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General Section

An inequality for coefficients of the real-rooted polynomials

Jeremy J.F. Guo

School of Mathematics, Tianjin University, Tianjin 300072, PR China

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A B S T R A C T

In this paper, we prove that if $f(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ is a polynomial with real zeros only, then the sequence $\{a_k\}_{k=0}^{n}$ satisfies the following inequalities $a_{k+1}^2(1 - \sqrt{1-c_k})^2/a_k^2 \leq (a_{k+1}^2 - a_k a_{k+2})/(a_k^2 - a_{k-1}a_{k+1}) \leq a_{k+1}^2(1 + \sqrt{1-c_k})^2/a_k^2$, where $c_k = a_k a_{k+2}/a_{k+1}^2$. This inequality is equivalent to the higher order Turán inequality. It holds for the coefficients of the Riemann ξ-function, the ultraspherical, Laguerre and Hermite polynomials, and the partition function. Moreover, as a corollary, for the partition function $p(n)$, we prove that $p(n)^2 - p(n-1)p(n+1)$ is increasing for $n \geq 55$. We also find that for a positive and log-concave sequence $\{a_k\}_{k\geq 0}$, the inequality $a_{k+2}/a_k \leq (a_{k+1}^2 - a_k a_{k+2})/(a_k^2 - a_{k-1}a_{k+1}) \leq a_{k+1}/a_{k-1}$ is the sufficient condition for both the 2-log-concavity and the higher order Turán inequalities of $\{a_k\}_{k\geq 0}$. It is easy to verify that if $a_k^2 \geq r a_{k+1}a_{k-1}$, where $r \geq 2$, then the sequence $\{a_k\}_{k\geq 0}$ satisfies this inequality.

1. Introduction

In this paper, we give an inequality for the coefficients of real-rooted polynomials, which is equivalent to the higher order Turán inequality.

E-mail address: guo@tju.edu.cn.

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Theorem 1.1. *For a real-rooted polynomial* $f(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$, *if* $a_k a_{k+1}(a_k^2 - a_{k-1}a_{k+1}) \neq 0$, *then the inequality*

$$\frac{a_{k+1}^2}{a_k^2}(1-\sqrt{1-c_k})^2 \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}^2}{a_k^2}(1+\sqrt{1-c_k})^2 \tag{1.1}$$

holds for $1 \leq k \leq n-2$, *where* $c_k = \frac{a_k a_{k+2}}{a_{k+1}^2}$. *This inequality is equivalent to the higher order Turán inequality as follows*

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1}a_{k+2})^2 \geq 0. \tag{1.2}$$

We say a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ is *real-rooted*, if all its zeros are real. The inequality (1.1) gives an upper and lower bound for the ratio $\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}}$.

Recall that a sequence $\{a_k\}_{k\geq 0}$ is said to be *log-concave* if for all $k \geq 1$,

$$a_k^2 - a_{k-1}a_{k+1} \geq 0. \tag{1.3}$$

Note that for a positive sequence $\{a_k\}_{k\geq 0}$, it is log-concave if and only if the ratio a_{k+1}/a_k is decreasing. We also say that the sequence $\{a_k\}_{k\geq 0}$ satisfies the *Turán inequalities*, if it satisfies the inequality (1.3).

For the sequence $\{a_k\}_{k\geq 0}$ satisfying the Turán inequalities, we consider the higher order Turán inequalities as follows. A sequence $\{a_k\}_{k\geq 0}$ is said to satisfy the *higher order Turán inequalities* if for $k \geq 1$,

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1}a_{k+2})^2 \geq 0. \tag{1.4}$$

Recall that a real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.5}$$

is said to be in the Laguerre-Pólya class, denoted $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k}, \tag{1.6}$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. These functions are only ones which are uniform limits of polynomials whose zeros are real. We refer to [13] and [18] for the background on the theory of the $\mathcal{LP}$ class.

For a real entire function $\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ in the $\mathcal{LP}$ class, the Maclaurin coefficients γ_k satisfy both the Turán inequalities, proved by Pólya and Schur [17], and the higher order Turán inequalities, proved by Dimitrov [8]. As a corollary, the ultraspherical, Laguerre and Hermite polynomials satisfy both the Turán inequalities and the higher order Turán inequalities, see [8].

Since the inequality (1.1) is equivalent to the higher order Turán inequality, then we get that for a real entire function $\psi(x)$ in the $\mathcal{LP}$ class, the Maclaurin coefficients satisfy the inequality (1.1). Consequently, the ultraspherical, Laguerre and Hermite polynomials satisfy the inequality (1.1).

To prove the higher order Turán inequalities for the Maclaurin coefficients, Dimitrov applied a theorem of Mařík [14] as follows.

Theorem 1.2. *If the real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k/(k!(n-k)!)$ of degree $n \geq 3$ has only real zeros, then the inequality*

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1}a_{k+2})^2 \geq 0$$

holds for $1 \leq k \leq n-1$.

It is well known that the Riemann hypothesis holds if and only if the Riemann ξ-function belongs to the $\mathcal{LP}$ class. Let ζ denote the Riemann zeta-function and Γ be the gamma-function. The Riemman ξ-function is defined by

$$\xi(iz) = \frac{1}{2}(z^2 - \frac{1}{2})\pi^{-z/2-1/4}\Gamma(\frac{z}{2} + \frac{1}{4})\zeta(z + \frac{1}{2}), \tag{1.7}$$

see, for example, Boas [1]. Hence, if the Riemann hypothesis is true, then the Maclaurin coefficients of the Riemann ξ-function satisfy both the Turán inequalities and the higher order Turán inequalities. Csordas, Norfolk and Varga [7] proved that the coefficients of the Riemann ξ-function satisfy the Turán inequalities, confirming a conjecture of Póly [16]. Dimitrov and Lucas [9] showed that the coefficients of the Riemann ξ-function satisfy the higher order Turán inequalities without resorting to the Riemann hypothesis. As a corollary, we conclude that the coefficients of the Riemann ξ-function satisfy the inequality (1.1).

For the partition function $p(n)$, Chen, Jia and Wang [4] proved that it satisfies the higher order Turán inequalities for $n \geq 95$. As a corollary, the inequality (1.1) holds for partition function $p(n)$ for $n \geq 95$.

Through the discussion about the lower bound $a_{k+1}^2(1-\sqrt{1-c_k})^2/a_k^2$ in the inequality (1.1), we prove that for the partition function $p(n)$, $p(n)^2 - p(n-1)p(n+1)$ is increasing for $n \geq 55$.

Go back to the log-concavity of the sequence $\{a_k\}_{k\geq 0}$. We consider the 2-*log-concavity*, which is equivalent to the decreasing property of the ratio $\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}}$. Moreover, we could define the *infinitely log-concave sequence* as follows.

Define an operator $\mathcal{L}$ on a sequence $\{a_k\}_{k\geq 0}$ by $\mathcal{L}(\{a_k\}_{k\geq 0}) = \{b_k\}_{k\geq 0}$, where $b_0 = a_0^2$ and $b_k = a_k^2 - a_{k-1}a_{k+1}$. This definition makes sense for finite sequences by regarding these as infinite sequences with finitely many nonzero entries. Hence a sequence $\{a_k\}_{k\geq 0}$ is log-concave if and only if $\mathcal{L}(\{a_k\}_{k\geq 0})$ is a nonnegative sequence. We say that a sequence $\{a_k\}_{k\geq 0}$ is *k-log-concave* if $\mathcal{L}^j(\{a_k\}_{k\geq 0})$ is nonnegative for all $0 \leq j \leq k$. A sequence $\{a_k\}_{k\geq 0}$ is *infinitely log-concave* if it is k-log-concave for all $k \geq 1$.

The notion of infinite log-concavity was introduced by Boros and Moll [2]. For the sequence $\{\binom{n}{k}\}_{k=0}^{n}$, they asked whether it is infinitely log-concave. The following result was independently conjectured by Fisk [10], McNamara-Sagan [15] and Stanley [19], and proved by Brändén [3].

Theorem 1.3. *If $f(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial with real- and nonpositive zeros only, then so is*

$$\mathcal{L}(f) = \sum_{k=0}^{n}(a_k^2 - a_{k+1}a_{k-1})x^k.$$

In particular, the sequence $\{a_k\}_{k=0}^{n}$ is infinitely log-concave.

It follows immediately that the sequence $\{\binom{n}{k}\}_{k=0}^{n}$ is infinitely log-concave.

There is also a simple criterion [6,15] that if

$$a_k^2 \geq r a_{k-1}a_{k+1}, \quad \textit{for all } k \geq 1,$$

where $r \geq (3+\sqrt{5})/2 \approx 2.62$, then the sequence $\{a_k\}_{k\geq 0}$ is infinitely log-concave.

We are interested in the connection between the 2-log-concavity and the higher order Turán inequalities. Based on the inequality (1.1), if we can find sharper bounds $l(n)$ and $u(n)$ for the ratio $\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}}$ such that

$$\frac{a_{k+1}^2}{a_k^2}(1-\sqrt{1-c_k})^2 \leq l(k) \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq u(k) \leq \frac{a_{k+1}^2}{a_k^2}(1+\sqrt{1-c_k})^2,$$

and for each $k \geq 1$, either $\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \geq u(k+1)$ or $\frac{a_{k+2}^2 - a_{k+1} a_{k+3}}{a_{k+1}^2 - a_k a_{k+2}} \leq l(k)$, then the sequence $\{a_k\}_{k\geq 0}$ is 2-log-concave, as well as satisfies the higher order Turán inequalities.

In Section 3, we will prove the following theorem.

Theorem 1.4. *For a log-concave positive sequence $\{a_k\}_{k\geq 0}$, if it satisfies the following inequalities*

$$\frac{a_{k+2}}{a_k} \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}}, \tag{1.8}$$

for $k \geq 1$. Then $\{a_k\}_{k\geq 0}$ is 2-log-concave and satisfies the higher order Turán inequalities for $k \geq 1$.

It is easy to verify that the sequence $\{\binom{n}{k}\}_{k=0}^{n}$ satisfies inequality (1.8), as well as the sequence $\{a_k\}_{k\geq 0}$, which satisfies $a_k^2 \geq ra_{k+1}a_{k-1}$, where $r \geq 2$.

Finally, in Section 4, we will discuss a problem we will consider in the further work.

2. Main theorem

In this section, we will give the proof of the Theorem 1.1.

Proof. Applying Theorem 1.2, we have

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) \geq (a_k a_{k+1} - a_{k-1}a_{k+2})^2. \tag{2.9}$$

Multiplying both sides of $1/a_k^2 a_{k+1}^2$ and simplifying, we obtain

$$4(1 - \frac{a_{k-1}a_{k+1}}{a_k^2})(1 - \frac{a_k a_{k+2}}{a_{k+1}^2}) \geq (1 - \frac{a_{k-1}a_{k+1}}{a_k^2}\frac{a_k a_{k+2}}{a_{k+1}^2})^2. \tag{2.10}$$

Substitute $c_n = \frac{a_n a_{n+2}}{a_{n+1}^2}$ for $n = k$ and $n = k-1$, and we get

$$4(1 - c_{k-1})(1 - c_k) \geq (1 - c_{k-1}c_k)^2. \tag{2.11}$$

Observe that

$$1 - c_{k-1}c_k = 1 - c_k + c_k - c_{k-1}c_k = 1 - c_k + c_k(1 - c_{k-1}). \tag{2.12}$$

It follows that

$$4(1 - c_{k-1})(1 - c_k) \geq (1 - c_k + c_k(1 - c_{k-1}))^2. \tag{2.13}$$

Since $a_k^2 - a_{k-1}a_{k+1} \neq 0$, $1 - c_{k-1} \neq 0$. Multiply both sides of $1/(1 - c_{k-1})^2$, and we get

$$4\frac{1 - c_k}{1 - c_{k-1}} \geq (\frac{1 - c_k}{1 - c_{k-1}} + c_k)^2. \tag{2.14}$$

Set $x = (1 - c_k)/(1 - c_{k-1})$. Then we obtain

$$x^2 - (4 - 2c_k)x + c_k^2 \leq 0. \tag{2.15}$$

Immediately we conclude that

$$\frac{4 - 2c_k - \sqrt{(4 - 2c_k)^2 - 4c_k^2}}{2} \leq x \leq \frac{4 - 2c_k + \sqrt{(4 - 2c_k)^2 - 4c_k^2}}{2}. \tag{2.16}$$

It leads to the following inequality after simplified

$$(1 - \sqrt{1 - c_k})^2 \leq x \leq (1 + \sqrt{1 - c_k})^2. \tag{2.17}$$

Since

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} = \frac{a_{k+1}^2}{a_k^2}\frac{1-c_k}{1-c_{k-1}} = \frac{a_{k+1}^2}{a_k^2}\cdot x, \tag{2.18}$$

we reach the inequality (1.1) as we want.

Through the proof above, we can easily get that if $a_k a_{k+1}(a_k^2 - a_{k-1}a_{k+1}) \neq 0$, the inequality (1.1) is equivalent to the higher order Turán inequality. □

As corollaries, we get the following results.

Corollary 2.1. *The inequality (1.1) holds for the ultraspherical, Laguerre and Hermite polynomials*

Corollary 2.2. *The inequality (1.1) holds for the coefficients of the Riemann ξ-function.*

Corollary 2.3. *The inequality (1.1) holds for the partition function $p(n)$ for $n \geq 95$.*

Recall that a sequence $\{a_k\}_{k\geq 0}$ is said to be *convex* if for $k \geq 1$,

$$a_{k+1} - a_k \geq a_k - a_{k-1}. \tag{2.19}$$

For the lower bound function $l(n) = \frac{a_{n+1}^2}{a_n^2}(1-\sqrt{1-c_n})^2$ in the inequality (1.1), we have the following result.

Lemma 2.4. *For the log-concave, increasing, positive sequence $\{a_k\}_{k\geq 0}$ which satisfies the inequality (1.1), if $\{a_k\}_{k\geq 0}$ is convex, then*

$$\frac{a_{k+1}^2}{a_k^2}(1-\sqrt{1-c_k})^2 \geq 1. \tag{2.20}$$

Proof. Since

$$\frac{a_{k+1}^2}{a_k^2}(1-\sqrt{1-c_k})^2 = \frac{1}{a_k^2}(a_{k+1} - a_{k+1}\sqrt{1-c_k})^2 = \frac{1}{a_k^2}(a_{k+1} - \sqrt{a_{k+1}^2 - a_k a_{k+2}})^2,$$

we only need to prove that

$$(a_{k+1} - \sqrt{a_{k+1}^2 - a_k a_{k+2}})^2 \geq a_k^2. \tag{2.21}$$

For $\{a_k\}_{k\geq 0}$ is an increasing, positive sequence, it is sufficient to prove that

$$a_{k+1}^2 - a_k a_{k+2} \leq (a_{k+1} - a_k)^2. \tag{2.22}$$

Since $\{a_k\}_{k\geq 0}$ is convex, for $k \geq 0$

$$a_{k+2} - a_{k+1} \geq a_{k+1} - a_k. \tag{2.23}$$

Thus

$$a_{k+2} \geq 2a_{k+1} - a_k. \tag{2.24}$$

It follows that

$$a_k a_{k+2} \geq a_k(2a_{k+1} - a_k). \tag{2.25}$$

Since

$$a_{k+1}^2 - a_k(2a_{k+1} - a_k) = (a_{k+1} - a_k)^2, \tag{2.26}$$

immediately we get the inequality (2.22). □

Combining the Theorem 1.1 and the Lemma 2.4, we get the following theorem.

Theorem 2.5. *For the log-concave, increasing, positive sequence $\{a_k\}_{k\geq 0}$ which satisfies the inequality (1.1), if $\{a_k\}_{k\geq 0}$ is convex, then the sequence $\{a_{k+1}^2 - a_k a_{k+2}\}_{k\geq 0}$ is increasing.*

For the partition function $p(n)$, $p(n)$ satisfies the inequality as follows [11]

$$2p(n) \leq p(n+1) + p(n-1). \tag{2.27}$$

Hence we have the corollary as follows.

Corollary 2.6. *For the partition function $p(n)$, $p(n)^2 - p(n-1)p(n+1)$ is increasing for $n \geq 55$.*

Proof. Applying the Theorem 2.5 and the Corollary 2.3, we get that for $n \geq 95$, $p(n)^2 - p(n-1)p(n+1)$ is increasing. For $55 \leq n \leq 95$, we can easily verify that $p(n)^2 - p(n-1)p(n+1)$ is also increasing. □

3. 2-log-concavity

In this section, we will give the proof of the Theorem 1.4.

Proof. Applying the inequalities (1.8), it follows immediately that

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}} \leq \frac{a_k^2 - a_{k-1}a_{k+1}}{a_{k-1}^2 - a_{k-2}a_k} \tag{3.28}$$

for $k \geq 1$. Hence $\{a_k\}_{k\geq 0}$ is 2-log-concave.

On the other hand, for $k \geq 1$, consider the right inequality of (1.8) and we have

$$0 \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}}. \tag{3.29}$$

Hence

$$a_{k-1}(a_{k+1}^2 - a_k a_{k+2}) \leq a_{k+1}(a_k^2 - a_{k-1}a_{k+1}). \tag{3.30}$$

Since $a_{k+1} > 0$, multiply both sides of a_{k+1}, and we obtain

$$a_{k-1}a_{k+1}(a_{k+1}^2 - a_k a_{k+2}) \leq a_{k+1}^2(a_k^2 - a_{k-1}a_{k+1}). \tag{3.31}$$

It leads to

$$-a_{k-1}a_k a_{k+1}a_{k+2} \leq a_k^2 a_{k+1}^2 - 2a_{k+1}^3 a_{k-1}. \tag{3.32}$$

Thus

$$-a_{k-1}a_k a_{k+1}a_{k+2} + a_k^2 a_{k+1}^2 \leq a_k^2 a_{k+1}^2 - 2a_{k+1}^3 a_{k-1} + a_k^2 a_{k+1}^2,$$

i.e.

$$a_k a_{k+1}(a_k a_{k+1} - a_{k-1}a_{k+2}) \leq 2a_{k+1}^2(a_k^2 - a_{k-1}a_{k+1}). \tag{3.33}$$

Similarly, for $k \geq 1$, consider the left inequality of (1.8) and we get the following inequality

$$a_k a_{k+1}(a_k a_{k+1} - a_{k-1}a_{k+2}) \leq 2a_k^2(a_{k+1}^2 - a_k a_{k+2}). \tag{3.34}$$

Note that $\{a_k\}_{k\geq 0}$ is log-concave. It is easy to verify that

$$a_k a_{k+1} - a_{k-1}a_{k+2} \geq 0. \tag{3.35}$$

Consequently, combine inequalities (3.33) and (3.34), and we get

$$a_k^2 a_{k+1}^2(a_k a_{k+1} - a_{k-1}a_{k+2})^2 \leq 4a_k^2 a_{k+1}^2(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}). \tag{3.36}$$

Hence $\{a_k\}_{k\geq 0}$ satisfies the higher order Turán inequalities. $\square$

For a log-concave positive sequence $\{a_k\}_{k\geq 0}$, to prove the inequality (1.8), we need the following two lemmas.

Lemma 3.1. *For a log-concave positive sequence* $\{a_k\}_{k\geq 0}$*, if the sequence* $\{\frac{a_{k+1}}{a_k}\}_{k\geq 0}$ *is convex, then*

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}}$$

for $k \geq 1$*.*

Proof. Since $\{a_k\}_{k\geq 0}$ is log-concave, $\{\frac{a_{k+1}}{a_k}\}_{k\geq 0}$ is decreasing. Then the convexity of $\{\frac{a_{k+1}}{a_k}\}_{k\geq 0}$ leads to

$$0 \geq \frac{a_{k+2}}{a_{k+1}} - \frac{a_{k+1}}{a_k} \geq \frac{a_{k+1}}{a_k} - \frac{a_k}{a_{k-1}},$$

i.e.

$$0 \leq \frac{a_{k+1}}{a_k} - \frac{a_{k+2}}{a_{k+1}} \leq \frac{a_k}{a_{k-1}} - \frac{a_{k+1}}{a_k}. \tag{3.37}$$

Observe that

$$a_{k+1}^2 - a_k a_{k+2} = a_{k+1}a_k(\frac{a_{k+1}}{a_k} - \frac{a_{k+2}}{a_{k+1}}), \tag{3.38}$$

and

$$a_k^2 - a_{k-1}a_{k+1} = a_k a_{k-1}(\frac{a_k}{a_{k-1}} - \frac{a_{k+1}}{a_k}). \tag{3.39}$$

It follows immediately that

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}}. \quad \square$$

Similarly, we have the following lemma.

Lemma 3.2. *For a log-concave positive sequence* $\{a_k\}_{k\geq 0}$*, if the sequence* $\{\frac{a_k}{a_{k+1}}\}_{k\geq 0}$ *is convex, then*

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \geq \frac{a_{k+2}}{a_k}$$

for $k \geq 1$*.*

Now we are ready to prove that the sequence $\{\binom{n}{k}\}_{k=0}^n$ satisfies the inequality (1.8).

Theorem 3.3. *The sequence* $\{\binom{n}{k}\}_{k=0}^n$ *satisfies the inequality (1.8).*

Proof. It's easy to prove that

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}. \tag{3.40}$$

Since

$$\frac{n-k}{k+1} - \frac{n-k+1}{k} = -\frac{n+1}{k(k+1)} \geq -\frac{n+1}{(k-1)k} = \frac{n-k+1}{k} - \frac{n-k+2}{k-1}, \tag{3.41}$$

and

$$\begin{aligned} \frac{k+1}{n-k} - \frac{k}{n-k+1} &= \frac{n+1}{(n-k)(n-k+1)} \geq \frac{n+1}{(n-k+1)(n-k+2)} \\ &= \frac{k}{n-k+1} - \frac{k-1}{n-k+2}, \end{aligned} \tag{3.42}$$

we conclude that the sequences $\{\frac{n-k}{k+1}\}_{k\geq 0}^{n}$ and $\{\frac{k+1}{n-k}\}_{k\geq 0}^{n}$ are both convex. Hence the sequence $\{\binom{n}{k}\}_{k=0}^{n}$ satisfies the inequality (1.8). □

Consequently, we have found a sufficient condition for both the 2-log-concavity and the higher order Turán inequalities of the sequence $\{\binom{n}{k}\}_{k=0}^{n}$.

Recall that there is a simple criterion on a nonnegative sequence $\{a_k\}_{k=0}^{\infty}$ that guarantees infinite log-concavity. Namely

$$a_k^2 \geq r a_{k-1} a_{k+1},$$

where $r \geq (3+\sqrt{5})/2$, for all $k \geq 1$.

For the inequality (1.8), we have the following result.

Theorem 3.4. *The positive sequence $\{a_k\}_{k=0}^{\infty}$ satisfies the inequality (1.8), if*

$$a_k^2 \geq r a_{k-1} a_{k+1}, \tag{3.43}$$

where $r \geq 2$, for all $k \geq 1$.

Proof. Applying the inequality (3.43), we have

$$\frac{a_k}{a_{k-1}} \geq 2\frac{a_{k+1}}{a_k}. \tag{3.44}$$

Since $a_k \geq 0$ for $k \geq 0$, we easily get

$$\frac{a_{k+2}}{a_{k+1}} + \frac{a_k}{a_{k-1}} \geq 2\frac{a_{k+1}}{k}, \tag{3.45}$$

which means the sequence $\{\frac{a_{k+1}}{a_k}\}_{k\geq 0}$ is convex.

Similarly, we can prove that the sequence $\{\frac{a_k}{a_{k+1}}\}_{k\geq 0}$ is also convex. □

Notice that in the proof of Theorem 1.4, we did not apply the Theorem 1.1. In the last part of this section, we will prove the following result.

Theorem 3.5. *For a log-concave positive sequence $\{a_k\}_{k\geq 0}$, if it satisfies the following inequalities*

$$\frac{a_{k+2}}{a_k} \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}}{a_{k-1}}, \tag{3.46}$$

for $k \geq 1$. Then

$$\frac{a_{k+1}^2}{a_k^2}(1-\sqrt{1-c_k})^2 \leq \frac{a_{k+2}}{a_k} \leq \frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \leq \frac{a_{k+1}^2}{a_k^2}(1+\sqrt{1-c_k})^2 \tag{3.47}$$

where $c_k = a_k a_{k+2}/a_{k+1}^2$, for $k \geq 1$.

Proof. Since

$$a_{k+1}^2 - a_k a_{k+2} = a_{k+1}^2(1-c_k) \tag{3.48}$$

and $\{a_k\}_{k\geq 0}$ is a positive sequence, the inequality (3.46) is equivalent to

$$\frac{a_{k+2}}{a_k} \leq \frac{a_{k+1}^2(1-c_k)}{a_k^2(1-c_{k-1})} \leq \frac{a_{k+1}}{a_{k-1}},$$

i.e.

$$c_k \leq \frac{1-c_k}{1-c_{k-1}} \leq \frac{1}{c_{k-1}}. \tag{3.49}$$

And the inequality (3.47) is equivalent to

$$(1-\sqrt{1-c_k})^2 \leq c_k \leq \frac{1-c_k}{1-c_{k-1}} \leq (1+\sqrt{1-c_k})^2. \tag{3.50}$$

We aim to prove the inequality (3.50).

First we will prove that

$$(1-\sqrt{1-c_k})^2 \leq c_k. \tag{3.51}$$

Since $\{a_k\}_{k\geq 0}$ is a log-concave positive sequence, we have $0 \leq c_k \leq 1$. Hence

$$0 \leq \sqrt{1-c_k} \leq 1. \tag{3.52}$$

Multiplying $\sqrt{1-c_k}$, we get

$$0 \leq 1-c_k \leq \sqrt{1-c_k}. \tag{3.53}$$

Note that $c_k = 1 - (1 - c_k)$ and we obtain

$$c_k \geq 1 - \sqrt{1 - c_k} \geq (1 - \sqrt{1 - c_k})^2. \tag{3.54}$$

Now we proceed to prove that

$$\frac{1 - c_k}{1 - c_{k-1}} \leq (1 + \sqrt{1 - c_k})^2. \tag{3.55}$$

For $0 \leq c_k \leq \frac{\sqrt{5}-1}{2}$, we have $1 - c_k \geq \frac{3-\sqrt{5}}{2}$. Applying the right inequality of (3.49), we get

$$\frac{3 - \sqrt{5}}{2(1 - c_{k-1})} \leq \frac{1}{c_{k-1}}. \tag{3.56}$$

Hence $c_{k-1} \leq \frac{2}{5-\sqrt{5}}$. And $1 - c_{k-1} \geq \frac{3-\sqrt{5}}{5-\sqrt{5}}$. It follows that

$$\frac{1 - c_k}{1 - c_{k-1}} \leq \frac{5 - \sqrt{5}}{3 - \sqrt{5}}(1 - c_k). \tag{3.57}$$

Therefore, it is sufficient to prove

$$\frac{5 - \sqrt{5}}{3 - \sqrt{5}}(1 - c_k) \leq (1 + \sqrt{1 - c_k})^2. \tag{3.58}$$

Set $t = \sqrt{1 - c_k}$. Then $\frac{\sqrt{5}-1}{2} \leq t \leq 1$, $c_k = 1 - t^2$, and inequality (3.58) is equivalent to

$$\frac{5 - \sqrt{5}}{3 - \sqrt{5}}t^2 \leq (1 + t)^2. \tag{3.59}$$

It is easy to verify that the inequality (3.59) holds for

$$\frac{1}{2}(3 - \sqrt{5} - \sqrt{20 - 8\sqrt{5}}) \leq t \leq \frac{1}{2}(3 - \sqrt{5} + \sqrt{20 - 8\sqrt{5}}). \tag{3.60}$$

And we can verify that

$$\frac{1}{2}(3 - \sqrt{5} - \sqrt{20 - 8\sqrt{5}}) \leq \frac{\sqrt{5} - 1}{2} \leq 1 \leq \frac{1}{2}(3 - \sqrt{5} + \sqrt{20 - 8\sqrt{5}}). \tag{3.61}$$

Consequently, we have finished the proof for the inequality (3.55) for $0 \leq c_k \leq \frac{\sqrt{5}-1}{2}$.

For $\frac{\sqrt{5}-1}{2} < c_k \leq 1$, actually we could prove that

$$\frac{1 - c_k}{1 - c_{k-1}} \leq \frac{1}{c_{k-1}} \leq (1 + \sqrt{1 - c_k})^2. \tag{3.62}$$

Apply the left inequality of (3.49), multiply $\frac{1-c_{k-1}}{c_k}$, and we get

$$1 - c_{k-1} \leq \frac{1-c_k}{c_k}. \tag{3.63}$$

It follows that

$$c_{k-1} \geq \frac{2c_k - 1}{c_k}, \tag{3.64}$$

Thus

$$\frac{1}{c_{k-1}} \leq \frac{c_k}{2c_k - 1}. \tag{3.65}$$

We aim to prove that for $\frac{\sqrt{5}-1}{2} < c_k \leq 1$,

$$\frac{c_k}{2c_k - 1} \leq (1 + \sqrt{1 - c_k})^2. \tag{3.66}$$

Set $t = \sqrt{1 - c_k}$. Then $0 \leq t \leq \frac{\sqrt{5}-1}{2}$, $c_k = 1 - t^2$, and inequality (3.66) is equivalent to

$$\frac{1-t^2}{1-2t^2} \leq (1+t)^2. \tag{3.67}$$

Multiplying $\frac{1-2t^2}{1+t}$ with both sides, we get

$$1 - t \leq (1+t)(1-2t^2),$$

i.e.

$$t(t^2 + t - 1) \leq 0. \tag{3.68}$$

Obviously, the inequality (3.68) holds for $t \leq -\frac{1+\sqrt{5}}{2}$ or $0 \leq t \leq \frac{\sqrt{5}-1}{2}$. Hence we complete the proof. □

Remarks. In fact, based on the Theorem 3.5, the Theorem 1.4 is a corollary of Theorem 1.1. And in the proof above, for $0 \leq c_k \leq \frac{\sqrt{5}-1}{2}$, we could not determine whether $\frac{1}{c_{k-1}} \leq (1 + \sqrt{1 - c_k})^2$ is true. We ask for an answer to this question.

4. Further work

In this section, we want to discuss a problem we will concern in the future work.

For the partition function $p(n)$, Hou and Zhang [12] proved that $p(n)$ is 2-log-concave for $n \geq 221$. The fact inspires us to consider the problem whether we can find a sufficient condition, similar to the inequality (1.8), for both the 2-log-concavity and the higher order Turán inequalities for $p(n)$ for $n \geq 221$.

Actually, Chen, Wang and Xie [5] proved that $\{p(n+1)/p(n)\}_{n\geq 116}$ is log-convex. Hence $\{p(n+1)/p(n)\}_{n\geq 116}$ is convex. Applying Lemma 3.1, then we get that

$$\frac{p(k+1)^2 - p(k)p(k+2)}{p(k)^2 - p(k-1)p(k+1)} \leq \frac{p(k+1)}{p(k-1)} \tag{4.69}$$

holds for $k \geq 116$. However, it seems that we can not find an integer $N \geq 0$ to make sure that the inequality

$$\frac{a_{k+1}^2 - a_k a_{k+2}}{a_k^2 - a_{k-1}a_{k+1}} \geq \frac{a_{k+2}}{a_k}$$

holds for $p(k)$ for $k \geq N$.

In deed, set

$$f(n) = \frac{p(n+1)^2 - p(n)p(n+2)}{p(n)^2 - p(n-1)p(n+1)}, \tag{4.70}$$

and

$$g_k(n) = \frac{p(n+k+2)}{p(n+k)}, \tag{4.71}$$

then we can verify that, for $224 \leq n \leq 225$,

$$g_{20}(n) \leq f(n) \leq g_{20}(n-1),$$

for $244 \leq n \leq 261$,

$$g_{21}(n) \leq f(n) \leq g_{21}(n-1),$$

for $268 \leq n \leq 291$,

$$g_{22}(n) \leq f(n) \leq g_{22}(n-1),$$

for $296 \leq n \leq 323$,

$$g_{23}(n) \leq f(n) \leq g_{23}(n-1),$$

for $326 \leq n \leq 355$,

$$g_{24}(n) \leq f(n) \leq g_{24}(n-1),$$

for $356 \leq n \leq 389$,

$$g_{25}(n) \leq f(n) \leq g_{25}(n-1),$$

and for $390 \leq n \leq 425$,

$$g_{26}(n) \leq f(n) \leq g_{26}(n-1).$$

Based on the verification above, we guess for any integer $n \geq 326$, we can find the $k = k(n)$ to satisfies the inequality

$$\frac{p(n+k+2)}{p(n+k)} \leq \frac{p(n+1)^2 - p(n)p(n+2)}{p(n)^2 - p(n-1)p(n+1)} \leq \frac{p(n+k+1)}{p(n+k-1)}. \tag{4.72}$$

For $k \geq 3$, we can easily prove that

$$\frac{p(n+k+1)}{p(n+k-1)} \leq \frac{p(n+1)^2}{p(n)^2}\left(1 - \sqrt{1 + \frac{p(n)p(n+2)}{p(n+1)^2}}\right)^2. \tag{4.73}$$

Then if we can prove that

$$\frac{p(n+k+2)}{p(n+k)} \geq \frac{p(n+1)^2}{p(n)^2}\left(1 - \sqrt{1 - \frac{p(n)p(n+2)}{p(n+1)^2}}\right)^2, \tag{4.74}$$

we will find the sufficient condition for both the 2-log-concavity and the higher order Turán inequalities for $p(n)$ for $n \geq 326$.

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