathrm{rig}}$ over C is the unique rigid analytic variety over C whose underlying set is $(C^{n+1} \setminus \{0\})/C^\times$ and such that $\mathbb{A}_C^{n,\mathrm{rig}} \to \mathbb{P}_C^{n,\mathrm{rig}}$, $z \mapsto [(z,1)]$ is an open immersion and the natural $\mathrm{GL}_{n+1}(C)$-action on it is through rigid analytic isomorphisms. If V is a C-vector space of finite dimension $n+1$, denote by $\mathbb{P}_V^{\mathrm{rig}}$ the rigid analytic variety over C with underlying set $(V \setminus \{0\})/C^\times$ such that one, and hence any, C-linear isomorphism $V \to C^{n+1}$ induces an isomorphism $\mathbb{P}_V^{\mathrm{rig}} \to \mathbb{P}_C^{n,\mathrm{rig}}$.

Proposition 2.22. *Let Z be the product of any affine with any projective rigid analytic variety. Then the intersection of finitely many affinoid subsets of Z is again affinoid.*

Proof. Both affine and projective rigid analytic varieties and hence their products are separated in the sense of [8, Definition 9.6.1.1]. The proposition then holds by [8, Proposition 9.6.1.6]. $\square$

Definition 2.23. [8, Definition 9.5.2.1] A subset $Y \subset X$ of a rigid analytic variety X is called *analytic* if there exists an admissible covering $(U_i)_{i \in I}$ of X such that $Y \cap U_i$ is the zero-locus of finitely many elements in $\mathcal{O}_X(U_i)$ for all $i \in I$.

Definition 2.24. [8, Section 9.5.3] A morphism $f \colon Y \to X$ between rigid analytic varieties is a *closed immersion* if there exists an admissible affinoid covering $(U_i)_{i \in I}$ of X such that $f^{-1}(U_i)$ is affinoid and such that the ring homomorphism belonging to the restriction $f^{-1}(U_i) \to U_i$ of f is surjective for all $i \in I$.

Definition 2.25. A morphism of rigid analytic varieties is called a *locally closed immersion* if the underlying map is injective and the induced homomorphisms on stalks are surjective.

Proposition 2.26. *[8, Proposition 9.5.3.5] A morphism $f \colon Y \to X$ of rigid analytic varieties is a closed immersion if and only if*

i) it is a locally closed immersion,
ii) its image is an analytic subset of X and
iii) there exists an admissible affinoid covering $(X_i)_{i\in I}$ of X and, for each $i \in I$, a finite admissible affinoid covering of $f^{-1}(X_i)$.

Proposition 2.27. *(Maximum Modulus Principle) [8, Lemma 9.1.4.6] Consider any affinoid algebra A and any $f \in A$. Then there exists $c > 0$ with $|f(x)| \leq c$ for any $x \in X := \mathrm{Sp}(A)$. Moreover, if f vanishes nowhere on X, then there exists $\delta > 0$ with $|f(x)| \geq \delta$ for any $x \in X$.*

Theorem 2.28. *(Bartenwerfer's Riemann extension theorem) [2, Section 3] Consider any normal quasi-compact rigid analytic variety Y, any closed subvariety $Z \subsetneq Y$ which is everywhere of positive codimension and any regular function $s\colon Y \setminus Z \to C$. Then the following are equivalent:*

i) s extends uniquely to a regular function $Y \to C$,
ii) s extends uniquely to a morphism $Y \to \mathbb{A}_C^{1,\mathrm{rig}}$ of Grothendieck topological spaces whose restriction to Z is regular,
iii) s is bounded.

Proposition 2.29 *(Kisin). For any affinoid algebra A over C, any admissible $U \subset X := \mathrm{Sp}(A)$ and any $a_1, \ldots, a_n \in A$ whose common zeroes lie in U exists an $\varepsilon > 0$ such that $\{x \in X |\ \forall 1 \leq i \leq n\colon |a_i(x)| \leq \varepsilon\} \subset U$.*

Proof. See [11, after Remark 5.2.9] for Conrad's short proof via Berkovich spaces. □

2.3. On some quotients of rigid analytic varieties

Consider any group Γ of C-linear automorphisms of any rigid analytic variety Y over C. Let

$$p\colon Y \to \Gamma\backslash Y$$

be the quotient morphism, where $\Gamma\backslash Y$ is endowed with the structure of Grothendieck ringed space induced by the quotient map, that is, a subset (resp. a covering of a subset) of $\Gamma\backslash Y$ is admissible precisely when its preimage is admissible and the sections on an admissible subset of $\Gamma\backslash Y$ are the Γ-invariant sections on its preimage.

Proposition 2.30. *Suppose that $Y = \mathrm{Sp}(A)$ is the affinoid variety associated with any affinoid variety A and suppose that Γ is finite. Then the subalgebra $A^\Gamma \subset A$ of Γ-invariant elements is affinoid and induces an isomorphism of affinoid varieties*

$$\Gamma\backslash \mathrm{Sp}(A) \to \mathrm{Sp}(A^\Gamma).$$

Moreover, A is a finite A^Γ-module and if A is normal, then so is A^Γ.

Proof. See [21, Thm. 1.3] for the first and [20, Prop. 2.33] for the second part. □

We will use the following generalization of Proposition 2.30.

Proposition 2.31. *Suppose that Y is separated (see [8, Definition 9.6.1.1.]). Consider any admissible affinoid covering $(Y_n)_{n\geq 1}$ of Y and finite subgroups $(\Gamma_n)_{n\geq 1}$ of Γ such that*

i) $\forall n' \geq n \geq 1\colon \Gamma_n \subset \Gamma_{n'} \wedge Y_n \subset Y_{n'}$,
ii) $\forall n \geq 1, \forall \gamma \in \Gamma_n\colon \gamma(Y_n) = Y_n$
iii) and any $n \geq 1$ admits an $n' \geq 1$ such that $\forall \gamma \in \Gamma \setminus \Gamma_{n'}: \gamma(Y_n) \cap Y_n = \emptyset$.

Then $(p(Y_n))_{n\geq 1}$ is an admissible covering of $\Gamma\backslash Y$ and any $p(Y_n)$ is admissibly covered by finitely many affinoid varieties. In particular, $\Gamma\backslash Y$ is a rigid analytic variety. Moreover, if Y is normal, then so is $\Gamma\backslash Y$.

Proof. Consider any $n \geq 1$ and choose any $n' \geq n \geq 1$ satisfying the property in iii). Let I be a set of representatives of $\Gamma/\Gamma_{n'}$. Set

$$\forall \gamma \in I\colon U_\gamma := \bigcup_{\gamma' \in \Gamma_{n'}} (\gamma\gamma')(Y_n).$$

Then the U_γ are pairwise disjoint and they cover $U := p^{-1}(p(Y_n))$. We claim that $U \subset Y$ is admissible and admissibly covered by the U_γ and, in particular, that $p(Y_n) \subset \Gamma\backslash Y$ is admissible. In order to prove the claim, it is enough, since $(Y_k)_{k\geq 1}$ is an admissible covering of Y, to check for any $k \geq 1$ that $U \cap Y_k \subset Y_k$ is admissible and admissibly covered by $(U_\gamma \cap Y_k)_{\gamma\in I}$. Consider any such k. Since Y is separated, the intersection of any finitely many affinoid subsets of Y is again affinoid [8, Proposition 9.6.1.6]. As U_γ is the union of finitely many admissible affinoid subsets, thus so is $U_\gamma \cap Y_k$ for any $\gamma \in I$. Moreover, iii) provides a $k' \geq 1$ such that $U_\gamma \cap Y_k = \emptyset$ for any $\gamma \in I \setminus \Gamma_{k'}$. Hence $U \cap Y_k$ is the union of finitely many admissible affinoid subsets and hence an admissible subset of Y_k and the covering $(U_\gamma \cap Y_k)_{\gamma\in I}$ has the finite affinoid, and thus admissible, refinement $(U_\gamma \cap Y_k)_{\gamma\in I\cap\Gamma_{k'}}$ and is thus itself admissible. This yields the claim.

As $\Gamma_{n'}$ is finite and acts on the affinoid $Y_{n'}$ by ii), Proposition 2.30 yields that $\Gamma_{n'}\backslash Y_{n'}$ is an affinoid variety and that its admissible subsets are precisely those whose preimages in $Y_{n'}$ are admissible. Let $\gamma_0 \in I$ represent the identity. By i), U_{γ_0} is the union of finitely many affinoid subsets of $Y_{n'}$, and hence quasi-compact, and $\Gamma_{n'}$-invariant. Hence its image $\Gamma_{n'}\backslash U_{\gamma_0}$ in $\Gamma_{n'}\backslash Y_{n'}$ is an admissible quasi-compact subset or, equivalently, the union of finitely many admissible affinoid subsets. As the U_γ are pairwise disjoint and form an admissible covering, the inclusion morphism $U_{\gamma_0} \to U$ induces an isomorphism $\Gamma_{n'}\backslash U_{\gamma_0} \to \pi(Y_n)$ of Grothendieck ringed spaces. Thus $p(Y_n)$ is indeed admissibly covered by finitely many affinoid varieties. Moreover, if Y is normal, then so is $Y_{n'}$ and hence $\Gamma_{n'}\backslash Y_{n'}$ by Proposition 2.30 and hence $\Gamma_{n'}\backslash U_{\gamma_0}$ and hence $p(Y_n)$.

It remains to be checked that the covering $(p(Y_n))_{n\geq 1}$ of $\Gamma\backslash Y$ is admissible. Using that $(Y_k)_{k\geq 1}$ is an admissible covering of Y, it suffices to check for any $k \geq 1$ that the covering $(p^{-1}(p(Y_n)) \cap Y_k)_{n\geq 1}$ of Y_k is admissible. But the latter covering has as admissible refinement the covering given by the single subset $p^{-1}(p(Y_k)) \cap Y_k$, i.e., by Y_k, and is thus itself admissible. $\square$

2.4. On lattices over admissible coefficient subrings

Suppose that the characteristic of C is finite.

Definition 2.32. A subset $S \subset C$ is called *strongly discrete* if its intersection with every ball of finite radius is finite.

Definition 2.33. We call a subring $A \subset C$ an *admissible coefficient subring* if it is strongly discrete and if it is a Dedekind domain that is finitely generated over a finite subfield of C.

Any subring $A \subset C$ as in the introduction is an admissible coefficient subring and vice versa (see for instance Harder's [22, Vol. 2, Sect. 9.1-3]).

Example 2.34. Consider any finite subfield $\mathbb{F}_q \subset C$ and any $t \in C$ with $|t| > 1$. Then $\mathbb{F}_q[t]$ is a polynomial ring over $\mathbb{F}_q$ and an admissible coefficient subring of C.

Proof. As the norm of C is non-Archimedean, as $|x| = 1$ for any $0 \neq x \in \mathbb{F}_q$ and as $|t| > 1$, any polynomial of degree $n \geq 0$ over $\mathbb{F}_q$ evaluated at t has norm $|t|^n$ in C. This implies both that $\mathbb{F}_q[t]$ is a polynomial ring and that it is strongly discrete in C. That any polynomial ring in one variable over a field is a Dedekind domain, is a classical fact. $\square$

Definition 2.35. Consider any admissible coefficient subring $A \subset C$ and let E be the completion of its quotient field. A finitely generated projective A-submodule $\Lambda \subset C$ is an *A-lattice* if the natural homomorphism $\Lambda \otimes_A E \to C$ is injective.

Proposition 2.36. *Any A-lattice $\Lambda \subset C$ is strongly discrete.*

Proof. See for instance [20, Prop. 2.51]. $\square$

Definition 2.37. Consider any admissible coefficient subring $A \subset C$, any projective A-module Λ of finite rank $d > 0$ and any norm $|\cdot|$ on Λ_E in the sense of Section 4.1, where E is the completion of the quotient field of A. For any $1 \leq i \leq d$ call

$$\mu_i(\Lambda) := \inf\{\max\{|\lambda_1|, \ldots, |\lambda_i|\} \mid \lambda_1, \ldots, \lambda_i \in \Lambda \text{ linearly independent}\}$$

the i-th *successive minimum* of Λ. Set $\mu_{\max}(\Lambda) := \mu_d(\Lambda)$.

Definition-Proposition 2.38. *Let $A = \mathbb{F}_q[t]$ be as in Example 2.34. Consider any A-module Λ and any norm $|\cdot|$ as in Definition 2.37. Then there exists a* minimal reduced *basis of Λ, i.e., an ordered basis $(\lambda_1, \dots, \lambda_d)$ of Λ such that $(|\lambda_1|, \dots, |\lambda_d|) = (\mu_1(\Lambda), \dots, \mu_d(\Lambda))$ and such that*

$$\forall a \in A^d \colon \left| \sum_{1 \le i \le d} a_i \lambda_i \right| = \max_{1 \le i \le d} |a_i| \cdot |\lambda_i|.$$

Moreover, $|\lambda_i| = \inf_{\lambda \in \Lambda} |\lambda_i + t \cdot \lambda|$ for any λ_i in any such basis.

Proof. Up to the last assertion, this is [7, Theorem 2.2.8]. Consider then any λ_i in any minimal reduced basis $(\lambda_1, \dots, \lambda_n)$ of Λ. Any $\lambda = \sum_{1=j}^{d} a_j \cdot \lambda_j \in \Lambda$ then satisfies as desired that

$$|\lambda_i + t\lambda| = \max_{j \ne i} \{ |(1 + t \cdot a_i) \cdot \lambda_i|, |t \cdot a_j \cdot \lambda_j| \} \ge |(1 + t \cdot a_i) \cdot \lambda_i| \ge |\lambda_i|. \quad \square$$

For any subset S of any normed vector space $\mathcal{V}$ set

$$\mathrm{d}(S) := \inf_{0 \ne s \in S} |s|;$$

this measures the distance of $S \setminus \{0\}$ to the origin.

Corollary 2.39. *Consider any $A = \mathbb{F}_q[t]$, any Λ and any norm on $\Lambda \otimes_A E$ as in Definition-Proposition 2.38. Consider any direct summand $0 \ne L \subset \Lambda$. Consider the projection $\pi \colon t^{-1}\Lambda \to \overline{\Lambda} := t^{-1}\Lambda/\Lambda$ and set $\overline{L} := t^{-1}L/L \subset \overline{\Lambda}$. Then*

$$\max_{\alpha \in \overline{L}} \mathrm{d}(\pi^{-1}(\alpha)) \le \mu_{\max}(L). \tag{3}$$

Moreover, choose a minimal reduced basis $(\lambda_1, \dots, \lambda_n)$ of $t^{-1}\Lambda$. Let $L' \subset t^{-1}\Lambda$ be the submodule generated by the λ_i with $|\lambda_i| < \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L}))$. Set $\overline{L}' := \pi(L')$. If

$$\max_{\alpha \in \overline{L}} \mathrm{d}(\pi^{-1}(\{\alpha\})) < \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L})), \tag{4}$$

then $\overline{L}' = \overline{L}$ and $\mathrm{d}(t^{-1}\Lambda \setminus L') = \mathrm{d}(\pi^{-1}(\overline{\Lambda} \setminus \overline{L}))$.

Proof. This is directly checked. For details, see [20, Cor. 2.54]. $\square$

Definition-Proposition 2.40. *For any strongly discrete subgroup $\Lambda \subset C$ the formula*

$$e_\Lambda(T) := T \cdot \prod_{0 \ne \lambda \in \Lambda} \left(1 - \frac{T}{\lambda}\right)$$

defines a morphism $e_\Lambda\colon \mathbb{A}^{1,\mathrm{rig}}_C \to \mathbb{A}^{1,\mathrm{rig}}_C$ *that is a surjective homomorphism with kernel* Λ*. Moreover,*

$$\forall c \in C \setminus \Lambda\colon \frac{1}{e_\Lambda(c)} = \sum_{\lambda\in\Lambda} \frac{1}{c+\lambda}.$$

Proof. This is explained for instance in [13, Chapter 2, Section 1] up to the last part. The last part follows from logarithmic differentiation using that $\frac{d}{dT}\exp_\Lambda(T) = 1$. □

Proposition 2.41. *Consider any* A*-lattice* $\Lambda \subset C$ *and any* $0 \neq c, c' \in C$ *such that* $|c| < |\lambda|$ *and* $|c'| \leq |c' + \lambda|$ *for every* $0 \neq \lambda \in \Lambda$*. Then*

$$\left|\frac{c'}{c}\right| \leq \left|\frac{e_\Lambda(c')}{e_\Lambda(c)}\right| \leq \left|\frac{c'}{c}\right|^{\left|\frac{c'}{c}\right|\cdot q\cdot \mathrm{rank}_{\mathbb{F}_q[t]}(\Lambda)}$$

for any polynomial ring $\mathbb{F}_q[t] \subset A$ *over any finite field with* q *elements.*

Proof. Use Definition-Proposition 2.38. For details, see [20, Prop. 2.56]. □

3. On stratifications of rigid analytic varieties by global sections

Consider any reduced rigid analytic variety Z over an algebraically closed complete non-Archimedean field C and any finite set S of global sections of an invertible sheaf on Z. With any $T \subset S$ and any $\varepsilon \in |C^\times|$ associate the reduced Zariski open, resp. admissible, resp. locally closed subvariety

$$\begin{aligned}
\mathcal{U}(T) &:= \{z \in Z \mid \forall t \in T : t(z) \neq 0\} \subset Z,\\
\mathcal{U}(T,\varepsilon) &:= \{z \in \mathcal{U}(T) \mid \forall s \in S \setminus T,\ \forall t \in T : \left|\frac{s}{t}(z)\right| \leq \varepsilon\} \subset \mathcal{U}(T),\\
\Omega(T) &:= \{z \in \mathcal{U}(T) \mid \forall s \in S \setminus T,\ \forall t \in T : \frac{s}{t}(z) = 0\} \subset Z.
\end{aligned}$$

This yields a stratification of Z by locally closed subvarieties

$$Z = \dot{\bigcup_{T\subset S}} \Omega(T).$$

3.1. Characterization of the Grothendieck topology

Proposition 3.1. *A subset* $X \subset Z$ *is admissible if and only if any* $T \subset S$ *with* $\Omega(T) \neq \emptyset$ *satisfies that*

i) the subset $X \cap \Omega(T) \subset \Omega(T)$ *is admissible and that* *ii)* *any admissible quasi-compact* $U \subset \mathcal{U}(T)$ *with* $U \cap \Omega(T) \subset X$ *admits an* $\varepsilon \in |C^\times|$ *with* $U \cap \mathcal{U}(T,\varepsilon) \subset X$*.*

Moreover, a covering of an admissible $X \subset Z$ *by admissible subsets is admissible if and only if its intersection with* $X \cap \Omega(T)$ *is admissible for any* $T \subset S$*.*

Proof. This is essentially a formal consequence of Proposition 2.29. For details, see [20, Proposition 3.1]. □

For the remainder of this section further assume for any $\Omega(T) \neq \emptyset$ the existence and choice of a morphism

$$\rho_T : \mathcal{U}(T) \to \Omega(T) \tag{5}$$

such that $\rho_T|_{\Omega(T)} = \mathrm{id}_{\Omega(T)}$ and such that

$$\mathcal{U}(O,\varepsilon) := \rho_T^{-1}(O) \cap \mathcal{U}(T,\varepsilon)$$

is quasi-compact for any quasi-compact $O \subset \Omega(T)$ and any $\varepsilon \in |C^\times|$.

Example 3.2. Let for example S be a C-basis of the global sections of the first twisting sheaf of any standard projective space Z over C and let ρ_T be the natural projection for any $\emptyset \neq T \subset S$. Consider for any $t \in T \subset S$ the isomorphism

$$i_t : \mathcal{U}(T) \to \Omega(T) \times \mathbb{A}_C^{(S\setminus T),\mathrm{rig}}, q \mapsto \left(\rho_T(q), (\frac{s}{t}(q))_{s \in S\setminus T}\right).$$

For any $\emptyset \neq T \subset S$, any $O \subset \Omega(T)$ and any $\varepsilon \in |C^\times|$ then

$$\mathcal{U}(O,\varepsilon) = \bigcap_{t \in T} i_t^{-1}(O \times B_\varepsilon).$$

In particular, such $\mathcal{U}(O,\varepsilon)$ is quasi-compact, resp. affinoid, whenever O is.

Proposition 3.1 may then be reformulated as follows.

Corollary 3.3. *Consider any* Z*,* S *and morphisms* ρ_T *as in (5). Let* $Y \subset Z$ *be any closed subvariety. Then a subset* $X \subset Y$ *is admissible if and only if any* $T \subset S$ *with* $\Omega(T) \cap Y \neq \emptyset$ *satisfies that*

i) *the subset* $X \cap \Omega(T) \subset Y \cap \Omega(T)$ *is admissible and that*
ii) *any admissible quasi-compact* $O \subset \Omega(T)$ *with* $O \cap Y \subset X$ *admits an* $\varepsilon(O) \in |C^\times|$ *with* $\mathcal{U}(O,\varepsilon(O)) \cap Y \subset X$*.*

Moreover, a covering of an admissible $X \subset Y$ by admissible subsets is admissible if and only if its intersection with $X \cap \Omega(T)$ is admissible for any $T \subset S$.

Proof. See [20, Cor. 3.4]. □

Corollary 3.4. *Consider any rigid analytic variety R and any integer $n \geq 0$. Let $Y \subset R \times \mathbb{A}_C^{n,\mathrm{rig}}$ be any closed subvariety. Then a subset $X \subset Y$ is admissible if and only if*

i) the subset $X \setminus R \times \{0\} \subset Y \setminus R \times \{0\}$ is admissible,
ii) the subset $X \cap R \times \{0\} \subset Y \cap R \times \{0\}$ is admissible and
iii) for any admissible quasi-compact $O \subset R$ with $O \times \{0\} \subset X$ exists an $\varepsilon > 0$ such that $(O \times B_\varepsilon) \cap Y \subset X$.

Moreover, a covering of an admissible subset $X \subset Y$ by admissible subsets is admissible if and only if both its intersection with $X \setminus R \times \{0\}$ and its intersection with $X \cap R \times \{0\}$ is admissible.

Proof. See [20, Cor. 3.5]. □

Corollary 3.5. *Let R be any separated rigid analytic variety. Consider any admissible subset $X \subset R \times \mathbb{A}_C^{1,\mathrm{rig}}$ and any regular function $s\colon X \setminus R \times \{0\} \to C$. Then there exist unique regular functions $s_i\colon X \cap R \times \{0\} \to C$ such that*

$$s((o,z)) = \sum_{i \in \mathbb{Z}} s_i(o,0) z^i \quad \text{for any } (o,z) \in O \times B_\varepsilon \setminus O \times \{0\}$$

for any admissible affinoid $O \times \{0\} \subset X \cap R \times \{0\}$ and any $\varepsilon \in |C^\times|$ with $O \times B_\varepsilon \subset X$. Moreover, the following statements are equivalent:

i) s extends to a regular function $X \to C$.
ii) s extends to a morphism $X \to \mathbb{A}_C^{1,\mathrm{rig}}$ of Grothendieck topological spaces whose restriction to $X \cap R \times \{0\}$ is regular.
iii) Any admissible affinoid $O \times \{0\} \subset X \cap R \times \{0\}$ admits an $\varepsilon \in |C^\times|$ with $O \times B_\varepsilon \subset X$ and such that s is bounded on $O \times B_\varepsilon \setminus O \times \{0\}$.
iv) $\forall i < 0 : s_i = 0$.

Moreover, the extension in i), resp. ii), is unique if it exists.

Proof. Via Corollary 3.4 and Proposition 2.22, the corollary is reduced to the case where R is affinoid, in which it is Bartenwerfer's [2, Satz 12]. For details, see [20, Cor 3.6]. □

Proposition 3.6. *Let Z and S and the ρ_T be as in Example 3.2. Consider the natural left-action on Z of any subgroup G of the symmetric group of S. Then for any G-invariant* *closed subvariety $Y \subset Z$ the quotient $G\backslash Y$ is a rigid analytic variety and it is normal if Y is.*

Proof. For any $0 \neq T \subset S$ and any $r \in |C|$ set

$$O(T,r) := \left\{ z \in \Omega(T) \mid \forall t,t' \in T : \left| \frac{t'}{t}(z) \right| \leq r \right\} \subset \Omega(T)$$

and for any further $\varepsilon \in |C^\times|$ set $\mathcal{U}(T,r,\varepsilon) := \mathcal{U}(O(T,r),\varepsilon) \subset \mathcal{U}(T)$. By Example 3.2 and the construction, any such $\mathcal{U}(T,r,\varepsilon)$ is a G_T-invariant admissible affinoid subvariety of Z, where G_T denotes the stabilizer of T in G. Fix any $1 > \varepsilon \in |C^\times|$. The construction yields for any $T' \subset S$ with $T' \not\subset T \not\subset T'$ and any $g \in G$ that

$$\mathcal{U}(T,\varepsilon) \cap \mathcal{U}(T',\varepsilon) = \emptyset \quad \text{and that} \quad g(\mathcal{U}(T,r,\varepsilon)) = \mathcal{U}(g(T),r,\varepsilon).$$

Hence the G-invariant subvariety

$$G(\mathcal{U}(T,r,\varepsilon)) = \bigcup_{g \in G} \mathcal{U}(g(T),r,\varepsilon) \subset Z$$

is a disjoint union of finitely many admissible affinoids; in particular, it is itself admissible affinoid. Finally, let $\mathcal{C}$ be the covering of Z by the $G(\mathcal{U}(T,r,\varepsilon))$ for varying $\emptyset \neq T \subset S$ and $r \in |C|$. Its intersection with any $\Omega(T) \neq \emptyset$ is refined by the admissible covering of $\Omega(T)$ by the $O(T,r)$, for varying r, so that it is itself admissible. By Proposition 3.3, thus $\mathcal{C}$ is admissible. In particular, the intersection of $\mathcal{C}$ with any G-invariant closed subvariety $Y \subset Z$ is an admissible covering by G-invariant affinoids. The proposition then follows from Proposition 2.30. □

3.2. Stratification and normalization

Consider first a general reduced rigid analytic variety X. We refer to Conrad's [10] for the definition of the normalization of X and a proof that it uniquely exists. Conrad uses it to define the irreducible components of X as the images of the connected components under the normalization morphism [10, Def. 2.2.2]. The irreducible components are then the maximal irreducible Zariski closed subsets of X [10, Thm. 2.2.4.(2)]. If X is the analytification of an algebraic variety X' over C, then its normalization, resp. its irreducible components, are the analytification of the normalization of X', resp. of the irreducible components of X' [10, Thm. 2.1.3, resp. 2.3.1].

Denote then by $\mathcal{O}_Z$ the structure sheaf of Z. Consider the Grothendieck ringed space $(Z, \tilde{\mathcal{O}}_Z)$ whose underlying Grothendieck topological space coincides with the one underlying $(Z, \mathcal{O}_Z)$ and whose section on any admissible $U \subset Z$ are precisely the functions $f : U \to C$ that are continuous with respect to the canonical topologies, that are bounded on any admissible affinoid subset of U and that restrict to regular functions $U \cap \Omega(T) \to C$ for any $T \subset S$. Consider the morphism of Grothendieck ringed spaces

$$n_Z\colon (Z, \tilde{\mathcal{O}}_Z) \to (Z, \mathcal{O}_Z) \tag{6}$$

whose underlying topological morphism is the identity and whose homomorphism $\mathcal{O}_Z(U) \to \tilde{\mathcal{O}}_Z(U)$ for any admissible $U \subset Z$ is the natural injection by means of the Maximum Modulus Principle, i.e., Proposition 2.27.

Theorem 3.7. *Suppose that*

i) Z is irreducible,
ii) the Zariski open subvariety $\Omega(S) \subset Z$ is normal,
iii) $Z \setminus \Omega(S)$ is of everywhere positive codimension in Z.
iv) any function $f\colon X \to C$ on any admissible $X \subset Z$ which is continuous with respect to the canonical topology and restricts to a regular function on $X \cap \Omega(S)$ restricts to a regular function on $X \cap \Omega(T)$ for any $T \subset S$ and
v) any $z \in Z$ has a fundamental basis of admissible neighborhoods U such that $U \cap \Omega(S)$ is connected and, in particular, non-empty.

Then n_Z is the normalization morphism in the sense of Conrad [10]. In particular, $(Z, \tilde{\mathcal{O}}_Z)$ is a normal rigid analytic variety.

Proof. Consider the normalization morphism

$$(n, n^{\#})\colon (\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to (Z, \mathcal{O}_Z).$$

We shall show that $(n, n^{\#})$ induces an isomorphism

$$(n, n^{+})\colon (\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to (Z, \tilde{\mathcal{O}}_Z)$$

whose composition with n_Z is $(n, n^{\#})$; this will then yield the theorem.

For any $T \subset S$ let $\tilde{T} := n^{+}(T)$ denote the set of global sections on $\tilde{Z}$ obtained by pulling back the elements of T by (n, n^{+}). Analogously as for Z and S, this yields a stratification of $\tilde{Z}$ by reduced locally closed subvarieties $\Omega(\tilde{T}) \subset \tilde{Z}$ for various $T \subset S$. Then $\Omega(\tilde{T}) = n^{-1}(\Omega(T))$ for any $T \subset S$; let

$$(n_T, n_T^{\#})\colon \Omega(\tilde{T}) \to \Omega(T)$$

be the morphism induced by $(n, n^{\#})$. Abbreviate $\Omega := \Omega(S)$ and $\tilde{\Omega} := \Omega(\tilde{S})$.

We first show that n is bijective. As it underlies a normalization morphism, it is surjective. We then consider any $z \in Z$ and claim that $|n^{-1}(z)| = 1$. As any normalization morphism is finite, $n^{-1}(z)$ is finite. From Proposition 2.29 and the fact that A is noetherian (see [8, Prop. 6.1.1.3]) then follows that the natural homomorphism

$$(n_* \mathcal{O}_{\tilde{Z}})_z \to \prod_{y \in n^{-1}(z)} \mathcal{O}_{\tilde{Z},y}$$

is an isomorphism (for more details see [10, Proof of Thm. 1.1.3] or [20, Lemma 3.9]). It thus suffices to show that $(n_*\mathcal{O}_{\tilde{Z}})_z$ is integral. As $\Omega(S)$ is normal by ii), its irreducible and its connected subsets coincide. Assumptions v) and ii) thus provide a fundamental system $\mathcal{F}$ of admissible open neighborhoods $U \subset Z$ of z such that $U \cap \Omega$ is irreducible or, equivalently, such that $\mathcal{O}_Z(U \cap \Omega)$ is integral. As Ω is normal by ii), $(n_S, n_S^{\#})$ is an isomorphism so that $(n_*\mathcal{O}_{\tilde{Z}})(U \cap \Omega)$ is integral, too, for any $U \in \mathcal{F}$. Assumption i) implies that $\tilde{Z}$ is irreducible. Assumption v) implies that $\Omega \neq \emptyset$ if $Z \neq \emptyset$. Thus the Zariski open subvariety $\tilde{\Omega}$ of the irreducible $\tilde{Z}$ is dense. Consequently, the restriction homomorphism

$$(n_*\mathcal{O}_{\tilde{Z}})(U) \to (n_*\mathcal{O}_{\tilde{Z}})(U \cap \Omega)$$

is injective for any $U \in \mathcal{F}$ so that, in fact, $(n_*\mathcal{O}_{\tilde{Z}})(U)$ is integral. Since $\mathcal{F}$ is a fundamental system of admissible neighborhoods of z, this implies that $(n_*\mathcal{O}_{\tilde{Z}})_z$ is indeed integral. We have thus shown that n is bijective.

Since, furthermore, $(n, n^{\#})$ is finite, n is a homeomorphism with respect to the canonical topologies by [8, Lemma 9.5.3.6].

Let us then define n^+. Consider first any admissible affinoid $U \subset Z$ and set $\tilde{U} := n^{-1}(U)$. Let $n^+(U)$ be the composition

$$\tilde{\mathcal{O}}_Z(U) \hookrightarrow \tilde{\mathcal{O}}_Z(U \cap \Omega)^b = \mathcal{O}_Z(U \cap \Omega)^b \xrightarrow{\cong} \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega})^b \xrightarrow{\cong} \mathcal{O}_{\tilde{Z}}(\tilde{U}),$$

where $(\cdot)^b$ denotes the operator that associates the subalgebra of bounded elements and where the arrows are defined as follows: The first arrow is the restriction homomorphism; it is injective since $\Omega \subset Z$ is dense. As Ω is normal, the homomorphism $n_S^{\#}(U \cap \Omega)$ is an isomorphism. The second arrow is the restriction of this isomorphism to the subalgebra of bounded elements. Finally, we claim that the restriction homomorphism

$$R := \mathcal{O}_{\tilde{Z}}(\tilde{U}) \to \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega}) =: S$$

induces an isomorphism onto S^b; the last arrow is then defined to be the induced inverse. As n is finite and U is affinoid, its preimage $\tilde{U}$ is affinoid too by [8, Proposition 9.4.4.1]. The Maximum modulus principle thus yields the boundedness of any element in R and hence of its image in S. Conversely, any element in S^b extends uniquely to an element in R: Indeed, by normality of $\tilde{U}$ and the Riemann extension theorem (see Theorem 2.28), this holds true if $\tilde{U} \backslash \tilde{\Omega}$ is of everywhere positive codimension in $\tilde{U}$. But the latter condition is guaranteed by iii) since n is finite. This shows the claim and thus finishes the definition of $n^+(U)$.

In fact, $n^+(U)$ is surjective and hence, by the above, an isomorphism. Indeed, consider any $\tilde{f} \in \mathcal{O}_{\tilde{Z}}(\tilde{U})$. As n is a homeomorphism,

$$f := \tilde{f} \circ n^{-1}|_U : U \to C$$

is continuous with respect to the canonical topologies. As $\mathcal{O}_{\tilde{Z}}(\tilde{U})$ is affinoid, the Maximum Modulus Principle (see Proposition 2.27) yields that $\tilde{f}$, and hence f, is bounded. In order to show that $f \in \tilde{\mathcal{O}}_Z(U)$, it remains to be checked that the restriction f_T of f to $U \cap \Omega(T)$ is regular for any $T \subset S$. Since f_S corresponds to the restriction of $\tilde{f}$ to $\tilde{U} \cap \tilde{\Omega}$ via the isomorphism $\mathcal{O}_Z(U \cap \Omega)^b \to \mathcal{O}_{\tilde{Z}}(\tilde{U} \cap \tilde{\Omega})^b$, it is regular. The regularity of an arbitrary f_T then follows from Assumption iv). Hence $n^+(U)$ is indeed surjective.

For an arbitrary admissible subset $X \subset Z$, the homomorphism $n^+(X)$ is then defined in the natural way by means of the admissible covering of X by all its admissible affinoid subsets using the sheaf property of $\mathcal{O}_{\tilde{Z}}$; by the affinoid case above, it is an isomorphism as well.

It remains to be shown that n is an isomorphism of Grothendieck topological spaces. We first consider any $T \subset S$ with $\Omega(T) \neq \emptyset$ and show that $(n_T, n_T^\#)$ is an isomorphism. As $(n, n^\#)$ is finite and a homeomorphism with respect to the canonical topologies, so is $(n_T, n_T^\#)$. In order to see that the latter is an isomorphism, it thus suffices, by Proposition 2.26, to show that $n_T^\#$ induces isomorphisms on stalks. Consider any $z \in \Omega(T)$ and set $\tilde{z} := n_T^{-1}(z)$. As n_T is surjective and $\Omega(T)$ is reduced, the homomorphism on stalks $\mathcal{O}_{\Omega(T),z} \to \mathcal{O}_{\Omega(\tilde{T}),\tilde{z}}$ is injective. In order to see that it is also surjective, consider any $\tilde{g} \in \mathcal{O}_{\Omega(\tilde{T}),\tilde{z}}$ and choose, using that n is a homeomorphism, an admissible affinoid $U \subset \mathcal{U}(T)$ containing z such that $\tilde{g}$ is defined on $\tilde{U} \cap \Omega(\tilde{T})$, where $\tilde{U} := n^{-1}(U)$. As n is finite, also $\tilde{U}$ is affinoid. Thus we may and do choose an $\tilde{f} \in \mathcal{O}_{\tilde{Z}}(\tilde{U})$ that restricts to $\tilde{g}$ on the Zariski closed affinoid subvariety $\tilde{U} \cap \Omega(\tilde{T}) \subset \tilde{U}$. Let $f \in \tilde{\mathcal{O}}_Z(U)$ correspond to $\tilde{f}$ under the isomorphism $n^+(U)$ discussed above. In particular, f restricts to a regular function g on $U \cap \Omega(T)$. By continuity of n and the construction, then $n_T^\#(g) = \tilde{g}$. This yields surjectivity of the above map on stalks. We have thus shown that $(n_T, n_T^\#)$ is an isomorphism.

That n is an isomorphism, then follows from the fact that the preimage under the finite morphism $(n, n^\#)$ of any quasi-compact is quasi-compact and from applying Proposition 3.1 once as stated and once to Z and S replaced by $\tilde{Z}$ and $\tilde{S}$ using that $\mathcal{U}(\tilde{T}, \varepsilon) = n^{-1}(\mathcal{U}(T, \varepsilon))$ for any $T \subset S$ and any $\varepsilon \in |C^\times|$. This finishes the proof. $\square$

4. Quotients of Drinfeld's period domain by discrete groups

Consider any non-Archimedean local field E. Denote by $\mathcal{O}_E$ the ring of integers of E. Choose a prime element $\pi \in \mathcal{O}_E$ and set $q := |\frac{1}{\pi}|$. Consider any finite dimensional E-vector space $\mathcal{V} \neq 0$. Set $\mathcal{G} := \mathrm{Aut}_E(\mathcal{V})$ and $\mathcal{PG} := \mathrm{PGL}(\mathcal{V}) = \mathcal{G}/E^\times$. Let C be any algebraically closed complete non-Archimedean valued field containing E as a valued subfield. Let $\mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$ be the projective rigid analytic variety over C with underlying set

$$(\mathrm{Hom}_C(\mathcal{V}_C, C) \setminus \{0\})/C^\times$$

(see Example 2.21). Drinfeld's *period domain* is the $\mathcal{PG}$-invariant subset

$$\Omega_{\mathcal{V}} \subset \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$$

of those $C^\times$-classes $[l]$ of C-linear maps $l\colon \mathcal{V}_C \to C$ for which $\mathrm{Ker}(l) \cap \mathcal{V} = 0$.

Most parts of Sections 4.1 and 4.2 may in fact be done more generally when $\mathrm{Hom}_C(\mathcal{V}_C, C)$ is replaced by $\mathrm{Hom}_C(\mathcal{V}_C, C^k)$ for any $k \geq 1$ but to the cost of additional complexity in proofs (see [20, Sections 5.1 and 5.2]).

4.1. Admissibility of the period domain

Denote by $\mathrm{N}_\mathcal{V}$ the set of *norms* on $\mathcal{V}$, i.e., of maps $\nu\colon \mathcal{V} \to \mathbb{R}_{\geq 0}$ for which

- $\forall e \in E, \forall v \in \mathcal{V}\colon \nu(e \cdot v) = |e| \cdot \nu(v)$,
- $\forall v, v' \in \mathcal{V}\colon \nu(v + v') \leq \max\{\nu(v), \nu(v')\}$,
- $\forall v \in \mathcal{V}\colon \nu(v) = 0 \Leftrightarrow v = 0$.

Proposition 4.1. *[19, Prop. 1.1] Any $\nu \in \mathrm{N}_\mathcal{V}$ admits a basis β of $\mathcal{V}$ such that*

$$\forall (e_w)_{w\in\beta} \in E^\beta\colon \nu\left(\sum_{w\in\beta} e_w \cdot w\right) = \max_{w\in\beta}|e_w| \cdot \nu(w).$$

The natural action of $\mathbb{R}_{>0}$ on $\mathbb{R}_{\geq 0}$ induces an action on $\mathrm{N}_\mathcal{V}$. Set

$$\forall [\nu], [\nu'] \in \mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V}\colon \rho([\nu], [\nu']) := \max_{0\neq v, v' \in \mathcal{V}} \frac{\nu'(v)}{\nu(v)} \cdot \frac{\nu(v')}{\nu'(v')}.$$

It is directly checked that the induced map $\rho\colon \mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V} \times \mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V} \to \mathbb{R}_{\geq 1}$ is a metric, i.e., that $\rho(x, y) = 1 \Leftrightarrow x = y$ and $\rho(x, y) \leq \rho(x, z) \cdot \rho(z, y)$ for all $x, y, z \in \mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V}$,

Proposition 4.2. *[19, Theorem 2.3] With respect to the metric ρ, any closed ball in $\mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V}$ is compact.*

Denote by $U_\nu \subset \mathcal{V}$ the unit ball with respect to any $\nu \in \mathrm{N}_\mathcal{V}$. Denote by $\mathrm{S}_\mathcal{V}$ the set of free $\mathcal{O}_E$-submodules $m \subset \mathcal{V}$ of maximal rank. It is naturally acted by $E^\times$ by dilation.

Definition-Proposition 4.3. *For any $m \in \mathrm{S}_\mathcal{V}$ denote by $\nu_m \in \mathrm{N}_\mathcal{V}$ the norm $v \mapsto \inf\{|e| \in |E^\times|\colon v \in e \cdot m\}$; then $U_{\nu_m} = m \setminus \pi m$ and for arbitrary $\nu' \in \mathrm{N}_\mathcal{V}$:*

$$\rho([\nu'], [\nu]) = \frac{\max_{u\in U_{\nu_m}} \nu'(u)}{\min_{u\in U_{\nu_m}} \nu'(u)}. \tag{7}$$

Proof. This is directly checked. □

Consider the map $\lambda_\mathcal{V}\colon \Omega_\mathcal{V} \to \mathbb{R}_{>0}\backslash \mathrm{N}_\mathcal{V}$ that sends any $[l]$ to the class of the norm $v \mapsto |l(v)|$.

Proposition 4.4. *(Drinfeld [12, Prop. 6.1]) Let $m_1, \ldots, m_t \in \mathrm{S}_{\mathcal{V}}$. Set*

$$X(c) := \lambda_{\mathcal{V}}^{-1}\left(\left\{[\nu] \in \mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}} \colon \prod_{1 \le s \le t} \rho([\nu], [\nu_{m_s}]) \le c\right\}\right) \subset \Omega_{\mathcal{V}}$$

for any $1 \le c \in |C|$. Then $\Omega_{\mathcal{V}} \subset \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}^*_C}$ is admissible and any such $X(c) \subset \Omega_{\mathcal{V}}$ is an admissible affinoid subset. Moreover, for any unbounded sequence $(c_n)_{n \ge 1}$ in $|C|$ the covering $(X(c_n))_{n \ge 1}$ of $\Omega_{\mathcal{V}}$ is admissible.

Remark 4.5. In the case where $t = 1$, the subsequent proof specializes to the one given by Schneider and Stuhler in [35, Section 1, Proof of Prop. 4], where they denote $X(q^n)$ by $\overline{\Omega}_n$ for any positive integer n.

Proof of Proposition 4.4. Set $\mathbb{P} := \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}^*_C}$. Set $U_s := U_{\nu_{m_s}}$ for any $1 \le s \le t$. Set $m := \prod_{1 \le s \le t} m_s$ and $U := \prod_{1 \le s \le t} U_s$. Let $1 \le c \in |C|$. For any $u \in U$ set

$$X(c, u) := \left\{[l] \in \mathbb{P} \;\middle|\; c \cdot \prod_{1 \le s \le t} |l(u_s)| \ge \prod_{1 \le s \le t} \max\nolimits_{u'_s \in U_s}(|l(u'_s)|)\right\}. \tag{8}$$

From (7) and the fact that $\mathcal{V} \setminus \{0\} = E^\times \cdot U_s$ for any $1 \le s \le t$ follows that

$$X(c) = \bigcap_{u \in U} X(c, u); \tag{9}$$

this is in fact a finite intersection since any $X(c, u)$ depends only on u mod $\pi^n m$ for any $n \ge 1$ with $q^n > c$. In order to see that any $X(c)$ is admissible affinoid, it thus suffices, by Proposition 2.22, to show that any such subset $X(c, u) \subset \mathbb{P}$ is admissible affinoid. Consider any $u \in U$ and for every $1 \le s \le t$ the admissible subset

$$X(u_s) := \{[l] \in \mathbb{P} \colon l(u_s) \neq 0\} \subset \mathbb{P};$$

any basis β of $\mathcal{V}$ containing u_s yields the rigid analytic isomorphism

$$i_{u_s, \beta} \colon X(u_s) \to \mathbb{A}_C^{(\beta \setminus \{u_s\}), \mathrm{rig}}, [l] \mapsto \left(\frac{l(v)}{l(u_s)}\right)_{v \neq u_s}.$$

Let $X(u) \subset \mathbb{P}$ be the intersection of the $X(u_s)$ for every $1 \le s \le t$. Choose any $\mathcal{O}_E$-basis β_s of m_s that contains u_s for every $1 \le s \le t$. As any of the maxima in (8) is attained at an element of such a β_s, thus

$$X(c, u) = \bigcap_{v \in \prod_{1 \le s \le t} \beta_s} \left\{[l] \in X(u) \;\middle|\; c \ge \prod_{1 \le s \le t} \left|\frac{l(v_s)}{l(u_s)}\right|\right\}. \tag{10}$$

In particular, for any $1 \leq s \leq t$ then $X(c,u)$ is contained in the affinoid $i^{-1}_{u_s,\beta_s}(B_c)$, where B_c denotes the closed polydisc of radius c around 0. Denote by $X'(c,u) \subset \mathbb{P}$ the intersection of all $i^{-1}_{u_s,\beta_s}(B_c)$ for all $1 \leq s \leq t$; it is again affinoid by Proposition 2.22 and satisfies that $X(c,u) \subset X'(c,u) \subset X(u)$. In particular, the equality in (10) remains valid if $X(u)$ is replaced by $X'(c,u)$. Thus $X(c,u)$ is an admissible affinoid subset of the admissible affinoid subset $X'(c,u) \subset \mathbb{P}$ and hence itself an admissible affinoid subset of $\mathbb{P}$. As argued before, thus $X(c) \subset \mathbb{P}$ is an admissible affinoid subset. Moreover, the covering $(X(q^n,u))_{n\geq 1}$ of $X(u)$ is admissible as the image of any morphism $Z \to X(u)$ from an affinoid Z is already contained in some $X(q^n,u)$ by the Maximum Modulus Principle (see Proposition 2.27) applied to the composition of φ with any of the products in (10).

Consider then any unbounded sequence $(c_n)_{n\geq 1}$ in $|C|$. Consider an arbitrary morphism $\varphi\colon Z \to \mathbb{P}$ from an affinoid variety Z whose image is contained in $\Omega_{\mathcal{V}}$. In order to show that $\Omega_{\mathcal{V}} \subset \mathbb{P}$ is admissible and admissibly covered by the $X(c_n)$, it suffices (see [8, Prop. 9.1.4.2]) to show that the image of φ is already contained in some $X(c_n)$. Since $\Omega_{\mathcal{V}} \subset X(u)$ for any $u \in U$ and since $(X(q^n,u))_{n\geq 1}$ is an admissible covering of $X(u)$, the image of φ is contained in $X(q^{n_u},u)$ for some $n_u \geq 1$. Choose such an $n_u \geq 1$ for any $u \in U$. By means of the quasi-compactness of U, choose a finite subset $U_0 \subset U$ such that U is covered by the $u + \pi^{n_u+1}m$ for all $u \in U_0$. Choose an $n \geq 1$ such that $c_n \geq q^{n_u}$ for all $u \in U_0$. Any $u' \in U$ thus admits an $u \in U_0$ for which $u' - u \in \pi^{n_u+1}m$ and hence

$$X(c_n,u') \supset X(q^{n_u},u') = X(q^{n_u},u) \supset \mathrm{Im}(\varphi).$$

Hence the image of φ is contained in $X(c_n)$ by (9) as desired. □

Corollary 4.6. *For any $O \subset \Omega_{\mathcal{V}}$ the following are equivalent:*

i) O is contained in an admissible affinoid subset of $\Omega_{\mathcal{V}}$.
ii) O is contained in an admissible quasi-compact subset of $\Omega_{\mathcal{V}}$.
iii) $\lambda_{\mathcal{V}}(O)$ is bounded.
iv) $\exists \kappa > 0\colon \forall\ [l],[l'] \in O, \forall\ 0 \neq x,y \in \mathcal{V}:\ \frac{|l(y)|}{|l(x)|} \leq \kappa \cdot \frac{|l'(y)|}{|l'(x)|}$.

Proof. Condition iv) is a reformulation of iii) (cf. Def-Proposition 4.2). The Corollary then directly follows from Proposition 4.4. □

Corollary 4.7. *Consider any $O \subset \Omega_{\mathcal{V}}$ that is contained in an admissible quasi-compact subset of $\Omega_{\mathcal{V}}$ and consider any non-empty discrete subset $\Lambda \subset \mathcal{V}$. Then there exists a finite subset of Λ in which every $[l] \in O$ attains $\inf_{\lambda\in\Lambda} |l(\lambda)|$.*

Proof. As $\Lambda \subset \mathcal{V}$ is discrete, $l(\Lambda)$ is strongly discrete (see for instance [20, Ex. 2.48 and Lemma 2.49]). Hence for any $[l] \in \Omega_{\mathcal{V}}$ the infimum $i(l) := \inf_{\lambda\in\Lambda} |l(\lambda)|$ is attained at some element of Λ. Assume without loss of generality that $O \neq \emptyset$ and that $0 \notin \Lambda$. Choose any $\kappa > 0$ satisfying the property in Corollary 4.6, iv) for O. Choose any $[l_0] \in O$ and any $\lambda_0 \in \Lambda$ at which $i(l_0)$ is attained. Consider any further $[l] \in O$ and any $\lambda \in \Lambda$ at which $i(l)$ is attained. Then

$$\frac{|l_0(\lambda)|}{|l_0(\lambda_0)|} \leq \kappa \cdot \frac{|l(\lambda)|}{|l(\lambda_0)|} \leq \kappa$$

and hence $|l_0(\lambda)| \leq |l_0(\lambda_0)|$. As $l_0(\Lambda)$ is strongly discrete and as $l_0|_{\mathcal{V}}$ is injective, the last inequality requires λ to lie in a finite subset of Λ that depends only on $[l_0]$ and λ_0 and κ. This yields the corollary. □

Lemma 4.8. *Any fiber of $\lambda_{\mathcal{V}}$ is open with respect to the canonical topology.*

Proof. Set $\mathbb{P} := \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}^*_C}$. Let $[l] \in \Omega_{\mathcal{V}}$. By Proposition 4.1, choose a basis β of $\mathcal{V}$ with

$$\forall (\mu_w)_{w\in\beta} \in E^\beta \colon \left| l\left(\sum_{w\in\beta} \mu_w \cdot w\right)\right| = \max_{w\in\beta} |\mu_w| \cdot |l(w)|. \tag{11}$$

Choose any $w_0 \in \beta$ for which $l(w_0) \neq 0$. We further choose an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the closed ball

$$B_{\varepsilon,\beta}([l]) := \left\{ [l'] \in \mathbb{P} \;\middle|\; l'(w_0) \neq 0 \wedge \forall w \in \beta \colon \left| \frac{l'(w)}{l'(w_0)} - \frac{l(w)}{l(w_0)} \right| \leq \varepsilon \right\}$$

around $[l]$ is contained in $\Omega_{\mathcal{V}}$, where we use that such balls form a basis of neighborhoods of $[l] \in \mathbb{P}$ and that $\Omega_{\mathcal{V}} \subset \mathbb{P}$ is admissible. Consider any $0 < \varepsilon < \varepsilon_0$ such that

$$\forall w \in \beta \colon \varepsilon < \left| \frac{l(w)}{l(w_0)} \right|. \tag{12}$$

We claim that $\lambda_{\mathcal{V}}(B_{\varepsilon,\beta}([l])) = \lambda_{\mathcal{V}}([l])$; this will then directly yield that $\lambda_{\mathcal{V}}^{-1}(\lambda_{\mathcal{V}}([l]))$ is indeed open. It suffices to show that

$$\forall [l'] \in B_{\varepsilon,\beta}([l]), \forall v \in \mathcal{V} \colon \left| \frac{l'(v)}{l'(w_0)} \right| = \left| \frac{l(v)}{l(w_0)} \right|; \tag{13}$$

indeed, any such $[l']$ then gives rise to the same class of norms on $\mathcal{V}$ as $[l]$. Consider any such $[l']$ and v and write $v = \sum_{w\in\beta} \mu_w \cdot w$ with $\mu_w \in E$. Then

$$\begin{aligned} \left| \frac{l'(v)}{l'(w_0)} - \frac{l(v)}{l(w_0)} \right| &\leq \max_{w\in\beta} |\mu_w| \cdot \left| \frac{l'(w)}{l'(w_0)} - \frac{l(w)}{l(w_0)} \right| \\ &\leq \max_{w\in\beta} |\mu_w| \cdot \varepsilon \overset{(12)}{<} \max_{w\in\beta} |\mu_w| \cdot \left| \frac{l(w)}{l(w_0)} \right| \overset{(11)}{=} \left| \frac{l(v)}{l(w_0)} \right|. \end{aligned}$$

As the norm on C is non-Archimedean, this yields (13) as desired. □

4.2. Quotients by discrete subgroups

The natural left $\mathcal{G}$-action on $\mathcal{V}$ induces one on each $\mathrm{S}_{\mathcal{V}}$ and $\mathrm{N}_{\mathcal{V}}$ for which $\mathrm{S}_{\mathcal{V}} \to \mathrm{N}_{\mathcal{V}}, m \mapsto \nu_m$ is equivariant. The action of $\mathcal{G}$ on $\mathrm{S}_{\mathcal{V}}$, resp. $\mathrm{N}_{\mathcal{V}}$, is compatible with the one of $E^\times$, resp. $\mathbb{R}_{>0}$, and thus induces an action of $\mathcal{PG}$ on $E^\times \backslash \mathrm{S}_{\mathcal{V}}$, resp. $\mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}}$.

Let $\mathrm{I}_{\mathcal{V}}$ be the set of non-empty subsets $\Delta = \{s_1, \ldots, s_t\} \subset E^\times \backslash \mathrm{S}_{\mathcal{V}}$ admitting $m_1, \ldots, m_t \in \mathrm{S}_{\mathcal{V}}$ such that $[m_1] = s_1, \ldots, [m_t] = s_t$ and

$$m_1 \supsetneq m_2 \supsetneq \cdots \supsetneq m_t \supsetneq \pi m_1. \tag{14}$$

The $\mathcal{PG}$-action on $E^\times \backslash \mathrm{S}_{\mathcal{V}}$ induces one on $\mathrm{I}_{\mathcal{V}}$.

For any $\Delta \in \mathrm{I}_{\mathcal{V}}$ and any $0 < \varepsilon < 1$ set

$$V_\Delta^\varepsilon := \left\{ [\nu] \in \mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}} \;\middle|\; \prod_{[m] \in \Delta} \rho([\nu], [\nu_m]) \le c' \wedge \forall [m] \in \Delta : \rho([\nu], [\nu_m]) \le c \right\},$$

where $c' = q^{|\Delta| - 1 - \frac{1+\varepsilon}{4\#\Delta}}$ and $c = q^{1 - \frac{3-\varepsilon}{4\#\Delta}}$.

Proposition 4.9. *[12, Proof of Prop. 6.2] Let $0 < \varepsilon < 1$. Then $(V_\Delta^\varepsilon)_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is a covering of $\mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}}$ such that $\forall g \in \mathcal{PG}, \Delta \in \mathrm{I}_{\mathcal{V}}\colon g(V_\Delta^\varepsilon) = V_{g(\Delta)}^\varepsilon$ and*

$$\forall \Delta, \Delta' \in \mathrm{I}_{\mathcal{V}}\colon V_\Delta^\varepsilon \cap V_{\Delta'}^\varepsilon \neq \emptyset \Leftrightarrow \Delta \subset \Delta' \vee \Delta \supset \Delta'.$$

Proof. We refer to [20, Beginning of Section 5.2] for details. □

Recall the $\mathcal{PG}$-equivariant map $\lambda_{\mathcal{V}}\colon \Omega_{\mathcal{V}} \to \mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}}$ that sends any $[l]$ to the class of the norm $v \mapsto |l(v)|$. For any $\Delta \in \mathrm{I}_{\mathcal{V}}$ and any $0 < \varepsilon < 1$ set

$$U_\Delta^\varepsilon := \lambda_{\mathcal{V}}^{-1}(V_\Delta^\varepsilon).$$

Proposition 4.10. *[12, Prop. 6.2] Let $0 < \varepsilon < 1$ be rational. Then $(U_\Delta^\varepsilon)_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is an admissible affinoid covering of $\Omega_{\mathcal{V}}$ and $\forall g \in \mathcal{PG}, \Delta \in \mathrm{I}_{\mathcal{V}}\colon g(U_\Delta^\varepsilon) = U_{g(\Delta)}^\varepsilon$ and*

$$\forall \Delta, \Delta' \in \mathrm{I}_{\mathcal{V}}\colon U_\Delta^\varepsilon \cap U_{\Delta'}^\varepsilon \neq \emptyset \Leftrightarrow \Delta \subset \Delta' \vee \Delta \supset \Delta'.$$

Proof. By Proposition 4.4 and since $q^{\mathbb{Q}} \subset |C|$, any U_Δ^ε is the intersection of finitely many affinoid subsets of $\Omega_{\mathcal{V}}$ and is thus, by Proposition 2.22, itself an affinoid subset of $\Omega_{\mathcal{V}}$.

By Proposition 4.9, we are thus reduced to showing that the covering $(U_\Delta^\varepsilon)_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ of $\Omega_{\mathcal{V}}$ is admissible. Consider any closed ball $B \subset \mathbb{R}_{>0} \backslash \mathrm{N}_{\mathcal{V}}$ around any $[\nu_m]$ for any $m \in \mathrm{S}_{\mathcal{V}}$, set $X := \lambda_{\mathcal{V}}^{-1}(B)$. By Proposition 4.4, we are further reduced to showing that the affinoid X is admissibly covered by the $U_\Delta^\varepsilon \cap X$ or, equivalently, that X is covered by finitely many of the U_Δ^ε or, equivalently, that B is covered by finitely many of the V_Δ^ε. The latter follows from the quasi-compactness of B guaranteed by Proposition 4.2: Indeed, for any

$\Delta \in \mathrm{I}_{\mathcal{V}}$ let $\mathring{V}_{\Delta}^{\varepsilon}$ be defined like V_{Δ}^{ε} upon replacing $\leq$ by $<$ everywhere. Then $V_{\Delta}^{\varepsilon'} \subset \mathring{V}_{\Delta}^{\varepsilon}$ for any $\Delta \in \mathrm{I}_{\mathcal{V}}$ and any $0 < \varepsilon' < \varepsilon$. Hence the open $\mathring{V}_{\Delta}^{\varepsilon}$ for all $\Delta \in \mathrm{I}_{\mathcal{V}}$ cover $\mathrm{I}_{\mathcal{V}}(\mathbb{R}_{>0})$ as well. The quasi-compact B is thus covered by finitely many of the $\mathring{V}_{\Delta}^{\varepsilon}$ and hence by finitely many of the V_{Δ}^{ε} as desired. The remaining assertions of the proposition follow directly from the discussion preceding it. □

Let $\Gamma \subset \mathcal{PG}$ be any subgroup which is discrete with respect the locally profinite topology on $\mathcal{PG}$ (see e.g. [20, Section 2.4]). Consider the quotient map

$$p_{\Gamma} \colon \Omega_{\mathcal{V}} \to \Gamma \backslash \Omega_{\mathcal{V}} =: \Omega_{\Gamma}$$

and endow its target with the structure of Grothendieck ringed space which it induces, that is, a subset (resp. a covering of a subset) of Ω_{Γ} is admissible precisely when its preimage is admissible and the sections on an admissible subset of Ω_{Γ} are the Γ-invariant sections on its preimage.

Lemma 4.11. *The stabilizer $\{\gamma \in \Gamma \colon \gamma(\Delta) = \Delta\}$ of any $\Delta \in \mathrm{I}_{\mathcal{V}}$ in Γ is finite.*

Proof. The stabilizer $\{g \in \mathcal{PG} \colon g([m]) = [m]\}$ of any $[m] \in E^{\times} \backslash \mathrm{S}_{\mathcal{V}}$ in $\mathcal{PG}$ equals $\mathrm{Aut}_{\mathcal{O}_E}(m) \cdot E^{\times}/E^{\times}$; its intersection with the discrete Γ is thus finite (see for instance [20, Lemma 2.41]). Now use that any $\Delta \in \mathrm{I}_{\mathcal{V}}$ is a finite subset of $E^{\times} \backslash \mathrm{S}_{\mathcal{V}}$. □

Lemma 4.12. *For any quasi-compact $U, U' \subset \Omega_{\mathcal{V}}$ the set $\{\gamma \in \Gamma \colon U' \cap \gamma(U) \neq \emptyset\}$ is finite.*

Proof. Consider any rational $0 < \varepsilon < 1$. As the covering $(U_{\Delta}^{\varepsilon})_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ of $\Omega_{\mathcal{V}}$ is admissible by Proposition 4.10, any quasi-compact subset of $\Omega_{\mathcal{V}}$ is covered by finitely many of its elements. It thus suffices to show for any $\Delta, \Delta' \in \mathrm{I}_{\mathcal{V}}$ that $\{\gamma \in \Gamma \colon U_{\Delta'}^{\varepsilon} \cap \gamma(U_{\Delta}^{\varepsilon}) \neq \emptyset\}$ is finite. However, this follows via Proposition 4.10 from Lemma 4.11. □

Proposition 4.13. *The Grothendieck ringed space Ω_{Γ} is a normal rigid analytic variety over C and $(p_{\Gamma}(U_{\Delta}^{\varepsilon}))_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is an admissible affinoid covering of Ω_{Γ} for any rational $0 < \varepsilon < 1$.*

Proof. Consider any such ε. For any $\Delta \in \mathrm{I}_{\mathcal{V}}$ set $U_{\Delta} := U_{\Delta}^{\varepsilon}$ and

$$\Gamma U_{\Delta} := \bigcup_{\gamma \in \Gamma} \gamma(U_{\Delta}) = p_{\Gamma}^{-1}(p_{\Gamma}(U_{\Delta})).$$

From Propositions 2.22 and 4.10 and Lemma 4.12 follows that $(U_{\Delta'} \cap \Gamma U_{\Delta})_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is a system of admissible subsets of $U_{\Delta'}$ for any $\Delta' \in \mathrm{I}_{\mathcal{V}}$; it is then in fact an admissible covering since it is refined by $(U_{\Delta'} \cap U_{\Delta})_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ which is an admissible covering by Proposition 4.10. As $(U_{\Delta'})_{\Delta' \in \mathrm{I}_{\mathcal{V}}}$ is an admissible covering of $\Omega_{\mathcal{V}}$, thus $(\Gamma U_{\Delta})_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is an admissible covering of $\Omega_{\mathcal{V}}$ and, equivalently, $(p_{\Gamma}(U_{\Delta}))_{\Delta \in \mathrm{I}_{\mathcal{V}}}$ is an admissible covering of Ω_{Γ}.

Consider any $\Delta \in \mathrm{I}_{\mathcal{V}}$. It remains to be shown that any $(p_\Gamma(U_\Delta))$ is an affinoid rigid analytic variety over C. The covering $(\gamma(U_\Delta))_{\gamma\in\Gamma}$ of ΓU_Δ is admissible since, by Propositions 2.22 and 4.10 and Lemma 4.12, its intersection with any element of the admissible covering $(U_{\Delta'})_{\Delta'\in\mathrm{I}_{\mathcal{V}}}$ of $\Omega_{\mathcal{V}}$ is admissible. Denote by Γ_Δ the stabilizer of Δ in Γ; it is finite by Lemma 4.11. By Proposition 4.10, then $\gamma(U_\Delta) = U_\Delta$ for any $\gamma \in \Gamma_\Delta$ and $\gamma(U_\Delta) \cap U_\Delta = \emptyset$ for any $\gamma \in \Gamma \setminus \Gamma_\Delta$. The inclusion $U_\Delta \to \Gamma U_\Delta$ thus induces an isomorphism of Grothendieck ringed spaces $\Gamma_\Delta \backslash U_\Delta \to p_\Gamma(U_\Delta)$. As $\Gamma_\Delta\backslash U_\Delta$ is a normal affinoid rigid analytic variety over C, thus so is $p_\Gamma(U_\Delta)$ as desired. □

Proposition 4.14. *Consider any $\omega \in \Omega_{\mathcal{V}}$ and denote by Γ_ω its stabilizer in Γ. Then there exists a basis of admissible affinoid neighborhoods U of ω such that*

i) $\forall \gamma \in \Gamma_\omega : \ \gamma(U) = U$ *and*
ii) $\forall \gamma \in \Gamma \setminus \Gamma_\omega : \ \gamma(U) \cap U = \emptyset$.

Proof. Consider the fiber $f := \lambda_{\mathcal{V}}^{-1}(\lambda_{\mathcal{V}}(\omega))$. Let $S \subset \Gamma$ be the subset of elements γ for which $\gamma(f) \cap f \neq \emptyset$; it is finite by Proposition 4.4 and Lemma 4.12. Using that the canonical topology on $\Omega_{\mathcal{V}}$ is Hausdorff and that $f \subset \Omega_{\mathcal{V}}$ is open by Lemma 4.8, we choose for any $\gamma \in S \setminus \Gamma_\omega$ an admissible affinoid neighborhood $U_\gamma \subset f$ of ω for which $\gamma(U_\gamma) \cap U_\gamma = \emptyset$. For any neighborhood U' of ω then

$$U := \left(\bigcap_{\gamma'\in\Gamma_\omega} \bigcap_{\gamma\in S\setminus\Gamma_\omega} \gamma'(U_\gamma) \right) \cap \left(\bigcap_{\gamma\in\Gamma_\omega} \gamma(U') \right)$$

is an neighborhood of ω that is contained in U' and satisfies i) and ii). If such an U' is affinoid, then U is the intersection of finitely many affinoid subsets and hence, by Proposition 2.22, itself affinoid. As the affinoid neighborhoods of ω form a basis of neighborhoods of ω, this yields the proposition. □

Corollary 4.15. *The morphism p_Γ is open with respect to the canonical topologies and, if Γ acts fixed-points free on $\Omega_{\mathcal{V}}$, it induces isomorphisms on the stalks and the stalks on Ω_Γ are then regular.*

Example 4.16. Consider any admissible coefficient subring $A \subset C$ such that the completion of its quotient field is E. Consider any projective A-submodule $\Lambda \subset \mathcal{V}$ for which the natural homomorphism $\Lambda \otimes_A E \to \mathcal{V}$ is an isomorphism. Let $0 \neq I \subset A$ be an ideal. Then the kernel of the natural homomorphism

$$\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(I^{-1}\Lambda\backslash\Lambda)$$

has discrete image in $\mathcal{PG}$ and, if $I \neq A$, its action on $\Omega_{\mathcal{V}}$ is fixed-point free.

Proof. As $\Lambda \otimes_A E \to \mathcal{V}$ is an isomorphism, $\Lambda \subset \mathcal{V}$ is discrete (see for instance [20, Lemma 2.48]). Hence the image in $\mathcal{PG}$ of $\mathrm{Aut}_A(\Lambda)$ is discrete, too (see for instance [20, Example 2.42]). Consider any $[l] \in \Omega_{\mathcal{V}}$ and any γ in the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(I^{-1}\Lambda \backslash \Lambda)$ with $\gamma[l] = [l]$. Hence $\gamma l = c \cdot l$ for some $c \in C^{\times}$. Using Proposition 2.36, choose an $0 \neq \lambda \in \Lambda$ for which $|l(\lambda)|$ is minimal among $|l(\Lambda)| \setminus \{0\}$. Then $|c \cdot l(\lambda)|$ is minimal among $|c \cdot l(\Lambda)| \setminus \{0\}$. As $l(\Lambda) = c \cdot l(\Lambda)$, thus $|l(\lambda)| = |c \cdot l(\lambda)| = |(\gamma l)(\lambda)| = |l(\lambda) + l(\gamma^{-1}\lambda - \lambda)|$ and hence

$$|l(\gamma^{-1}\lambda - \lambda)| \leq |l(\lambda)|$$

Moreover, $\gamma^{-1}\lambda - \lambda \in I\Lambda$ by definition of Γ. If $I \neq A$, then the smallest non-zero element of $l(I\Lambda)$ is strictly larger than the one of $l(\Lambda)$. In this case, thus $\gamma^{-1}\lambda - \lambda = 0$ and hence $c \cdot l(\lambda) = (\gamma l)(\lambda) = l(\lambda)$ and hence $c = 1$ and hence $\gamma l = l$. As $l|_{\mathcal{V}}$ is injective, thus γ is the identity as desired. $\square$

4.3. Some connected subsets of Drinfeld's period domain

Consider any E-subspace $\mathcal{W} \subset \mathcal{V}$ and any discrete subset $\Lambda \subset \mathcal{V}$ such that $\Lambda \cap \mathcal{W}$ contains a non-zero element. For any $O \subset \Omega_{\mathcal{W}}$ and any $r \in |C|$ set

$$\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) := \left\{ [l] \in \Omega_{\mathcal{V}} \,\middle|\, [l|_{\mathcal{W}_C}] \in O \wedge \inf_{\lambda \in \Lambda \backslash \mathcal{W}} |l(\lambda)| \geq r \cdot \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)| \right\}.$$

Lemma 4.17. *Consider any $r \in |C|$ and any admissible $O \subset \Omega_{\mathcal{W}}$. For any admissible affinoid $U \subset \Omega_{\mathcal{V}}$ then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U \subset U$ is admissible and, if O is quasi-compact, quasi-compact. Moreover, $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \subset \Omega_{\mathcal{V}}$ is admissible.*

Proof. Let $U \subset \Omega_{\mathcal{V}}$ be affinoid. We first show that the intersection of

$$\mathcal{U}(r) := \left\{ [l] \in \Omega_{\mathcal{V}} \,\middle|\, \inf_{\lambda \in \Lambda \backslash \mathcal{W}} |l(\lambda)| \geq r \cdot \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)| \right\} \subset \Omega_{\mathcal{V}}$$

with U is a quasi-compact admissible subset of U. Suppose without loss of generality that $\Lambda \setminus \mathcal{W} \neq \emptyset$; otherwise $\mathcal{U}(r) = \Omega_{\mathcal{V}}$ and then the intersection equals the affinoid U. Corollary 4.7 provides a finite subset $S_1 \subset \Lambda \setminus \mathcal{W}$, respectively $S_2 \subset \Lambda \cap \mathcal{W} \setminus \{0\}$, in which every $[l] \in U$ attains

$$\inf_{\lambda \in \Lambda \backslash \mathcal{W}} |l(\lambda)|, \quad \text{respectively} \inf_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l(\lambda)|.$$

Hence

$$\mathcal{U}(r) \cap U = \bigcup_{\mu \in S_2} \bigcap_{\lambda \in S_1} \{[l] \in U \colon \; |l(\lambda)| \geq r \cdot |l(\mu)|\}.$$

By [8, Prop. 7.2.3.7], thus the subset $\mathcal{U}(r) \cap U \subset U$ is the union of finitely many rational subdomains and hence quasi-compact admissible as desired.

Consider the morphism $\pi\colon \Omega_{\mathcal{V}} \to \Omega_{\mathcal{W}}, [l] \mapsto [l|_{\mathcal{W}_C}]$. Then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U$ is the intersection of the admissible $\mathcal{U}(r) \cap U$ with the admissible $\pi^{-1}(O)$ and is thus itself admissible. Since U was arbitrary, the admissibility of $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ follows by virtue of an admissible affinoid covering of $\Omega_{\mathcal{V}}$.

Suppose then that O is quasi-compact. By means of Corollary 4.6, choose an admissible affinoid $O' \subset \Omega_{\mathcal{W}}$ such that $\pi(U) \subset O'$ and $O \subset O'$. Then π restricts to a morphism $\mathcal{U}(r) \cap U \to O'$ from a quasi-compact to an affinoid variety. By [8, Proposition 7.2.2.4], the preimage $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r) \cap U$ of the affinoid O under this restriction is thus quasi-compact as desired. $\square$

The following definition and proposition concerning connected varieties is due to Conrad [10, Below Theorem 2.1.3] except that we furthermore ask them to be non-empty.

Definition 4.18. A rigid analytic variety X is *connected* if it is non-empty and if any admissible covering $\{U, U'\}$ of X satisfies that

$$U \cap U' = \emptyset \Rightarrow U = \emptyset \vee U' = \emptyset.$$

Proposition 4.19. *A non-empty rigid analytic variety X is connected if and only if any $x, x' \in X$ admit connected admissible subvarieties $X_1, \ldots, X_n$ of X such that $x \in X_1$ and $x' \in X_n$ and $X_i \cap X_{i+1} \neq \emptyset$ for any $1 \leq i < n$; in this case, such X_i can in fact be chosen to be affinoid.*

Theorem 4.20. *Suppose that $\Lambda \subset \mathcal{V}$ is a discrete subgroup such that*

$$\Lambda = \underbrace{(\Lambda \cap \mathcal{W})}_{\neq 0} \oplus (\Lambda \cap E \cdot v_1) \oplus \cdots \oplus (\Lambda \cap E \cdot v_k)$$

for some $0 \neq v_i \in \mathcal{V}$ such that $\mathcal{V} = \mathcal{W} \oplus E \cdot v_1 \oplus \cdots \oplus E \cdot v_k$. Then $\mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ is connected for any connected admissible $O \subset \Omega_{\mathcal{W}}$ and any $1 \leq r \in |C|$.

We shall prove Theorem 4.20 at the end of this section. First, we apply it: If $\Lambda \subset \mathcal{W}$, then $\mathcal{U}_{\mathcal{V}}(\Lambda, \Omega_{\mathcal{W}}, r) = \Omega_{\mathcal{V}}$ for any $r \in |C|$. If $\dim_E(\mathcal{W}) = 1$, then $\Omega_{\mathcal{W}}$ is a point and thus connected. Theorem 4.20 thus specializes to

Corollary 4.21. *Drinfeld's period domain $\Omega_{\mathcal{V}}$ is connected.*

Corollary 4.22. *The quotient of Drinfeld's period domain by any discrete subgroup of $\mathcal{PG}$ is irreducible.*

Proof. As such a quotient is a normal rigid analytic variety by Proposition 4.13, it is irreducible if and only if it is connected (see Conrad [10, Definition 2.2.2]). However, any quotient of a connected variety is connected. □

The proof below of Theorem 4.20 is inspired by van der Put's [38, Example 3.5] and builds on the following results. In the case of Corollary 4.21, it in fact specializes to a variation of the proof that van der Put outlines there.

Proposition 4.23. *(Bosch, Lütkebohmert [9, Corollary 5.11]) Let $p\colon X \to Y$ be a flat morphism between quasi-compact rigid analytic varieties over C. Then the image under p of any admissible quasi-compact subset is admissible quasi-compact.*

Corollary 4.24. *Consider any flat morphism $p\colon X \to Y$ between quasi-compact rigid analytic varieties over C. Suppose that Y and every fiber of p is connected. Then X is connected.*

Proof. By assumption, every fiber of p lies in a connected component of X. Thus the images under p of the connected components of X are disjoint and, by surjectivity of p, cover Y. By Proposition 4.23, this covering of Y is admissible. The connectedness of Y then yields that X has only one connected component, i.e., that X is connected. □

Definition 4.25. A subset S of the projective line $\mathbb{P}_C^{1,\mathrm{rig}}$ is a *closed ball* if it is the image of the closed unit ball of the affine line $\mathbb{A}_C^{1,\mathrm{rig}}$ under an element of $\mathrm{PGL}_2(C)$.

Proposition 4.26. *A subset $S \subset \mathbb{P}_C^{1,\mathrm{rig}}$ is a closed ball if and only if it equals*

$$\{z \in \mathbb{A}_C^{1,\mathrm{rig}}\colon |z-c| \leq |c'|\} \quad \textit{or} \quad \{z \in \mathbb{A}_C^{1,\mathrm{rig}}\colon |z-c| \geq |c'|\} \cup (\mathbb{P}_C^{1,\mathrm{rig}} \setminus \mathbb{A}_C^{1,\mathrm{rig}}) \tag{15}$$

for some $0 \neq c', c \in \mathbb{A}_C^{1,\mathrm{rig}}$.

Proof. That any subset as in (15) is a closed ball is directly checked. We consider then any $g = (a, b; c, d) \in \mathrm{GL}_2(C)$ and need to show that

$$B_g := \{z \in \mathbb{A}_C^{1,\mathrm{rig}}\colon |az+b| \leq |cz+d|\}$$

is a subset of $\mathbb{A}_C^{1,\mathrm{rig}}$ of the first kind in (15) if $|a| > |c|$, resp. of the second if $|a| \leq |c|$. If $a = 0$ or $c = 0$, this is directly checked. Thus assume that $a \neq 0 \neq c$. Let $z \in \mathbb{A}_C^{1,\mathrm{rig}}$ and set $z' := cz + d$ and $\mu := \frac{bc-ad}{c}$. Then $az + b = \frac{a}{c}z' + \mu$ and

$$z \in B_g \Leftrightarrow \left|\frac{a}{c}z' + \mu\right| \leq |z'|.$$

If $|a| \leq |c|$, thus $z \in B_g \Leftrightarrow |\mu| \leq |z'|$. If $|a| > |c|$, then

$$\left|\frac{a}{c}z' + \mu\right| \leq |z'| \Leftrightarrow \left|\frac{a}{c}z' + \mu\right| \leq \left|\frac{c}{a}\mu\right|$$

since both sides imply that $|\mu| = |\frac{a}{c}z'|$. From this is directly checked that B_g is of the desired form in both cases. $\square$

Proposition 4.27. *Any non-empty intersection of any finitely many closed balls in* $\mathbb{P}_C^{1,\mathrm{rig}}$ *is connected.*

Proof. Consider any such intersection I. The connectedness of $\mathbb{P}_C^{1,\mathrm{rig}}$ yields the proposition in the case where I is the intersection over the empty set. Suppose then that I is contained in a ball. Then the image of I under the transformation by a suitable element of $\mathrm{PGL}_2(C)$ is in $\mathbb{A}_C^{1,\mathrm{rig}}$. We thus assume without loss of generality that $I \subset \mathbb{A}_C^{1,\mathrm{rig}}$. By Proposition 4.26 and since any non-empty intersection of finitely many closed balls in $\mathbb{A}_C^{1,\mathrm{rig}}$ is again a closed ball, there exist a $0 \leq k \leq n-1$ and some $c_0, c_0' \ldots, c_k, c_k' \in \mathbb{A}_C^{1,\mathrm{rig}}$ such that

$$I = \{z \in \mathbb{A}_C^{1,\mathrm{rig}} \colon |z - c_0| \leq |c_0'| \wedge \forall 1 \leq j \leq k \colon |z - c_j| \geq |c_j'|\}.$$

By [8, Theorem 9.7.2.2], any non-empty such set is connected. $\square$

Proof of Theorem 4.20. Let $1 \leq r \in |C|$ and set $\mathcal{U}(O) := \mathcal{U}_{\mathcal{V}}(\Lambda, O, r)$ for any admissible $O \subset \Omega_{\mathcal{W}}$. We shall show that $\mathcal{U}(O)$ is connected in the case where $O \subset \Omega_{\mathcal{W}}$ is any connected admissible and affinoid subset. In particular, $\mathcal{U}(O)$ is then non-empty for any non-empty admissible $O \subset \Omega_{\mathcal{W}}$ since the latter can be covered by connected subsets. Given this affinoid case, the theorem thus directly follows: Indeed, for an arbitrary connected admissible $O \subset \Omega_{\mathcal{W}}$ use that any admissible affinoid $O', O'' \subset O$ with $O' \cap O'' \neq \emptyset$ satisfy that $\mathcal{U}(O') \cap \mathcal{U}(O'') = \mathcal{U}(O' \cap O'') \neq \emptyset$ and that, by the affinoid case, both $\mathcal{U}(O')$ and $\mathcal{U}(O'')$ are connected if O' and O'' are.

Consider thus any connected admissible affinoid $O \subset \Omega_{\mathcal{W}}$. Choose any free $\mathcal{O}_E$-submodule of $m_0 \subset \mathcal{V}$ of maximal rank such that $\Lambda \cap m_0 \neq 0$ and any v_i as in the theorem and, using that Λ is discrete, such that

$$\forall 1 \leq i \leq k \colon\ [\Lambda \cap E \cdot v_i \neq 0 \ \Rightarrow\ \Lambda \cap \mathcal{O}_E \cdot v_i \neq 0 = \Lambda \cap \mathcal{O}_E \cdot \pi \cdot v_i]. \tag{16}$$

For any $0 \leq i \leq k$ set $m_i := m_0 \oplus \mathcal{O}_E \cdot v_1 \oplus \cdots \oplus \mathcal{O}_E \cdot v_i$ and $U_i := m_i \setminus \pi m_i$ and $\mathcal{W}_i := E \cdot m_i$ and for any linear $l \colon (\mathcal{W}_i)_C \to C$ set

$$\mu_i(l) := \max_{u \in U_i} |l(u)|. \tag{17}$$

Any $\Omega_{\mathcal{W}_i}$ is admissibly covered by the ascending affinoid subsets

$$\Omega_i^n := \{[l] \in \mathbb{P}^{\mathrm{rig}}_{(\mathcal{W}_i)_C^*} \mid \ \forall u \in U_i \colon |l(u)| \geq |\pi^n| \cdot \mu_i(l)\}$$

for all $n \geq 1$ by Lemma 4.4 and Definition-Proposition 4.3. Consider the morphism

$$p_i\colon \Omega_{\mathcal{W}_i} \to \Omega_{\mathcal{W}_{i-1}}, [l] \mapsto [l|_{(\mathcal{W}_{i-1})_C}]$$

for any $1 \leq i \leq k$. Choose any $j \geq 1$ for which $|\pi|^{-j} > r$. Define $\underline{\Omega}_0^n := \Omega_0^n$ for any $n \geq 1$ and iteratively

$$\forall 1 \leq i \leq k, \forall n > i \cdot j\colon \; \underline{\Omega}_i^n := p_i^{-1}(\underline{\Omega}_{i-1}^{n-j}) \cap \Omega_i^n.$$

Since, by construction, $p_i(\Omega_i^n) \subset \Omega_{i-1}^n$ for any $n \geq 1$ and since the preimage of an affinoid subset under an affinoid morphism is affinoid by [8, Proposition 7.2.2.4], any such $\underline{\Omega}_i^n \subset \Omega_i^n$ is an affinoid subset. Moreover, being cofinal with $(\Omega_i^n)_{n\geq 1}$, the system $(\underline{\Omega}_i^n)_{n>i\cdot j}$ of ascending subsets is an admissible covering of $\Omega_{\mathcal{W}_i}$ for any $0 \leq i \leq k$. Set

$$\forall 0 \leq i \leq k, \forall n > i \cdot j\colon \; Y_i^n := \underline{\Omega}_i^n \cap \underbrace{\mathcal{U}_{\mathcal{W}_i}(\Lambda \cap \mathcal{W}_i, O, r)}_{=:\mathcal{U}_i(O)}.$$

Thus $\mathcal{U}(O) = \mathcal{U}_k(O)$ is admissibly covered by the ascending subsets Y_k^n for all $n > k \cdot j$. It thus suffices to show that Y_k^n is connected for any large enough n. We choose, by means of Corollary 4.6, any $n_0 \geq 1$ such that $O \subset \Omega_0^{n_0}$. We are reduced to showing that Y_i^n is connected for any $0 \leq i \leq k$ and any $n \geq n_0 + i \cdot j$. We prove this by induction on i. If $i = 0$, it follows directly from the assumption on O using that $Y_0^n = O$ for any $n \geq n_0$. Consider then any $i > 0$ and any $n \geq n_0 + i \cdot j$ and suppose by induction hypothesis that Y_{i-1}^{n-j} is connected. By construction, p_i restricts to a morphism

$$p\colon Y_i^n \to Y_{i-1}^{n-j}.$$

As both $\underline{\Omega}_i^n$ and $\underline{\Omega}_{i-1}^{n-j}$ are affinoid, Lemma 4.17 yields that both Y_i^n and Y_{i-1}^{n-j} are quasi-compact. Being admissible subvarieties of standard projective spaces, they are further regular. Let $[l'] \in Y_{i-1}^{n-j}$. In view of Corollary 4.24, it remains to show that $p^{-1}([l'])$ is isomorphic to a connected admissible subvariety of $\mathbb{P}_C^{1,\mathrm{rig}}$. In view of Proposition 4.27, this follows from the following lemmas.

Lemma 4.28. $p^{-1}([l']) \neq \emptyset$.

Proof. Use that $|C|$ contains $|\pi|^{\mathbb{Q}}$, that $|l'(W_{i-1})|$ is the union in $|C|$ of finitely many translates of $|\pi|^{\mathbb{Z}}$ and that $|\pi|^{-j} > r$ in order to choose a linear form $l\colon (\mathcal{W}_i)_C \to C$ such that

i) $l|_{(\mathcal{W}_{i-1})_C} = l'$,
ii) $|l(v_i)| \in |C| \setminus |l'(\mathcal{W}_{i-1})|$,
iii) $|\pi|^{-j} \cdot \mu_{i-1}(l') \overset{(*)}{\geq} |l(v_i)| \overset{(**)}{\geq} r \cdot \mu_{i-1}(l')$.

We show that $[l] \in p^{-1}([l'])$. By i), it suffices to show that $[l] \in Y_i^n$. As $r \geq 1$, Condition $(**)$ implies that $|l(v_i)| \geq \mu_{i-1}(l')$ and hence that

$$\mu_i(l) = \max\{|l(v_i)|, \mu_{i-1}(l')\} = |l(v_i)|, \tag{18}$$

where the first equality holds true, as $\mu_i(l)$ is attained by an element of any basis of m_i, so for instance of a basis consisting of v_i and a basis of m_{i-1}. From ii) it follows, as $|\cdot|$ is non-Archimedean, that

$$\forall e \in E, \forall w' \in \mathcal{W}_{i-1}\colon\ |l(e \cdot v_i + w')| = \max\{|e| \cdot |l(v_i)|, |l'(w')|\}. \tag{19}$$

As $[l'] \in \underline{\Omega}_{i-1}^{n-j}$, that $[l] \in \underline{\Omega}_i^n$ is equivalent to $[l] \in \Omega_i^n$, i.e., to

$$\forall u \in U_i\colon\ |l(u)| \geq |\pi|^n \cdot \mu_i(l).$$

Consider any $u \in U_i$ and write $u = e \cdot v_i + w'$ for some $e \in \mathcal{O}_E$ and some $w' \in m_{i-1}$. If $w' \in U_{i-1}$, then

$$|l(u)| \overset{(19)}{\geq} |l'(w')| \overset{[l'] \in \Omega_{i-1}^{n-j}}{\geq} |\pi|^{n-j} \cdot \mu_{i-1}(l') \overset{(*)\wedge(18)}{\geq} |\pi|^n \cdot \mu_i(l).$$

If $w' \notin U_{i-1}$, then $e \in \mathcal{O}_E^\times$ as $u \in U_i$; in this case, thus

$$|l(u)| \overset{(19)}{\geq} |e| \cdot |l(v_i)| = |l(v_i)| \overset{(18)}{=} \mu_i(l) \geq |\pi|^n \mu_i(l).$$

Hence $[l] \in \underline{\Omega}_i^n$. It remains to show that $[l] \in \mathcal{U}_i(O)$, i.e., that

$$\forall \lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}\colon |l(\lambda)| \geq r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

Consider any $\lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$. Write $\lambda = e \cdot v_i + \lambda'$ for some $e \in E$ and $\lambda' \in \mathcal{W}_{i-1}$. By assumption on Λ, both $e \cdot v_i$ and λ' lie in Λ. If $\lambda' \notin \mathcal{W}$, then

$$|l(\lambda)| \overset{(19)}{\geq} |l'(\lambda')| \overset{[l'] \in \mathcal{U}_{i-1}(O)}{\geq} r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

If $\lambda' \in \mathcal{W}$, then $e \cdot v_i \neq 0$ as $\lambda \notin \mathcal{W}$; in this case, $|e| \geq 1$ by (16) and hence

$$|l(\lambda)| \overset{(19)}{\geq} |e| \cdot |l(v_i)| \overset{(**)}{\geq} r \cdot \mu_{i-1}(l') \overset{\Lambda \cap m_0 \neq 0}{\geq} r \cdot \min_{0 \neq w \in \Lambda \cap m_0} |l'(w)| \geq r \cdot \min_{0 \neq w \in \Lambda \cap \mathcal{W}} |l'(w)|.$$

Hence $[l] \in \mathcal{U}_i(O)$. As argued before, thus $[l] \in p^{-1}([l'])$ as desired. $\square$

Lemma 4.29. *The fiber $p^{-1}([l'])$ is isomorphic to the intersection of finitely many closed balls of $\mathbb{P}_C^{1,\mathrm{rig}}$.*

Proof. Denote by A the set of $C^\times$-classes of linear forms $l\colon (\mathcal{W}_i)_C \to C$ for which $[l|_{(\mathcal{W}_{i-1})_C}] = [l']$. Then $p^{-1}([l']) = \mathcal{U}_i(O) \cap \Omega_i^n \cap A$. Choose any $0 \neq w_0 \in \mathcal{W}$ and consider the isomorphism

$$\varphi\colon A \to \mathbb{A}_C^{1,\mathrm{rig}},\ [l] \mapsto \frac{l(v_i)}{l(w_0)}.$$

We first show that $\varphi(\Omega_i^n \cap A)$ is a closed ball of $\mathbb{P}_C^{1,\mathrm{rig}}$ and then that so is $\varphi(\mathcal{U}_i(O) \cap \Omega_i^n \cap A)$. For any $[l] \in A$ set

$$\mu_{i-1}(l) := \mu_{i-1}(l|_{(W_{i-1})_C}) \overset{(17)}{:=} \max_{u \in U_{i-1}} |l(u)|.$$

Choose a finite set of representatives S of U_i modulo $\pi^{n+1} m_i$, respectively S' of $(U_i \backslash U_{i-1})$ modulo $\pi^{n+1} m_i$. Using that $[l'] \in \Omega_{i-1}^{n-j} \subset \Omega_{i-1}^n$ and that $\mu_i(l) = \max\{|l(v_i)|, \mu_{i-1}(l)\}$ by the same reason as in (17), then

$$\Omega_i^n \cap A = \{[l] \in A |\ \forall u \in S\colon |l(u)| \geq |\pi^n l(v_i)| \wedge \forall u' \in S'\colon |l(u')| \geq |\pi^n| \mu_{i-1}(l)\}.$$

Write any element u of S (resp. of S') in the form $e_u \cdot v_i + w_u$ for some $w_u \in m_{i-1}$ and some (non-zero) $e_u \in \mathcal{O}_E$ such that $w_u \in U_{i-1}$ or $e_u \in \mathcal{O}_E^\times$ and set

$$c_u := \frac{l'(w_u)}{l'(w_0)} = \frac{l(w_u)}{l(w_0)} \text{ and } c := \frac{\mu_{i-1}(l')}{l'(w_0)} = \frac{\mu_{i-1}(l)}{l(w_0)}$$

for any $[l] \in A$. Then $\varphi(\Omega_i^n \cap A)$ equals

$$\{z \in \mathbb{A}_C^{1,\mathrm{rig}} |\ \forall u \in S\colon |e_u \cdot z + c_u| \overset{(*)}{\geq} |\pi^n \cdot z| \wedge \forall u' \in S'\colon |e_{u'} \cdot z + c_{u'}| \geq |\pi^n \cdot c|\}.$$

We may and do assume that $e_u = 0$ for some $u \in S$. For such a u then $\overset{(*)}{\geq}$ defines a closed ball in $\mathbb{P}_C^{1,\mathrm{rig}}$ which is already contained in $\mathbb{A}_C^{1,\mathrm{rig}}$. By Proposition 4.26, thus $\varphi(\Omega_i^n \cap A)$ is a closed ball in $\mathbb{P}_C^{1,\mathrm{rig}}$.

Choose then a $\lambda_0 \in \Lambda \cap \mathcal{W}$ for which $|l'(\lambda_0)| = \min_{0 \neq \lambda \in \Lambda \cap \mathcal{W}} |l'(\lambda)|$. Then

$$\forall [l] \in A\colon [l] \in \mathcal{U}_i(O) \Leftrightarrow \forall \lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}\colon \left|\frac{l(\lambda)}{l(w_0)}\right| \geq r \cdot \left|\frac{l'(\lambda_0)}{l'(w_0)}\right|. \tag{20}$$

By Corollary 4.7, the affinoid Ω_i^n admits a finite subset $T \subset \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$ in which the infimum of the $|l(\lambda)|$ for all $\lambda \in \Lambda \cap \mathcal{W}_i \setminus \mathcal{W}$ is attained for any $[l] \in \Omega_i^n$. Choose such a T. Write any $\lambda \in T$ in the form $e_\lambda \cdot v_i + w_\lambda$ for some $e_\lambda \in E$ and some $w_\lambda \in \mathcal{W}_{i-1}$ and set

$$c_\lambda := \frac{l'(w_\lambda)}{l'(w_0)} = \frac{l(w_\lambda)}{l(w_0)} \text{ and } c_0 := \frac{l'(\lambda_0)}{l'(w_0)}$$

for any $[l] \in A$. Then

$$\varphi(p^{-1}([l'])) = \varphi(\mathcal{U}_i(O) \cap \Omega_i^n \cap A) = \{z \in \varphi(\Omega_i^n \cap A) | \ \forall \lambda \in T\colon |e_\lambda \cdot v_i + c_\lambda| \geq r \cdot |c_0|\}.$$

As $\varphi(\Omega_i^n \cap A)$ is a closed ball of $\mathbb{P}_C^{1,\mathrm{rig}}$ which is already contained in $\mathbb{A}_C^{1,\mathrm{rig}}$, thus so is $\varphi(p^{-1}([l']))$ by Proposition 4.26. This yields the lemma. □

As argued before Lemmas 4.28 and 4.29, they finish the proof. □

4.4. Quotients by discrete subgroups of codimension 1 *vector subspaces*

Suppose that $d := \dim_E(\mathcal{V}) \geq 2$. Consider any E-subspace $\mathcal{W} \subset \mathcal{V}$ of codimension 1, any $0 \neq w \in \mathcal{W}$, any $v \in \mathcal{V} \setminus \mathcal{W}$ and any discrete subgroup $\Gamma \subset \mathrm{Aut}_E(\mathcal{V})$ such that any $\gamma \in \Gamma$ restricts to the identity on $\mathcal{W}$ and satisfies that $\gamma(v) - v \in \mathcal{W}$.

If $\mathrm{Aut}_E(\mathcal{V})$ is identified with $\mathrm{GL}_d(E)$ via the choice of an ordered basis of $\mathcal{V}$ whose first $d-1$ vectors are an ordered basis of $\mathcal{W}$, then any $\gamma \in \Gamma$ is of the form

$$\begin{pmatrix} \mathrm{id} & * \\ 0 & 1 \end{pmatrix}.$$

Consider the admissible subvariety $\mathcal{E} \subset \mathbb{P}^{\mathrm{rig}}_{\mathcal{V}_C^*}$ of those $[l]$ for which $[l|_{\mathcal{W}_C}] \in \Omega_{\mathcal{W}}$; it is isomorphic to $\Omega_{\mathcal{W}} \times \mathbb{A}_C^{1,\mathrm{rig}}$ via

$$i\colon \mathcal{E} \to \Omega_{\mathcal{W}} \times \mathbb{A}_C^{1,\mathrm{rig}},\ [l] \mapsto \left([l|_{\mathcal{W}_C}], \frac{l(v)}{l(w)}\right). \tag{21}$$

For any $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$ set

$$\mathcal{E}(O) := i^{-1}(O \times \mathbb{A}_C^{1,\mathrm{rig}}) \text{ and } \mathcal{E}(O,n) := i^{-1}(O \times B_n),$$

where $B_n \subset \mathbb{A}_C^{1,\mathrm{rig}}$ denotes the closed ball of radius n around the origin. Thus $(\mathcal{E}(O,n))_{n\geq 1}$ is an admissible affinoid covering of $\mathcal{E}(O)$ for any admissible affinoid $O \subset \Omega_{\mathcal{W}}$. By construction, Γ acts on $\mathcal{E}$. Consider the quotient map

$$p_\Gamma\colon \mathcal{E} \to \Gamma\backslash\mathcal{E}$$

and endow its target with the structure of Grothendieck ringed space induced by p_Γ, that is, a subset (resp. a covering of a subset) of $\Gamma\backslash\mathcal{E}$ is admissible precisely when its preimage is admissible and the sections on an admissible subset of $\Gamma\backslash\mathcal{E}$ are the Γ-invariant sections on its preimage. Thus p_Γ restricts to the quotient map $\Omega_{\mathcal{V}} \to \Omega_\Gamma$ considered in Section 4.14 and $\Gamma\backslash\mathcal{E}$ contains the rigid analytic variety Ω_Γ as a Grothendieck ringed subspace. By Lemma 4.33 below, $\Gamma\backslash\mathcal{E}$ is in fact itself a rigid analytic variety.

Denote by $v_\Gamma \subset \mathcal{W}$ the image of the injective continuous homomorphism

$$\Gamma \to \mathcal{W},\ \gamma \mapsto v_\gamma := \gamma(v) - v;$$

it is a discrete subgroup of $\mathcal{W}$ as Γ is discrete in $\mathrm{Aut}_E(\mathcal{V})$. For any $[l] \in \Omega_{\mathcal{W}}$ thus $l(v_\Gamma) \subset C$ is strongly discrete, i.e., its intersection with every ball of finite radius is finite (see for instance [20, Ex. 2.48 and Lemma 2.49]). Set

$$\forall [l] \in \mathcal{E} \colon e([l]) := \frac{\mathrm{e}_{l(v_\Gamma)}(l(v))}{l(w)} = \mathrm{e}_{\frac{l(v_\Gamma)}{l(w)}}\left(\frac{l(v)}{l(w)}\right),$$

where $e_{l(v_\Gamma)} \colon \mathbb{A}_C^{1,\mathrm{rig}} \to \mathbb{A}_C^{1,\mathrm{rig}}$ is the analytic surjective group homomorphism with kernel $l(v_\Gamma)$ defined in Definition-Proposition 2.40. We are thus given a bijective map

$$\mathrm{e}_\Gamma \colon \Gamma\backslash\mathcal{E} \to \Omega_{\mathcal{W}} \times \mathbb{A}_C^{1,\mathrm{rig}}, p_\Gamma([l]) \mapsto ([l|_{\mathcal{W}_C}], e([l])).$$

Lemma 4.30. *The map $e \colon \mathcal{E} \to \mathbb{A}_C^{1,\mathrm{rig}}, [l] \mapsto e([l])$ is morphism of rigid analytic varieties over C.*

Proof. Any admissible affinoid covering $\mathcal{C}$ of $\Omega_{\mathcal{W}}$ yields via (21) the admissible affinoid covering $(\mathcal{E}(O,n))_{O\in\mathcal{C},n\geq 1}$ of $\mathcal{E}$. Consider any admissible affinoid non-empty $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$. It thus suffices to show that the restriction $Y := \mathcal{E}(O,n) \to C$ of e is regular; indeed, if the restriction is regular, its image is contained in an affinoid subvariety of $\mathbb{A}_C^{1,\mathrm{rig}}$ by Proposition 2.27 and hence it is a morphism of rigid analytic varieties by [8, Prop. 9.3.1.1]. Choose any $[l_0] \in O$. Set $L := v_\Gamma \subset \mathcal{W}$. Then $\frac{1}{l_0(w)} \cdot l_0(L) \subset C$ is strongly discrete. For any integer $k \geq 1$ thus

$$L_k := \left\{\lambda \in L \colon \left|\frac{l_0(\lambda)}{l_0(w)}\right| \leq k\right\} \subset L$$

is finite and hence the function

$$e_k \colon Y \to C, [l] \mapsto \frac{l(v)}{l(w)} \cdot \prod_{0\neq\lambda\in L_k}\left(1 - \frac{l(v)}{l(\lambda)}\right)$$

is a finite product of regular functions and thus regular. As the sup-norm on the ring of regular functions on Y is complete by [8, Theorem 6.2.4.1], it thus suffices to show that the e_k for all $k \geq 1$ converge uniformly to e. By means of Corollary 4.6, choose a $\kappa > 0$ such that

$$\forall [l], [l'] \in O, \forall 0 \neq x, y \in \mathcal{W} \colon \left|\frac{l(y)}{l(x)}\right| \leq \kappa \cdot \left|\frac{l'(y)}{l'(x)}\right|.$$

For any $k \geq 1$, any $\lambda \in L \setminus L_k$ and any $[l] \in Y$ then

$$\left|\frac{l(v)}{l(\lambda)}\right| \leq \left|\frac{l(v)}{l(w)}\right| \cdot \left|\frac{l(w)}{l(\lambda)}\right| \leq n \cdot \kappa \cdot \left|\frac{l_0(w)}{l_0(\lambda)}\right| < \frac{n \cdot \kappa}{k}$$

and, if $\frac{n \cdot \kappa}{k} < 1$, hence $\left|1 - \frac{l(v)}{l(\lambda)}\right| = 1$ and

$$\left|1 - \prod_{\lambda \in L \setminus L_k} \left(1 - \frac{l(v)}{l(\lambda)}\right)\right| \leq \frac{n \cdot \kappa}{k}. \tag{22}$$

Choose a $k_0 > n \cdot \kappa$. Using Proposition 2.27 and that Y is affinoid, choose a $c_0 > 0$ which bounds e_{k_0}. For any $k \geq k_0$ and any $[l] \in Y$ thus $|e_k([l])| = |e_{k_0}([l])| \leq c_0$ and, further using (22), hence

$$|e_k([l]) - e([l])| = |e_k([l])| \cdot \left|1 - \prod_{\lambda \in L \setminus L_k} \left(1 - \frac{l(v)}{l(\lambda)}\right)\right| \leq c_0 \cdot \frac{n \cdot \kappa}{k}.$$

This shows as desired that the e_k converge uniformly to e. □

Proposition 4.31. *The map* e_Γ *is an isomorphism of rigid analytic varieties. In particular, it restricts to an open immersion on* Ω_Γ*.*

In order to prove Proposition 4.31 we need the following lemmas. Since any $[l] \in \Omega_{\mathcal{V}}$ satisfies that $l(v) \notin l(v_\Gamma)$ and hence that $e([l]) \neq 0$, Proposition 4.31 will directly yield

Definition-Proposition 4.32. *The map*

$$q_\Gamma \colon \Omega_\Gamma \to \Omega_{\mathcal{W}} \times (\mathbb{A}^{1,\mathrm{rig}}_C \setminus \{0\}), p_\Gamma([l]) \mapsto \left([l|_{\mathcal{W}_C}], \frac{1}{e([l])}\right)$$

is an open immersion.

Lemma 4.33. *Consider any admissible affinoid covering* $\mathcal{C}$ *of* $\Omega_{\mathcal{W}}$*. Then*

$$(p_\Gamma(\mathcal{E}(O,n)))_{O \in \mathcal{C}, n \geq 1}$$

is an admissible covering of $\Gamma \backslash \mathcal{E}$ *and any* $p_\Gamma(\mathcal{E}(O,n))$ *is admissibly covered by finitely many affinoid varieties. In particular,* $\Gamma \backslash \mathcal{E}$ *is a rigid analytic variety.*

Proof. The covering $(\pi_\Gamma(\mathcal{E}(O)))_{O \in \mathcal{C}}$ of $\Gamma \backslash \mathcal{E}$ is the preimage of $\mathcal{C}$ under the natural morphism $\Gamma \backslash \mathcal{E} \to \Omega_{\mathcal{W}}$ and hence admissible. We consider any admissible affinoid $O \subset \Omega_{\mathcal{W}}$, set $Y := \mathcal{E}(O)$ and $Y_n := \mathcal{E}(O,n)$ for any $n \geq 1$ and are thus reduced to showing the claim that $\Gamma \backslash Y$ is admissibly covered by the $p_\Gamma(Y_n)$ and that each of them is admissibly covered by finitely many affinoid varieties. In order to prove the claim, we shall apply

Proposition 2.31 to the following setting: For any $n \geq 1$ denote by $\Gamma_n \subset \Gamma$ the subgroup of those elements γ such that

$$\forall\ [l] \in O : \left|\frac{l(v_\gamma)}{l(w)}\right| \leq n;$$

it is finite as $\frac{l(v_\Gamma)}{l(w)} \subset C$ is strongly discrete for any $[l] \in O$. Moreover, any Y_n is Γ_n-invariant. Furthermore, as $\Omega_{\mathcal{W}}$ and $\mathbb{A}_C^{1,\mathrm{rig}}$ are both separated, so is their product and hence $\mathcal{E}$ and hence the admissible subvariety $Y \subset \mathcal{E}$. It remains to verify the remaining Condition iii) of Proposition 2.31; i.e., that any $n \geq 1$ admits an $n' \geq n$ such that

$$\forall \gamma \in \Gamma \setminus \Gamma_{n'} : \gamma(Y_n) \cap Y_n = \emptyset. \tag{23}$$

In order to do so, choose, by means of Corollary 4.6, a $\kappa > 0$ such that

$$\forall\ [l], [l'] \in O, \forall\ 0 \neq x, y \in \mathcal{W} :\ \left|\frac{l(y)}{l(x)}\right| \geq \kappa \cdot \left|\frac{l'(y)}{l'(x)}\right|.$$

Consider any $n \geq 1$, choose any $n' \geq \frac{n}{\kappa}$ and consider any $\gamma \in \Gamma \setminus \Gamma_{n'}$. Thus

$$\exists\ [l'] \in O : \left|\frac{l'(v_\gamma)}{l'(w)}\right| > n'$$

which implies that

$$\forall\ [l] \in O : \left|\frac{l(v_\gamma)}{l(w)}\right| > \kappa \cdot n' \geq n$$

and hence that

$$\forall\ [l] \in Y_n :\ \left|\frac{l(\gamma v)}{l(w)}\right| = \left|\frac{l(v_\gamma)}{l(w)} + \frac{l(v)}{l(w)}\right| > n$$

or, equivalently, that $\gamma(Y_n) \cap Y_n = \emptyset$ as desired. $\square$

Lemma 4.34. *Any $[l] \in \mathcal{E}$ admits a basis of admissible neighborhoods such that $\gamma(U) \cap U = \emptyset$ for any U in this basis and any $\mathrm{id} \neq \gamma \in \Gamma$ and such that $(\gamma(U))_{\gamma \in \Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}$.*

Proof. Let $[l] \in \mathcal{E}$ and set $l_0 := l|_{\mathcal{W}_C}$. Associate with any admissible neighborhood O of $[l_0]$ in $\Omega_{\mathcal{W}}$ and any $\varepsilon \in |C^\times|$ the admissible neighborhood

$$X(O, \varepsilon) := \left\{[l'] \in \mathcal{E} \colon [l'|_{\mathcal{W}_C}] \in O \wedge \left|\frac{l'(v)}{l'(w)} - \frac{l(v)}{l(w)}\right| \leq \varepsilon\right\}$$

of $[l]$ in $\mathcal{E}$. Using that $l_0(v_\Gamma) \subset C$ is strongly discrete, we choose an $\varepsilon_0 > 0$ such that 0 is the only element in $l_0(v_\Gamma)$ whose norm is $\leq \varepsilon_0 \cdot |l_0(w)|$. Moreover, by Lemma 4.8, $[l_0]$ admits an admissible affinoid neighborhood O such that all elements in O induce the same class of norms on $\mathcal{W}$. Then the $X(O, \varepsilon)$ for all such O and all $\varepsilon_0 \geq \varepsilon \in |C^\times|$ form a desired basis of admissible neighborhoods of $[l]$. Indeed, consider any such O and ε and any $[l'] \in X(O, \varepsilon)$ and $\gamma \in \Gamma$ such that $\gamma[l'] \in X(O, \varepsilon)$. Then

$$\left|\frac{l_0(\gamma v - v)}{l_0(w)}\right| = \left|\frac{l'(\gamma v - v)}{l'(w)}\right| = \left|\frac{(\gamma^{-1}l')(v)}{(\gamma^{-1}l')(w)} - \frac{l(v)}{l(w)} + \frac{l(v)}{l(w)} - \frac{l'(v)}{l'(w)}\right| \leq \varepsilon$$

so that $\gamma v - v = 0$. As γ further restricts to the identity on $\mathcal{W}$, it is the identity as desired. In order to see that $(\gamma(X(O, \varepsilon))_{\gamma \in \Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}(O)$, it suffices, as $(\mathcal{E}(O, n))_{n \geq 1}$ is an admissible covering of $\mathcal{E}(O)$, to show for any $n \geq 1$ that $(\gamma(X(O, \varepsilon)) \cap \mathcal{E}(O, n))_{\gamma \in \Gamma}$ is an admissible covering of an admissible subset of $\mathcal{E}(O, n)$. However, this holds true for any $n \geq 1$ since, by Proposition 2.22, the intersection of the affinoid $\gamma(X(O, \varepsilon))$ with the affinoid $\mathcal{E}(O, n)$ is again affinoid for any $\gamma \in \Gamma$ and, by (23), empty for all but finitely many $\gamma \in \Gamma$. □

Lemma 4.35. *Consider any admissible $O \subset \Omega_{\mathcal{W}}$ and any integer $n \geq 1$. Then*

$$\mathrm{e}_\Gamma(p_\Gamma(\mathcal{E}(O, n))) \supset O \times B_n.$$

Moreover, if O is affinoid, then there exists an $m \geq 1$ with

$$\mathrm{e}_\Gamma(p_\Gamma(\mathcal{E}(O, n))) \subset O \times B_m.$$

Proof. Set $\mathcal{L} := v_\Gamma$. As $|\cdot|$ is non-Archimedean, any $[l] \in \mathcal{E}$ satisfies that

$$|e([l])| = \left|\frac{l(v)}{l(w)} \cdot \prod_{0 \neq \lambda \in \mathcal{L}} \frac{l(v) + l(\lambda)}{l(\lambda)}\right| = \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0 \neq \lambda \in \mathcal{L} \\ |l(\lambda)| \leq |l(v)|}} \left|\frac{l(v) + l(\lambda)}{l(\lambda)}\right|. \tag{24}$$

As $l(\mathcal{L})$ is strongly discrete for any $l \in \tilde{\Omega}_L$, any $x \in \Gamma \backslash \mathcal{E}$ is represented by some $[l] \in \mathcal{E}$ such that $|l(v)| \leq |l(v) + l(\lambda)|$ for any $\lambda \in \mathcal{L}$ and hence, by (24), such that $|\mathrm{e}_\Gamma(x)| = |e([l])| \geq \left|\frac{l(v)}{l(w)}\right|$. As e_Γ is surjective, this shows the first part. Suppose then that O is affinoid. By (24), any $[l] \in \mathcal{E}$ satisfies that

$$|\mathrm{e}([l])| \leq \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0 \neq \lambda \in \mathcal{L} \\ |l(\lambda)| \leq |l(v)|}} \left|\frac{l(v)}{l(\lambda)}\right| \leq \left|\frac{l(v)}{l(w)}\right| \cdot \prod_{\substack{0 \neq \lambda \in \mathcal{L} \\ |l(\lambda)| \leq |l(v)|}} \left|\frac{l(v)}{l(w)}\right| \cdot \left|\frac{l(w)}{l(\lambda)}\right|. \tag{25}$$

Moreover, any $[l] \in \mathcal{E}(O, n)$ and any $\lambda \in \mathcal{L}$ with $|l(\lambda)| \leq |l(v)|$ satisfy that $\left|\frac{l(\lambda)}{l(w)}\right| \leq \left|\frac{l(v)}{l(w)}\right| \leq n$. By (25), it thus suffices to show that for any $[l] \in \mathcal{E}(O, n)$ both the number of $\lambda \in \mathcal{L}$ with $\left|\frac{l(\lambda)}{l(w)}\right| \leq n$ as well as the norm $\left|\frac{l(w)}{l(\lambda)}\right|$ for any such λ is bounded from above by a constant depending only on O and n. Since O is affinoid, Corollary 4.6 provides a $\kappa > 0$ such that

$$\forall [l'], [l] \in O, \forall \lambda \in \mathcal{L}\colon \left|\frac{l'(\lambda)}{l'(w)}\right| \leq \kappa \cdot \left|\frac{l(\lambda)}{l(w)}\right|.$$

From this thus follows the second part as any $[l] \in O$ admits only finitely many $\lambda \in \mathcal{L}$ with $\left|\frac{l(\lambda)}{l(w)}\right| \leq \kappa \cdot n$ as $\frac{l(\mathcal{L})}{l(w)} \subset C$ is strongly discrete. $\square$

Proof of Proposition 4.31. By Lemma 4.33, e_Γ is a morphism between reduced rigid analytic varieties. As argued above, it is bijective. By Proposition 2.26, it thus remains to be shown that e_Γ induces isomorphism on the stalks and that there exists an admissible affinoid covering of $\Omega_\mathcal{W} \times \mathbb{A}_C^{1,\mathrm{rig}}$ such that the preimage under e_Γ of any of its elements is a finite union of affinoids. As

$$\frac{d}{dT}\left(e_{\frac{l(v_\Gamma)}{l(w)}}(T)\right) = 1,$$

the tangent map of $\mathrm{e}_\Gamma \circ p_\Gamma$ at any point is a triangular matrix with only ones on the diagonal with respect to a suitable basis and thus an isomorphism; thus it induces isomorphisms on the stalks (see [34, Part 2, Chapter 3.9, Theorem 2]). By Lemma 4.34, the quotient morphism p_Γ induces isomorphism on the stalks, too. Hence so does e_Γ. Moreover, the $O_n := O \times B_n$ for all admissible affinoid subsets $O \subset \Omega_\mathcal{W}$ and all $n \geq 1$ form an admissible affinoid covering of $\Omega_\mathcal{W} \times \mathbb{A}_C^{1,\mathrm{rig}}$. Consider any such O_n and set $X := p_\Gamma(\mathcal{E}(O, n))$. By means of Lemma 4.35 choose an $m \geq n$ with $O_n \subset \mathrm{e}_\Gamma(X) \subset O_m$. For any affinoid $X' \subset X$ then $\mathrm{e}_\Gamma^{-1}(O_n) \cap X'$ is the preimage of the affinoid subset O_n under the morphism $X' \to O \times B_m$ between affinoid varieties induced by e_Γ and is thus [8, Proposition 7.2.2.4] itself affinoid. As, by Lemma 4.33, X is admissibly covered by finitely many such affinoid subvarieties X', the preimage $\mathrm{e}_\Gamma^{-1}(O_n)$ is thus a finite union of affinoid subsets. As argued before, e_Γ is thus an isomorphism. Finally, by Lemma 4.4, the Γ-invariant $\Omega_\mathcal{V} \subset \mathcal{E}$ is an admissible subvariety. Hence so is $\Omega_\Gamma \subset \Gamma\backslash\mathcal{E}$. Thus the restriction of e_Γ to Ω_Γ is an open immersion. $\square$

Proposition 4.36. *Let A and $\Lambda \subset \mathcal{V}$ be as in Example 4.16 and suppose that $b \cdot L \subset v_\Gamma$ for some $0 \neq b \in A$. Let $O \subset \Omega_\mathcal{W}$ be admissible affinoid. Then*

i) any $\varepsilon > 0$ admits an $r > 0$ such that $q_\Gamma^{-1}(O \times B_\varepsilon) \supset p_\Gamma(\mathcal{U}_\mathcal{V}(\Lambda, O, r))$,
ii) any $r > 0$ admits an $\varepsilon > 0$ such that $q_\Gamma^{-1}(O \times B_\varepsilon) \subset p_\Gamma(\mathcal{U}_\mathcal{V}(\Lambda, O, r))$,

where $\mathcal{U}_\mathcal{V}(\Lambda, O, r) \subset \Omega_\mathcal{V}$ is the subset defined before Lemma 4.17.

Proof. Set $L := \Lambda \cap \mathcal{W}$ and $\mathcal{L} := v_\Gamma \subset L$ and $\mathcal{U}(O, r) := \mathcal{U}_\mathcal{V}(\Lambda, O, r)$ for any $r \in |C|$. As O is affinoid, Corollary 4.6 provides a $\kappa > 0$ such that

$$\forall [l],[l'] \in O, \forall w'', w' \in \mathcal{W}\setminus\{0\}\colon \left|\frac{l(w'')}{l(w')}\right| \leq \kappa \cdot \left|\frac{l'(w'')}{l'(w')}\right|. \tag{26}$$

Choose for any $[l] \in O$ an $0 \neq w_l \in L$ such that

$$|l(w_l)| = \inf_{0\neq\lambda\in L} |l(\lambda)|.$$

using that $l(L)$ is strongly discrete by Proposition 2.36. Then

$$\forall [l],[l'] \in O\colon \left|\frac{l(w)}{l(w_l)}\right| \leq \kappa \cdot \left|\frac{l'(w)}{l'(w_l)}\right| \leq \kappa \cdot \left|\frac{l'(w)}{l'(w_{l'})}\right|.$$

In particular, there exists an $s > 0$ such that

$$\forall [l] \in O\colon \left|\frac{l(w)}{l(w_l)}\right| \leq s.$$

For any $[l] \in \mathcal{E}(O)$ set $w_l := w_{l|_{\mathcal{W}_C}}$ and choose a $v_l \in \Lambda \setminus L$ such that

$$|l(v_l)| = \inf_{\lambda\in\Lambda\setminus L} |l(\lambda)|.$$

Part i) holds true as for any $r \in |C^\times|$ and any $[l] \in \mathcal{U}(O,r)$ holds that

$$\left|\frac{l(w)}{l(v+\lambda)}\right| \leq \left|\frac{l(w)}{l(v_l)}\right| = \left|\frac{l(w_l)}{l(v_l)}\cdot\frac{l(w)}{l(w_l)}\right| \leq \frac{s}{r}$$

and hence that

$$\left|\frac{1}{e([l])}\right| = \left|\frac{l(w)}{e_{l(\mathcal{L})}(l(v))}\right| = \left|\sum_{\lambda\in\mathcal{L}}\frac{l(w)}{l(v+\lambda)}\right| \leq \frac{s}{r}.$$

Let us then show Part ii). As $\frac{1}{l(w_l)} \cdot l(L)$ is co-compact in $l(\mathcal{W})$ for any $[l] \in O$, there exists an $r([l]) > 0$ such that

$$\forall x \in \mathcal{W}\ \exists \lambda \in L\colon \left|\frac{l(x-\lambda)}{l(w_l)}\right| \leq r([l]). \tag{27}$$

Using (26) and that $|l'(w_l)| \geq |l'(w_{l'})|$ for any $[l],[l'] \in O$, we may and do choose the $r([l])$ to be uniformly bounded. As $\mathcal{U}(O,r) \subset \mathcal{U}(O,r')$ for any $r \geq r' > 0$, it thus suffices to show Part ii) only for any $r \in |C|$ such that $r > r([l])$ for any $[l] \in O$. Consider any such r. By surjectivity of e_Γ, it suffices to find an $\varepsilon > 0$ such that

$$p_\Gamma(\mathcal{E}(O)\setminus\mathcal{U}(O,r)) \subset \mathrm{e}_\Gamma^{-1}(O \times B_{\frac{1}{\varepsilon}}).$$

By the second part of Lemma 4.35, it thus suffices to find an $n \geq 1$ with

$$\mathcal{E}(O) \setminus \mathcal{U}(O,r) \subset \bigcup_{\gamma\in\Gamma} \gamma(\mathcal{E}(O,n)). \tag{28}$$

Using the assumption, choose $0 \neq b \in A$ such that $b \cdot L \subset \mathcal{L}$. Using that $\Lambda/(A \cdot v + L)$ is a torsion A-module, we further choose a $c \in A$ such that $c \cdot \Lambda \subset A \cdot v + L$. We claim that any $n \geq \max\{|c|, |b|\} \cdot r$ satisfies (28). Consider any $[l] \in \mathcal{E}(O) \setminus \mathcal{U}(O,r)$. Then

$$\left|\frac{l(v_l)}{l(w_l)}\right| < r;$$

indeed, if $[l] \in \Omega_\Lambda$, this follows from the definition of $\mathcal{U}(O,r)$ and if $[l] \notin \Omega_\Lambda$, then $l(\mathcal{V}) = l(\mathcal{W})$ so that (27) provides for any $x \in \mathcal{W}$ with $l(v_l) = l(x)$ an $\lambda \in L$ with $\left|\frac{l(x-\lambda)}{l(w_l)}\right| < r$ so that

$$|l(v_l)| \leq |l(v_l - \lambda)| = |l(x-\lambda)| < r \cdot |l(w_l)|.$$

Let $a \in A$ and $\lambda' \in L$ be such that $c \cdot v_l = a \cdot v + \lambda'$. Using (27), choose a $\lambda \in \mathcal{L}$ with $\left|\frac{l(\frac{\lambda'}{a}-\lambda)}{l(w_l)}\right| \leq |b| \cdot r$. Write $\lambda = \gamma(v) - v$ for a unique $\gamma \in \Gamma$. Then

$$\begin{aligned}\left|\frac{(\gamma^{-1}l)(v)}{(\gamma^{-1}l)(w)}\right| &= \left|\frac{l(v+\lambda)}{l(w)}\right| \leq \left|\frac{l(v+\lambda)}{l(w_l)}\right| = \left|\frac{l(v+\frac{\lambda'}{a}+\lambda-\frac{\lambda'}{a})}{l(w_l)}\right| \\ &\leq \max\left\{\frac{|c|}{|a|}\cdot\left|\frac{l(v_l)}{l(w_l)}\right|, \left|\frac{l(\frac{\lambda'}{a}-\lambda)}{l(w_l)}\right|\right\} \leq \max\{|c|,|b|\}\cdot r\end{aligned}$$

which yields the claim and hence Part ii). $\square$

5. Compactification of analytic irreducible components

Consider any algebraically closed complete non-Archimedean valued field C of finite characteristic, any admissible coefficient subring $A \subset C$ (see Definition 2.33), any finitely generated projective A-module $\Lambda \neq 0$ and any congruence subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. Let $E \subset C$ be the smallest local field containing A.

Notation 5.1. For any direct summand $L \subset \Lambda$ denote by Γ_L, resp. $\mathring{\Gamma}_L$, the normalizer, resp. centralizer, of L in Γ, i.e.,

- $\Gamma_L := \{\gamma \in \Gamma\colon\ \gamma(L) = L\} \subset \Gamma$, resp.
- $\mathring{\Gamma}_L := \{\gamma \in \Gamma_L\colon\ \gamma|_L = \mathrm{id}_L\} \subset \Gamma_L$, and set
- $\overline{\Gamma}_L := \{\gamma|_L \in \mathrm{Aut}_A(L)\colon \gamma \in \Gamma_L\}$.

5.1. Grothendieck topology on the pre-quotient

Let Ω^*_Λ be the set-theoretic disjoint union of all the $\Omega_L := \Omega_{L_E}$ for all direct summands $0 \neq L \subset \Lambda$. In this section we endow Ω^*_Λ with the structure of Grothendieck topological space whose quotient Ω^*_Γ by some natural Γ-action, introduced in the next section, will be the Grothendieck topological space underlying the desired compactification of Ω_Γ.

Definition 5.2. For any subset $S \subset C$ let $\mathrm{d}(S) := \inf\limits_{0 \neq s \in S} |s|$.

Definition 5.3. For any direct summand $0 \neq L \subset \Lambda$ and any admissible $O \subset \Omega_L$ and any $r \in |C|$ denote by

$$\mathcal{U}(\Lambda, O, r)$$

the subset of Ω^*_Λ of all elements $[l]$ with $[l] \in \Omega_{L'}$ for some $L \subset L' \subset \Lambda$ for which

i) $[l|_{L_C}] \in O$ and
ii) $\mathrm{d}(l(L' \setminus L)) \geq r \cdot \mathrm{d}(l(L))$.

Definition-Proposition 5.4. *Endow Ω^*_Λ with the following Grothendieck topology: A subset $Y \subset \Omega^*_\Lambda$ is* admissible *if for every direct summand $0 \neq L \subset \Lambda$*

i) the subset $Y \cap \Omega_L \subset \Omega_L$ is admissible and
ii) if every affinoid $O \subset Y \cap \Omega_L$ admits an $r \in |C|$ with $\mathcal{U}(\Lambda, O, r) \subset Y$.

Moreover, a covering of an admissible subset of Ω^*_Λ by admissible subsets is admissible *if its intersection with every Ω_L is admissible.*

Proof. All properties required by Definition 2.2 follow directly from the corresponding ones of the Grothendieck topological spaces Ω_L and the fact that any admissible covering of an affinoid subset has a finite subcovering. □

Example 5.5. Any subset $\mathcal{U}(\Lambda, O, r)$ as in Definition 5.3 is admissible.

Proof. Consider any admissible $O \in \Omega_L$ and any $r \in |C|$. By Lemma 4.17, the intersection of $U := \mathcal{U}(\Lambda, O, r)$ with any $\Omega_{L'}$ is an admissible subset of $\Omega_{L'}$ for any further direct summand $L \subset L' \subset \Lambda$. Consider then any affinoid $O' \subset U \cap \Omega_{L'}$ for any such L'. We shall show that $\mathcal{U}(\Lambda, O', r') \subset U$ for some $r' \in |C|$. Using Corollaries 4.6 and 4.7 and that O' is affinoid, choose an $\varepsilon > 0$ such that

$$\forall [l'] \in O' \colon\ \mathrm{d}(l'(L')) \geq \varepsilon \cdot \mathrm{d}(l'(L)).$$

Let $\frac{r}{\varepsilon} \leq r' \in |C|$ and $[l] \in \mathcal{U}(\Lambda, O', r') \cap \Omega_{L''}$ for any $L' \subset L'' \subset \Lambda$. Then

$$\begin{aligned}\mathrm{d}(l(L''\setminus L)) &\geq \min\{\mathrm{d}(l(L''\setminus L')), \mathrm{d}(l(L'\setminus L))\}\\ &\geq \min\{r'\cdot \mathrm{d}(l(L')), r\cdot \mathrm{d}(l(L))\} \geq r\cdot \mathrm{d}(l(L)).\end{aligned}$$

As $[l|_{L'}] \in O' \subset U$, also $[l|_L] \in O$. Thus $[l] \in U$. Hence $\mathcal{U}(\Lambda, O, r') \subset U$. Hence U is indeed admissible. □

Corollary 5.6. *Consider any admissible $Y \subset \Omega^*_\Lambda$ and for any direct summand $0 \neq L \subset \Lambda$ an admissible covering $\mathcal{C}_L$ of $Y \cap \Omega_L$ and an $r_O \in |C|$ for any $O \in \mathcal{C}_L$ such that $U_O := \mathcal{U}(\Lambda, O, r_O) \subset Y$. Then the covering $\mathcal{C}$ of Y by all these U_O is admissible.*

Proof. By Example 5.5, any U_O is an admissible subset of Y. Moreover, the intersection of $\mathcal{C}$ with any boundary component Ω_L is refined by the admissible $\mathcal{C}_L$ and is thus, by Property vii) of Definition 2.2, itself admissible. Hence $\mathcal{C}$ is indeed admissible. □

Corollary 5.7. *For any direct summand $0 \neq L \subset \Lambda$, any $[l] \in \Omega_L$ and any countable neighborhood basis $(O_n)_{n\geq 1}$ of $[l]$ in Ω_L the system $(\mathcal{U}(\Lambda, O_n, r_n))_{n\geq 1}$ is a countable neighborhood basis of $[l]$ in Ω^*_Λ for any unbounded sequence $\{r_n\}_{n\geq 1} \subset |C|$.*

Proof. This follows directly from Example 5.5 and Definition 5.4, i). □

Corollary 5.8. *The canonical topology on Ω^*_Λ is first countable.*

Corollary 5.9. *Let $Y \subset \Omega^*_\Lambda$ be admissible. With respect to the canonical topologies, then a function $f\colon Y \to C$ is continuous if and only if it is sequentially continuous.*

5.2. Structure of Grothendieck graded ringed space

For any direct summand $0 \neq L \subset \Lambda$ denote by $\tilde{\Omega}_L$ the preimage of Ω_L under the quotient-by-$C^\times$ morphism $\mathbb{A}^{\mathrm{rig}}_{L^*_C} \setminus \{0\} \to \mathbb{P}^{\mathrm{rig}}_{L^*_C}$; it consists precisely of the C-linear maps $l\colon L_C \to C$ for which $\mathrm{Ker}(l) \cap L_E = 0$.

Definition 5.10. Let $\tilde{\Omega}^*_\Lambda$ be the set-theoretic disjoint union of all such $\tilde{\Omega}_L$ equipped with the induced $C^\times$-action and the $C^\times$-equivariant action of $\mathrm{Aut}_A(\Lambda)$ defined by

$$\forall \gamma \in \mathrm{Aut}_A(\Lambda), l \in \tilde{\Omega}_L\colon \gamma(l) := l \circ (\gamma^{-1}|_{(\gamma(L))_C}) \in \tilde{\Omega}_{\gamma(L)}.$$

The induced action of $\mathrm{Aut}_A(\Lambda)$ on Ω^*_Λ is by isomorphism of Grothendieck topological spaces by

Lemma 5.11. *For any $\mathcal{U}(\Lambda, O, r) \subset \Omega^*_\Lambda$ as in Definition 5.3 holds that*

$$\forall \gamma \in \mathrm{Aut}_A(\Lambda)\colon \gamma(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, \gamma(O), r). \tag{29}$$

Proof. This is directly checked. □

Definition 5.12. Consider the quotient map

$$p_\Gamma \colon \Omega_\Lambda^* \to \Gamma\backslash\Omega_\Lambda^* =: \Omega_\Gamma^*$$

and endow its target with the structure of Grothendieck topological space which it induces, that is, a subset (resp. a covering of a subset) of Ω_Γ^* is admissible precisely when its preimage is admissible.

Remark 5.13. The induced canonical topology on Ω_Γ^* was introduced by Kapranov [25] in the case, where A is a polynomial ring.

Example 5.14. Consider any direct summand $0 \neq L \subset \Lambda$, any admissible quasi-compact $O \subset \Omega_L$ and any $1 < r \in |C|$. Then the $\gamma(\mathcal{U}(\Lambda, O, r))$ for all $\gamma \in \Gamma$ form an admissible covering of an admissible subset of Ω_Λ. In particular, $p_\Gamma(\mathcal{U}(\Lambda, O, r)) \subset \Omega_\Gamma^*$ is admissible. Moreover, if O is connected, then so is $p_\Gamma(\mathcal{U}(\Lambda, O, r)) \cap \Omega_\Gamma \subset \Omega_\Gamma$.

Proof. Set $U := \mathcal{U}(\Lambda, O, r)$. Let us show the first assertion. By construction, any $\gamma(U)$ depends only on γ modulo $\mathring{\Gamma}_L$. By means of Example 5.5, Lemma 4.17 and any admissible affinoid covering of any stratum $\Omega_{L'}$, it suffices to show for any admissible affinoid subset O' of any $\Omega_{L'}$ that $\gamma(U) \cap O' = \emptyset$ for all but finitely many $\gamma \in \Gamma$ modulo $\mathring{\Gamma}_L$. Thus consider any such admissible affinoid $O' \subset \Omega_{L'}$. We assume that $L' = \Lambda$ and that $U \cap O' \neq \emptyset$; the general case is directly reduced to this case. Using that O, resp. O', is quasi-compact, choose a $\kappa > 0$, resp. $\kappa' > 0$, which satisfies the property in Corollary 4.6, iv) with respect to O and L_E, resp. O' and Λ_E. Choose any basis β of L_F that is contained in L and choose any $[l] \in U \cap O'$. Choose a $\lambda \in L$ such that $|l(\lambda)| = \mathrm{d}(l(L))$. Consider any $\gamma \in \Gamma$ for which $\gamma(U) \cap O' \neq \emptyset$ and choose any $[l'] \in \gamma(U) \cap O'$. Choose a $\lambda' \in \gamma L$ with $|l'(\lambda')| = \mathrm{d}(l'(\gamma L)) \overset{r \geq 1}{=} \mathrm{d}(l'(\Lambda))$. For any $v \in \beta$ then

$$\begin{aligned}
\left|\frac{l(\gamma v)}{l(\lambda)}\right| &\leq \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\lambda)}\right| \leq \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\lambda')}\right| = \kappa' \cdot \left|\frac{l'(\gamma v)}{l'(\gamma\lambda)}\right| \cdot \left|\frac{l'(\gamma\lambda)}{l'(\lambda')}\right| \\
&\overset{*}{\leq} \kappa' \cdot \kappa^2 \cdot \left|\frac{(\gamma l)(\gamma v)}{(\gamma l)(\gamma\lambda)}\right| \cdot \left|\frac{(\gamma l)(\gamma\lambda)}{(\gamma l)(\lambda')}\right| = \kappa' \cdot \kappa^2 \cdot \left|\frac{l(v)}{l(\lambda)}\right| \cdot \left|\frac{(l(\lambda)}{l(\gamma^{-1}\lambda')}\right| \\
&\leq \kappa' \cdot \kappa^2 \cdot \left|\frac{l(v)}{l(\lambda)}\right|,
\end{aligned}$$

where at $\overset{*}{\leq}$ we have used that $[(\gamma l)|_{(\gamma L)_C}] \in \gamma(O)$ as $[l|_{L_C}] \in O$ and that, as is directly checked, the constant κ also satisfies the property in Corollary 4.6, iv) for $\gamma(O)$ and $(\gamma L)_E$. As $l(\Lambda)$ is strongly discrete, thus $l(\gamma(\beta))$ lies in a finite subset of $l(\Lambda)$ that depends not on γ. As such a γ modulo $\mathring{\Gamma}_L$ is uniquely determined by its action on β, there exists thus indeed only finitely many γ modulo $\mathring{\Gamma}_L$ satisfying the above inequality and hence that $\gamma(U) \cap O' \neq \emptyset$. Moreover, if O is connected, then so is $U \cap \Omega_\Lambda$ by Theorem 4.20 and hence so is its admissible image $p_\Gamma(U) \cap \Omega_\Gamma = p_\Gamma(U \cap \Omega_\Lambda) \subset \Omega_\Gamma$. $\square$

Corollary 5.15. *The map p_Γ is open with respect to the canonical topologies.*

Proof. This follows from Definition-Proposition 5.4 and Examples 5.5 and 5.14. $\square$

Corollary 5.16. *Any point in Ω_Γ^* has a fundamental basis of admissible neighborhoods whose intersection with Ω_Γ is connected and irreducible.*

Proof. As Ω_Γ is normal by Proposition 4.13, its irreducible and connected subsets coincide by [10, Definition 2.2.2]. The corollary thus follows from Example 5.14 and Corollary 5.7 using that any point in any stratum Ω_L has a basis of connected admissible affinoid neighborhoods in Ω_L. $\square$

Example 5.17. Consider any admissible subset $X \subset \Omega_\Gamma^*$. For any direct summand $0 \neq L \subset \Lambda$ choose an admissible affinoid covering $\mathcal{C}_L$ of $p_\Gamma^{-1}(X) \cap \Omega_L$ and for any $O \in \mathcal{C}_L$ an $r_O \in |C|$ for which $\mathcal{U}(\Lambda, O, r_O) \subset p_\Gamma^{-1}(X)$. Then the covering $\mathcal{C}$ of X by the $p_\Gamma(\mathcal{U}(\Lambda, O, r_O))$ for all O in all $\mathcal{C}_L$ is admissible.

Proof. By Example 5.14, any element of $\mathcal{C}$ is admissible. Let $\mathcal{D}$ be the covering of $p_\Gamma^{-1}(X)$ defined as the preimage of $\mathcal{C}$ under p_Γ. It remains to check that the intersection of $\mathcal{D}$ with any Ω_L is an admissible covering of $p_\Gamma^{-1}(X) \cap \Omega_L$. However, by construction, such an intersection is refined by $\mathcal{C}_L$ and is thus admissible as desired. $\square$

Proposition 5.18. *Consider any direct summand $0 \neq L \subset \Lambda$, any $\omega \in \Omega_L$ and any admissible affinoid neighborhood $O \subset \Omega_L$ of ω satisfying the properties in Proposition 4.14, i.e., that*

$$\forall \gamma \in \Gamma_\omega \colon \gamma(O) = O \text{ and } \forall \gamma \in \Gamma \setminus \Gamma_\omega \colon \gamma(O) \cap O = \emptyset.$$

Then there exists $r_0 > 0$ such that for any $r_0 < r \in |C|$ the subset $U_r := \mathcal{U}(\Lambda, O, r) \subset \Omega_\Lambda^*$ satisfies that

$$\forall \gamma \in \Gamma_\omega \colon \gamma(U_r) = U_r \text{ and } \forall \gamma \in \Gamma \setminus \Gamma_\omega \colon \gamma(U_r) \cap U_r = \emptyset.$$

Proof. From the equality $\gamma(U_r) = \mathcal{U}(\Lambda, \gamma(O), r)$ for all $\gamma \in \Gamma$ directly follows that $\gamma(U_r) = U_r$ for all $\gamma \in \Gamma_\omega$ and $\gamma(U_r) \cap U_r = \emptyset$ for all $\gamma \in \Gamma_L \setminus \Gamma_\omega$. Using Corollary 4.6 and that O is affinoid, choose for the remaining assertion a $\kappa > 0$ such that

$$\forall [l], [l'] \in O \colon \forall 0 \neq x, y \in L \colon \frac{|l(y)|}{|l(x)|} \leq \kappa \cdot \frac{|l'(y)|}{|l'(x)|}.$$

Choose any $[l_0] \in O$ and let $0 \neq \lambda_0 \in L$ be such that $\mathrm{d}(l_0(L)) = |l_0(\lambda_0)|$. For any $[l]$ and any $0 \neq \lambda \in L$ with $\mathrm{d}(l(L)) = |l(\lambda)|$ then

$$\frac{|l(\lambda_0)|}{\mathrm{d}(l(L))} \leq \kappa \cdot \frac{|l_0(\lambda_0)|}{|l_0(\lambda)|} \leq \kappa \tag{30}$$

and, denoting by $\mu_{\max}$ the last successive minimum (see Definition 2.37),

$$\frac{\mu_{\max}(l(L))}{\mathrm{d}(l(L))} \overset{(30)}{\leq} \kappa \cdot \frac{\mu_{\max}(l(L))}{|l(\lambda_0)|} \leq \kappa^2 \cdot \frac{\mu_{\max}(l_0(L))}{|l_0(\lambda_0)|} =: r_0.$$

Let $\gamma \in \Gamma \setminus \Gamma_L$ and $r_0 < r \in |C|$ and suppose, by contradiction, the existence and choice of an $[l] \in \gamma(U_r) \cap U_r$. Then $L' := \gamma(L) \not\subset L \not\subset L'$. Choose $\lambda \in L \setminus L'$ with $|l(\lambda)| \leq \mu := \mu_{\max}(l(L))$ and $\lambda' \in L' \setminus L$ with $|l(\lambda')| \leq \mu' := \mu_{\max}(l(L'))$. If $\mu' \leq \mu$, we get the contradiction that

$$|l(\lambda')| \leq \mu' \leq \mu < r \cdot \mathrm{d}(l(L)) \overset{[l]\in U_r}{\leq} \mathrm{d}(l(L' \setminus L)) \leq |l(\lambda')|.$$

By symmetry, we also get a contradiction if $\mu \leq \mu'$ using that $\mu' \leq r_0 \cdot \mathrm{d}(l(L'))$; the latter holds true since all $[l'] \in \gamma(O)$ satisfy that

$$\mu_{\max}(l'(L')) = \mu_{\max}(\gamma^{-1}(l')(L)) \leq r_0 \cdot \mathrm{d}(\gamma^{-1}(l')(L)) = r_0 \cdot \mathrm{d}(l'(L')))$$

and since $[l] \in \gamma(U_r) = \mathcal{U}(\Lambda, \gamma(O), r)$. $\square$

Definition-Proposition 5.19. *For any orbit $\mathfrak{O}$ of the natural Γ-action on the set of non-zero direct summands let*

$$\Omega_{\mathfrak{O}} := p_\Gamma\left(\bigcup_{L\in\mathfrak{O}} \Omega_L\right)$$

be equipped with the structure of Grothendieck ringed space turning the natural map $\Omega_{\overline{\Gamma}_L} \to \Omega_{\mathfrak{O}}$ into an isomorphism for every $L \in \mathfrak{O}$. Then a subset $X \subset \Omega_\Gamma^*$ is admissible if and only if for every such orbit $\mathfrak{O}$:

i) $X \cap \Omega_{\mathfrak{O}} \subset \Omega_{\mathfrak{O}}$ is admissible and
ii) every admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$ with $Y \subset X$ admits an $r \geq 0$ with

$$\mathcal{U}(\Lambda, Y, r) := p_\Gamma\left(\bigcup_{L\in\mathfrak{O}} \mathcal{U}(\Lambda, p_\Gamma^{-1}(Y) \cap \Omega_L, r)\right) \subset X.$$

Moreover, a covering of an admissible subset $X \subset \Omega_\Gamma^*$ by admissible subsets is admissible precisely if its intersection with $X \cap \Omega_{\mathfrak{O}}$ is admissible for every orbit $\mathfrak{O}$.

Proof. Consider any $X \subset \Omega_\Gamma^*$. For any orbit $\mathfrak{O}$ the subset $X \cap \Omega_{\mathfrak{O}} \subset \Omega_{\mathfrak{O}}$ is admissible if and only if for every $L \in \mathfrak{O}$ the subset $p_\Gamma^{-1}(X) \cap \Omega_L$ is admissible. Thus i) holds true for every orbit $\mathfrak{O}$ if and only if $p_\Gamma^{-1}(X) \cap \Omega_L \subset \Omega_L$ is admissible for every direct summand $0 \neq L \subset \Lambda$. We assume that these equivalent statements hold true. Similarly, it directly follows that a covering of X by admissible subsets is admissible if and only if its intersection with $X \cap \Omega_{\mathfrak{O}}$ is admissible for every such $\mathfrak{O}$. We are thus reduced to showing that ii) holds true for every such $\mathfrak{O}$ if and only if every such L and every affinoid $O \subset p_\Gamma^{-1}(X) \cap \Omega_L$ admit an $r \in |C|$ for which $\mathcal{U}(\Lambda, O, r) \subset p_\Gamma^{-1}(X)$. First assume the first property and consider any such affinoid $O \subset \Omega_L$ and denote by $\mathfrak{O}$ the Γ-orbit of L. For any $r \in |C|$ then

$$p_\Gamma(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, p_\Gamma(O), r)$$

is admissible by Example 5.14; in particular, $p_\Gamma(O) \subset \Omega_{\mathfrak{O}}$ is admissible and, being the image of O, quasi-compact. Thus ii) provides a desired $r \in |C|$. Conversely assume the second property and consider any admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$. Choose any $L \in \mathfrak{O}$ and an admissible affinoid covering $\mathcal{C}_L$ of $p_\Gamma^{-1}(Y) \cap \Omega_L$ and for any $O \in \mathcal{C}_L$ an $r_O \in |C|$ such that $\mathcal{U}(\Lambda, O, r_O) \subset p_\Gamma^{-1}(X)$. Then $(p_\Gamma(O))_{O \in \mathcal{C}_L}$ is a covering of Y; it is in fact admissible since any of its elements is admissible by Example 5.14 since the preimage in Ω_L of the covering has the admissible refinement $\mathcal{C}_L$ and is thus itself admissible. By quasi-compactness of Y, we may thus choose finitely many $p_\Gamma(O_1), \ldots, p_\Gamma(O_n)$ which cover Y. For any $r \in |C|$ greater than the r_{O_i} for all $1 \leq i \leq n$ then

$$\mathcal{U}(\Lambda, Y, r) \subset \bigcup_{1 \leq i \leq n} \mathcal{U}(\Lambda, p_\Gamma(O_i), r_{O_i}) = p_\Gamma\left(\bigcup_{1 \leq i \leq n} \mathcal{U}(\Lambda, O_i, r_{O_i})\right) \subset X.$$

This yields the converse direction and finishes the proof. □

Definition-Proposition 5.20. *Consider any integer k and any admissible $Y \subset \Omega_\Lambda^*$ with preimage $\tilde{Y} \subset \tilde{\Omega}_\Lambda^*$. A function $f \colon \tilde{Y} \to C$ is called* weight k regular *if*

i) $\forall c \in C^\times, l \in \tilde{Y} \colon f(c \cdot l) = c^{-k} \cdot f(l)$

ii) and every direct summand $0 \neq L \subset \Lambda$, every admissible affinoid $O \subset Y \cap \Omega_L$ and one, and hence every, $0 \neq \lambda \in L$, admit an $r \in |C|$ such that $\mathcal{U} := \mathcal{U}(\Lambda, O, r) \subset Y$ and

$$\mathcal{U} \to C, [l] \mapsto f(l) \cdot l(\lambda)^k \tag{31}$$

is bounded, continuous with respect to the canonical topologies and restricts to a regular (see Definition 2.17) function $\mathcal{U} \cap \Omega_{L'} \to C$ for every direct summand $0 \neq L' \subset \Lambda$.

Proof. If for some $0 \neq \lambda \in L$ the function in (31) is bounded and continuous, then the same holds true for any such λ since for any $0 \neq \lambda', \lambda \in L$ the regular function

$$\mathcal{U} \to C, [l] \mapsto \frac{l(\lambda)}{l(\lambda')}$$

is continuous and bounded; indeed the function is continuous as it factors through the continuous restriction morphism $\mathcal{U} \to O$ and it is bounded by Proposition 2.27 applied to the affinoid O. □

Definition-Proposition 5.21. *For any admissible $X \subset \Omega_\Gamma^*$ and any integer k let $\mathcal{O}_\Gamma^*(k)(X)$ be the set of Γ-invariant weight k regular functions $\pi_\Gamma^{-1}(X) \to C$, where*

$$\pi_\Gamma \colon \tilde{\Omega}_\Lambda^* \to \Omega_\Gamma^* = \Gamma \backslash \tilde{\Omega}_\Lambda^* / C^\times$$

is the double quotient map. By means of the ring structure on C, then

i) the $\mathcal{O}_\Gamma^*(0)(X)$ for all admissible subsets $X \subset \Omega_\Gamma^*$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}_\Gamma^*$ of rings on Ω_Γ^*, called structure sheaf *on Ω_Γ^*, and*
ii) for any integer k the $\mathcal{O}_\Gamma^*(k)(X)$ for all admissible $X \subset \Omega_\Gamma^*$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}_\Gamma^*(k)$ of $\mathcal{O}_\Gamma^*$-modules on Ω_Γ^*, called k-th twisting $\mathcal{O}_\Gamma^*$-module *and*
iii) a sheaf $\mathcal{R}_\Gamma^*$ of graded $\mathcal{O}_\Gamma^*$-algebras on Ω_Γ^* is formed by the

$$\mathcal{R}_\Gamma^*(X) := \sum_{k \in \mathbb{Z}} \mathcal{O}_\Gamma^*(k)(X)$$

for all admissible $X \subset \Omega_\Gamma^*$ together with the natural restriction homomorphisms.

In particular, $(\Omega_\Gamma^*, \mathcal{O}_\Gamma^*)$ (resp. $(\Omega_\Gamma^*, \mathcal{R}_\Gamma^*)$) is a Grothendieck (graded) ringed space containing the rigid analytic variety Ω_Γ as an admissible Grothendieck ringed subspace.

Proof. This is directly checked. □

Proposition 5.22. *Consider any admissible $X \subset \Omega_\Gamma^*$. Then precomposition with the restriction $\pi_\Gamma^{-1}(X) \to X$ of π_Γ induces a bijection to $\mathcal{O}_\Gamma^*(X)$ from the set $\mathcal{O}_\Gamma'(X)$ of functions $s \colon X \to C$ that are continuous with respect to the canonical topologies, that restrict to a regular function on $X \cap \Omega_{\mathfrak{O}}$ for every Γ-orbit $\mathfrak{O}$ and that are bounded on $\mathcal{U}(\Lambda, Y, r)$ for every admissible quasi-compact $Y \subset \Omega_{\mathfrak{O}}$ and every $r \in |C|$ for which $\mathcal{U}(\Lambda, Y, r) \subset X$.*

Proof. Consider first any direct summand $0 \neq L \subset \Lambda$ in any orbit $\mathfrak{O}$. As already argued in the proof of Definition-Proposition 5.19, for any admissible quasi-compact $O \subset L$ then $p_\Gamma(O) \subset \Omega_{\mathfrak{O}}$ is admissible quasi-compact and

$$p_\Gamma(\mathcal{U}(\Lambda, O, r)) = \mathcal{U}(\Lambda, p_\Gamma(O), r)$$

for any $r \in |C|$. Using this, it directly follows that precomposition with π_Γ defines an injective map $\mathcal{O}_\Gamma'(X) \to \mathcal{O}_\Gamma^*(X)$. On the other hand, consider any $f \in \mathcal{O}_\Gamma^*(X)$. Being $C^\times$- and Γ-invariant, it induces a function $s \colon X \to C$ which, by Corollary 5.15, is continuous with respect to the canonical topologies since f is. By construction of the $\Omega_{\mathfrak{O}}$, moreover, s restricts stratawise to a regular function since f does. The boundedness property for s follows from the one of f via the argument at the end of the proof of Definition-Proposition 5.19. $\square$

Proposition 5.23. *For any Γ-orbit $\mathfrak{O} = \Gamma \cdot L \neq \{0\}$ the composition of the canonical bijection $\Omega_{\overline{\Gamma}_L} \to \Omega_{\mathfrak{O}}$ with the inclusion $\Omega_{\mathfrak{O}} \subset \Omega_\Gamma^*$ is an injective morphism of Grothendieck ringed spaces whose induced morphisms on stalks are surjective.*

Proof. That the composition is a morphism follows directly from Definition-Proposition 5.19 and Proposition 5.22. By construction, it is injective. It remains to show that it is surjective on stalks. Let $\omega \in \Omega_L$. Using Proposition 4.14, choose a basis S of admissible affinoid neighborhoods O of ω in Ω_L such that $\gamma(O) = O$ for all $\gamma \in \Gamma_\omega$ and $\gamma(O) \cap O = \emptyset$ for all $\gamma \in \Gamma \setminus \Gamma_\omega$. By Example 5.14 applied to the case $\Lambda = L$, for any $O \in S$ the subset $p_{\overline{\Gamma}_L}(O) \subset \Omega_{\overline{\Gamma}_L}$ is admissible and hence the subset $p_\Gamma(O) \subset \Omega_{\mathfrak{O}}$ corresponding to $p_{\overline{\Gamma}_L}(O)$ under the isomorphism $\Omega_{\overline{\Gamma}_L} \cong \Omega_{\mathfrak{O}}$ is admissible. Since S is a basis, the $p_\Gamma(O)$ for all $O \in S$ are in fact a basis of admissible neighborhoods of $p_\Gamma(\omega)$. By Example 5.14 and Definition-Proposition 5.19, the subsets $p_\Gamma(\mathcal{U}(\Lambda, O, r)) \subset \Omega_\Gamma^*$ for all $O \in S$ and all $r \in |C|$ form a basis of admissible neighborhoods of $p_\Gamma(\omega)$ in Ω_Γ^*. Consider any $O \in S$ and set $U_r := \mathcal{U}(\Lambda, O, r)$ for all $r \in |C|$. It thus suffices to show for large enough $r \in |C|$ that every section on $p_\Gamma(O) = p_\Gamma(U_r) \cap \Omega_{\mathfrak{O}}$ extends to a section on $p_\Gamma(U_r)$. However, if r is large enough, then

$$\forall \gamma \in \Gamma_\omega \colon \gamma(U_r) = U_r \text{ and } \forall \gamma \in \Gamma \setminus \Gamma_\omega \colon \gamma(U_r) \cap U_r = \emptyset$$

by Proposition 5.18. For such r the maps $\gamma(U_r) \to \gamma(O), [l] \mapsto [l|_{\gamma(L)_C}]$ for all $\gamma \in \Gamma$ are thus the restrictions of a well-defined map

$$\rho \colon \bigcup_{\gamma \in \Gamma} \gamma(U_r) \to \bigcup_{\gamma \in \Gamma} \gamma(O).$$

It is directly checked that ρ is Γ-equivariant, continuous with respect to the canonical topologies and that its restriction to the intersection of its domain with $\Omega_{L'}$ for any direct summand $0 \neq L' \subset \Lambda$ is a morphism of rigid analytic varieties. The map $\rho_\Gamma \colon p_\Gamma(U_r) \to p_\Gamma(O)$ induced by ρ is then also continuous with respect to the canonical topologies and restricts to a morphism of rigid analytic varieties $p_\Gamma(U_r) \cap \Omega_{\mathfrak{O}'} \to p_\Gamma(O)$ for all Γ-orbits $\mathfrak{O}' \neq \{0\}$. Under the identification in Proposition 5.22, any regular function $s \colon p_\Gamma(O) \to C$ thus extends to the regular function $s \circ \rho_\Gamma$ on $p_\Gamma(U_r)$ using moreover that s is bounded; in fact, by construction, s may be interpreted as a Γ_ω-invariant regular function on the affinoid variety O and is thus bounded by Proposition 2.27. This shows surjectivity on stalks. $\square$

5.3. Eisenstein series

Denote by F the quotient field of A and by $\pi\colon \Lambda_F \to \Lambda_F/\Lambda$ the quotient homomorphism. Let $\alpha \in \Lambda_F/\Lambda$ and set $L(\alpha) := \pi^{-1}(\alpha) \cap L_F$ for any direct summand $0 \neq L \subset \Lambda$. Let k be any positive integer. Consider the sum

$$E_{\Lambda,\alpha,k}\colon \tilde{\Omega}^*_\Lambda \to C,\ l \mapsto \sum_{0\neq\lambda\in L(\alpha)} \frac{1}{l(\lambda)^k}, \quad \text{if } l \in \tilde{\Omega}_L.$$

Proposition 5.24. *$E_{\Lambda,\alpha,k}$ converges everywhere and, if Γ fixes α, is in $\mathcal{O}^*_\Gamma(k)(\Omega^*_\Gamma)$.*

Proof. Consider any direct summand $0 \neq L_0 \subset \Lambda$, any affinoid $O \subset \Omega_{L_0}$, any $0 \neq \lambda_0 \in L_0$ and any $r \in |C|$. Set $U := \mathcal{U}(\Lambda, O, r)$ and consider the sum

$$E\colon U \to C, [l] \mapsto E_{\Lambda,\alpha,k}(l) \cdot l(\lambda_0)^k.$$

We first show via the following lemmas that for every further direct summand $L_0 \subset L \subset \Lambda$ the sum E converges to a regular function on $U_L := U \cap \Omega_L$ and that E is continuous with respect to the canonical topologies and bounded.

Lemma 5.25. *On every U_L the sum E converges to a regular function.*

Proof. By means of an admissible affinoid covering of any such U_L, it suffices to show that the restriction $E_{O'}$ of E to every admissible affinoid $O' \subset U_L$ converges to a regular function. Consider any such $O' \subset U_L$ and choose any $[l] \in O'$. As $L(\alpha) \subset L_E$ is discrete, where E is the completion of F, the subset $l(L(\alpha)) \subset C$ is strongly discrete (see for instance [20, Ex. 2.48 and Lemma 2.49]) so that for any integer $m \geq 1$ the subset $L(\alpha)_m \subset L(\alpha)$ of those λ for which $\left|\frac{l(\lambda)}{l(\lambda_0)}\right| \leq m$ is finite; thus

$$E_{O',m}\colon O' \to C, [l'] \mapsto \sum_{0\neq\lambda\in L(\alpha)_m} \frac{l'(\lambda_0)^k}{l'(\lambda)^k}$$

is a regular function. As the ring of regular functions on O' is complete with respect to the sup-norm by Proposition 2.19, it suffices to show that the $E_{O',m}$ for all $m \geq 1$ form a Cauchy-sequence; indeed, as $L(\alpha)$ is covered by the $L(\alpha)_m$ for all $m \geq 1$, their limit must then be $E_{O'}$. Applying Corollary 4.6 to the affinoid O', we choose a $\kappa' > 0$ such that

$$\forall m \geq 1, \forall [l'] \in O', \forall \lambda \in L(\alpha) \setminus L(\alpha)_m\colon \left|\frac{l'(\lambda_0)}{l'(\lambda)}\right| \leq \kappa' \cdot \left|\frac{l(\lambda_0)}{l(\lambda)}\right| \leq \frac{\kappa'}{m}.$$

This directly yields that the $E_{O',m}$ indeed form a Cauchy-sequence. □

Lemma 5.26. *E is continuous with respect to the canonical topologies and bounded.*

Proof. As for continuity, it suffices, by Corollary 5.9, to show that E is sequentially continuous. Consider thus any $[l] \in U$ and any sequence $\{[l_n]\}_{n\geq 1} \subset U$ converging to $[l]$ and let us show that

$$\lim_{n\to\infty} E([l_n]) = E([l]). \tag{32}$$

Let $L \supset L_0$, resp. $L_n \supset L_0$, be such that $[l] \in \Omega_L$, resp. $[l_n] \in \Omega_{L_n}$ for any $n \geq 1$. Choose a fundamental basis of admissible affinoid neighborhoods $(O_n)_{n\geq 1}$ of $[l]$ in Ω_L such that $O_n \supset O_{n+1}$ for all $n \geq 1$. Using Corollary 5.7, we choose a sequence $\{r_n\}_{n\geq 1} \subset |C|$ converging to infinity and an $n_0 \geq 1$ such that $[l_n] \in U_n := \mathcal{U}(\Lambda, O_n, r_n)$ for every $n \geq n_0$. The choice of the O_n and the regularity of the restriction E_{O_1} of E to O_1 by Lemma 5.25 imply that

$$\lim_{n\to\infty} E([l_n|_{L_C}]) = \lim_{n\to\infty} E_{O_1}([l_n|_{L_C}]) = E_{O_1}([l]) = E([l]).$$

It thus remains to show that $E([l_n]) - E([l_n|_{L_C}])$ converges to 0 for $n \to \infty$. Applying Corollary 4.7 and Proposition 2.27 to the affinoid O_1, we choose an $s > 0$ such that $|l(\lambda_0)| \leq s \cdot \mathrm{d}(l(L))$ for every $[l] \in O_1$. Choose any $f \in F$ for which $\pi^{-1}(\alpha) \subset f \cdot \Lambda$. For any $n \geq n_0$ and any $\lambda \in L_n(\alpha) \setminus L_0$ then

$$|l_n(\lambda)| \geq \mathrm{d}(l_n(L_n(\alpha)\setminus L_0)) \geq |f|\cdot \mathrm{d}(l_n(L_n\setminus L_0)) \overset{[l_n]\in U_n}{\geq} |f|\cdot r_n \cdot \mathrm{d}(l_n(L)) \geq \frac{|f|\cdot r_n}{s}\cdot |l_n(\lambda_0)|.$$

For any $n \geq n_0$, as $O_1 \supset O_n$, thus

$$|E([l_n]) - E([l_n|_{L_C}])| = \left| \sum_{\lambda\in L_n(\alpha)\setminus L} \frac{l_n(\lambda_0)^k}{l_n(\lambda)^k} \right| \leq \frac{s^k}{(|f|\cdot r_n)^k} \xrightarrow[n\to\infty]{} 0.$$

Thus E is indeed sequentially continuous. In order to see that E is bounded, we use the preceding calculations in the case, where $O_1 = O$ and $n_0 = 1$ and $r_1 = r$, in order to see that

$$\forall [l] \in U = U_1\colon |E([l]) - E_O([l|_{L_C}])| = |E([l]) - E([l|_{L_C}])| \leq \frac{s^k}{(|f|\cdot r)^k}.$$

Moreover, as E_O is regular by Lemma 5.25, it is bounded by Proposition 2.27. Thus E is indeed bounded. □

By Lemma 5.25, the sum E converges everywhere on O. Hence $E_{\Lambda,\alpha,k}$ converges everywhere on O and thus, as O was arbitrary, on $\tilde{\Omega}^*_\Lambda$. The construction further yields that $E_{\Lambda,\alpha,k}(c\cdot l) = c^{-k}\cdot E_{\Lambda,\alpha,k}(l)$ for any $c \in C^\times$ and any $l \in \tilde{\Omega}^*_\Lambda$ and, if Γ fixes α, that $E_{\Lambda,\alpha,k}$ is Γ-invariant. Jointly with Lemmas 5.25 and 5.26, this yields the proposition. □

5.4. Fourier expansion of weak modular forms

Definition 5.27. For any integer k the sections in $\mathcal{O}^*_\Gamma(k)(\Omega_\Gamma)$ are called *weak modular forms* of weight k with respect to Γ.

This is a coordinate free version of Basson, Breuer and Pink's [3, Def. 3.1.7, 3.3.1] (see Goss' [17, Cor. 1.40, Prop. 1.43] or see [20, Remark 6.40]).

Suppose that $d := \operatorname{rank}_A(\Lambda) \geq 2$. Consider any direct summand $L \subset \Lambda$ of corank 1. Choose any $v \in \Lambda \setminus L$. Denote by $\mathring{\Gamma} \subset \mathring{\Gamma}_L$ the subgroup of elements γ such that $\gamma(v) - v \in L$. Thus $\mathring{\Gamma}$ is a discrete subgroup of $\operatorname{Aut}_E(\Lambda_E)$ of the form considered in Section 4.4 and

$$\mathring{\Gamma} \to L,\ \gamma \mapsto v_\gamma := \gamma(v) - v$$

is a continuous injective group homomorphism. Denote by $v_{\mathring{\Gamma}} \subset L$ its image and set

$$\forall l \in \tilde{\Omega}_\Lambda \colon u(l) := \frac{1}{\mathrm{e}_{l(v_{\mathring{\Gamma}})}(l(v))}.$$

Let $\Omega_{\mathring{\Gamma}}, p_{\mathring{\Gamma}}, \pi_{\mathring{\Gamma}}$ be as defined in Section 5.2 upon replacing Γ by $\mathring{\Gamma}$.

Definition 5.28. $\mathcal{U}_{\mathring{\Gamma}} := \Omega_L \cup \Omega_{\mathring{\Gamma}} = p_{\mathring{\Gamma}}(\mathcal{U}(\Lambda, \Omega_L, 0))$.

Choose any $0 \neq w \in L$ and consider the map

$$q \colon \mathcal{U}_{\mathring{\Gamma}} \to \Omega_L \times \mathbb{A}^{1,\mathrm{rig}}_C, p_{\mathring{\Gamma}}([l]) \mapsto \begin{cases} ([l|_{L_C}], l(w) \cdot u(l)) & \text{if } [l] \in \Omega_\Lambda \\ ([l], 0) & \text{if } l \in \Omega_L. \end{cases}$$

Proposition 5.29. *The map q is an open immersion of regular rigid analytic varieties.*

Proof. Set $\mathcal{V} := \Lambda_E$ and $\mathcal{W} := L_E$. Then q restricts to the open immersion $q_{\mathring{\Gamma}} \colon \Omega_{\mathring{\Gamma}} \to \Omega_{\mathcal{W}} \times (\mathbb{A}^{1,\mathrm{rig}}_C \setminus \{0\})$ provided by Definition-Proposition 4.32. Moreover, by assumption, Γ contains a principal congruence subgroup of some level $0 \neq (b) \subsetneq A$. But $b \cdot L \subset v_{\mathring{\Gamma}}$ for any such b. Hence Proposition 4.36 applies and yields, jointly with Corollary 3.4 and the fact that the restriction $q_{\mathring{\Gamma}}$ of q is an open immersion, that q is an isomorphism of Grothendieck topological spaces onto an admissible subset of $\Omega_L \times \mathbb{A}^{1,\mathrm{rig}}_C$. Jointly with Corollary 3.5, this implies that q is an isomorphism of Grothendieck ringed spaces onto an admissible subvariety of $\Omega_L \times \mathbb{A}^{1,\mathrm{rig}}_C$. $\square$

For any $\varepsilon \in |C^\times|$ denote by $B_\varepsilon \subset \mathbb{A}^{1,\mathrm{rig}}_C$ the closed disc around 0 of radius ε.

Corollary 5.30. *Any admissible $X \subset \mathcal{U}_{\mathring{\Gamma}}$ and any admissible affinoid $O \subset X \cap \Omega_L$ admit an $\varepsilon \in |C^\times|$ such that $O \times B_\varepsilon \subset q(X)$.*

Proof. This follows directly from Proposition 5.29 and Corollary 3.4. □

For any $O \subset \Omega_L$ and any $\varepsilon \in |C^\times|$ set $\tilde{\mathcal{U}}(O,\varepsilon) := \tilde{\Omega}_\Lambda \cap (q \circ \pi_{\mathring{\Gamma}})^{-1}(O \times B_\varepsilon)$.

Corollary 5.31. *Any admissible $X \subset \mathcal{U}_{\mathring{\Gamma}}$, any $k \in \mathbb{Z}$ and any $f \in \mathcal{O}_{\mathring{\Gamma}}(k)(X \cap \Omega_{\mathring{\Gamma}})$ admit unique weight $k-i$ regular functions $f_i \colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L) \to C$ for all $i \in \mathbb{Z}$ such that for every admissible affinoid $O \times B_\varepsilon \subset q(X)$ holds that*

$$\forall l \in \tilde{\mathcal{U}}(O,\varepsilon)\colon\ \ f(l) = \sum_{i \in \mathbb{Z}} f_i(l|_{L_C})u(l)^i \tag{33}$$

and for which the following are equivalent:

i) $\forall i < 0 : f_i = 0$.
ii) *f extends to an element in $\mathcal{O}_{\mathring{\Gamma}}(k)(X)$ which restricts to f_0 on $\pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L)$.*
iii) *The section $\left[g\colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_{\mathring{\Gamma}}) \to C \quad l \mapsto f(l) \cdot l(w)^k\right] \in \mathcal{O}_{\mathring{\Gamma}}(0)(X \cap \Omega_{\mathring{\Gamma}})$ extends to a morphism of Grothendieck topological spaces $\pi_{\mathring{\Gamma}}^{-1}(X) \to \mathbb{A}_C^{1,\mathrm{rig}}$ whose restriction to $\pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L)$ is in $\mathcal{O}_{\{\mathrm{id}\}}(0)(X \cap \Omega_L)$.*
iv) *g is bounded on $\tilde{\mathcal{U}}(O,\varepsilon)$ for any admissible affinoid $O \times B_\varepsilon \subset q(X)$.*

Proof. Consider any such f. Proposition 5.29 and Corollaries 3.4 and 3.5 yield unique weight $-i$ regular functions $g_i \in\colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L) \to C$ for all $i \in \mathbb{Z}$ that satisfy the desired properties when f is replaced by the section g defined in iii) and k by 0. It is directly checked that then the

$$f_i\colon \pi_{\mathring{\Gamma}}^{-1}(X \cap \Omega_L) \to C \quad l \mapsto g_i([l|_{L_C}]) \cdot l(w)^{-k}$$

for all $i \in \mathbb{Z}$ satisfy the desired properties. □

Remark 5.32. By Basson [3, Prop. 3.2.7], any f_i is a weak modular form of weight $k-i$ with respect to some congruence subgroup of $\mathrm{Aut}_A(L)$. For a proof using the notation here, see [20, Cor. 6.45].

Remark 5.33. In [3, Sections 3.4 and 3.5], Basson computed the Fourier expansion of for instance the Eisenstein series from Section 5.3.

Remark 5.34. Suppose that $\mathring{\Gamma} = \mathring{\Gamma}_L$ and that the action of Γ on Ω_Λ and the action of $\overline{\Gamma}_L$ on Ω_L are both fixed-point free. From Proposition 5.29 follows that the composition

$$q(\mathcal{U}_{\mathring{\Gamma}}) \xrightarrow{q^{-1}} \mathcal{U}_{\mathring{\Gamma}} \xrightarrow{\pi} \mathcal{U}_\Gamma,$$

where π denotes the natural quotient morphism, induces isomorphisms on stalks. Consequently, $\mathcal{U}_\Gamma$ is regular. For details see [20, Cor. 6.48].

6. Compactification of analytic moduli spaces

Let $A \subset C$ be as in Section 5. Denote by $\hat{A}$ the profinite completion of A. Consider any finitely generated free $\hat{A}$-module $M \neq 0$ and any *congruence subgroup* $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$, i.e., $\mathcal{K}$ is a subgroup containing the kernel of the natural morphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(M/IM)$ for some ideal $0 \neq I \subset A$.

6.1. Structure of Grothendieck graded ringed space

Definition 6.1. An A-submodule $\Lambda \subset M$ is called an A-*structure* of M if the inclusion induces an $\hat{A}$-linear isomorphism $\Lambda_{\hat{A}} \to M$.

Proposition 6.2. *Any A-structure of M is finitely generated projective.*

Proof. As $\hat{A}$ is a faithfully flat A-algebra, an A-module Λ is finitely generated if and only if $\Lambda_{\hat{A}}$ is a finitely generated $\hat{A}$-module (see for instance [37, Tag 03C4]). Hence A-structures of M are finitely generated A-modules. Moreover, they are torsion free, since M is. Now use that A is a Dedekind domain. □

Definition 6.3. Consider any A-structure Λ of M. Define the natural bijections

i) $\Omega^*_{\{\Lambda\}} := \Omega^*_\Lambda \times \{\Lambda\} \to \Omega^*_\Lambda, ([l], \Lambda) \mapsto [l]$,
ii) $\tilde{\Omega}^*_{\{\Lambda\}} := \tilde{\Omega}^*_\Lambda \times \{\Lambda\} \to \tilde{\Omega}^*_\Lambda, (l, \Lambda) \mapsto l$.

Endow $\Omega^*_{\{\Lambda\}}$ with the Grothendieck topology for which the first bijection is an isomorphism with respect to the topology of Ω^*_Λ defined in Definition-Proposition 5.4. Endow $\tilde{\Omega}^*_{\{\Lambda\}}$ with the $C^\times$-action for which the second bijection is $C^\times$-equivariant.

Definition 6.4. Let Ω^*_M, resp. $\Omega_M \subset \Omega^*_M$, be the disjoint union of the Grothendieck topological spaces $\Omega^*_{\{\Lambda\}}$, resp. $\Omega_{\{\Lambda\}} := \Omega_\Lambda \times \{\Lambda\}$, for all A-structures Λ of M.

Definition 6.5. Let $\tilde{\Omega}^*_M$, resp. $\tilde{\Omega}_M \subset \tilde{\Omega}^*_M$, be the disjoint union of the $\tilde{\Omega}^*_{\{\Lambda\}}$, resp. $\tilde{\Omega}_{\{\Lambda\}} := \tilde{\Omega}_\Lambda \times \{\Lambda\}$, for all A-structures Λ of M.

Lemma 6.6. *Consider any A-structure Λ of M, any direct summand $0 \neq L \subset \Lambda$, any $l \in \tilde{\Omega}_L$ and any $g \in \mathrm{Aut}_{\hat{A}}(M)$. Then $g(\Lambda)$ is an A-structure of M and $g(L)$ is a direct summand of $g(\Lambda)$ and $g(l)\colon g(L)_C \to C, \lambda \mapsto l(g^{-1}\lambda)$ is in $\Omega_{g(L)}$.*

Proof. This is directly checked. □

Consider the $C^\times$-equivariant action of $\mathrm{Aut}_{\hat{A}}(M)$ on $\tilde{\Omega}^*_M$ given by

$$\forall g \in \mathrm{Aut}_{\hat{A}}(M), \forall (l, \Lambda) \in \tilde{\Omega}^*_M \colon\ g(l, \Lambda) := (g(l), g(\Lambda)).$$

By construction, for any A-structure Λ the induced action of $\mathrm{Aut}_A(\Lambda) \subset \mathrm{Aut}_{\hat{A}}(M)$ on $\tilde{\Omega}^*_\Lambda$ coincides with the one in Definition 5.10. By construction and the remark before Lemma 5.11, the induced action of $\mathrm{Aut}_{\hat{A}}(M)$ on Ω^*_M is via isomorphisms of Grothendieck topological spaces.

Definition 6.7. By means of Definition-Proposition 6.6, consider the quotient map

$$p_{\mathcal{K}} \colon \Omega^*_M \to \mathcal{K}\backslash\Omega^*_M =: \Omega^*_{\mathcal{K}}$$

and endow its target with the structure of Grothendieck topological space which it induces, that is, a subset (resp. a covering of a subset) of $\Omega^*_{\mathcal{K}}$ is admissible precisely when its preimage is admissible.

Denote by $\pi_{\mathcal{K}} \colon \tilde{\Omega}^*_M \to \mathcal{K}\backslash\tilde{\Omega}^*_M/C^\times = \Omega^*_{\mathcal{K}}$ the double quotient map.

Definition-Proposition 6.8. *For any admissible $X \subset \Omega^*_{\mathcal{K}}$ and any integer k let $\mathcal{O}^*_{\mathcal{K}}(k)(X)$ be the set of $\mathcal{K}$-invariant functions $\pi^{-1}_{\mathcal{K}}(X) \to C$ whose restriction to $\pi^{-1}_{\mathcal{K}}(X) \cap \tilde{\Omega}^*_{\{\Lambda\}}$ is weight k regular (in the sense of Definition-Proposition 5.20 via Definition 6.3, ii)) for every A-structure Λ of M. Then, by means of the ring structure on C,*

i) the $\mathcal{O}^*_{\mathcal{K}}(X) := \mathcal{O}^*_{\mathcal{K}}(0)(X)$ for all admissible subsets $X \subset \Omega^*_{\mathcal{K}}$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}^*_{\mathcal{K}}$ of rings on $\Omega^*_{\mathcal{K}}$, called structure sheaf *on $\Omega^*_{\mathcal{K}}$, and*

ii) for any integer k the $\mathcal{O}^*_{\mathcal{K}}(k)(X)$ over all admissible $X \subset \Omega^*_{\mathcal{K}}$ together with the natural restriction homomorphisms form a sheaf $\mathcal{O}^*_{\mathcal{K}}(k)$ of $\mathcal{O}^*_{\mathcal{K}}$-modules on $\Omega^*_{\mathcal{K}}$, called k-th twisting $\mathcal{O}^*_{\mathcal{K}}$-module *and*

iii) a sheaf $\mathcal{R}^*_{\mathcal{K}}$ of graded $\mathcal{O}^*_{\mathcal{K}}$-algebras on $\Omega^*_{\mathcal{K}}$ is formed by the

$$\mathcal{R}^*_{\mathcal{K}}(X) := \sum_{k \in \mathbb{Z}} \mathcal{O}^*_{\mathcal{K}}(k)(X)$$

for all admissible $X \subset \Omega^*_{\mathcal{K}}$ and the natural restriction homomorphisms.

In particular, $(\Omega^*_{\mathcal{K}}, \mathcal{O}^*_{\mathcal{K}})$ (resp. $(\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}})$) is a Grothendieck (graded) ringed space.

Proof. This is directly checked. □

Example 6.9. Denote by F the quotient field of A. Consider any $\alpha \in M_F/M$ and any integer $k \geq 1$ and associate with them the map

$$E_{M,\alpha,k} \colon \tilde{\Omega}^*_M \to C, \; (l, \Lambda) \mapsto E_{\Lambda,\alpha,k}(l),$$

where $E_{\Lambda,\alpha,k}$ is the Eisenstein series defined in Section 5.3 and where α is viewed in Λ_F/Λ via the natural isomorphism $\Lambda_F/\Lambda \cong M_F/M$. If $\mathcal{K}$ fixes α, then

$$E_{M,\alpha,k} \in \mathcal{O}^*_{\mathcal{K}}(k)(\Omega^*_{\mathcal{K}}).$$

Proof. This follows directly from the construction and Proposition 5.24. □

For any A-structure Λ of M set $\overline{\mathcal{K}}_\Lambda := \{\gamma \in \mathrm{Aut}_A(\Lambda) |\ \exists \kappa \in \mathcal{K}\colon \kappa|_\Lambda = \gamma\}$; since $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ is a congruence subgroup, so is $\overline{\mathcal{K}}_\Lambda \subset \mathrm{Aut}_A(\Lambda)$.

Proposition 6.10. *Any complete set S of representatives of the natural $\mathcal{K}$-action on the set of A-structures of M is finite and the natural maps $\Omega^*_\Lambda \to \Omega^*_{\{\Lambda\}} \to \Omega^*_{\{M\}}$ for all $\Lambda \in S$ induce an isomorphism of Grothendieck graded ringed spaces*

$$\coprod_{\Lambda\in S} (\Omega^*_{\overline{\mathcal{K}}_\Lambda}, \mathcal{R}^*_{\overline{\mathcal{K}}_\Lambda}) \longrightarrow (\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}}) \tag{34}$$

which restricts to an isomorphism between normal rigid analytic varieties over C

$$\coprod_{\Lambda\in S} \Omega_{\overline{\mathcal{K}}_\Lambda} \longrightarrow \Omega_{\mathcal{K}}. \tag{35}$$

Proof. As $\mathcal{K}$ contains a principal congruence subgroup, the first assertion follows from Corollary 6.20 below whose proof does not depend on this result. That the maps $\Omega^*_\Lambda \to \Omega^*_{\{\Lambda\}} \to \Omega^*_{\{M\}}$ induce isomorphisms (34) and (35) of Grothendieck (graded) ringed spaces, follows directly from the construction. Moreover, $\Omega_{\overline{\mathcal{K}}_\Lambda}$ is a normal rigid analytic over C for every $\Lambda \in S$ by Proposition 4.13 and hence so are, as S is finite, both spaces in (35). □

6.2. Case of principal congruence subgroups

Consider any ideal $0 \neq I \subset A$ and set $\overline{A} := A/I$. Associate with any A- or $\hat{A}$-module Q the $\overline{A}$-module $\overline{Q} := I^{-1}Q/Q$. Suppose that

$$\mathcal{K} = \mathrm{Ker}(\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_{\overline{A}}(\overline{M})).$$

Proposition 6.11. *Consider any free direct summand $N \subset M$, any $\hat{A}$-linear injective morphism $\Psi : N \to M$ onto a free direct summand of M and any $\epsilon \in \mathrm{Aut}_{\overline{A}}(\overline{M})$ whose restriction to $\overline{N}$ is the map $\overline{N} \to \overline{M}$ induced by Ψ. Then there exists a $\sigma \in \mathrm{Aut}_{\hat{A}}(M)$ that induces ϵ and restricts to Ψ.*

Proof. By means of the unique prime factorization of the non-zero ideals in the Dedekind domain A and by the Chinese remainder theorem, it is enough to show the statement of the proposition for $\hat{A}$ replaced by the $\mathfrak{p}$-adic completion $A_\mathfrak{p}$ of A at any prime ideal $\mathfrak{p} \subset A$ and for I replaced by any power $(\mathfrak{p}A_\mathfrak{p})^n$. In this case, choose a $\sigma \in \mathrm{Hom}_{A_\mathfrak{p}}(M, M)$ that induces ϵ and restricts to Ψ. Then the determinant of σ modulo $\mathfrak{p}^n$ equals the determinant of ϵ and is thus a unit. If $n \geq 1$, then the determinant of σ is thus itself a unit since $A_\mathfrak{p}$ is a discrete valuation ring so that σ is in fact an automorphism. If $n = 0$, then $\epsilon = 0$, so that σ may be *chosen* to further be an automorphism. $\square$

Corollary 6.12. *The natural morphism* $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_{\overline{A}}(\overline{M})$ *is surjective.*

Corollary 6.13. *Suppose that* $I \subsetneq A$. *Then* $N \mapsto \overline{N}$ *induces a bijection from the set of* $\mathcal{K}$*-orbits of free direct summands of* M *to the set of free* $\overline{A}$*-submodules of* $\overline{M}$.

Proof. Let $N, N' \subset M$ be free direct summands with $\overline{N} = \overline{N'}$. As $I \subsetneq A$, then

$$\mathrm{rank}_{\hat{A}}(N) = \mathrm{rank}_{\overline{A}}(\overline{N}) = \mathrm{rank}_{\overline{A}}(\overline{N'}) = \mathrm{rank}_{\hat{A}}(N)$$

so that N and N' are $\hat{A}$-linearly isomorphic. By Corollary 6.12, the natural homomorphism $\mathrm{Aut}_{\hat{A}}(N) \to \mathrm{Aut}_{\overline{A}}(\overline{N})$ is surjective. Hence there exists an isomorphism $\Psi\colon N \to N'$ inducing the identity on $\overline{N} = \overline{N'}$. By Proposition 6.11, such a Ψ extends to an element in $\mathcal{K}$. This shows injectivity. Consider then any $\overline{A}$-submodule $X \subset \overline{M}$. By means of a basis of M choose a free direct summand $M' \subset M$ with

$$\mathrm{rank}_{\overline{A}}(\overline{M'}) = \mathrm{rank}_{\hat{A}}(M') = \mathrm{rank}_{\overline{A}}(X).$$

Choose an $\epsilon \in \mathrm{Aut}_A(\overline{M})$ with $\epsilon(\overline{M'}) = X$. Proposition 6.11 then provides a lift $\sigma \in \mathrm{Aut}_{\hat{A}}(M)$ of ϵ. Then $\overline{\sigma(M')} = X$ which shows surjectivity. $\square$

We further need the following consequences of Prasad's theorem [32, Theorem A] on strong approximation for semi-simple groups over function fields.

Proposition 6.14. *Let* F *be the quotient field of* A. *Consider any finitely generated projective* A*-module* Λ. *Surjective is then the natural homomorphism*

$$\mathrm{SL}_A(\Lambda) := \mathrm{Aut}_A(\Lambda) \cap \mathrm{SL}_F(\Lambda_F) \to \mathrm{SL}_{\overline{A}}(\overline{\Lambda}).$$

Proof. By Prasad's theorem [32, Theorem A], the subgroup

$$\mathrm{SL}_F(\Lambda_F) \subset \mathrm{SL}_{\hat{A}_F}(\Lambda_{\hat{A}_F})$$

is dense. Since $\mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}})$ is open in $\mathrm{Aut}_{\hat{A}_F}(\Lambda_{\hat{A}_F})$, then also the subgroup

$$\mathrm{SL}_A(\Lambda) = \mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}}) \cap \mathrm{SL}_F(\Lambda_F) \subset \mathrm{Aut}_{\hat{A}}(\Lambda_{\hat{A}}) \cap \mathrm{SL}_{\hat{A}_F}(\Lambda_{\hat{A}_F}) = \mathrm{SL}_{\hat{A}}(\Lambda_{\hat{A}})$$

is dense. As, by Proposition 6.11, the natural continuous group homomorphism

$$\mathrm{SL}_{\hat{A}}(\Lambda_{\hat{A}}) \to \mathrm{SL}_{\overline{A}}(\overline{\Lambda})$$

with discrete target is surjective, so is its restriction to $\mathrm{SL}_A(\Lambda)$. $\square$

Corollary 6.15. *The determinant induces an isomorphism*

$$\mathrm{Aut}_{\overline{A}}(\overline{\Lambda})/\,\mathrm{Aut}_A(\Lambda) \xrightarrow{\cong} \overline{A}^{\times}/A^{\times}. \tag{36}$$

Corollary 6.16. *Consider any finitely generated projective A-module Λ, any direct summand $L \subset \Lambda$, any injective A-linear map $\psi\colon L \to \Lambda$ onto a direct summand of Λ and any $\epsilon \in \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$ with $\det(\epsilon) \in A^{\times}$ whose restriction to $\overline{L}$ is the map $\overline{L} \to \overline{\Lambda}$ induced by ψ. Then there exists a $\gamma \in \mathrm{Aut}_A(\Lambda)$ that induces ϵ and restricts to ψ.*

Proof. Using direct complements, choose an extension $\gamma' \in \Gamma := \mathrm{Aut}_A(\Lambda)$ of ψ. It is enough to find a $\gamma'' \in \Gamma$ restricting to the identity on L and whose induced $\overline{\gamma''} \in \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$ equals $\overline{\gamma'}^{-1} \circ \epsilon$; indeed, $\gamma := \gamma' \circ \gamma''$ is then a desired automorphism. We are thus reduced to the case where ψ is the inclusion, i.e., to showing surjectivity of the natural morphism

$$C := \{\gamma \in \Gamma\colon \gamma|_L = \mathrm{id}_L\} \to \overline{C} := \{\varepsilon \in \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})\colon \varepsilon|_{\overline{L}} = \mathrm{id}_{\overline{L}} \wedge \det(\varepsilon) \in A^{\times}\}.$$

Choose a direct complement L' of L in Λ. Set $G := \mathrm{Aut}_A(L')$, $H := \mathrm{Hom}_A(L', L)$, $\overline{G} := \{\varepsilon \in \mathrm{Aut}_{\overline{A}}(\overline{L'})\colon\ \det(\varepsilon) \in A^{\times}\}$ and $\overline{H} := \mathrm{Hom}_{\overline{A}}(\overline{L'}, \overline{L})$. Under the natural isomorphisms

$$C \cong H \rtimes G \quad\text{and}\quad \overline{C} \cong \overline{H} \rtimes \overline{G},$$

the map $C \to \overline{C}$ restricts to the natural morphism $G \to \overline{G}$, which is surjective by Corollary 6.15, and to the natural morphism $H \to \overline{H}$, which is surjective by projectivity of L'. Hence $C \to \overline{C}$ is surjective, too. □

Proposition 6.17. *Any projective module Λ of finite rank $d \geq 1$ over any Dedekind ring A admits a unique class $[J] \in \mathrm{Pic}(A)$ such that $\Lambda \cong A^{d-1} \oplus J$.*

Proof. See for instance [28, Theorems 1.32 and 1.39]. □

Corollary 6.18. *Consider any finitely generated projective A-modules L and Λ and any injective non-surjective $\overline{A}$-linear map $\tau : \overline{L} \to \overline{\Lambda}$. Then there exists an injective A-linear map $L \to \Lambda$ onto a direct summand of Λ which induces τ.*

Proof. By the properties of τ, the rank of Λ is greater than the rank of L. By means of Proposition 6.17, we thus assume that L is a proper direct summand of Λ. Using that $\overline{L} \subsetneq \overline{\Lambda}$, we choose an extension $\rho \in \mathrm{SL}_{\overline{A}}(\overline{\Lambda})$ of τ. Proposition 6.14 then provides a desired $\sigma \in \mathrm{SL}_A(\Lambda)$ inducing ρ. □

Corollary 6.19. *Consider any direct summands $L, L' \subset \Lambda$ such that $\overline{L} \subsetneq \overline{L}'$. Then there exists a $\gamma \in \mathrm{Ker}(\mathrm{Aut}_A(\Lambda)) \to \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$ for which $\gamma(L) \subset L'$.*

Proof. Using Corollary 6.18, choose an injective A-linear map $\psi\colon L \to L'$ onto a direct summand of L' whose induced map $\overline{L} \to \overline{L}'$ is the inclusion. Apply Corollary 6.16 to ψ

and $\mathrm{id}_{\overline{\Lambda}}$ to get an extension of ψ in $\mathrm{Ker}(\mathrm{Aut}_A(\Lambda)) \to \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$. Such an extension maps L to L'. $\square$

Corollary 6.20. *Denote by $h(A)$ the class number of A. For any complete set S of representatives for the $\mathcal{K}$-action on the set of A-structures as in Proposition 6.10 then holds that*

$$|S| = h(A) \cdot \left|\overline{A}^{\times}/A^{\times}\right|.$$

Proof. Set $\mathcal{K}' := \mathrm{Aut}_{\hat{A}}(M)$. If any A-structures Λ, Λ' lie in the same $\mathcal{K}'$-orbit, i.e., if $\Lambda = \kappa'(\Lambda')$ for some $\kappa' \in \mathcal{K}'$, then such a κ' restricts to an A-linear isomorphism $\Lambda' \to \Lambda$. Conversely, any A-linear isomorphism $\varphi\colon \Lambda' \to \Lambda$ between any A-structures Λ, Λ' induces an automorphism

$$M \cong \Lambda'_{\hat{A}} \xrightarrow{\varphi_{\hat{A}}} \Lambda_{\hat{A}} \cong M$$

in $\mathcal{K}'$. In the case $I = A$, the corollary thus follows from Proposition 6.17. Consider any A-structure Λ. In the general case, it thus suffices to show that the number $n(\Lambda)$ of $\mathcal{K}$-orbits in the $\mathcal{K}'$-orbit of Λ equals $|\overline{A}^{\times}/A^{\times}|$. Set $\Gamma' := \mathrm{Aut}_A(\Lambda)$ and let $\Gamma \subset \Gamma'$ be its principal congruence subgroup of level I. The orbit $\mathcal{K}' \cdot \Lambda$, resp. $\mathcal{K} \cdot \Lambda$, is then in a natural bijection with $\mathcal{K}'/\Gamma'$, resp. $\mathcal{K}/\Gamma$. Via the isomorphism $\overline{\Lambda} \cong \overline{M}$ induced by $\Lambda \subset M$, then as desired

$$\begin{aligned} n(\Lambda) = |(\mathcal{K}'/\Gamma')/(\mathcal{K}/\Gamma)| = |(\mathcal{K}'/\mathcal{K})/(\Gamma'/\Gamma)| &\overset{(6.11)}{=} |\,\mathrm{Aut}_{\overline{A}}(\overline{M})/(\Gamma'/\Gamma)| \\ = |\,\mathrm{Aut}_{\overline{A}}(\overline{\Lambda})/(\Gamma'/\Gamma)| = |\,\mathrm{Aut}_{\overline{A}}(\overline{\Lambda})/\Gamma'| &\overset{(36)}{=} \left|(\overline{A})^{\times}/A^{\times}\right|. \quad \square \end{aligned}$$

Suppose finally that $I \subsetneq A$ and consider any free $\overline{A}$-submodule $0 \neq W \subset \overline{M}$. For any direct summand $0 \neq N \subset M$ view Ω_N as a disjoint union of the Grothendieck ringed spaces $\Omega_{\{L\}}$ for all A-structures L of N. Consider the disjoint union of Grothendieck ringed spaces

$$\Omega_{M,W} := \coprod_{\substack{N \subset M \\ \overline{N} = W}} \Omega_N$$

being naturally acted by $\mathcal{K}$; let $\Omega_{\mathcal{K},W}$ be its quotient by $\mathcal{K}$.

Proposition 6.21. *For any free direct summand $0 \neq N \subset M$ with $\overline{N} = W$ the inclusion $\Omega_N \subset \Omega_{M,W}$ induces an isomorphism of Grothendieck ringed spaces*

$$\Omega_{\overline{\mathcal{K}}_N} \cong \Omega_{\mathcal{K},W},$$

where $\overline{\mathcal{K}}_N := \{\kappa' \in \mathrm{Aut}_{\hat{A}}(N) |\ \exists \kappa \in \mathcal{K}\colon \kappa|_N = \kappa'\}$.

Proof. That it induces an injective morphism $\Omega_{\overline{\mathcal{K}}_N} \to \Omega_{\mathcal{K},W}$ follows directly from the construction. Surjectivity follows from Corollary 6.13. $\square$

Consider further any A-structure Λ of M and identify $\overline{\Lambda}$ with $\overline{M}$ as above. Set $\Gamma := \overline{\mathcal{K}}_\Lambda$; it is the kernel of the natural homomorphism $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_{\overline{A}}(\overline{\Lambda})$. Consider the disjoint union of Grothendieck ringed spaces

$$\Omega_{\Lambda,W} := \coprod_{\substack{L\subset\Lambda \\ \overline{L}=W}} \Omega_L$$

being naturally acted by Γ; let $\Omega_{\Gamma,W}$ be its quotient by Γ.

Proposition 6.22. *Suppose that $W \subsetneq V$. Then the injections $\Omega_L \to \Omega_{L_{\hat{A}}}, [l] \mapsto ([l], L)$ for all $L \subset \Lambda$ with $\overline{L} = W$ induce an isomorphism of Grothendieck ringed spaces $\Omega_{\Gamma,W} \cong \Omega_{\mathcal{K},W}$.*

Proof. That it induces an injective morphism $\Omega_{\Gamma,W} \to \Omega_{\mathcal{K},W}$ follows directly from the construction. Let us show that it is also surjective. Consider any direct summand $N \subset M$ with $\overline{N} = W$ and any A-structure L' of N. As $W \subsetneq V$, Corollary 6.18 provides a direct summand $0 \neq L \subsetneq \Lambda$ such that $\overline{L} = \overline{L'}$ via the canonical inclusions or identifications $\overline{L} \subset \overline{\Lambda} = \overline{M} \supset \overline{N} = \overline{L'}$ and an A-linear isomorphism $\rho\colon L \to L'$ that induces the identity map $\overline{L} \to \overline{L'}$. Proposition 6.11 then provides a $\kappa \in \mathcal{K}$ that restricts to the tensor product of ρ by $\hat{A}$ and hence restricts to ρ. Then $\kappa(\Omega_L) = \Omega_{\kappa(L)} = \Omega_{L'}$ from which surjectivity follows. $\square$

Following Definition 2.35, a finitely generated projective A-submodule $Y \subset C$ is an *A-lattice* if the natural homomorphism $Y_E \to E \cdot Y$ is injective. Let $d := \mathrm{rank}_{\hat{A}}(M)$. By a *level-$I$-structure* of an A-lattice $Y \subset C$ of rank d, we mean an $\overline{A}$-linear isomorphism $i\colon \overline{M} \to \overline{Y}$. By an isomorphism from any such (Y, i) to any further such tuple (Y', i'), we mean an element $c \in C^\times$ such that multiplication by c maps Y onto Y' and such that the induced isomorphism $\overline{Y} \to \overline{Y'}$ is compatible with the level structures.

For any A-structure Λ of M identify $\overline{\Lambda}$ with $\overline{M}$ via the isomorphism induced by the inclusion $\Lambda \subset M$. Corollary 6.12 essentially implies

Corollary 6.23. *A bijection between $\Omega_{\mathcal{K}}$ and the set of isomorphism classes of A-lattices in C of rank d with level I-structure is induced by associating with any $([l], \Lambda) \in \Omega_M$ the class of $l(\Lambda) \subset C$ with level-I-structure $\bar{l}\colon \overline{M} \to \overline{l(\Lambda)}$ induced by l.*

Proof. It is directly checked for any $([l], \Lambda) \in \Omega_M$ that $(l(\Lambda), \bar{l})$ is an A-lattice with level-I-structure and that its isomorphism class depends only on its class in $\Omega_{\mathcal{K},k}$. Consider any $([l], \Lambda), ([l'], \Lambda') \in \Omega_M$ whose associated isomorphism classes coincide and let us show the claim that their images in $\Omega_{\mathcal{K}}$ coincide. Without loss of generality, we assume the representatives l, l' to be such that $l(\Lambda) = l'(\Lambda')$. Thus also the free $\hat{A}$-modules $l(\Lambda)_{\hat{A}}$

and $l'(\Lambda')_{\hat{A}}$ of rank d are equal. There exists thus a unique $\kappa \in \mathrm{Aut}_{\hat{A}}(M)$ whose composition with the tensor product $(l|_\Lambda)_{\hat{A}}$ equals $(l'|_{\Lambda'})_{\hat{A}}$ via the isomorphisms $\Lambda_{\hat{A}} \cong M \cong \Lambda'_{\hat{A}}$. As $\bar{l} = \overline{l'}$, in fact $\kappa \in \mathcal{K}$. This directly yields the claim. Consider then any A-lattice $Y \subset C$ of rank d with any level I-structure $i\colon \overline{M} \to \overline{Y}$. Then $Y_{\hat{A}}$ is a free $\hat{A}$-lattice of rank d and hence isomorphic to M. Further using Corollary 6.12, we choose an $\hat{A}$-linear isomorphism $\eta\colon M \to Y_{\hat{A}}$ inducing i. Then $\Lambda := \eta^{-1}(Y)$ is an A-structure of M. As Y is an A-lattice, $\eta|_\Lambda$ induces an isomorphism $\Lambda_E \to E \cdot Y$ and thus extends uniquely to a C-linear map $l\colon \Lambda_C \to C$ for which $\mathrm{Ker}(l) \cap \Lambda_E = 0$. Thus $[l] \in \Omega_\Lambda$ and, by construction, $(l(\Lambda), \bar{l}) = (Y, i)$ as desired. $\square$

7. Compactifications of algebraic moduli spaces

Let $A \subset C$ be as in Sections 5 and 6. Denote by F the quotient field of A, by p the characteristic of F and by $\mathbb{F}_p$ the field with p elements. For any $0 \neq a \in A$ set $\deg(a) := \dim_{\mathbb{F}_p}(A/(a))$. For any line bundle E over any scheme S over $\mathbb{F}_p$ denote by $\tau\colon E \to E^p, x \mapsto x^p$ the Frobenius homomorphism.

7.1. Pink's normal compactification

Proposition 7.1. *(Drinfeld [12, Proposition 2.1]) Consider any line bundle E over any field K of characteristic p and any homomorphism*

$$\varphi\colon A \to \mathrm{End}(E), a \mapsto \varphi_a := \sum_{i \geq 0} \varphi_{a,i}\tau^i,$$

where any $\varphi_{a,i}$ is in the one-dimensional K-vector space $\Gamma(\mathrm{Spec}(K), E^{1-p^i})$ and any $\varphi_{a,0}$ is the image of a under the structure homomorphism $A \to K$. Then there exists a unique integer $r \geq 0$ such that $\varphi_{a,i} = 0$ for any $i > r \cdot \deg(a)$ and such that $\varphi_{a,r\cdot\deg(a)} \neq 0$ for any $0 \neq a \in A$ with $r \cdot \deg(a) > 0$.

Definition 7.2. Any φ as in Proposition 7.1 with $r > 0$ is called a *Drinfeld A-module* over K of *rank* r.

Let S be a scheme over F.

Definition 7.3. (Pink [29, Definition 3.1]) A *generalized Drinfeld A-module* over S is a pair (E, φ) consisting of a line bundle E over S and a ring homomorphism

$$\varphi\colon A \to \mathrm{End}(E), a \mapsto \varphi_a = \sum_i \varphi_{a,i}\tau^i$$

with $\varphi_{a,i} \in \Gamma(S, E^{1-p^i})$ satisfying the following conditions:

- The derivative $d\varphi\colon A \to \varphi_{a,0}$ is the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.
- Over any point $s \in S$ the map φ defines a Drinfeld A-module of some rank $r_s \geq 1$ in the sense of Definition 7.2.

A generalized Drinfeld A-module is of rank $\leq r$ if

$$\forall a \in A, \forall i > r \cdot \deg(a)\colon \varphi_{a,i} = 0.$$

An isomorphism of generalized Drinfeld A-modules is an isomorphism of line bundles that is equivariant with respect to the action of A on both sides.

Definition 7.4. (Pink [29, Definition 3.2]) A generalized Drinfeld A-module over S of rank $\leq r$ with $r_s = r$ everywhere is a *Drinfeld A-module* of rank r over S.

Lemma 7.5. *If $S = \operatorname{Spec}(R)$ is affine, then giving a Drinfeld A-module of rank r as in Definition 7.4 is equivalent to giving, as in the introduction, a ring homomorphism*

$$\varphi\colon A \to R\{\tau\}, a \mapsto \varphi_a = \sum_{0 \leq i \leq d \cdot \deg(a)} \varphi_{a,i}\tau^i$$

for which $\varphi_{a,0} = \iota(a)$, where $\iota\colon A \to R$ is the structure morphism, and for which $\varphi_{a,d\cdot\deg(a)} \in R^\times$ for any $0 \neq a \in A$.

Proof. See Pink's [29, Proposition 3.4 and its proof]. $\square$

Consider any ideal $0 \neq I \subsetneq A$ and any free A/I-module $V \neq 0$ of finite rank d.

Notation 7.6. Denote by $\underline{V}$ the constant group scheme over S with fibers V.

Definition 7.7. A *level I structure* on a Drinfeld A-module $\varphi\colon A \to \operatorname{End}(E)$ of rank d is an isomorphism of group schemes

$$\underline{V} \longrightarrow \bigcap_{a \in I} \operatorname{Ker}(\varphi_a).$$

Lemma 7.8. *Suppose that $S = \operatorname{Spec}(R)$ is affine with structure morphism $\iota\colon A \to R$ and that $I = (t)$ for some $0 \neq t \in A$. Giving a level (t) structure on a Drinfeld A-module φ over R is then equivalent to giving, as in the introduction, a map $\lambda\colon V \to R$ for which $\lambda(V \setminus \{0\}) \subset R^\times$ and*

$$\varphi_t(T) = \iota(t) \cdot T \prod_{0 \neq v \in V} \left(1 - \frac{T}{\lambda(v)}\right) \tag{37}$$

for which the induced map $\lambda\colon V \to \operatorname{Ker}(R \xrightarrow{\varphi_t} R)$ is an A-linear isomorphism.

Proof. This is directly checked. For details, see [20, Lemma 8.8]. □

Denote by $X^d_{A,I}$ Drinfeld's [12, Section 5] fine moduli space over $\operatorname{Spec}(F)$ of Drinfeld A-modules of rank d with level I structure; it is an irreducible smooth affine algebraic variety of dimension $d-1$ of finite type over $\operatorname{Spec}(F)$.

For the remainder of this section consider any congruence subgroup $\mathcal{K} \subset \operatorname{Aut}_{\hat{A}}(M)$ as in Section 6 and suppose that $V = I^{-1}M/M$ and that $\mathcal{K}$ contains the kernel $\mathcal{K}(I)$ of $\operatorname{Aut}_{\hat{A}}(M) \to \operatorname{Aut}_A(V)$.

Definition 7.9. The subgroup $\mathcal{K} \subset \operatorname{Aut}_{\hat{A}}(M)$ is called *fine* if for some maximal ideal $\mathfrak{p} \subset A$ the image of $\mathcal{K}$ in $\operatorname{Aut}_A(\mathfrak{p}^{-1}M/M)$ is unipotent.

Definition-Proposition 7.10. *The natural action of $\mathcal{K}$ on level I structures induces an action on $X^d_{A,I}$ that factors through the finite group $\mathcal{K}/\mathcal{K}(I)$. Denote its quotient by*

$$X^d_{A,\mathcal{K}} := (\mathcal{K}/\mathcal{K}(I))\backslash X^d_{A,I}.$$

If $\mathcal{K}$ is fine, then the universal family on $X^d_{A,I}$ descends to a Drinfeld A-module on $X^d_{A,\mathcal{K}}$ which is called the universal family *on $X^d_{A,\mathcal{K}}$. Moreover, $X^d_{A,\mathcal{K}}$ and, if $\mathcal{K}$ is fine, its universal family are, up to a natural isomorphism, independent of the choice of such I.*

Proof. See Pink's [29, (1.1)-(1.3) and Proposition 1.5]. □

Definition 7.11. (Pink [29, Def. 3.9]) A generalized Drinfeld A-module (E,φ) over S is called *weakly separating* if for any Drinfeld A-module (E',φ') over any field L containing F, at most finitely many fibers of (E,φ) over L-valued points of S are isomorphic to (E',φ').

Theorem 7.12. *(Pink [29, Theorem 4.2]) If $\mathcal{K}$ is fine, then there exists a normal projective algebraic variety $\overline{X}^d_{A,\mathcal{K}}$ over F together with an open embedding*

$$X^d_{A,\mathcal{K}} \to \overline{X}^d_{A,\mathcal{K}}$$

and a weakly separating generalized Drinfeld A-module $(\overline{E},\overline{\varphi})$ on $\overline{X}^d_{A,\mathcal{K}}$ extending the universal family on $X^d_{A,\mathcal{K}}$; moreover, such $\overline{X}^d_{A,\mathcal{K}}$ and $(\overline{E},\overline{\varphi})$ are unique up to unique isomorphism.

7.2. Moduli space of A-reciprocal maps

Using that A is finitely generated, choose any $0 \neq t \in A$ whose divisors

$$\operatorname{Div}_A(t) := \{a \in A | t \in (a)\} \tag{38}$$

generate A. Consider any free $A/(t)$-module $V \neq 0$ of finite rank.

For any ideal $\mathfrak{a} \subset A$ consider the $\mathfrak{a}$-torsion submodule

$$T_{\mathfrak{a}}(V) := \{v \in V |\ \forall a \in \mathfrak{a} \colon a \cdot v = 0\} \subset V.$$

Set $T_a(V) := T_{(a)}(V)$ for any $a \in A$. For any $W \subset V$ set

$$\mathring{W} := W \setminus \{0\}.$$

With any invertible sheaf $\mathcal{L}$ on any scheme S associate the graded ring of global sections

$$R(S, \mathcal{L}) := \bigoplus_{n \geq 0} \Gamma(S, \mathcal{L}^n),$$

where any $\Gamma(S, \mathcal{L}^n)$ denotes the space of global sections of $\mathcal{L}^n$.

Definition 7.13. A map $\mathring{V} \to \Gamma(S, \mathcal{L})$ is called *fiberwise non-zero*, resp. *fiberwise injective*, if for any point $s \in S$ the composite $\mathring{V} \to \Gamma(S, \mathcal{L}) \to \mathcal{L} \otimes_{\mathcal{O}_S} k(s)$ is non-zero, resp. injective.

Definition 7.14. Consider any invertible sheaf $\mathcal{L}$ on any scheme S over $\mathrm{Spec}(A)$. A map $\rho \colon \mathring{V} \to \Gamma(S, \mathcal{L})$ is *A-reciprocal* if for all $a \in \mathrm{Div}_A(t)$ and all $v, v' \in \mathring{V}$:

i) $a \cdot v \in \mathring{V} \Rightarrow \rho(v)^{|T_a(V)|} = a \cdot \rho(a \cdot v) \cdot \prod_{0 \neq w \in T_a(V)} (\rho(v) - \rho(w))$,
ii) $v + v' \in \mathring{V} \Rightarrow \rho(v) \cdot \rho(v') = \rho(v + v') \cdot (\rho(v) + \rho(v'))$,
iii) and if there exists a ring homomorphism $\varphi^\rho \colon A \to R(S, \mathcal{L})\{\tau\} = \mathrm{End}(\mathcal{L}^{-1})$ restricting to

$$\mathrm{Div}_A(t) \to R(S, \mathcal{L})[T],\ a \mapsto \varphi_a(T) := a \cdot T \cdot \prod_{0 \neq v \in T_a(V)} (1 - \rho(v) \cdot T).$$

Consider the polynomial ring $A_{\mathring{V}} := A[(Y_v)_{v \in \mathring{V}}]$. Let $I_{\mathring{V}} \subset A_{\mathring{V}}$ be the smallest homogeneous ideal for which

$$\sigma_V \colon \mathring{V} \to A_{\mathring{V}}, v \mapsto Y_v$$

induces an A-reciprocal map

$$\rho_V \colon \mathring{V} \to \Gamma(Q_V, \mathcal{O}_{Q_V}(1)) \subset A_{\mathring{V}}/I_{\mathring{V}},$$

where $\mathcal{O}_{Q_V}(1)$ denotes the first twisting sheaf of $Q_V := \mathrm{Proj}(A_{\mathring{V}}/I_{\mathring{V}})$. Denote by

$$\Omega_V \subset Q_V$$

the open subscheme defined as the non-vanishing locus of $\{\rho_V(v)|v \in \mathring{V}\}$.

Proposition 7.15. *The scheme Q_V, resp. Ω_V, with the universal family $(\mathcal{O}_{Q_V}(1), \rho_V)$, resp. $(\mathcal{O}_{Q_V}(1)|_{\Omega_V}, \rho_V|_{\Omega_V})$, represents the functor which associates with any scheme S over* $\mathrm{Spec}(A)$ *the set of isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ on S and a fiberwise non-zero, resp. fiberwise injective, A-reciprocal map $\rho\colon \mathring{V} \to \Gamma(S, \mathcal{L})$.*

Proof. Denote by $\mathcal{O}_{\mathring{V}}(1)$ the first twisting sheaf of $P_{\mathring{V}} := \mathrm{Proj}(A_{\mathring{V}})$. By [21, Chapter 2, Theorem 7.1], the scheme $P_{\mathring{V}}$ with the universal family $(\mathcal{O}_{\mathring{V}}(1), \sigma_V)$ represents the functor which associates with any scheme over $\mathrm{Spec}(A)$ the set of isomorphism classes of pairs $(\mathcal{L}, \rho)$ consisting of an invertible sheaf $\mathcal{L}$ on S and a fiberwise non-zero map $\rho\colon \mathring{V} \to \Gamma(S, \mathcal{L})$. The relations defining $I_{\mathring{V}}$ are precisely those that require such a ρ to be A-reciprocal. The proposition then follows by construction of Q_V and $\Omega_{\mathcal{V}}$. $\square$

Consider any free $A/(t)$-submodule $0 \neq W \subset V$. Extending any fiberwise non-zero A-reciprocal map $\rho\colon \mathring{W} \to \Gamma(S, \mathcal{L})$ to $\mathring{V}$ by setting $\rho(v) := 0$ for any $v \in V \setminus W$ yields a fiberwise non-zero A-reciprocal map. This defines a closed embedding $\mathrm{Q}_W \to Q_V$ between the moduli schemes by means of which we identify Q_W with a closed subscheme of Q_V.

Theorem 7.16.

i) Q_V is the disjoint union of the locally closed subschemes Ω_W for all free $A/(t)$-submodules $0 \neq W \subset V$.

ii) Consider the functor which associates with any scheme S over F the set of isomorphism classes of triples (E, φ, λ), where E is a line bundle on S and $\varphi\colon A \to \mathrm{End}(E)$ is a Drinfeld A-module of rank d over S and $\lambda\colon \underline{V} \to \mathrm{Ker}(\varphi(t))$ is a level (t)-structure. Mapping any such (E, φ, λ) to $(\mathcal{L}, \rho)$, where $\mathcal{L}$ is the inverse of the invertible sheaf on S dual to E and where $\rho\colon \mathring{V} \to \Gamma(S, \mathcal{L}), v \mapsto \frac{1}{\lambda(v)}$, induces an isomorphism of functors whose image is the functor in Proposition 7.15 represented by the pullback $\Omega_{V,F}$ of Ω_V to F.

Proof. The assertion in ii) is local in S. Consider any ring homomorphism $\iota\colon A \to R$. Via Lemma 7.5, giving a Drinfeld A-module φ of rank d over R is equivalent to giving for any $a \in \mathrm{Div}_A(t)$ a polynomial

$$\varphi_a = \sum_{0 \leq i \leq d \cdot \deg(a)} \varphi_{a,i} \tau^i \in R\{\tau\}$$

with $\varphi_{a,0} = \iota(a)$ and $\varphi_{a,d\cdot\deg(a)} \in R^\times$ such that

$$\mathrm{Div}_A(t) \to R\{\tau\}, a \mapsto \varphi_a \tag{39}$$

extends to a ring homomorphism $A \to R\{\tau\}$. Via Lemma 7.8, giving a level (t)-structure for such φ is equivalent to giving an injection $\lambda\colon V \to R$ for which $\lambda(\mathring{V}) \subset R^\times$ and for which (37) holds and such that

$$\forall a \in \operatorname{Div}_A(t)\colon \varphi_a \circ \lambda = \lambda \circ a \quad \text{and} \quad \forall v, v' \in V\colon \lambda(v+v') = \lambda(v) + \lambda(v'). \tag{40}$$

Let $a \in \operatorname{Div}_A(t)$. We claim that any such level (t) structure satisfies that

$$\varphi_a(T) = \iota(a) \cdot T \prod_{0 \neq v \in T_a(V)} \left(1 - \frac{T}{\lambda(v)}\right). \tag{41}$$

Indeed, (40) implies that $\lambda(T_a(V))$ is contained in the set of zeroes of φ_a. As $T_a(V)$ is a free A/a-module of rank d, moreover

$$|\lambda(T_a(V))| = |T_a(V)| = q^{d \cdot \deg(a)} = q^{\deg_\tau(\varphi_a)}.$$

Hence the left and right hand side of (41) coincide up to an element in $R^\times$. This element must be 1 since the constant coefficient of each side is $\iota(a)$ which is non-zero as R is over F. This yields the claim. From this characterization of Drinfeld A-modules over R with level (t) structure, Part ii) is directly deduced.

Consider then any $s \in Q_V$ with A-reciprocal map

$$\rho^s\colon \mathring{V} \to \mathcal{O}_{Q_V}(1) \otimes \mathcal{O}_{Q_V} k(s) =: K$$

induced by ρ. As ρ^s is non-zero by assumption, the ring homomorphism $\varphi\colon A \to K\{\tau\}$ induced by ρ^s does not coincide with the structure homomorphism $A \to K$; as K is a field, it is thus, by Proposition 7.1 a Drinfeld A-module of some rank $1 \leq d' \leq d$. Then $\operatorname{Ker}(\varphi_t)$ is a free $A/(t)$-module scheme of rank d' (see e.g. [27, Proposition 4.1]). Let

$$W := \{0\} \cup \{v \in \mathring{V} | \rho^s(v) \neq 0\}. \tag{42}$$

Properties i) and ii) in Definition 7.14 of ρ^s imply that

$$\mathring{W} \to \operatorname{Ker}(\varphi_t), w \mapsto \frac{1}{\rho^s(w)}$$

extends to an A-linear isomorphism $W \to \operatorname{Ker}(\varphi_t)$. Hence $W \subset V$ is a free $A/(t)$-submodule of rank d' and $s \in \Omega_W$. By (42), moreover, $s \notin \Omega_{W'}$ for any other free non-zero $A/(t)$-submodule $W' \subset V$. This yields Part i). □

Remark 7.17. Lemma 7.8 and Theorem 7.16,ii) work more generally for schemes S over $\operatorname{Spec}(A[\frac{1}{t}])$. Thus already over $\operatorname{Spec}(A[\frac{1}{t}])$, the scheme Q_V is a compactification of Drinfeld's moduli scheme of Drinfeld A-modules of rank r with level (t) structure.

Proposition 7.18. *The pullback of Q_V to F is irreducible.*

Proof. Property v) in the proof of Theorem 8.14, which does not depend on this proposition, implies that $\Omega_V(C)$ is dense in $Q_V(C)$. Hence the pullback of Ω_V to F is dense in the pullback of Q_V to F. That the latter is irreducible, thus follows via Theorem 7.16,ii) from the irreducibility of Drinfeld's moduli scheme over F. $\square$

Definition 7.19. A subgroup $\Delta \subset \mathrm{Aut}_A(V)$ is called *fine* if it has unipotent image in $\mathrm{Aut}_A(T_{\mathfrak{p}}(V))$ for some maximal ideal $\mathfrak{p} \subset A$ containing t.

Proposition 7.20. *Consider any fine subgroup $\Delta \subset \mathrm{Aut}_A(V)$ by means of some maximal ideal $\mathfrak{p} \subset A$ and consider any free $A/(t)$-submodule $0 \neq W \subset V$. Then the stabilizer $\Delta_W := \{\delta \in \Delta |\ \delta(W) = W\}$ of W in Δ is a fine subgroup of $\mathrm{Aut}_A(W)$ by means of $\mathfrak{p}$ and it has a non-zero fixed point in $T_{\mathfrak{p}}(W)$, and hence in W, under the natural action.*

Proof. The first assertion is directly checked. Let us show the second assertion. The assumption that $t \in \mathfrak{p}$ implies that $T_{\mathfrak{p}}(W) \neq 0$. It then suffices to show that the image G of Δ_W in $\mathrm{Aut}_A(T_{\mathfrak{p}}(W))$ is a p-group, where p is the characteristic of A. Suppose, by contradiction, the existence of a non-trivial $g \in G$ of order k not divisible by p. Let χ, resp. m, be the characteristic, resp. minimal, polynomial of g over $A/\mathfrak{p}$ and set $r(X) := X^k - 1$. Since g is unipotent, χ is a power of $(X-1)$. Moreover, r is separable since p does not divide k. As m divides both χ and r, it thus equals $X-1$. This implies that g is trivial and thus yields a contradiction as desired. $\square$

For any subgroup $\Delta \subset \mathrm{Aut}_A(V)$ view $\Delta\backslash Q_V(C)$ with its structure of projective rigid analytic variety. For any integer k denote by $\mathcal{O}(k)$ the analytification of the pullback of the k-th twisting $\mathcal{O}_{Q_V}$-module to $Q_V(C)$ under $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Pink's [29, Lemma 4.4 and its proof] inspired

Proposition 7.21. *Consider any fine subgroup $\Delta \subset \mathrm{Aut}_A(V)$ and any integer k. Then the subsheaf of Δ-invariants of $\mathcal{O}(k)$ is an invertible sheaf on the projective rigid analytic variety $\Delta\backslash Q_V(C)$ and its pullback to $Q_V(C)$ is $\mathcal{O}(k)$. Moreover, if $k > 0$, then the subsheaf of Δ-invariants of $\mathcal{O}(k)$ is ample.*

Proof. Let $\mathcal{O}_\Delta(k)$ denote the subsheaf of $\mathcal{O}(k)$ of Δ-invariants. For any free $A/(t)$-submodule $0 \neq W \subset V$ consider the Zariski open subset $U_W \subset Q_V$ defined as the union of the $\Omega_{W'}$ for all free $A/(t)$-submodules $W \subset W' \subset V$. Choose then a $1 > \varepsilon \in |C^\times|$. For any such W consider the admissible subset

$$U(W,\varepsilon) := \left\{ [(y_\alpha)_{\alpha \in \mathring{V}}] \in U_W(C) \mid \forall \alpha \in \mathring{W}, \forall \beta \in V \setminus W \colon \left| \frac{y_\beta}{y_\alpha} \right| \leq \varepsilon \right\} \subset Q.$$

By Theorem 7.16,i) and since $\Omega_W(C) \subset U(W,\varepsilon)$ for any such W, the rigid analytic variety Q is covered by the $U(W,C)$ for all such W. As this covering is finite, it is admissible. As $\varepsilon < 1$, it holds that $U(W,\varepsilon) \cap U(W',\varepsilon) = \emptyset$ for any free submodules $W, W' \subset V$ with $W \not\subset W' \not\subset W$. Moreover, $g(U(W,\varepsilon)) = U(g(W),\varepsilon)$ for any such W and any $g \in \mathrm{Aut}_A(W)$. Consequently, any such $U(W,\varepsilon)$ is invariant under $\Delta_W := \{\delta \in \Delta | \delta(W) = W\}$ and satisfies that $\delta(U(W,\varepsilon)) \cap U(W,\varepsilon) = \emptyset$ for any $\delta \in \Delta \setminus \Delta_W$. In order to see that $\mathcal{O}_\Delta(k)$ is an invertible sheaf, it thus suffices to show that for any such W the subsheaf $\mathcal{O}_W(k)$ of Δ_W-invariants of the restriction of $\mathcal{O}(k)$ to $U(W,\varepsilon)$ is an invertible sheaf on $\Delta_W \backslash U(W,\varepsilon)$. Consider such a W. Then Proposition 7.20 provides an $0 \neq \alpha \in W$ that is fixed by Δ_W. For such an α the restriction of the global section $(Y_\alpha)^k$ to $\mathcal{O}_W(k)$ thus induces a nowhere vanishing global section in the quotient $\Delta_W \backslash \mathcal{O}_W(k)$ and hence yields a trivialization of it as desired. Let $\mathcal{F}$ denote the pullback of $\mathcal{O}_\Delta(k)$ under the quotient morphism. Using the above trivialization, it is directly checked that the natural morphism $\mathcal{F} \to \mathcal{O}(k)$ of $\mathcal{O}(0)$-modules is an isomorphism. By [24, Chapter 2, Proposition 5.12, (c)], $\mathcal{O}_{Q_V}(k)$ is the inverse image of the k-th twisting sheaf on $\mathrm{Proj}(A_{\check{V}})$ and is thus [24, Chapter 2, Proposition 5.12,(b) and Theorem 7.6] ample if $k > 0$. If $k > 0$, thus $\mathcal{O}(k)$ is ample and, since it is the pullback under the finite quotient map of the invertible sheaf $\mathcal{O}_\Delta(k)$, also $\mathcal{O}_\Delta(k)$ is ample by [23, Chapter 1, Proposition 4.4.] via Köpf's GAGA-Theorem [26, Satz 5.1]. $\square$

For the remainder of this section consider any congruence subgroup $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ as in Section 6 and suppose that $V = t^{-1}M/M$ and that $\mathcal{K}$ contains the kernel $\mathcal{K}(t)$ of $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(V)$ and denote by Δ the image of $\mathcal{K}$ in $\mathrm{Aut}_A(V)$. Let $d := \mathrm{rank}_{A/(t)}(V)$.

Lemma 7.22. *$\mathcal{K}$ is fine (in the sense of Definition 7.9) if and only if Δ is fine.*

Proof. Suppose first that $\mathcal{K}$ is fine. Choose any maximal ideal $\mathfrak{p} \subset A$ such that the image in $\mathrm{Aut}_A(\mathfrak{p}^{-1}M/M)$ of $\mathcal{K}$ is unipotent. As $\mathcal{K} \supset \mathcal{K}(t)$, then $t \in \mathfrak{p}$. The natural morphism $T_\mathfrak{p}(V) \to \mathfrak{p}^{-1}M/M$ is thus an isomorphism which maps the image of Δ in $\mathrm{Aut}_A(T_\mathfrak{p}(V))$ onto the image of $\mathcal{K}$ in $\mathrm{Aut}_A(\mathfrak{p}^{-1}M/M)$. In particular, Δ is fine. The converse direction follows similarly from a suitable isomorphism as before. $\square$

Proposition 7.23. *The correspondence in Theorem 7.16,ii) induces an isomorphism between normal quasi-projective varieties*

$$X^d_{A,\mathcal{K}} \to \Delta \backslash \Omega_{V,F}. \tag{43}$$

Proof. Theorem 7.16,ii) provides an isomorphism $X^d_{A,\mathcal{K}(t)} \to \Omega_{V,F}$ between smooth quasi-projective varieties which is equivariant with respect to $\Delta \cong \mathcal{K}/\mathcal{K}(t)$. Its induced morphism on quotients is thus as desired. $\square$

Denote by E_V the line bundle on Q_V dual to the minus first twisting $\mathcal{O}_{Q_V}$-module and view φ^{ρ_V} as ring homomorphism $A \to \mathrm{End}(E_V)$. Denote by $Q_{V,F}$, resp. $E_{V,F}$, resp. $\varphi^{\rho_V}_F$, the pullback of Q_V, resp. E_V, resp. φ^{ρ_V}, to F. Denote by $n\colon Q^n_{V,F} \to Q_{V,F}$ the normalization morphism. The action of Δ on Q_V induces an action on the projective variety $Q^n_{V,F}$; denote by Q^n_Δ its quotient viewed as projective algebraic variety.

The following corollary follows from Theorem 7.16,ii) in the case where $\Delta = 0$ and then by Pink's [29, Lemma 4.4 and its proof] in the general case.

Corollary 7.24. *Suppose that $\mathcal{K}$ and Δ are fine. Then the pullback of $(E_{V,F}, \varphi_F^{\rho_V})$ under n descends to a weakly separating Drinfeld A-module $(E^n_\Delta, \varphi^\Delta)$ over Q^n_Δ which extends, via (43), the universal family of the open subscheme $\Delta\backslash\Omega_V \subset Q^n_\Delta$.*

Proof. Theorem 7.16 implies that $(E_{V,F}, \varphi_F^{\rho_V})$ is a weakly separating Drinfeld A-module over $Q_{V,F}$ that extends the universal family over $\Omega_{V,F}$. By construction, it is Δ-invariant. Moreover, by means of the normality of $\Omega_{V,F}$, we identify $\Omega_{V,F}$ with its preimage under n. Hence also the pullback of $(E_{V,F}, \varphi_F^{\rho_V})$ under the finite morphism n is a Δ-invariant weakly separating Drinfeld A-module over the projective scheme $Q^n_{V,F}$ that extends the universal family over $\Omega_{V,F}$. By Pink's [29, Lemma 4.4 and its proof], as Δ is fine, then the quotient E^n_Δ of the pullback of $E^n_{V,F}$ under n by Δ is a line bundle over Q^n_Δ and the pullback of $(E_{V,F}, \varphi_F^{\rho_V})$ descends to a weakly separating Drinfeld A-module $(E^n_\Delta, \varphi^\Delta)$ over the projective scheme Q^n_Δ that extends the universal family over $\Delta\backslash\Omega_{V,F}$. □

Corollary 7.25. *Suppose that $\mathcal{K}$ and Δ are fine. Then Q^n_Δ and $(E^n_\Delta, \varphi^\Delta)$ coincide up to unique isomorphism with $\overline{X}^d_{A,\mathcal{K}}$ and $(\overline{E}, \overline{\varphi})$ from Theorem 7.12.*

Proof. This follows from Corollary 7.24 and the uniqueness property in Theorem 7.12. □

8. Comparison of algebraic and analytic compactifications

Let $A \subset C$ be as in Sections 5, 6, 7. Let $0 \neq t \in A$ be such that $\mathrm{Div}_A(t)$ generates A as in Section 7.2. Consider any free $\hat{A}$-module $M \neq 0$ of finite rank, set $V := t^{-1}M/M$ and $\mathring{V} := V\backslash\{0\}$ and let $\mathcal{K}$ be the kernel of the natural homomorphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(V)$. Consider the closed subvariety

$$Q := Q_V(C) \subset P := \mathrm{Proj}(C[(Y_v)_{v\in\mathring{V}}])$$

provided by Section 7.2 with its structure of reduced rigid analytic variety over C. By Theorem 7.16,i), Q is stratified by the locally closed subvarieties $\Omega_W(C)$ for all free $A/(t)$-submodules $0 \neq W \subset V$; any $\Omega_W(C)$ is the intersection of the non-vanishing locus in Q of $(Y_\alpha)_{0\neq\alpha\in W}$ with the vanishing locus in Q of $(Y_\alpha)_{\alpha\in V\backslash W}$. Set $\Omega := \Omega_V(C)$. Recall the Eisenstein series $E_\alpha := E_{M,\alpha,1}$ for all $\alpha \in \mathring{V}$ from Example 6.9.

Theorem 8.1. *The $(E_\alpha)_{\alpha\in\mathring{V}}$ define a morphism of Grothendieck ringed spaces $E_\mathcal{K} \colon \Omega^*_\mathcal{K} \to Q$ which is the normalization morphism (in the sense of Conrad's [10]) of Q and restricts to Drinfeld's isomorphism $\Omega_\mathcal{K} \to \Omega$ between normal rigid analytic varieties. Moreover,* *the morphism of Grothendieck topological spaces underlying $E_{\mathcal{K}}$ restricts to isomorphisms between irreducible components.*

We prove Theorem 8.1 at the end of this section. We first recall Drinfeld's correspondence between level structures of A-lattices and level structures of Drinfeld A-modules as well as the induced isomorphism between moduli spaces.

Theorem 8.2. *(Drinfeld [12, Proposition 3.1]) Consider any integer $d \geq 1$ and any ideal $0 \neq I \subset A$. For any A-lattice $Y \subset C$ of rank d with level I-structure $i\colon (I^{-1}/A)^d \to I^{-1}Y/Y$ (see Corollary 6.23) the map*

$$\varphi\colon A \to C\{\tau\}, a \mapsto \varphi_a := a \cdot T \cdot \prod_{0 \neq [x] \in I^{-1}Y/Y} (1 - \frac{T}{e_Y(x)})$$

is a Drinfeld module of rank r with level I-structure

$$(I^{-1}A/A)^d \to I^{-1}Y/Y, v \mapsto e_Y(i(v))$$

satisfying $e_Y \circ a = \varphi_a \circ e_Y$ for any $a \in A$. This induces a bijection from the set of isomorphism classes of A-lattices in C of rank d with I-level structure to the set of rank d Drinfeld A-modules over C with level I-structure.

Proposition 8.3. *(Drinfeld [12, Prop. 6.6]) The rule*

$$\Omega_{\mathcal{K}} \to \Omega, \pi_{\mathcal{K}}(l, \Lambda) \mapsto \left[(E_\alpha(l, \Lambda))_{\alpha \in \mathring{V}}\right] \tag{44}$$

defines an isomorphism of rigid analytic varieties over C.

Proof. By Definition-Proposition 2.40, for any $\alpha \in \mathring{V}$ and any lift $\tilde{\alpha} \in t^{-1}\Lambda$ of α holds that

$$\forall (l, \Lambda) \in \tilde{\Omega}_M\colon E_\alpha(l, \Lambda) = \frac{1}{e_{l(\Lambda)}(l(\tilde{\alpha}))}.$$

From Proposition 6.23 and Theorem 8.2 applied to the case $I = (t)$ and from Theorem 7.16,ii) thus follows that the rule in (44) defines a bijective map. For more details, we refer to [12, Prop. 6.6] or [20, Prop. 9.5]. □

Corollary 8.4. *Consider any A-structure Λ of M, set $\Gamma := \overline{\mathcal{K}}_\Lambda$ and view Ω^*_Γ as a subspace of $\Omega^*_{\mathcal{K}}$ via Proposition 6.10. Consider any free $A/(t)$-submodule $0 \neq W \subsetneq V$ and recall the rigid analytic variety $\Omega_{\Gamma,W}$ defined before Proposition 6.22. Then*

$$\Omega_{\Gamma,W} \to \Omega_W(C), \pi_\Gamma(l) \mapsto \left[(E_\alpha(l, \Lambda))_{\alpha \in \mathring{W}}\right]$$

defines an isomorphism of rigid analytic varieties.

Proof. By means of Corollary 6.13, choose a free direct summand $N \subset M$ such that $t^{-1}N/N = W$. By Proposition 8.3,

$$\Omega_{\overline{\mathcal{K}}_N} \to \Omega_W(C), \pi_{\overline{\mathcal{K}}_N}(l, L) \mapsto \left[(E_{\alpha,N,1}(l, L))_{\alpha\in\mathring{W}}\right]$$

defines an isomorphism of rigid analytic varieties. It is directly checked that the precomposition of this isomorphism with the isomorphisms $\Omega_{\Gamma,W} \to \Omega_{\mathcal{K},W}$ and $\Omega_{\mathcal{K},W} \to \Omega_{\overline{\mathcal{K}}_N}$ provided by Propositions 6.21 and 6.22 defines the desired isomorphism. □

Choose a finite set of representatives $\{\Lambda_i\}_{i\in I}$ of the orbits of the $\mathcal{K}$-action on the set of A-structures of M and recall from Proposition 6.10 the isomorphism

$$\coprod_{i\in I} \Omega^*_{\Gamma_i} \to \Omega^*_{\mathcal{K}},$$

where $\Gamma_i := \overline{\mathcal{K}}_{\Lambda_i}$ for every $i \in I$. For any $i \in I$ denote by Ω_i the image of Ω_{Γ_i} under the isomorphism $\Omega_{\mathcal{K}} \to \Omega$ between normal rigid analytic varieties in Proposition 8.3. By Corollary 4.22, the Ω_{Γ_i} are the irreducible components of $\Omega_{\mathcal{K}}$ and hence the Ω_i are the irreducible components of Ω.

Definition 8.5. Set $Q_i := \Omega_i \cup (Q \setminus \Omega) \subset Q$ for any $i \in I$.

Lemma 8.6. *Any $Q_i \subset Q$ is Zariski closed and any $\Omega_i \subset Q_i$ is Zariski open.*

Proof. By Proposition 6.10, any Ω_{Γ_i} is Zariski closed and open in $\Omega_{\mathcal{K}}$. Hence any Ω_i is Zariski closed and open in Ω. As, furthermore, Ω is Zariski open in Q, thus any Ω_i is Zariski open in Q_i and any Q_i is Zariski closed in Q. □

Proposition 8.7. *For any $i \in I$ the rule*

$$E_i\colon \Omega^*_{\Gamma_i} \to Q_i, \pi_{\Gamma_i}(l) \mapsto (E_\alpha(l, \Lambda_i))_{\alpha\in\mathring{V}}$$

defines an isomorphism of Grothendieck topological spaces which restricts to a map $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ underlying an isomorphism of rigid analytic varieties for any free $A/(t)$-submodule $0 \neq W \subsetneq V$.

We will prove Proposition 8.7 before Corollary 8.11 after more preparation. However, we may already see that the rule $\pi_{\Gamma_i}(l) \mapsto (E_\alpha(l, \Lambda_i))_{\alpha\in\mathring{V}}$ defines a bijective map $E_i\colon \Omega^*_{\Gamma_i} \to Q_i$ which restricts to isomorphisms $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ of rigid analytic varieties: Indeed, this follows from Corollary 8.4 as well as the facts that $Q \setminus \Omega$, resp. $\Omega^*_{\Gamma_i} \setminus \Omega_{\Gamma_i}$, is the disjoint union the $\Omega_W(C)$ for all such $0 \neq W \subsetneq V$ by Theorem 7.16, resp. of the $\Omega_{\Gamma_i,W}$ by construction.

Let us recall the content of Proposition 3.3 in the present setup. Consider any subset $T \subset \mathring{V}$ and any $\varepsilon \in |C^\times|$ and associate with it the Zariski open, resp. admissible, resp. Zariski closed subvariety

$$\mathcal{U}(T) := \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in P \mid \forall\alpha \in T : y_\alpha \neq 0\} \subset P,$$
$$\mathcal{U}(T,\varepsilon) := \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in \mathcal{U}(T) \mid \forall\alpha' \in \mathring{V}\setminus T,\ \forall\alpha \in T : \left|\frac{y_{\alpha'}}{y_\alpha}\right| \leq \varepsilon\} \subset \mathcal{U}(T),$$
$$\Omega(T) := \{[(y_\alpha)_{\alpha\in\mathring{V}}] \in \mathcal{U}(T) \mid \forall\alpha' \in \mathring{V}\setminus T,\ \forall\alpha \in T : \frac{y_{\alpha'}}{y_\alpha} = 0\} \subset \mathcal{U}(T).$$

Then $\Omega(T) \neq \emptyset \Leftrightarrow T \neq \emptyset$; in this case, denote by $\rho_T : \mathcal{U}(T) \to \Omega(T)$ the natural projection morphism and for any $O \subset \Omega(T)$ set

$$\mathcal{U}(O,\varepsilon) := \rho_T^{-1}(O) \cap \mathcal{U}(T,\varepsilon).$$

Proposition 8.8. *Consider any closed subvariety $P' \subset P$. Then a subset $X \subset P'$ is admissible if and only if for any $T \subset \mathring{V}$ with $\Omega(T) \cap P' \neq \emptyset$:*

i) the subset $X \cap \Omega(T) \subset P' \cap \Omega(T)$ is admissible and
ii) any admissible quasi-compact $O \subset \Omega(T)$ with $O \cap P' \subset X$ admits an $\varepsilon > 0$ such that $\mathcal{U}(O,\varepsilon) \cap P' \subset X$.

A covering of an admissible subset $X \subset P'$ by admissible subsets is admissible if and only if its intersection with $X \cap \Omega(T)$ is admissible for any $T \subset \mathring{V}$.

Proof. The present setup is a special case of Example 3.2. Hence the proposition is an instance of Proposition 3.3. □

Proposition 8.9. *Let $T \subset \mathring{V}$. If $T = \mathring{W}$ for some free $A/(t)$-submodule $0 \neq W \subset V$, then $\Omega(T) \cap Q = \Omega_W(C)$. Otherwise, $\Omega(T) \cap Q = \emptyset$. Moreover, $\Omega(\mathring{V}) \cap Q_i = \Omega_i$ for any $i \in I$.*

Proof. This follows directly from Theorem 7.16,i). The last assertion follows directly from the definition of the Q_i. □

For any $i \in I$ and any free $A/(t)$-submodule $0 \neq W \subset V$ denote by $\mathrm{Orb}(i,W)$ the finite set of orbits $\mathfrak{O}$ of the Γ_i-action on the set of direct summand $L \subset \Lambda_i$ for which $t^{-1}L/L = W$. In the notation of Definition-Proposition 5.19 and Proposition 6.22, for any $i \in I$ we have a disjoint union

$$\Omega_{\Gamma_i,W} = \coprod_{\mathfrak{O}\in\mathrm{Orb}(i,W)} \Omega_{\mathfrak{O}} \tag{45}$$

of rigid analytic varieties and for any $Y \subset \Omega_{\Gamma_i,W}$ and any $r \in |C|$ we set

$$\mathcal{U}(\Lambda_i, Y, r) := \bigcup_{\mathfrak{O} \in \mathrm{Orb}(i,W)} \mathcal{U}(\Lambda_i, Y \cap \Omega_{\mathfrak{O}}, r) \subset \Omega^*_{\Gamma_i}. \tag{46}$$

Lemma 8.10. *Consider any $i \in I$, any free $A/(t)$-submodule $0 \neq W \subsetneq V$, any admissible quasi-compact $O \subset \Omega(\mathring{W})$ and any finite field $\mathbb{F}_q \subset A$ with q elements. Then there exist $c, r_O > 0$ such that for any $r_O < r \in |C|$:*

$$E_i^{-1}(\mathcal{U}(O, r^{-r \cdot q \cdot \mathrm{rank}_{\mathbb{F}_q[t]}(\Lambda)})) \subset \mathcal{U}(\Lambda_i, E_i^{-1}(O), r) \subset E_i^{-1}(\mathcal{U}(O, \frac{c}{r}))$$

Proof. Using the quasi-compactness of O, choose a $c > 1$ such that

$$\forall [y] = [(y_\beta)_{\beta \in \mathring{V}}] \in O, \forall \alpha, \alpha' \in \mathring{W}\colon \quad \left| \frac{y_{\alpha'}}{y_\alpha} \right| \leq c.$$

Choose a basis $a_1, \ldots, a_k$ of the $\mathbb{F}_q[t]$-module A and set

$$c' := c \cdot \max_{1 \leq i \leq k} |a_i|.$$

Using [20, Cor. 2.29], choose a $\delta > 0$ such that for any $[y], [z] \in \Omega(\mathring{W})$:

$$\left[\forall \alpha, \alpha' \in \mathring{W}\colon \left| \frac{y_{\alpha'}}{y_\alpha} - \frac{z_{\alpha'}}{z_\alpha} \right| < \delta \right] \Rightarrow [[y] \in O \Leftrightarrow [z] \in O]. \tag{47}$$

Set $r_O := \max\{c', \frac{c^2}{\delta}\}$. Consider any $r_O < r \in |C|$ and set $\varepsilon := r^{-r \cdot \mathrm{rank}_{\mathbb{F}_q[t]}(\Lambda)}$. Set $\Lambda := \Lambda_i$ and $\Gamma := \Gamma_i$ and $E := E_i$. Denote by $\pi\colon t^{-1}\Lambda \to V$ the quotient morphism. Moreover, for any subset $S \subset C$ set as before

$$\mathrm{d}(S) := \inf_{0 \neq s \in S} |s|.$$

Consider any $l \in \tilde{\Omega}^*_\Lambda$, say $l \in \tilde{\Omega}_L$. Set $\mathcal{L} := l(L)$ and $n := \mathrm{rank}_{\mathbb{F}_q[t]}(\mathcal{L})$. Choose an $x_l \in t^{-1}\mathcal{L} \setminus \{0\}$ of minimal norm and let $\alpha_l := \pi(l^{-1}(x_l))$. Proposition 2.41 then yields for any further $x' \in t^{-1}\mathcal{L}$ which is non-zero modulo $\mathcal{L}$ and of minimal norm in $x' + \mathcal{L}$, with $\alpha' := \pi(l^{-1}(x'))$, that

$$\left| \frac{x'}{x_l} \right| \leq \left| \frac{e_{\mathcal{L}}(x')}{e_{\mathcal{L}}(x_l)} \right| = \left| \frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \right| \leq \left| \frac{x'}{x_l} \right|^{\left| \frac{x'}{x_l} \right| \cdot q \cdot n}. \tag{48}$$

Suppose first that $\pi_\Gamma(l) \in E^{-1}(\mathcal{U}(O, \varepsilon))$. Then $\alpha_l \in W$; indeed, if α_l was not in W, then we could choose an x' and α' as in (48) with $\alpha' \in \mathring{W}$ and apply the assumption that then $\left| \frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \right| \leq \varepsilon < 1$ contradicting the fact that $|x_l| \leq |x'|$ via the first inequality of (48). Consider then any x'_1 and α'_1, resp. x'_2 and α'_2, as in (48) such that $\alpha'_1 \in \mathring{W}$, resp. $\alpha'_2 \notin \mathring{W}$. The first, resp. second, inequality of (48) then yields that

$$\left|\frac{x_1'}{x_l}\right| \leq \left|\frac{E_{\alpha_l}(l)}{E_{\alpha_1'}(l)}\right| \leq c, \quad \text{resp.} \quad r^{r\cdot q\cdot n} \leq \frac{1}{\varepsilon} \leq \left|\frac{E_{\alpha_l}(l)}{E_{\alpha_2'}(l)}\right| \leq \left|\frac{x_2'}{x_l}\right|^{\left|\frac{x_2'}{x_l}\right|\cdot q\cdot n}, \tag{49}$$

and, in particular, that $|x_1'| < |x_2'|$ since $r > c'$. We have thus verified condition (4) of Corollary 2.39 in the following case: Let $x_1, \ldots, x_n \in t^{-1}\mathcal{L}$ be a minimal reduced $\mathbb{F}_q[t]$-basis of $t^{-1}\mathcal{L}$ and let $L' \subset \Lambda$ be the $\mathbb{F}_q[t]$-submodule generated by the $t \cdot l^{-1}(x_i)$ for all x_i with $|x_i| < d := \mathrm{d}(l(\pi^{-1}(V \setminus W)) \cap L)$. Then $t^{-1}L'/L' = W$ and $\mathrm{d}(l(t^{-1}L \setminus t^{-1}L']) = d$ by Corollary 2.39. Hence

$$\frac{\mathrm{d}(l(L \setminus L'))}{\mathrm{d}(l(L'))} = \frac{\mathrm{d}(t^{-1}l(L \setminus L'))}{\mathrm{d}(t^{-1}l(L'))} = \frac{d}{|x_l|} \overset{(49)}{\geq} r. \tag{50}$$

In fact, $L' \subset \Lambda$ is an A-submodule and, as such, a direct summand: Indeed, the first inequality of (49) and the definition of c' and r imply that

$$\forall 1 \leq j \leq k, 1 \leq i \leq n\colon \quad |a_j \cdot x_i| \leq c' \cdot |x_l| < r \cdot |x_l| \leq d = \mathrm{d}(l(t^{-1}L \setminus t^{-1}L'))$$

and thus, as $t^{-1}\mathcal{L}$ is an A-module, that $a_j \cdot x_i \in t^{-1}l(L')$ for any such i, j. The basis property of bot the a_j and the x_i then yields that $t^{-1}l(L') \subset t^{-1}\mathcal{L}$ and hence $L' \subset L \subset \Lambda$ are A-submodules. Moreover, as $L' \subset L$ and $L \subset \Lambda$ are direct summands as $\mathbb{F}_q[t]$-submodules, the quotient Λ/L' is torsion-free as $\mathbb{F}_q[t]$-module and hence also as A-module. In particular, Λ/L' is a projective A-module. The short exact sequence $0 \to L' \to \Lambda \to \Lambda/L' \to 0$ thus splits; equivalently, the A-submodule $L' \subset \Lambda$ is a direct summand.

Set $l' := l|_{L_C'}$. Let $\mathfrak{O}$ be the Γ-orbit of L'. As argued above, $t^{-1}L'/L' = W$. Hence $\mathfrak{O} \in \mathrm{Orb}(i, W)$. We claim that $E(\pi_\Gamma(l')) \in O$ and hence, in view of (50), that $[l] \in \mathcal{U}(\Lambda, p_\Gamma^{-1}(E^{-1}(O)) \cap \Omega_{L'}, r)$ so that as desired

$$\pi_\Gamma(l) \in \mathcal{U}(\Lambda, E^{-1}(O), r).$$

For the claim, it suffices, by (47) and since $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$, to show that

$$\forall \alpha, \alpha' \in \mathring{W}\colon \quad \left|\frac{E_{\alpha'}(l)}{E_\alpha(l)} - \frac{E_{\alpha'}(l')}{E_\alpha(l')}\right| < \delta.$$

For any $\beta \in \mathring{W}$ set

$$E_\beta = E_\beta(l) \text{ and } E_\beta' := E_\beta(l') \text{ and } \epsilon_\beta := E_\beta - E_\beta' = \sum_{\lambda \in l(\pi^{-1}(\beta) \cap L \setminus t^{-1}L')} \frac{1}{\lambda}. \tag{51}$$

We then have for any $\alpha, \alpha' \in \mathring{W}$ that

$$\left|\frac{\epsilon_{\alpha'}}{E_\alpha}\right| \overset{(50)}{\leq} \frac{1}{r \cdot |x_l| \cdot |E_\alpha|} = \frac{1}{r} \cdot \left|\frac{e_{\mathcal{L}}(x_l)}{x_l}\right| \cdot \left|\frac{E_{\alpha_l}}{E_\alpha}\right| \leq \frac{1}{r} \cdot \prod_{0 \neq \lambda \in \mathcal{L}} \left|\frac{x_l + \lambda}{\lambda}\right| \cdot c \overset{|x_l| \leq |\lambda|}{=} \frac{c}{r}$$

and hence as desired that

$$\left|\frac{E_{\alpha'}}{E_\alpha} - \frac{E'_{\alpha'}}{E'_\alpha}\right| = \left|\frac{E_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'} - \epsilon_{\alpha'}}{E_\alpha - \epsilon_\alpha}\right| = \left|\frac{\frac{\epsilon_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'}}{E_\alpha} \cdot \frac{\epsilon_\alpha}{E_\alpha}}{1 - \frac{\epsilon_\alpha}{E_\alpha}}\right|$$
$$\overset{1 > \frac{c}{r}}{=} \left|\frac{\epsilon_{\alpha'}}{E_\alpha} - \frac{E_{\alpha'}}{E_\alpha} \cdot \frac{\epsilon_\alpha}{E_\alpha}\right| \leq \frac{c^2}{r} < \delta.$$

This shows the claim and hence the first inclusion stated in the lemma.

Conversely, assume that $\pi_\Gamma(l) \in \mathcal{U}(\Lambda, E^{-1}(O), r)$. Thus

$$[l] \in \mathcal{U}(\Lambda, p^{-1}(E^{-1}(O)) \cap \Omega_{L'}, r)$$

for some $L' \in \mathfrak{O} \in \mathrm{Orb}(i, W)$. Choose such an L' and set $l' := l|_{L'_C} \in \tilde{\Omega}_{L'}$ and define E_β, E'_β and ϵ_β for any $\beta \in \mathring{W}$ as in (51). Using (47) similarly as before, we shall first show that $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$. The assumption implies that $E(\pi_\Gamma(l')) \in O$ and, as $r > 1$, that $x_l \in l(t^{-1}L')$. For any $\alpha, \alpha' \in \mathring{W}$ thus follows that

$$\left|\frac{\epsilon_{\alpha'}}{E'_\alpha}\right| \leq \frac{1}{r \cdot |x_l| \cdot |E'_\alpha|} = \frac{1}{r} \cdot \left|\frac{e_{l'(L')}(x_l)}{x_l}\right| \cdot \left|\frac{E'_{\alpha_l}}{E'_\alpha}\right|$$
$$\leq \frac{1}{r} \cdot \prod_{0 \neq \lambda \in l'(L')} \left|\frac{x_l + \lambda}{\lambda}\right| \cdot c \overset{|x_l| < |\lambda|}{=} \frac{c}{r}$$

and hence that

$$\left|\frac{E'_{\alpha'}}{E'_\alpha} - \frac{E_{\alpha'}}{E_\alpha}\right| = \left|\frac{E'_{\alpha'}}{E'_\alpha} - \frac{E'_{\alpha'} + \epsilon_{\alpha'}}{E'_\alpha + \epsilon_\alpha}\right| = \left|\frac{\frac{E'_{\alpha'}}{E'_\alpha} \cdot \frac{\epsilon_\alpha}{E'_\alpha} - \frac{\epsilon_{\alpha'}}{E'_\alpha}}{1 + \frac{\epsilon_\alpha}{E'_\alpha}}\right|$$
$$\overset{1 > \frac{c}{r}}{=} \left|\frac{E'_{\alpha'}}{E'_\alpha} \cdot \frac{\epsilon_\alpha}{E'_\alpha} - \frac{\epsilon_{\alpha'}}{E'_\alpha}\right| \leq \frac{c^2}{r} < \delta.$$

Hence $\rho_{\mathring{W}}(E(\pi_\Gamma(l))) \in O$ by (47) since $E(\pi_\Gamma(l')) \in O$. We finally show that

$$\forall \alpha' \in \mathring{W}, \forall \alpha \in V \setminus W\colon \quad \left|\frac{E_\alpha(l)}{E_{\alpha'}(l)}\right| \leq \frac{c}{r}. \tag{52}$$

Consider any α, α' as in (52). Suppose without loss of generality that $E_\alpha(l) \neq 0$ so that $\pi^{-1}(\alpha) \cap L \neq \emptyset$. Choose an $x \in l(\pi^{-1}(\alpha) \cap L)$ of minimal norm. Then

$$\left|\frac{E_\alpha(l)}{E_{\alpha'}(l)}\right| = \left|\frac{E_{\alpha_l}(l)}{E_{\alpha'}(l)} \cdot \frac{E_\alpha(l)}{E_{\alpha_l}(l)}\right| \leq c \cdot \left|\frac{e_{\mathcal{L}}(x_l)}{e_{\mathcal{L}}(x)}\right| \overset{(48)}{\leq} c \cdot \left|\frac{x_l}{x}\right| \leq \frac{c}{r}.$$

Hence $E(\pi_\Gamma(l)) \in \mathcal{U}(O, \frac{c}{r})$. This establishes the second inclusion. $\square$

Proof of Proposition 8.7. As argued after Proposition 8.7, it remains to be shown the claim that E_i induces an isomorphism of Grothendieck topologies. From Definition-Proposition 5.19 follows via (45), that a subset $X \subset \Omega^*_{\Gamma_i}$ is admissible if and only if for any free $A/(t)$-submodule $0 \neq W \subset V$ the subset $X \cap \Omega_{\Gamma_i,W} \subset \Omega_{\Gamma,i}$ is admissible and any admissible quasi-compact $Y \subset X \cap \Omega_{\Gamma_i,W}$ admits an $r \in |C|$ with $\mathcal{U}(\Lambda_i, Y, r) \subset X$. Moreover, for any such W the admissible quasi-compact subsets of $\Omega_W(C)$ are precisely the intersections with $\Omega_W(C)$ of the admissible quasi-compact subsets of $\Omega(\mathring{W})$ (see [20, Cor. 2.30]). As E_i restricts to an isomorphism $\Omega_{\Gamma_i} \to \Omega_i$ and to an isomorphism $\Omega_{\Gamma_i,W} \to \Omega_W(C)$ for any such $W \subsetneq V$ by Corollary 8.4, the claim directly follows from Proposition 8.8 applied to the case $P' = Q_i$ jointly with Proposition 8.9 and Lemma 8.10. □

Corollary 8.11. *The Q_i for all $i \in I$ are the irreducible components of Q. Moreover, Ω_i is dense in Q_i for every $i \in I$.*

Proof. By Lemma 8.6, any $Q_i \subset Q$ is Zariski-closed. By Corollary 5.16, any Ω_{Γ_i} is dense in $\Omega^*_{\Gamma_i}$. By Proposition 8.7, thus any Ω_i is dense in Q_i. Consequently, any Q_i contains the dense irreducible subset Ω_i and is thus itself irreducible. Moreover, for any irreducible Zariski closed subset $Y \subset Q$ the intersection $Y \cap \Omega$ with the Zariski open Ω is irreducible and thus contained in some Ω_i by maximality of the irreducible components Ω_i. Hence the Q_i are maximal among the irreducible Zariski closed subsets of Q and are thus the irreducible components. □

Corollary 8.12. *Consider any direct summand $0 \neq W \subset V$ and any irreducible component Y of $\Omega_W(C)$. Then Y is Zariski locally closed and irreducible subset of Q and its Zariski closure in Q consists of Y and all $\Omega_{W'}(C)$ for all $0 \neq W' \subsetneq W$.*

Proof. By Corollary 8.11 applied to $Q' := Q_W(C)$ instead of Q, the closure Z of Y in Q' is $Y \cup (Q' \setminus \Omega_W(C))$. By Lemma 8.6, $Y \subset Z$ is Zariski open. Now use that the closed embedding $Q_W \to Q_V$ defined before Theorem 7.16 identifies Z with the closure of Y in Q, and that Q' is the union of the $\Omega_{W'}(C)$ for all $0 \neq W' \subset W$. □

Let $i \in I$. Consider the Grothendieck ringed space $(Q_i, \tilde{\mathcal{O}}_{Q_i})$ whose underlying Grothendieck topological space coincides with the one underlying $(Q_i, \mathcal{O}_{Q_i})$ and whose sections on any admissible $U \subset Q_i$ are the functions $f\colon U \to C$ that are continuous with respect to the canonical topologies, that are bounded on any admissible affinoid subset of U and that restrict to regular functions $U \cap \Omega_W(C) \to C$ for any free (A/t)-submodule $0 \neq W \subset V$. Denote by

$$n_{Q_i}\colon (Q_i, \tilde{\mathcal{O}}_{Q_i}) \to (Q_i, \mathcal{O}_{Q_i})$$

the morphism of Grothendieck ringed spaces whose underlying morphism of Grothendieck topological spaces is the identity and whose homomorphism $\mathcal{O}_{Q_i}(U) \to \tilde{\mathcal{O}}_{Q_i}(U)$ for any admissible $U \subset Q_i$ is the natural injection by means of the Maximum Modulus Principle, i.e., Proposition 2.27.

Corollary 8.13. *The isomorphism E_i of Grothendieck topological spaces yields an isomorphism*

$$(\Omega^*_{\Gamma_i}, \mathcal{O}^*_{\Gamma_i}) \to (Q_i, \tilde{\mathcal{O}}_{Q_i})$$

of Grothendieck ringed spaces, where the homomorphisms on sections are given by precomposition with E_i.

Proof. This directly follows from Proposition 5.22 and the construction of $\tilde{\mathcal{O}}_{Q_i}$ via Proposition 8.7, Corollary 8.4, Lemma 8.10 and (45). □

Theorem 8.14. *The morphism n_{Q_i} is the normalization of Q_i.*

Proof of Theorems 8.1 and 8.14. By means of the isomorphism in Corollary 8.13, we identify $(\Omega^*_{\Gamma_i}, \mathcal{O}^*_{\Gamma_i})$ with $(Q_i, \tilde{\mathcal{O}}_{Q_i})$ and are reduced to showing that n_{Q_i} is the normalization morphism for Q_i. Set $Z := Q_i$. We want to apply Theorem 3.7 to the present case, i.e., where the global sections S on Z are the restrictions to Z of the Y_α for all $\alpha \in \mathring{V}$. Let us verify its conditions:

i) Z is irreducible,
ii) the Zariski open subvariety $\Omega(S) \subset Z$ is normal,
iii) $Z \setminus \Omega(S)$ is of everywhere positive codimension in Z.
iv) any function $f: X \to C$ on any admissible $X \subset Z$ which is continuous with respect to the canonical topology and restricts to a regular function on $X \cap \Omega(S)$ restricts to a regular function on $X \cap \Omega(T)$ for any $T \subset S$ and
v) any $z \in Z$ has a fundamental basis of admissible neighborhoods U such that $U \cap \Omega(S)$ is connected and, in particular, non-empty.

Condition i) follows from Corollary 8.11. Normality of $\Omega(S) = \Omega(\mathring{V}) \cap Z = \Omega_i$ follows from Theorem 4.13 via Theorem 8.3; this yields ii). By Proposition 8.3, Example 4.16 and Corollary 4.15, moreover, $\Omega_W(C)$ is everywhere of dimension $\mathrm{rank}_{A/(t)}(W) - 1$ for any free $A/(t)$-submodule $0 \neq W \subset V$. Via Proposition 8.9, thus follows that $Z \setminus \Omega(S)$ is of everywhere positive codimension which yields iii). As for Assumption iv), consider any admissible $X \subset Z$ and any function $f: X \to C$ which is continuous with respect to the canonical topologies and which restricts to a regular function on $X \cap \Omega(\mathring{V})$. The regularity of the restriction of f to $X \cap \Omega(\mathring{W})$ for an arbitrary free $A/(t)$-submodule $0 \neq W \subset V$ then follows from Proposition 5.31 by descending induction on the rank of W. Taking Proposition 8.9 again into account, this yields Condition iv). Finally, Corollary 5.16 provides Condition v). We may thus apply Theorem 3.7 which concludes the proof. □

Corollary 8.15. *Let $i \in I$ and set $\Gamma := \Gamma_i$. Consider any Γ-orbit $\mathfrak{O} = \Gamma \cdot L \neq \{0\}$ of direct summands of Λ_i. With respect to the Zariski topology, the subset $\Omega_{\mathfrak{O}} \subset \Omega_\Gamma^*$ is irreducible, locally closed and its closure is the union of all $\Omega_{\Gamma \cdot L'}$ for all $0 \neq L' \subset L$.*

Proof. Choose any $L \in \mathfrak{O}$ and set $W := t^{-1}L/L \subset V$. The image Y of $\Omega_{\mathfrak{O}}$ under E_i is an irreducible component of $\Omega_W(C)$ by Corollary 8.4 and (45). By Corollary 8.12, Y is Zariski locally closed and its closure Z is the union of Y and all $\Omega_{W'}(C)$ for all $0 \neq W' \subsetneq W$. As E_i is bijective, $\Omega_{\mathfrak{O}}$ is the preimage of Y and thus a Zariski locally closed irreducible subset in Ω_Γ^*. The closure of $\Omega_{\mathfrak{O}}$ in Ω_Γ^* is $E_i^{-1}(Z)$ since E_i is bijective and since, being a normalization morphism, E_i finite and thus [8, Prop. 9.6.2.5 and 9.6.3.3] sends Zariski closed subsets to Zariski closed subsets. However, $E_i^{-1}(Z)$ is the union of $\Omega_{\mathfrak{O}}$ and the $\Omega_{\mathfrak{O}'}$ for all Γ-orbits $\mathfrak{O}' = \Gamma \cdot L'$ with $0 \neq t^{-1}L'/L' \subsetneq W$ by Corollary 8.4. The corollary now follows from Corollary 6.19. □

9. Consequences of the comparison

Let $A \subset C$ be as in Sections 5, 6, 7, 8. Consider any congruence subgroup $\mathcal{K} \subset \mathrm{Aut}_{\hat{A}}(M)$ as in Section 6. Choose $0 \neq t \in A$ such that $\mathcal{K}$ contains the kernel $\mathcal{K}(t)$ of the natural homomorphism $\mathrm{Aut}_{\hat{A}}(M) \to \mathrm{Aut}_A(V)$, where $V := t^{-1}M/M$, and, using that A is finitely generated, such that $\mathrm{Div}_A(t)$ generates A as in Section 7.2. Identify $\Delta := \mathcal{K}/\mathcal{K}(t)$ with the image of $\mathcal{K}$ in $\mathrm{Aut}_A(V)$. Let $Q := Q_V(C)$ and $\Omega := \Omega_V(C)$ be as in Section 8.

Theorem 9.1. *The normalization morphism $E_{\mathcal{K}(t)}$ in Theorem 8.1 is Δ-equivariant and the induced morphism $E_{\mathcal{K}}\colon \Omega_{\mathcal{K}}^* \to \Delta\backslash Q$ is the normalization morphism of $\Delta\backslash Q$ and restricts to Drinfeld's isomorphism $\Omega_{\mathcal{K}} \to \Delta\backslash\Omega$ between normal rigid analytic varieties. Moreover, the morphism of Grothendieck topological spaces underlying $E_{\mathcal{K}}$ restricts to isomorphisms between irreducible components.*

Proof. By construction, $E_{\mathcal{K}(t)}$ is Δ-equivariant and thus induces a morphism $E_{\mathcal{K}}\colon \Omega_{\mathcal{K}}^* \to \Delta\backslash Q$ between their quotients. From Theorem 8.1 follows via Köpf's GAGA-theorem [26, Satz 5.1] that the quotient $\Omega_{\mathcal{K}}^*$ of $\Omega_{\mathcal{K}(t)}^*$ by the finite group Δ is a normal projective rigid analytic variety since $\Omega_{\mathcal{K}(t)}^*$ is. Moreover, $E_{\mathcal{K}}$ is finite since $E_{\mathcal{K}(t)}$ is. Moreover, as $E_{\mathcal{K}(t)}$ restricts to an isomorphism $\Omega_{\mathcal{K}(t)} \to \Omega$, also $E_{\mathcal{K}}$ restricts to an isomorphism $\Omega_{\mathcal{K}} \to \Delta\backslash\Omega$ between their quotients. Furthermore, Corollary 5.16 yields via Proposition 6.10 that both the Zariski closed complement of $\Delta\backslash\Omega$ in $\Delta\backslash Q$ and its preimage in $\Omega_{\mathcal{K}}^*$ are nowhere dense. By [10, Theorem 2.1.2], thus $E_{\mathcal{K}}$ is indeed the normalization morphism. Moreover, as the Grothendieck topological space on each side of $E_{\mathcal{K}}$ is the quotient by Δ of the respective side of $E_{\mathcal{K}(t)}$, the last assertion, too, follows from Theorem 8.1. □

Choose any complete set S of representatives of the natural $\mathcal{K}$-action on the set of A-structures of M and recall the isomorphism

$$\coprod_{\Lambda \in S} (\Omega^*_{\overline{\mathcal{K}}_\Lambda}, \mathcal{R}^*_{\overline{\mathcal{K}}_\Lambda}) \longrightarrow (\Omega^*_{\mathcal{K}}, \mathcal{R}^*_{\mathcal{K}}) \tag{53}$$

of Grothendieck graded ringed spaces provided by Proposition 6.10.

Corollary 9.2. *$\Omega^*_{\mathcal{K}}$ is a normal projective rigid analytic variety over C whose irreducible components are, via (53), the $\Omega^*_{\overline{\mathcal{K}}_\Lambda}$ for all $\Lambda \in S$.*

Proof. By [10, Theorem 2.1.3], the analytification functor commutes with the normalization functor. From Theorem 9.1 thus follows that $\Omega^*_{\mathcal{K}}$ is a normal projective rigid analytic variety. Moreover, via (53), the $\Omega^*_{\overline{\mathcal{K}}_\Lambda}$ are admissible subsets of $\Omega^*_{\mathcal{K}}$ and pairwise disjoint. It thus suffices to show that each of them is irreducible. Consider any $\Lambda \in S$. Then the admissible subvariety $\Omega_{\overline{\mathcal{K}}_\Lambda} \subset \Omega^*_{\overline{\mathcal{K}}_\Lambda}$ is irreducible by Proposition 4.13 and dense by Corollary 5.16. Thus $\Omega^*_{\overline{\mathcal{K}}_\Lambda}$ is itself irreducible as desired. $\square$

Consider any finitely generated projective A-module $\Lambda \neq 0$ and any congruence subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. For the remainder we consider the following special case of M, t, $\mathcal{K}$, S so that we may interpret Λ as an element of S and Γ to be $\overline{\mathcal{K}}_\Lambda$: Assume that $M = \Lambda \otimes_A \hat{A}$; then Λ is an A-structure of M. Using that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$ is a congruence subgroup, assume that $0 \neq t \in A$ is such that $\mathrm{Div}_A(t)$ generates A and that furthermore Γ contains the kernel of $\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_A(t^{-1}\Lambda/\Lambda)$. Identify $t^{-1}\Lambda/\Lambda$ with $V := t^{-1}M/M$ via the isomorphism induced by the inclusion $\Lambda \subset M$. Assume that $\mathcal{K}$ is the preimage in $\mathrm{Aut}_{\hat{A}}(M)$ of the image of Γ in $\mathrm{Aut}_A(V)$. Assume finally that S contains Λ. In this case indeed $\Gamma = \overline{\mathcal{K}}_\Lambda$.

Corollary 9.3. *The Grothendieck ringed space $(\Omega^*_\Gamma, \mathcal{O}^*_\Gamma)$ is an integral normal projective rigid analytic variety over C containing Ω_Γ as a dense admissible subvariety.*

Proof. Cororollary 9.2 yields the first part and Corollary 5.16 the second. $\square$

The subgroup $\Gamma \subset \mathrm{Aut}_A(\Lambda)$ is called *fine* if its image in $\mathrm{Aut}_A(\Lambda/\mathfrak{p}\Lambda)$ is unipotent for some maximal ideal $\mathfrak{p} \subset A$.

Corollary 9.4. *Suppose that Γ is fine and let $k \geq 0$ be any integer. Denote by E the restriction of $E_{\mathcal{K}}$ to Ω^*_Γ and by Y its image. Let $\mathcal{O}_Y(k)$ be the pullback under $Y \subset \Delta\backslash Q_V(C)$ of the invertible sheaf from Proposition 7.21. Then the morphism*

$$E^{-1}(\mathcal{O}_Y(k)) \otimes_{E^{-1}(\mathcal{O}_Y)} \mathcal{O}^*_\Gamma \to \mathcal{O}^*_\Gamma(k) \tag{54}$$

induced by E from the inverse image under E of $\mathcal{O}_Y(k)$ to $\mathcal{O}^*_\Gamma(k)$ is an isomorphism and the natural morphism $(\mathcal{O}^*_\Gamma(k))^{k'} \to \mathcal{O}^*_\Gamma(k \cdot k')$ is an isomorphism for any $k' \geq 0$. Consequently, if $k \geq 1$, then $\mathcal{O}^*_\Gamma(k)$ is an ample invertible $\mathcal{O}^*_\Gamma$-module.

Proof. The morphism of Grothendieck topological spaces underlying E is an isomorphism by Theorem 9.1. Thus

$$E^{-1}(\mathcal{O}_Y(k))(X') = \mathcal{O}_Y(k)(E(X')) \tag{55}$$

for any admissible $X' \subset \Omega_\Gamma^*$. Moreover, by construction of $\mathcal{O}_\Gamma^*$, any nowhere vanishing section in $\mathcal{O}_\Gamma(k)(X')$ is a basis for $\mathcal{O}_\Gamma(k)|_{X'}$ over $\mathcal{O}_\Gamma|_{X'}$ for any admissible $X' \subset \Omega_\Gamma^*$. Using that $\mathcal{O}_Y(k)$ is invertible, choose any admissible covering $\mathcal{C}$ of Y such that any $Y' \in \mathcal{C}$ admits a nowhere vanishing section in $\mathcal{O}_Y(k)(Y')$ which is a basis of $\mathcal{O}_Y(k)|_{Y'}$ over $\mathcal{O}_Y|_{Y'}$. Let $Y' \in \mathcal{C}$ and set $X' := E^{-1}(Y')$. Using (55) and that E sends any nowhere vanishing section in $\mathcal{O}_Y(k)(Y')$ to a nowhere vanishing section in $\mathcal{O}_\Gamma(k)(X')$, it is directly checked that (54) restricts to an isomorphism on X'. As the preimage of $\mathcal{C}$ under E is an admissible covering, this yields the first part. The second part holds true since moreover, by [24, Chapter 2, Prop. 5.12], the natural morphism $\mathcal{O}_Y(k)^{k'} \to \mathcal{O}_Y(k \cdot k')$ is an isomorphism for any $k' \geq 0$ and since the formation of tensor products and inverse images are compatible.

Suppose that $k \geq 1$. As $\mathcal{O}_Y(k)$ is ample invertible by Proposition 7.21, so is its inverse image under the finite morphism E by [23, Ch. 1, Prop. 4.4] using that E is the analytification of the algebraic normalization of Y by [10, Thm. 2.1.3]. Hence $\mathcal{O}_\Gamma^*(k)$ is ample invertible by the isomorphism (54). □

Corollary 9.5. *The C-algebra $\mathcal{R}_\Gamma^*(\Omega_\Gamma^*)$ is finitely generated with $\mathcal{O}_\Gamma^*(\Omega_\Gamma^*) = C$ and Ω_Γ^* is the analytification of* $\mathrm{Proj}(\mathcal{R}_\Gamma^*(\Omega_\Gamma^*))$.

Proof. By Köpf's GAGA-theorems [26, Sätze 4.7 und 5.1] and Corollaries 9.3 and 9.4, the variety Ω_Γ^* is the analytification of some normal integral projective algebraic variety X and, if Γ is fine, the ample invertible sheaf $\mathcal{O}_\Gamma^*(k)$ is the analytification of an ample invertible sheaf $\mathcal{L}_k$ on X for any $k \geq 0$, and the global sections on $\mathcal{O}_\Gamma^*(k)$ are naturally isomorphic to the ones of $\mathcal{L}_k$. If Γ is fine, thus $\mathcal{O}_\Gamma^*(\Omega_\Gamma^*) = C$ and the corollary follows using the isomorphisms $(\mathcal{O}_\Gamma^*(k))^{k'} \to \mathcal{O}_\Gamma^*(k \cdot k')$ for all $k, k' \geq 0$ as well as the fact (see [29, Theorem 5.7]) that the ring of sections in all powers of $\mathcal{L}_1$ is a finitely generated normal integral domain and that its Proj is X.

Via the choice of a fine normal subgroup $\Gamma' \subset \Gamma$, the general case is reduced to the previous case using that by Noether's theorem (see [36, Theorem 2.3.1]) the subring of invariants $\mathcal{R}_\Gamma^*(\Omega_\Gamma^*) \subset \mathcal{R}_{\Gamma'}^*(\Omega_{\Gamma'}^*)$ with respect to the C-linear action by the finite group Γ/Γ' is again finitely generated. □

Definition 9.6. For any integer $k \geq 0$ a weak modular form $f \in \mathcal{O}_\Gamma^*(k)(\Omega_\Gamma)$ (see Definition 5.27) is called a *modular form* if the negatively indexed coefficients of the Fourier expansions at all direct summands $0 \neq L \subset \Lambda$ of co-rank 1 all vanish; denote by $\mathcal{M}_\Gamma(k) \subset \mathcal{O}_\Gamma^*(k)(\Omega_\Gamma)$ the C-subspace of modular forms of weight k. Set

$$\mathcal{M}_\Gamma := \sum_{k\geq 0} \mathcal{M}_\Gamma(k).$$

Proposition 9.7. *The restriction homomorphism $\mathcal{R}^*_\Gamma(\Omega^*_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega_\Gamma)$ is injective with image $\mathcal{M}_\Gamma$.*

Proof. Let $\Omega^{<2}_\Lambda$ be the union of the Ω_L for all direct summands $0 \neq L \subset \Lambda$ with $\mathrm{rank}_A(L) \geq \mathrm{rank}_A(\Lambda) - 1$ and consider the admissible subset

$$\Omega^{<2}_\Gamma := p_\Gamma(\Omega^{<2}_\Lambda) \subset \Omega^*_\Gamma.$$

Corollary 5.31 applied to the various such L yields that the restriction homomorphism $\mathcal{R}^*_\Gamma(\Omega^{<2}_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega_\Gamma)$ is injective with image $\mathcal{M}_\Gamma$. We claim that, moreover, the restriction morphism

$$\mathcal{R}^*_\Gamma(\Omega^*_\Gamma) \to \mathcal{R}^*_\Gamma(\Omega^{<2}_\Gamma)$$

is bijective. Consider $\Gamma' := \overline{\mathcal{K}(t)}_\Lambda \subset \Gamma$. By construction of $\Omega^{<2}_{\Gamma'}$ and Ω^*_Γ as well as of $\mathcal{O}^*_{\Gamma'}$ and $\mathcal{O}^*_\Gamma$, the claim is directly reduced to showing the claim in the case $\Gamma = \Gamma'$. Thus assume that $\Gamma = \Gamma'$. By the Riemann extension theorem [2, Satz 10], the restriction morphism is bijective if Ω^*_Γ is normal and if $\Omega^*_\Gamma \backslash \Omega^{<2}_\Gamma \subset \Omega^*_\Gamma$ is Zariski-closed of codimension ≤ 2. From Corollary 9.3 follows the normality of Ω^*_Γ. The image of Ω^*_Γ under $E_\mathcal{K}$ is then an irreducible component Q_i of Q. We are thus reduced to showing that the image U of $\Omega^*_\Gamma \setminus \Omega^{<2}_\Gamma \subset \Omega^*_\Gamma$ under the isomorphism $E_i\colon (\Omega^*_\Gamma, \mathcal{O}^*_\Gamma) \to (Q_i, \tilde{\mathcal{O}}_{Q_i})$ provided by Corollary 8.13 is Zariski-closed in Q_i and of codimension ≤ 2. By Corollary 8.4, the image U is the union of the $\Omega_W(C)$ for all free direct summands $0 \neq W \subset V$ with $\mathrm{rank}_{A/(t)}(W) \leq \mathrm{rank}_{A/(t)}(V) - 2$. By Theorem 7.16,i), equivalently, U is the union of the $Q_W(C) \subset Q_i$ for all such W. For any such W, moreover, $Q_W(C)$ is Zariski-closed in $Q_V(C)$ and hence Zariski-closed in Q_i with respect to $\mathcal{O}_{Q_i}$ and thus also with respect to $\tilde{\mathcal{O}}_{Q_i}$. Being a finite union of Zariski-closed subsets, hence U itself is Zariski-closed. Moreover, by Theorem 7.16, ii), for any direct summand $0 \neq W \subset V$ the dimension of any irreducible component of $\Omega_W(C)$ equals $\mathrm{rank}_{A/(t)}(W) - 1$. Hence $U \subset Q_i$ is Zariski-closed of codimension ≤ 2. $\square$

We finally summarize, respectively conclude, the following results about the analytic Satake compactification of Ω_Γ.

Theorem 9.8. *(Analytic Satake compactification)*

i) The Grothendieck ringed space $(\Omega^*_\Gamma, \mathcal{O}^*_\Gamma)$ is an integral normal projective rigid analytic variety over C containing Ω_Γ as a dense admissible subvariety.

ii) If for some maximal ideal $\mathfrak{p} \subset A$ the image of Γ in $\mathrm{Aut}_A(\Lambda/\mathfrak{p}\Lambda)$ is unipotent, $\mathcal{O}^*_\Gamma(k)$ is ample invertible for any $k \geq 1$.

iii) *For any $k \geq 0$ the restriction morphism $\mathcal{O}^*_\Gamma(k)(\Omega^*_\Gamma) \to \mathcal{O}_\Gamma(k)(\Omega_\Gamma)$ is injective and its image is the space of modular forms $\mathcal{M}_\Gamma(k)$.*

iv) *The graded C-algebra $\mathcal{M}_\Gamma := \sum_{k\geq 0} \mathcal{M}_\Gamma(k)$ is finitely generated with $\mathcal{M}_\Gamma(0) = C$ and Ω^*_Γ is the analytification of* $\mathrm{Proj}(\mathcal{M}_\Gamma)$. *Moreover, $\mathcal{M}_\Gamma(k)$ is a finite dimensional vector space over C for any $k \geq 0$.*

v) *Consider any Γ-orbit $\mathfrak{O} = \Gamma \cdot L \neq \{0\}$. With respect to the Zariski topology, the subset $\Omega_\mathfrak{O} \subset \Omega^*_\Gamma$ is irreducible, locally closed and its closure is the union of all $\Omega_{\Gamma\cdot L'}$ for all direct summands $0 \neq L' \subset L$.*

vi) *Consider any direct summand $0 \neq L \subset \Lambda$ and set $\mathfrak{O} := \Gamma \cdot L$ and $\overline{\Gamma}_L := \{\gamma' \in \mathrm{Aut}_A(L) \mid \exists \gamma \in \Gamma \colon \gamma|_L = \gamma'\}$. The composition of the canonical bijection $\overline{\Gamma}_L\backslash\Omega_L \to \Omega_\mathfrak{O}$ with the inclusion $\Omega_\mathfrak{O} \subset \Omega^*_\Gamma$ is a locally closed immersion (in the sense of Definition 2.25) of rigid analytic varieties.*

Proof. Part i) is Corollary 9.3. Part ii) is the last statement of Corollary 9.4. Part iii) is equivalent to Proposition 9.7. The first statement of Part iv) is a combination of Corollary 9.5 and Proposition 9.7 and the second statement follows from the first by induction on k. Part v) is Corollary 8.15 in the case where Γ equals $\Gamma' := \mathrm{Ker}(\mathrm{Aut}_A(\Lambda) \to \mathrm{Aut}_{\overline{A}}(V))$. In general, the action of the finite group Γ/Γ' on $\Omega^*_{\Gamma'}$ yields a finite morphism $\pi\colon \Omega^*_{\Gamma'} \to \Omega^*_\Gamma$ (see [20, Corollary 9.23] for details). As π is finite (and in fact the quotient by a finite group), it maps Zariski closed (resp. open) subsets to Zariski closed (resp. open) subsets. Thus $\Omega_\mathfrak{O} = \Omega_{\Gamma\cdot L} = \pi(\Omega_{\Gamma'\cdot L})$ is locally closed and irreducible and the closure of $\Omega_\mathfrak{O}$ in Ω^*_Γ is the image under π of the closure of $\Omega_{\Gamma'\cdot L}$ in $\Omega^*_{\Gamma'}$. Since the closure of $\Omega_{\Gamma'\cdot L}$ is the union of the $\Omega_{\Gamma'\cdot L'}$ for all $0 \neq L' \subset L$ by Corollary 8.15 and since $\pi(\Omega_{\Gamma'\cdot L'}) = \Omega_{\Gamma\cdot L'}$, the closure of $\Omega_{\Gamma\cdot L}$ is the union of the $\Omega_{\Gamma\cdot L'}$ for all $0 \neq L' \subset L$. This shows part v). Finally, $\Omega_{\overline{\Gamma}_L}$, resp. Ω^*_Γ, is a rigid analytic variety by Proposition 4.13, resp. part i). Part vi) then follows from Proposition 5.23. □

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