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The first moment of L-functions of primitive forms on $\Gamma_0(p^\alpha)$ and a basis of old forms

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ABSTRACT

We consider the first moment of the value of automorphic L-functions at an arbitrary point on the critical line, a sum over primitive forms weighted by their Petersson's norm. In this paper, we obtain an asymptotic formula for it when weight k is an even integer satisfying $0 < k < 12$ or $k = 14$ and level is p^α, where p is a prime number. This formula yields a lower bound of the number of primitive forms, whose L-functions do not vanish at that point.

1. Introduction

In 1995, Duke studied the non-vanishing of the value $L_f(1/2, \chi)$ (see [5]), where this L-function is associated with a cusp form f. He obtained a lower bound of the number of orthogonal cusp forms for which $L_f(1/2, \chi)$ does not vanish, where cusp forms f are weight 2 for $\Gamma_0(N)$ and N is a prime number. He studied the first and second moments of the value $L_f(1/2, \chi)$ when f varies among an orthogonal basis of cusp forms. He obtained an asymptotic formula for the first moment and an upper bound of the second moment. These yield the above lower bound by using Cauchy's inequality. In the case considered by Duke, the space of cusp forms does not include old forms. (In this case, Kowalski and Michel expressed the above lower bound by the dimension of the space of cusp forms (see [12]).)

In 1999, Akbary extended Duke's results to more general situation (see [1]). He considered the space of cusp forms of weight k for $\Gamma_0(N)$, where N is a prime number. In his case, the space of cusp forms includes old forms and the level of them is 1. He obtained the lower bound of the number of new forms for which $L_f(1/2, \chi)$ does not vanish by using Duke's method and Pizer's results (see [15]).

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In 2000, Kamiya considered such type of lower bounds in general (see [11]). He was interested in the non-vanishing of $L_f(s,\chi)$ at a point on the central line for an orthogonal basis of the space of cusp forms of even weight k for $\Gamma_0(N)$, where N is a positive integer.

When N varies among a family of square-free integers satisfying $\phi(N) \sim N$, where ϕ is the Euler function, Iwaniec and Sarnak showed that at least 50% of the value $L_f(1/2)$ are positive among the set of even primitive forms of even weight and conductor N (see [10]).

In this paper, we are interested in an asymptotic formula for the first moment of the value of $L_f(s,\chi)$ at a point on the central line, where f runs over the primitive forms (normalized Hecke eigen new forms) of weight k for $\Gamma_0(N)$. We obtain it when k is an even integer satisfying $0<k<12$, or $k=14$ and $N=p^\alpha$, where p is a prime number and α is a positive integer.

First of all, we define some notations. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$. We denote the set of primitive forms of weight k for $\Gamma_0(N)$ by $H_k(N)$, which is an orthogonal basis of the space of new forms in $S_k(N)$. The inner product of the space $S_k(N)$ is defined by

$$\langle f,g\rangle_N = \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^{k-2}\,dx\,dy,$$

where $z=x+iy$. Let f be a cusp form and we write

$$f(z)=\sum_{n=1}^{\infty} a_{f,\infty}(n)e^{2\pi inz}.$$

This is the Fourier expansion at the cusp ∞. We put

$$a_{f,\infty}(n)=\lambda_{f,\infty}(n)n^{(k-1)/2}.$$

If $f\in H_k(N)$, we know that $|\lambda_{f,\infty}(n)|\leqslant d(n)$ by Deligne [4], where $d(n)$ is the divisor function, i.e. the number of positive divisors of n. Let χ be a primitive character of modulus q with $(q,N)=1$. The L-function $L_f(s,\chi)$ is defined by

$$L_f(s,\chi)=\sum_{n=1}^{\infty}\frac{\lambda_{f,\infty}(n)\chi(n)}{n^s}$$

for $\sigma>1$, where $s=\sigma+it$. This L-function can be analytically continued on the whole complex plane as a holomorphic function. The main result of this paper is as follows.

Theorem 1. *Let k be an even integer satisfying $0<k<12$ or $k=14$, p a prime number and α a positive integer. Let χ be a primitive character of modulus q with $(q,p)=1$. For any fixed real number y, we have*

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k(p^\alpha)}\frac{L_f(\frac{1}{2}+iy,\chi)}{\langle f,f\rangle_{p^\alpha}}
= 1-c_p(\alpha)+\begin{cases} O(p^{-\frac{k}{2}+\frac{1}{4}}q^{\frac{k}{2}}(1+|y|)^{\frac{k}{2}}) & \text{if } \alpha=1,\\ O(p^{-\frac{5}{4}}q^{\frac{k}{2}}(1+|y|)^{\frac{k}{2}}) & \text{if } \alpha=2,\\ O((\alpha+1)p^{-\frac{(\alpha-1)k}{2}+\frac{(\alpha-1)}{4}+\frac{k}{4}-1}q^{\frac{k}{2}}(1+|y|)^{\frac{k}{2}}) & \text{if } \alpha\geqslant 3,\end{cases}$$

where the implied constants are absolute. Here

$$c_p(\alpha) = \begin{cases} 0 & \text{if } \alpha = 1, \\ p(p^2-1)^{-1} & \text{if } \alpha = 2, \\ p^{-1} & \text{if } \alpha \geqslant 3. \end{cases}$$

Remark 1. For the proof of this theorem, we construct an orthogonal basis of the space of old forms in Section 2. But the technique does not apply to the old forms which come from the cusp forms of level 1. That is why we need to impose $0 < k < 12$ or $k = 14$ in Theorem 1. In the above asymptotic formula, the implied constants would a priori depend on k, if the limitation on k were removed, but not on any other parameter.

Remark 2. When $k = 2$ and $N = p$, Duke obtained an asymptotic formula which is same as Theorem 1 (see [5]). In [10], Iwaniec and Sarnak considered a first moment of the value $L_f(1/2, \chi)$ weighted by $\zeta_N(2)L(1, \bigvee^2 f)^{-1}$, when k is an even positive integer and N is a square-free positive integer. They obtained that the main term of it is same order as N, as N goes to infinity with $\phi(N) \sim N$. When $\phi(N) \sim N$, their weight $\zeta_N(2)L(1, \bigvee^2 f)^{-1}$ is similar to $N/\langle f, f\rangle$ because of (2.36) in [9]. Therefore their result means that the main term of the first moment of the value $L_f(1/2, \chi)$ weighted by Petersson's norm is same order as 1, as square-free integer N goes to infinity with $\phi(N) \sim N$. In Theorem 1, the main term of the asymptotic formula is $1 - c_p(\alpha)$ which is also same order as 1 for any prime p and α. Kamiya obtained an asymptotic formula of the first moment of the value $L_f(1/2 + it, \chi)$ over orthogonal basis whose norm is 1. In Theorem 1, since k is less than 12 or $k = 14$, there are no old forms if $\alpha = 1$. In this case, Proposition 1 in [11] is similar to our result. In fact, we can put $y = x^{-1}N(q\tau)^2$ and $x = N^{1-1/2k}q\tau$ in the proof of Proposition 1 in [11]. This yields that the error term of (15) in [11] is estimated by

$$O\left(d(N)N^{-\frac{k}{2}+\frac{1}{4}}(q\tau)^{\frac{k}{2}}\right).$$

In the error terms of the asymptotic formula in Theorem 1, the exponents of p are negative. And we know

$$\frac{1}{\langle f, f\rangle_N} \ll_k \frac{\log(kN+1)}{N}$$

for any new form $f \in H_k(N)$ (see [3] and [6]), where the implied constant can be explicitly written in terms of k. By using Theorem 1, Cauchy's inequality and Corollary 12 in [11] (Kamiya's results are obtained for the orthogonal basis $\mathcal{F}$ whose norm is 1), we can get the following result.

Corollary 1. *Let $0 < k < 12$ or $k = 14$ be an even integer, q be a fixed positive integer, p a prime number not dividing q, and α a positive integer. Then there exists a positive real number C such that*:

- *when α is fixed*:

$$\liminf_{p\to\infty} \frac{|\{f \in H_k(p^\alpha) \mid L_f(1/2+iy, \chi) \neq 0\}|}{(1-c(\alpha))^2 p^\alpha (\log p^\alpha)^{-2}} \geqslant C,$$

- *when p is fixed*:

$$\liminf_{\alpha\to\infty} \frac{|\{f \in H_k(p^\alpha) \mid L_f(1/2+iy, \chi) \neq 0\}|}{p^\alpha (\log p^\alpha)^{-2}} \geqslant Cp^{-2}.$$

We also see that there exists a positive integer M such that for $p + \alpha > M$, one has

$$\left|\left\{f \in H_k\left(p^\alpha\right) \,\middle|\, L_f(1/2+iy, \chi) \neq 0\right\}\right| > C\left(1-c(\alpha)\right)^2 \frac{p^\alpha}{(\log p^\alpha)^2}.$$

Remark 3. In Corollary 1, positive constants C are absolute (i.e. doesn't depend on q, p, α), but would a priori depend on k if the restriction of k were removed.

Theorem 1 is obtained by the following consideration. By using the 'approximate functional equation' (see (11) below), we see that it is sufficient to study the following two sums

$$\sum_{f\in H_k(p^\alpha)} \frac{\lambda_{f,\infty}(n)}{\langle f,f\rangle_{p^\alpha}} = \sum_{f\in H_k(p^\alpha)} \frac{\overline{\lambda_{f,\infty}(n)}\lambda_{f,\infty}(1)}{\langle f,f\rangle_{p^\alpha}},$$
$$\sum_{f\in H_k(p^\alpha)} \frac{\lambda_{f,0}(n)}{\langle f,f\rangle_{p^\alpha}} = \sum_{f\in H_k(p^\alpha)} \frac{\overline{\lambda_{f,0}(n)}\lambda_{f,\infty}(1)}{\langle f,f\rangle_{p^\alpha}}$$

for the proof of Theorem 1. Here $\lambda_{f,0}(n)$ is not defined yet, but it means the nth-Fourier coefficient at the cusp 0 (see Section 3). Firstly we construct an orthogonal basis of the space of old forms in $S_k(p^\alpha)$ by using the method of Iwaniec, Luo and Sarnak (see [9]). Secondly we show that the above sums are expressed by certain sums over the orthogonal bases of some spaces of cusp forms. Finally, we apply Petersson's formula (see (10) below) to them and obtain Theorem 1.

2. Orthogonal basis

Let k be an even positive integer and N a positive integer. In general we have a decomposition into a direct sum:

$$S_k(N) = \bigoplus_{N=ML} \bigoplus_{f\in H_k(M)} S_k(L;f),$$

where $S_k(L;f)$ is a linear space spanned by $\{f_{|\ell}\}$ which is defined by

$$f_{|\ell}(z) = \ell^{\frac{k}{2}} f(\ell z)$$

and ℓ runs over all positive divisors of L. We know $\dim S_k(L;f) = d(L)$. These facts are mentioned in [9] and we can also see them by using the result of Atkin and Lehner [2] and of Ogg [14]. This basis $\{f_{|\ell}\}$ is not always orthogonal. Let's mention some useful properties of Fourier coefficients of primitive form $f \in H_k(N)$. The $\lambda_{f,\infty}(n)$s are known to be real numbers for all n. We also have

$$\begin{cases} \lambda^2_{f,\infty}(p) = p^{-1} & \text{if } p \mid N \text{ and } p^2 \nmid N, \\ \lambda_{f,\infty}(p) = 0 & \text{if } p^2 \mid N \end{cases} \tag{1}$$

and

$$\begin{cases} \lambda_{f,\infty}(p^n) = \lambda_{f,\infty}(p)^n & \text{if } p \mid N, \\ \lambda_{f,\infty}(mn) = \lambda_{f,\infty}(m)\lambda_{f,\infty}(n) & \text{if } (m,n)=1, \end{cases} \tag{2}$$

where n and m are any integers and p is a prime number (see Lemma 4.5.7, 4.5.8 and Theorem 4.6.17 in [13]).

In the case when k is less than 12 or $k = 14$, since the set $S_k(1)$ is empty, we have

$$S_k(p^\alpha) = \bigoplus_{m=1}^{\alpha} \bigoplus_{f\in H_k(p^m)} S_k(p^{\alpha-m};f).$$

We start by constructing an orthogonal basis of $S_k(p^{\alpha-m}; f)$ for $f \in H_k(p^m)$. The method is similar to the one of Iwaniec, Luo and Sarnak (see [9]). They made an orthogonal basis of $S_k(L; f)$ when $N = ML$ is a square-free positive integer.

Put $\ell_i \mid L$ $(i = 1, 2)$ and consider the function

$$F(s) = \left\langle E(z, s) f(\ell_1 z), f(\ell_2 z)\right\rangle_{p^\alpha},$$

where $E(z, s)$ is the Eisenstein series defined by

$$E(z, s) = y^{s-k+1} \sum_{\gamma \in \Gamma_0(p^\alpha)_\infty \backslash \Gamma_0(p^\alpha)} \left|j(\gamma, z)\right|^{-2(s-k+1)}$$

and

$$j(\gamma, z) = cz + d \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(see (7.2.60) in [13]). We consider the residue of $F(s)$ by two methods, one uses the residue of $E(z, s)$ and the other uses the residue of the Rankin–Selberg L-function associated with f. The comparison of them yields the following lemma.

Lemma 1. *Let α and m be positive integers satisfying $1 \leqslant m \leqslant \alpha$, k an even integer and p a prime number. For positive integer ℓ_i dividing $p^{\alpha-m}$ and a primitive form f of weight k for $\Gamma_0(p^m)$, we have*

$$\langle f_{|\ell_1}, f_{|\ell_2}\rangle_{p^\alpha} = \lambda_{f,\infty}(\ell)\ell^{-\frac{1}{2}}\langle f, f\rangle_{p^\alpha},$$

where $\ell = \ell_1\ell_2(\ell_1, \ell_2)^{-2}$.

Proof. The method of the calculations in this proof is similar to [9]. We know that the residue of Eisenstein series $E(z, s)$ at $s = k$ is $3(\pi p^\alpha(1 + 1/p))^{-1}$ (see Theorem 7.2.17 in [13]) and we see that

$$\operatorname*{Res}_{s=k} F(s) = \frac{3}{\pi p^\alpha}\left(1 + \frac{1}{p}\right)^{-1}(\ell_1\ell_2)^{-\frac{k}{2}}\langle f_{|\ell_1}, f_{|\ell_2}\rangle_{p^\alpha} \tag{3}$$

from the definition of $F(s)$. Here the left-hand side of (3) means the residue of $F(s)$ at $s = k$. Next we calculate the residue of $F(s)$ by another method. Since f is a primitive form, we know that $a_{f,\infty}(n)$ is a real number and obtain

$$\begin{aligned} F(s) &= \int_0^\infty y^{s-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\ell_1 z)\overline{f(\ell_2 z)}\,dx\,dy \\ &= (4\pi)^{-s}\Gamma(s) \sum_{\substack{1 \leqslant n_1, n_2 \leqslant \infty \\ n_1\ell_1 = n_2\ell_2}} \frac{a_{f,\infty}(n_1)a_{f,\infty}(n_2)}{(\ell_1 n_1)^s}. \end{aligned}$$

Put $\ell' = \ell_1(\ell_1, \ell_2)^{-1}$ and $\ell'' = \ell_2(\ell_1, \ell_2)^{-1}$. We know $(\ell', \ell'') = 1$ and $n_1\ell' = n_2\ell''$. Therefore we can put $n_1 = n\ell''$, $n_2 = n\ell'$ $(1 \leqslant n \leqslant \infty)$ and obtain

$$F(s) = (4\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{\lambda_{f,\infty}(n\ell'')\lambda_{f,\infty}(n\ell')}{n^{s-k+1}}\frac{(\ell'\ell'')^{\frac{k-1}{2}}}{(\ell_1\ell'')^s}.$$

We recall that ℓ' and ℓ'' are 1 or power of p. We have

$$F(s) = (4\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{\lambda_{f,\infty}^2(n)}{n^{s-k+1}}\lambda_{f,\infty}\big(\ell'\ell''\big)\big(\ell'\ell''\big)^{\frac{k-1}{2}}\left(\frac{\ell_1\ell_2}{(\ell_1, \ell_2)}\right)^{-s} \tag{4}$$

from (2). We have the relation (2.31) in [9]. Iwaniec, Luo and Sarnak suppose that N is square-free before (2.31). But this relation holds in general, because of Ogg's result (see (4′) in [14]). Therefore we obtain

$$\operatorname*{Res}_{s=k}\left(\sum_{n=1}^{\infty}\frac{\lambda_{f,\infty}^2(n)}{n^{s-k+1}}\right) = \frac{(4\pi)^k}{\Gamma(k)}\frac{3}{p^m\pi}\left(1+\frac{1}{p}\right)^{-1}\langle f, f\rangle_{p^m}. \tag{5}$$

And we know

$$\langle f, f\rangle_M = \frac{M\prod_{p|M}(1+\frac{1}{p})}{N\prod_{p|N}(1+\frac{1}{p})}\langle f, f\rangle_N \tag{6}$$

for $f \in S_k(M)$, where $M \mid N$ (see Theorem 7.2.17 in [13]). We can obtain Lemma 1 from (3), (4), (5) and (6). □

Remark 4. In order to obtain Eq. (4), we have to produce $\lambda_{f,\infty}^2(n)$ from $\lambda_{f,\infty}(n\ell')\lambda_{f,\infty}(n\ell'')$ for any n. When ℓ' and ℓ'' are power of p, we can do this if p divide the level of $f \in H_k(p^m)$. This explains why we need the condition $m \geqslant 1$ in this lemma. From this lemma, we construct an orthogonal basis of $S_k(p^{\alpha-m}; f)$ by $f \in H_k(p^m)$. Since we cannot use this construction when $m = 0$, we have to impose the limitation $0 < k < 12$ or $k = 14$, which implies that $S_k(1)$ is $\{0\}$.

Remark 5. Iwaniec, Luo and Sarnak obtained the relation between $\langle f_{|\ell_1}, f_{|\ell_2}\rangle$ and $\langle f, f\rangle$, when N is a square-free integer (see [9]).

Remark 6. When $N = p^\alpha$ with $\alpha \geqslant 1$, for $f \in H_k(1)$ we can prove that

$$\begin{aligned}\langle f_{\ell_1}, f_{\ell_2}\rangle_{p^\alpha} = \ell^{-\frac{1}{2}}&\sum_{j=0}^{\infty}\frac{\lambda_{f,\infty}(p^j\ell'')\lambda_{f,\infty}(p^j\ell')}{p^j}\\ &\times\frac{p-1}{p+1}\left(1-\frac{\alpha_{f,\infty}^2(p)}{p}\right)\left(1-\frac{\beta_{f,\infty}^2(p)}{p}\right)\langle f, f\rangle_{p^\alpha},\end{aligned} \tag{7}$$

instead of Lemma 1, by the similar way to the above proof. Here

$$\ell' = \ell_1(\ell_1, \ell_2)^{-1}, \qquad \ell'' = \ell_2(\ell_1, \ell_2)^{-1} \quad \text{and} \quad \ell = \ell_1\ell_2(\ell_1, \ell_2)^{-2}.$$

The complex numbers $\alpha_{f,\infty}(p)$ and $\beta_{f,\infty}(p)$ appear in the Euler product of $L_f(s)$, which satisfy

$$\begin{cases} \alpha_{f,\infty}(p)+\beta_{f,\infty}(p)=\lambda_{f,\infty}(p), \\ \alpha_{f,\infty}(p)\beta_{f,\infty}(p)=1. \end{cases}$$

We want to obtain an orthogonal basis $\{f_{p^d}\}$ $(0 \leqslant d \leqslant \alpha)$ which satisfies $\langle f_{p^d}, f_{p^d}\rangle_{p^\alpha} = \langle f, f\rangle_{p^\alpha}$. We consider a matrix $T=(t_{a,b})$ $(0 \leqslant a, b \leqslant \alpha)$, where

$$t_{a,b}=\frac{\langle f_{|p^a}, f_{|p^b}\rangle_{p^\alpha}}{\langle f, f\rangle_{p^\alpha}}.$$

This is a real symmetric matrix and $t_{a,a}=1$. We denote a map $\{f_{p^d}\} \to \{f_{|p^d}\}$ $(0 \leqslant d \leqslant \alpha)$ by a matrix Z. Since $\{f_{p^d}\}$ and $\{f_{|p^d}\}$ are basis of $S_k(p^\alpha; f)$, $T = Z\,{}^t\overline{Z}$ is regular. Therefore there exists an orthogonal matrix Q which satisfies

$${}^tQ\,T\,Q=\mathrm{diag}\bigl(\Lambda(0), \Lambda(1), \ldots, \Lambda(\alpha)\bigr)$$

and $\Lambda(d)$s are non-zero real numbers. We denote $\{i, j\}$-entry of Q by $q_{i,j}$. Since Q is an orthogonal matrix, we know

$$\sum_{h=0}^{\alpha} q_{h,i}q_{h,j}=\sum_{h=0}^{\alpha} q_{i,h}q_{j,h}=\begin{cases} 1, & i=j, \\ 0, & i\neq j. \end{cases}$$

Then we can write

$$f_{p^d}=\Lambda(d)^{-1/2}\sum_{m=0}^{\alpha} q_{m,d}f_{|p^m}.$$

It is difficult to find $q_{m,d}$ explicitly, however we can estimate eigenvalues. We consider $\Lambda(j)$ for fixed j. There exists an index i satisfying

$$|q_{i,j}|=\max_{0\leqslant h\leqslant\alpha}\bigl\{|q_{h,j}|\bigr\}.$$

Since $TQ = Q\ \mathrm{diag}(\Lambda(0), \Lambda(1), \ldots, \Lambda(d))$, we have

$$q_{i,j}+\sum_{\substack{h=0\\ h\neq i}}^{\alpha} t_{i,h}q_{h,j}=q_{i,j}\Lambda(j)$$

thus

$$\bigl|\Lambda(j)-1\bigr| \leqslant \sum_{\substack{h=0\\ h\neq i}}^{\alpha} |t_{i,h}| \leqslant \alpha C_f(p,\alpha), \tag{8}$$

where $|t_{i,h}| < C_f(p,\alpha)$ $(i \neq h)$ and

$$C_f(p,\alpha) = p^{-1/2}\left((\alpha+1) + \frac{4p^3-3p+1+\alpha(2p^2-3p+1)}{(p-1)^3\log p}\right) \times \frac{p-1}{p+1}\left(1-\frac{\alpha_{f,\infty}^2(p)}{p}\right)\left(1-\frac{\beta_{f,\infty}^2(p)}{p}\right).$$

We can obtain $C_f(p,\alpha)$ by elementary calculation. From (7), when $a \neq b$, we have

$$t_{a,b} \leqslant p^{-1/2}\sum_{j=0}^{\infty}\frac{|\lambda_{f,\infty}(p^j\ell'')\lambda_{f,\infty}(p^j\ell')|}{p^j}\left(1-\frac{\alpha_{f,\infty}^2(p)}{p}\right)\left(1-\frac{\beta_{f,\infty}^2(p)}{p}\right).$$

If $a > b$, we know

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{|\lambda_{f,\infty}(p^j\ell'')\lambda_{f,\infty}(p^j\ell')|}{p^j} &= \left|\lambda_f\left(p^{a-b}\right)\right| + \sum_{j=1}^{\infty}\frac{|\lambda_{f,\infty}(p^j\ell'')\lambda_{f,\infty}(p^j\ell')|}{p^j}\\ &< d\left(p^{a-b}\right) + \sum_{j=1}^{\infty}\frac{d(p^j\ell'')d(p^j\ell')}{p^j}\\ &< (a-b+1) + \sum_{j=1}^{\infty}\frac{(j+1)(a-b+j)}{p^j}\\ &< (\alpha+1) + \frac{4p^3-3p+1+\alpha(2p^2-3p+1)}{(p-1)^3\log p}.\end{aligned}$$

For sufficiently large p, all eigenvalues are positive since T tends to identity matrix as $p \to \infty$, $\min\{\Lambda(d)\} \leqslant 1$ since $\mathrm{Tr}(T) = \alpha+1$, and $\alpha C_f(p,\alpha) < 1$. Therefore we have $1-\alpha C_f(p,\alpha) \leqslant \min\Lambda(d)$ from (8).

Lemma 1 yields the following lemma.

Lemma 2. *Let k be an even integer satisfying $0 < k < 12$ or $k = 14$, p a prime number and α a positive integer. Then we have an orthogonal decomposition*

$$S_k\left(p^\alpha\right) = \bigoplus_{m=1}^{\alpha}\bigoplus_{f\in H_k(p^m)}\bigoplus_{d\mid p^{\alpha-m}}\langle f_d\rangle,$$

where $f_1 = f$ and

$$f_d = \begin{cases} d^{\frac{k}{2}} f(dz) & \text{if } m \geqslant 2,\\ p\sqrt{p^2-1}^{-1}\left(d^{\frac{k}{2}} f(dz) - p^{-\frac{1}{2}}\lambda_f(p)(\frac{d}{p})^{\frac{k}{2}} f(\frac{dz}{p})\right) & \text{if } m = 1\end{cases}$$

for $d \neq 1$ and $f \in H_k(p^m)$. We also have

$$\lambda_{f_d,\infty}(n) = \begin{cases} d^{\frac{1}{2}}\lambda_{f,\infty}(\frac{n}{d}) & \text{if } m \geqslant 2,\\ d^{\frac{1}{2}} p\sqrt{p^2-1}^{-1}\left(\lambda_{f,\infty}(\frac{n}{d}) - p^{-1}\lambda_{f,\infty}(p)\lambda_{f,\infty}(\frac{np}{d})\right) & \text{if } m = 1\end{cases}$$

for $d \neq 1$ and $f \in H_k(p^m)$. Here if x is not an integer, we put $\lambda_{f,\infty}(x) = 0$. Moreover this orthogonal basis satisfies

$$\langle f_d, f_d \rangle_{p^\alpha} = \langle f, f \rangle_{p^\alpha}.$$

Proof. In this proof, the method is also the same as in [9]. We want to find an orthogonal basis $\{f_d\}$ of $S_k(p^{\alpha-m}; f)$ for $f \in H_k(p^m)$. Since the space $S_k(p^{\alpha-m}; f)$ is spanned by $\{f_{|\ell}\}$, let's define f_d by

$$f_d = \sum_{\ell \mid p^{\alpha-m}} x_d(\ell) f_{|\ell}.$$

Consider

$$\delta_f(d_1, d_2)_{p^\alpha} = \frac{\langle f_{d_1}, f_{d_2} \rangle_{p^\alpha}}{\langle f, f \rangle_{p^\alpha}}$$

for $d_i \mid p^{\alpha-m}$. We want to find $x_d(\ell)$ which yields

$$\delta_f(d_1, d_2)_{p^\alpha} = \begin{cases} 1 & \text{if } d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2. \end{cases}$$

In this proof, we put $L = p^{\alpha-m}$ and $N = p^\alpha$ for simplicity. We can see that

$$\begin{aligned} \delta_f(d_1, d_2)_N &= \sum_{\ell_1 \mid L} \sum_{\ell_2 \mid L} x_{d_1}(\ell_1) \overline{x_{d_2}(\ell_2)} \frac{\langle f_{|\ell_1}, f_{|\ell_2} \rangle_N}{\langle f, f \rangle_N} \\ &= \sum_{a \mid L} \sum_{d \mid \frac{L}{a}} \left(\sum_{\ell \mid \frac{L}{ad}} x_{d_1}(ad\ell) \lambda_{f,\infty}(d\ell) \ell^{-\frac{1}{2}} \right) \left(\sum_{\ell \mid \frac{L}{ad}} \overline{x_{d_2}(ad\ell) \lambda_{f,\infty}(d\ell) \ell^{-\frac{1}{2}}} \right) \frac{\mu(d)}{d} \end{aligned}$$

by using Lemma 1 and the calculations similar to [9]. We recall that the only prime factor of L is p. By using (1) and (2), we have

$$\begin{aligned} \delta_f(d_1, d_2)_N &= \begin{cases} \sum_{a \mid L} x_{d_1}(a) \overline{x_{d_2}(a)} & \text{if } m \geqslant 2, \\ \sum_{c \mid L} \sum_{d \mid c} \mu(d) d^{-1} \lambda_{f,\infty}(d)^2 \mathcal{Y}_{d_1}(c) \overline{\mathcal{Y}_{d_2}(c)} & \text{if } m = 1 \end{cases} \\ &= \begin{cases} \sum_{a \mid L} x_{d_1}(a) \overline{x_{d_2}(a)} & \text{if } m \geqslant 2, \\ \sum_{c \mid L} (\sqrt{\rho_f(c)} \mathcal{Y}_{d_1}(c)) \overline{(\sqrt{\rho_f(c)} \mathcal{Y}_{d_2}(c))} & \text{if } m = 1 \end{cases} \end{aligned}$$

where

$$\mathcal{Y}_d(c) = \sum_{\ell \mid \frac{L}{c}} x_d(c\ell) \lambda_{f,\infty}(\ell) \ell^{-\frac{1}{2}}$$

and

$$\rho_f(c) = \sum_{d \mid c} \frac{\mu(c)}{d} \lambda_{f,\infty}(d)^2 = \prod_{p \mid c} \left(1 - \frac{\lambda_{f,\infty}(p)^2}{p} \right) = \begin{cases} 1 & \text{if } c = 1, \\ 1 - p^{-2} & \text{if } c \neq 1. \end{cases}$$

Let's impose:

$$\begin{cases} x_d(a) = \delta_{d,a} & \text{if } m \geqslant 2, \\ \sqrt{\rho_f(d)}\mathcal{Y}_d(c) = \delta_{d,c} & \text{if } m = 1, \end{cases}$$

where $\delta_{*,\dagger}$ is Kronecker's delta symbol. This constraint is fulfilled by the choice

$$x_d(*) = \begin{cases} \delta_{d,*} & \text{if } m \geqslant 2, \\ \sqrt{\rho_f(d)}^{-1}\lambda_{f,\infty}(\frac{d}{*})(\frac{d}{*})^{-\frac{1}{2}}\mu(\frac{d}{*}) & \text{if } * \mid d,\ m = 1, \\ 0 & \text{if } * \nmid d,\ m = 1. \end{cases}$$

This ends the proof of Lemma 2. □

Remark 7. Iwaniec, Luo and Sarnak obtained an orthogonal basis of $S_k(M; f)$, which is a space of old forms in $S_k(N)$, when $M \mid N$ are square-free integers (see [9]). But it is difficult to obtain an asymptotic formula of the first moment of $L_f(1/2 + it, \chi)$ over primitive forms by using their orthogonal basis and the method in this paper, because we do not know the value of (2.40) in [9] which appear in their basis.

3. Petersson's formula

We recall Petersson's formula when k is a positive even integer and N is a positive integer. The facts in this section are explained in [8]. For a cusp form f in $S_k(N)$, we have the Fourier expansion of f at a cusp $\mathfrak{a}$ and we denote its nth-Fourier coefficient by $\lambda_{f,\mathfrak{a}}(n)n^{(k-1)/2}$. In this paper, we only consider two cusps ∞ and 0. For $f \in S_k(N)$, we denote

$$(f|_k\gamma)(z) = (\det\gamma)^{\frac{k}{2}} j(\gamma, z)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}).$$

And from definitions we can write

$$(f|_k\sigma_0)(z) = (f|_k\omega_N)(z) = \sum_{n=1}^{\infty} \lambda_{f,0}(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where

$$\sigma_0 = \sigma_{0,N} = \begin{pmatrix} 0 & -\sqrt{N}^{-1} \\ \sqrt{N} & 0 \end{pmatrix} \quad \text{and} \quad \omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

The cusp form $f|_k\omega_N$ appears in the functional equation of $L_f(s, \chi)$, which is

$$\Lambda_N(s; f, \chi) = i^k C_\chi \Lambda_N(1-s; f|_k\omega_N, \overline{\chi}), \tag{9}$$

where

$$\Lambda_N(s; f, \chi) = \left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L_f(s, \chi)$$

and C_χ depend on χ with $|C_\chi| = 1$ (see Theorem 4.3.11 in [13]). If f is a primitive form, we know that $f|_k\omega_N = C_f f$ and $C_f = \pm 1$ depending on the form f (see Theorem 4.6.15 in [13]). In Section 2 we mentioned that $\lambda_{f,\infty}(n)$ is a real number when $f \in H_k(N)$, so $\lambda_{f,0}(n)$ is also a real number.

In the proof of Theorem 1 below, we use Petersson's formula (see Theorem 3.6 in [8] and Proposition 2.1 in [9]). Let's define:

$$\Delta_{k,N}(m,n;\mathfrak{a},\mathfrak{b}) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f} \frac{\overline{\lambda_{f,\mathfrak{a}}(m)}\lambda_{f,\mathfrak{b}}(n)}{\langle f, f\rangle_N}$$

for $\mathfrak{a}$ and $\mathfrak{b}$ as ∞ or 0, where the sum is over an orthogonal basis $\{f\}$ of $S_k(N)$. This definition is independent of the choice of the orthogonal basis. Petersson's formula is as follows

$$\Delta_{k,N}(m,n;\mathfrak{a},\mathfrak{b}) = \delta_{m,n}\delta_{\mathfrak{a},\mathfrak{b}} + 2\pi i^{-k} \sum_{c\in\mathcal{C}(\mathfrak{a},\mathfrak{b})} c^{-1} S_{\mathfrak{a},\mathfrak{b}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{10}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are ∞ or 0. This formula is showed in Chapter 3 in [8]. We explain the notations in this formula. In the right-hand side of (10), $\delta_{*,\dagger}$ is Kronecker's delta symbol and the sum is over

$$\mathcal{C}(\mathfrak{a},\mathfrak{b}) = \begin{cases} \{c = \ell N\colon\ \ell \in \mathbb{N}\} & \text{if } \mathfrak{a} = \mathfrak{b}, \\ \{c = \ell\sqrt{N}\colon\ \ell \in \mathbb{N} \text{ and } (\ell, N) = 1\} & \text{if } \mathfrak{a} \neq \mathfrak{b} \end{cases}$$

(see Section 4.2 in [8]). The function J_{k-1} is a Bessel function and $S_{\mathfrak{a},\mathfrak{b}}$ means the Kloosterman sum as

$$S_{\mathfrak{a},\mathfrak{b}}(m,n;c) = \begin{cases} S(m,n,\ell N) & \text{if } \mathfrak{a} = \mathfrak{b}, \\ S(m\overline{N},n,\ell) & \text{if } \mathfrak{a} \neq \mathfrak{b}, \end{cases}$$

where $\overline{N}$ means $N\overline{N} \equiv 1 \bmod \ell$ (see Section 4.2 in [8]).

4. An approximate functional equation

In this section, we prove what is called an 'approximate functional equation' (11) when k and N are positive. From now on we define $s = \sigma + it$. Let ε be a small positive number. For any primitive form f in $H_k(N)$ and $X > 0$, we can obtain

$$L_f\left(\frac{1}{2} + iy, \chi\right) = \sum_{n=1}^{\infty} \frac{\lambda_{f,\infty}(n)\chi(n)}{n^{\frac{1}{2}+iy}} e^{-(\frac{n}{X})^h} - I, \tag{11}$$

where

$$I = \frac{i^k C_\chi}{2\pi i} \int_{(c_1)} \left(\frac{4\pi^2}{q^2 N}\right)^{s+iy} G_k\left(s + \frac{1}{2} + iy\right) X^s \frac{\Gamma(1+\frac{s}{h})}{s} \sum_{n=1}^{\infty} \frac{\lambda_{f,0}(n)\overline{\chi}(n)}{n^{\frac{1}{2}-s-iy}}\, ds,$$

$h = (k+1)/2$ and $c_1 = -k/2 - \varepsilon$. Here the path of integration (c) in the above integral is the vertical line $\{z \in \mathbb{C} \mid \sigma = c\}$. The function $G_k(s)$ is defined by

$$G_k(s) = \frac{\Gamma(\frac{k+1}{2} - s)}{\Gamma(s + \frac{k-1}{2})}.$$

Since the method to obtain (11) is known, we recall it briefly (see Section 4.4 in [7]). For $c > 0$ we know

$$e^{-X^h} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(1+\frac{s}{h})}{s} X^{-s}\, ds \tag{12}$$

and see

$$\frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} L_f\left(s + \frac{1}{2} + iy, \chi\right) X^s \frac{\Gamma(1+\frac{s}{h})}{s}\, ds = \sum_{n=1}^{\infty} \frac{\lambda_{f,\infty}(n)\chi(n)}{n^{\frac{1}{2}+iy}} e^{-(\frac{n}{X})^h}.$$

On the left-hand side of this equation, we move the path of integration from the line $\sigma = 1/2 + \varepsilon$ to the line $\sigma = -k/2 - \varepsilon$ by using the residue theorem, and we can obtain (11) by using the functional equation (9). To achieve that, we need estimates for $\Gamma(s)$ and $L_f(s, \chi)$ for $-k/2 - \varepsilon \leqslant \sigma \leqslant 1 + k/2 + \varepsilon$ and any t. For the gamma function, we know Stirling's formula

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O_\delta\left(|s|^{-1}\right) \tag{13}$$

for $-\pi + \delta < \arg s < \pi - \delta$. By using this, we can see

$$\left|\Gamma(s)\right| = \sqrt{2\pi}\, |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|}\left(1 + O\left(|t|^{-1}\right)\right)$$

for $-\pi + \delta < \arg s < \pi - \delta$, $|\sigma/t| \leqslant c_0$ and $c_1 \leqslant \sigma \leqslant c_2$, where c_i are absolute constants. We apply this estimate to $\Gamma(1 + s/h)$. Next let's give an estimate for $L_f(s)$. From Stirling's formula (13) we see

$$\begin{aligned} G_k(s) &= \prod_{j=0}^{k/2-1} \frac{s + (k-1)/2 + j}{(k+1)/2 - s + j} \times \frac{\Gamma(k - 1/2 - s)}{\Gamma(s + k + 1/2)} (k - 1/2 - s)(s + k - 1/2) \\ &\ll_k \left| \frac{\Gamma(k - 1/2 - s)}{\Gamma(s + k + 1/2)} \right| |k + it|^2 \ll_k \left(1 + |t|\right)^{1 - 2\sigma} \end{aligned} \tag{14}$$

for $-k/2 - 1/4 \leqslant \sigma \leqslant 1/2$. And we know

$$\left|L_f(1 + a + it, \chi)\right| \leqslant \sum_{n=1}^{\infty} \frac{d(n)}{n^{1+a}} = \zeta^2(1 + a) \ll \left(1 + a^{-1}\right)^2.$$

Therefore the functional equation (9) yields

$$L_f(-k/2 - \varepsilon + it, \chi) \ll_k \left(q\sqrt{N}\left(1 + |t|\right)\right)^{1+k+2\varepsilon}$$

for fixed small positive constant ε. By using these facts and the Phragmén–Lindelöf theorem or the maximum modulus principle, for $-k/2 - \varepsilon \leqslant \sigma \leqslant 1 + k/2 + \varepsilon$ we have

$$L_f(s, \chi) \ll_k \left(q\sqrt{N}\left(1 + |t|\right)\right)^{1 + \frac{k}{2} + \varepsilon - \sigma}.$$

5. The key lemma

In what follows we consider only the special case where k is an even positive integer satisfying $0 < k < 12$ or $k = 14$, p is a prime number with $(q, p) = 1$ and $N = p^\alpha$. We define:

$$\mathfrak{S}_k(p^\alpha) = \sum_{f \in H_k(p^\alpha)} \frac{L_f(\frac{1}{2} + iy, \chi)}{\omega_f(p^\alpha)},$$

where

$$\omega_f(p^\alpha) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle_{p^\alpha}.$$

By using the approximate functional equation (11), we have

$$\mathfrak{S}_k(p^\alpha) = \sum_{n=1}^{\infty} \frac{\chi(n) e^{-(\frac{n}{X})^h}}{n^{\frac{1}{2}+iy}} \sum_{f \in H_k(p^\alpha)} \frac{\lambda_{f,\infty}(n)}{\omega_f(p^\alpha)} - \frac{i^k C_\chi}{2\pi i} \int_{(c_1)} \left(\frac{4\pi^2}{q^2 p^\alpha} \right)^{s+iy} X^s \\ \times \frac{\Gamma(1 + \frac{s}{h})}{s} G_k\left(s + \frac{1}{2} + iy\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{\frac{1}{2}-s-iy}} \sum_{f \in H_k(p^\alpha)} \frac{\lambda_{f,0}(n)}{\omega_f(p^\alpha)} \, ds,$$

where $h = (k+1)/2$. We put

$$\mathfrak{s}_{k,p^\alpha}^{\mathfrak{a}}(n) = \sum_{f \in H_k(p^\alpha)} \frac{\lambda_{f,\mathfrak{a}}(n)}{\omega_f(p^\alpha)}$$

for $\mathfrak{a} = \infty$, 0 and we can write

$$\mathfrak{S}_k(p^\alpha) = \sum_{n=1}^{\infty} \frac{\chi(n) e^{-(\frac{n}{X})^h}}{n^{\frac{1}{2}+iy}} \mathfrak{s}_{k,p^\alpha}^{\infty}(n) - \frac{i^k C_\chi}{2\pi i} \int_{(c_1)} \left(\frac{4\pi^2}{q^2 p^\alpha} \right)^{s+iy} X^s \\ \times \frac{\Gamma(1 + \frac{s}{h})}{s} G_k\left(s + \frac{1}{2} + iy\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{\frac{1}{2}-s-iy}} \mathfrak{s}_{k,p^\alpha}^{0}(n) \, ds. \tag{15}$$

It is important to study $\mathfrak{s}_{k,p^\alpha}^{\mathfrak{a}}(n)$ for our aim.

For an integer β satisfying $0 < \beta \leqslant \alpha$, we put

$$B_k(p^\beta) = \bigcup_{m=1}^{\beta} \bigcup_{f \in H_k(p^m)} \bigcup_{d | p^{\beta-m}} \{f_d\}.$$

From Lemma 2 we can see that $B_k(p^\beta)$ is an orthogonal basis of $S_k(p^\beta)$.

In the case of $\mathfrak{s}_{k,p}$, since $H_k(p) = B_k(p)$, we obtain

$$\mathfrak{s}_{k,p}^{\mathfrak{a}}(n) = \sum_{f \in B_k(p)} \frac{\lambda_{f,\mathfrak{a}}(n) \lambda_{f,\infty}(1)}{\omega_f(p)} = \Delta_{k,p}(n, 1; \mathfrak{a}, \infty). \tag{16}$$

In the case of $\mathfrak{s}_{k,p^2}$, we have

$$\begin{aligned}
\mathfrak{s}^{\mathfrak{a}}_{k,p^2}(n) &= \sum_{f\in H_k(p^2)} \frac{\lambda_{f,\mathfrak{a}}(n)}{\omega_f(p^2)} = \sum_{f\in H_k(p^2)} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)}\lambda_{f,\infty}(1)}{\omega_f(p^2)} \\
&= \sum_{f\in B_k(p^2)} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)}\lambda_{f,\infty}(1)}{\omega_f(p^2)} \\
&\quad - \sum_{f\in H_k(p)} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)}\lambda_{f,\infty}(1)}{\omega_f(p^2)} - \sum_{f\in H_k(p)} \frac{\overline{\lambda_{f_p,\mathfrak{a}}(n)}\lambda_{f_p,\infty}(1)}{\omega_f(p^2)}. \qquad (17)
\end{aligned}$$

From Lemma 2, for $f_p \in S_k(p^2)$ which is related to $f \in H_k(p)$, we have

$$\lambda_{f_p,\infty}(n) = \frac{p}{\sqrt{p^2-1}}\left(p^{\frac{1}{2}}\lambda_{f,\infty}\left(\frac{n}{p}\right) - \lambda_{f,\infty}(pn)p^{-\frac{1}{2}}\right)$$

and also see that

$$\lambda_{f_p,\infty}(1) = \frac{p}{\sqrt{p^2-1}}\bigl(-\lambda_{f,\infty}(p)p^{-\frac{1}{2}}\bigr).$$

By (1) and (2) we obtain

$$\overline{\lambda_{f_p,\infty}(n)}\lambda_{f_p,\infty}(1) = \frac{1}{p^2-1}\lambda_{f,\infty}(n)\lambda_{f,\infty}(1) \times \begin{cases} 1 & \text{if } p\nmid n, \\ (1-p^2) & \text{if } p\mid n. \end{cases}$$

When $\mathfrak{a}=\infty$, we apply these relations to (17), and recall (6), then we obtain

$$\begin{aligned}
\mathfrak{s}^{\infty}_{k,p^2}(n) &= \Delta_{k,p^2}(n,1;\infty,\infty) - \sum_{f\in B_k(p)} \frac{\lambda_{f,\infty}(n)\lambda_{f,\infty}(1)}{\omega_f(p^2)} \\
&\quad - \sum_{f\in B_k(p)} \frac{\lambda_{f,\infty}(n)\lambda_{f,\infty}(1)}{\omega_f(p^2)} \frac{1}{(p^2-1)} \times \begin{cases} 1, & p\nmid n, \\ (1-p^2), & p\mid n \end{cases} \\
&= \Delta_{k,p^2}(n,1;\infty,\infty) - \begin{cases} \frac{p}{p^2-1}\Delta_{k,p}(n,1;\infty,\infty), & p\nmid n, \\ 0, & p\mid n. \end{cases} \qquad (18)
\end{aligned}$$

This is because $H_k(p) = B_k(p)$ and (6). We consider (17) when $\mathfrak{a}=0$. We have proven

$$f_p(z) = \frac{p}{\sqrt{p^2-1}}\left(\left(f\middle|\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\right)(z) - p^{-\frac{1}{2}}\lambda_{f,\infty}(p)f(z)\right)$$

for $f \in H_k(p)$ from Lemma 2. We can see that

$$(f_p|\sigma_{0,p^2})(z) = \frac{p}{\sqrt{p^2-1}}\bigl(C_f f(z) - p^{\frac{k-1}{2}}\lambda_{f,\infty}(p)C_f f(pz)\bigr).$$

This implies

$$\lambda_{f_p,0}(n) = \begin{cases} \frac{p}{\sqrt{p^2-1}} C_f \lambda_{f,\infty}(n), & p \nmid n, \\ 0, & p \mid n. \end{cases}$$

From (2) we obtain

$$\overline{\lambda_{f_p,0}(n)} \lambda_{f_p,\infty}(1) = \begin{cases} -\frac{p^{2-1/2}}{p^2-1} \lambda_{f,0}(np), & p \nmid n, \\ 0, & p \mid n \end{cases}$$

since $\lambda_{f,\infty}(n)$ are real and $C_f \lambda_{f,\infty}(n) = \lambda_{f,0}(n)$ for $f \in H_k(p)$. From (17) we have

$$\begin{aligned}
\mathfrak{s}^0_{k,p^2}(n) &= \Delta_{k,p^2}(n,1;0,\infty) - \sum_{f \in B_k(p)} \frac{\lambda_{f,0}(n)\lambda_{f,\infty}(1)}{\omega_f(p^2)} \\
&\quad + \sum_{f \in B_k(p)} \frac{\lambda_{f,0}(pn)\lambda_{f,\infty}(1)}{\omega_f(p^2)} \frac{p^{2-1/2}}{(p^2-1)} \times \begin{cases} 1, & p \nmid n, \\ 0, & p \mid n \end{cases} \\
&= \Delta_{k,p^2}(n,1;0,\infty) - \frac{1}{p} \Delta_{k,p}(n,1;0,\infty) \\
&\quad + \begin{cases} \frac{p^{1/2}}{p^2-1} \Delta_{k,p}(pn,1;0,\infty), & p \nmid n, \\ 0, & p \mid n \end{cases}
\end{aligned} \tag{19}$$

by using $H_k(p) = B_k(p)$ and (6). From (18) and (19) we have

$$\begin{aligned}
\mathfrak{s}^{\mathfrak{a}}_{k,p^2}(n) &= \Delta_{k,p^2}(n,1;\mathfrak{a},\infty) \\
&\quad + \begin{cases} -\frac{p}{p^2-1} \Delta_{k,p}(n,1;\infty,\infty), & p \nmid n,\ \mathfrak{a} = \infty, \\ 0, & p \mid n,\ \mathfrak{a} = \infty, \\ -\frac{1}{p} \Delta_{k,p}(n,1;0,\infty) + \frac{p^{1/2}}{p^2-1} \Delta_{k,p}(pn,1;0,\infty), & p \nmid n,\ \mathfrak{a} = 0, \\ -\frac{1}{p} \Delta_{k,p}(n,1;0,\infty), & p \mid n,\ \mathfrak{a} = 0. \end{cases}
\end{aligned} \tag{20}$$

In the case of $\mathfrak{s}_{k,p^\alpha}$ for $\alpha \geqslant 3$, we have

$$\begin{aligned}
\mathfrak{s}^{\mathfrak{a}}_{k,p^\alpha}(n) &= \sum_{f \in H_k(p^\alpha)} \frac{\lambda_{f,\mathfrak{a}}(n)}{\omega_f(p^\alpha)} = \sum_{f \in H_k(p^\alpha)} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)} \lambda_{f,\infty}(1)}{\omega_f(p^\alpha)} \\
&= \sum_{f \in B_k(p^\alpha)} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)} \lambda_{f,\infty}(1)}{\omega_f(p^\alpha)} - \sum_{f \in B_k(p^{\alpha-1})} \frac{\overline{\lambda_{f,\mathfrak{a}}(n)} \lambda_{f,\infty}(1)}{\omega_f(p^\alpha)} \\
&\quad - \sum_{m=1}^{\alpha-1} \sum_{f \in H_k(p^m)} \frac{\overline{\lambda_{f_{p^{\alpha-m}},\mathfrak{a}}(n)} \lambda_{f_{p^{\alpha-m}},\infty}(1)}{\omega_f(p^\alpha)}.
\end{aligned} \tag{21}$$

From Lemma 2, for $f \in H_k(p^m)$ we have

$$\lambda_{f_{p^{\alpha-m}},\infty}(n) = \begin{cases} p^{\frac{\alpha-m}{2}}\lambda_{f,\infty}(\frac{n}{p^{\alpha-m}}), & \alpha > m \geqslant 2, \\ \frac{p^{(\alpha+1)/2}}{\sqrt{p^2-1}}(\lambda_{f,\infty}(\frac{n}{p^{\alpha-1}}) - \lambda_{f,\infty}(p)\lambda_{f,\infty}(\frac{np}{p^{\alpha-1}})p^{-1}), & m = 1 \end{cases}$$

and also see that

$$\lambda_{f_{p^{\alpha-m}},\infty}(1) = 0.$$

From (21) and (6) we obtain

$$\mathfrak{s}^{\mathfrak{a}}_{k,p^\alpha}(n) = \Delta_{k,p^\alpha}(n,1;\mathfrak{a},\infty) - \frac{1}{p}\Delta_{k,p^{\alpha-1}}(n,1;\mathfrak{a},\infty). \tag{22}$$

In order to estimate (16), (20) and (22), we consider $\Delta_{k,M}(n,1;\mathfrak{a},\infty)$, where k and M are positive integers. Recall Weil's bound on Kloosterman's sums

$$\left|S(m,n;c)\right| \leqslant (m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}} d(c)$$

(see [17]) and we have

$$J_{k-1}(x) \ll_k x^{k-1}$$

for $x > 0$ (see (3) in [16, Section 2.3]). By using these estimates, we can see that

$$\sum_{\ell=1}^{\infty} \frac{S(n,1;\ell M)}{\ell M} J_{k-1}\left(\frac{4\pi\sqrt{n}}{\ell M}\right) \ll_k d(M) n^{\frac{k-1}{2}} M^{-k+\frac{1}{2}}$$

and

$$\sum_{\ell=1}^{\infty} \frac{S(n\overline{M},1;\ell)}{\ell\sqrt{M}} J_{k-1}\left(\frac{4\pi\sqrt{n}}{\ell\sqrt{M}}\right) \ll_k n^{\frac{k-1}{2}} M^{-\frac{k}{2}}.$$

From these estimates and Petersson's formula (10), we have

$$\Delta_{k,M}(n,1;\mathfrak{a},\infty) = \delta_{n,1}\delta_{\mathfrak{a},\infty} + \begin{cases} O_k(d(M) n^{\frac{k-1}{2}} M^{-k+\frac{1}{2}}), & \mathfrak{a} = \infty, \\ O_k(n^{\frac{k-1}{2}} M^{-\frac{k}{2}}), & \mathfrak{a} = 0. \end{cases}$$

By applying this to (16), (20) and (22), we obtain the following lemma.

Lemma 3. *Let k be an even integer satisfying $0 < k < 12$ or $k = 14$, p a prime number and α a positive integer. We denote the set of primitive forms of weight k for $\Gamma_0(p^\alpha)$ by $H_k(p^\alpha)$ and put*

$$\mathfrak{s}^{\mathfrak{a}}_{k,p^\alpha}(n) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in H_k(p^\alpha)} \frac{\lambda_{f,\mathfrak{a}}(n)}{\langle f, f\rangle_{p^\alpha}}$$

for $\mathfrak{a} = \infty, 0$. Then we have

$$\mathfrak{s}^{\mathfrak{a}}_{k,p^\alpha}(n) = \delta_{n,1}\delta_{\mathfrak{a},\infty}\bigl(1-c_p(\alpha)\bigr) + \begin{cases} O(n^{\frac{k-1}{2}}p^{-k+\frac{1}{2}}), & \alpha=1,\ \mathfrak{a}=\infty, \\ O(n^{\frac{k-1}{2}}p^{-\frac{k}{2}}), & \alpha=1,\ \mathfrak{a}=0, \\ O(n^{\frac{k-1}{2}}p^{-2}), & \alpha=2,\ \mathfrak{a}=0, \\ O((\alpha+1)n^{\frac{k-1}{2}}p^{-1-(\alpha-1)k+\frac{(\alpha-1)}{2}}), & \alpha\geqslant 2,\ \mathfrak{a}=\infty, \\ O(n^{\frac{k-1}{2}}p^{-1-(\alpha-1)\frac{k}{2}}), & \alpha\geqslant 3,\ \mathfrak{a}=0, \end{cases}$$

where the implied constants are absolute and

$$c_p(\alpha) = \begin{cases} 0, & \alpha=1, \\ p(p^2-1)^{-1}, & \alpha=2, \\ p^{-1}, & \alpha\geqslant 3. \end{cases}$$

Remark 8. The implied constants in Lemma 3 would a priori depend on k, if the limitation on k were removed, but not on any other parameter.

6. Proof of the theorem

By using Lemma 3, we can estimate the right-hand side of (15). We divide the right-hand side of (15) into three parts,

$$\mathfrak{S}_k\bigl(p^\alpha\bigr) = E_{p^\alpha}(X) - I_{1,p^\alpha}(X,Y) - I_{2,p^\alpha}(X,Y),$$

where

$$E_{p^\alpha}(X) = \sum_{n=1}^{\infty} \frac{\chi(n)e^{-(\frac{n}{X})^h}}{n^{\frac{1}{2}+iy}} \mathfrak{s}^{\infty}_{k,p^\alpha}(n),$$

$$I_{1,p^\alpha}(X,Y) = \frac{i^k C_\chi}{2\pi i}\int_{(c_1)} \left(\frac{4\pi^2}{q^2p^\alpha}\right)^{s+iy} \frac{X^s\Gamma(1+\frac{s}{h})}{s} G_k\left(s+\frac{1}{2}+iy\right)\sum_{n>Y}\frac{\overline{\chi}(n)\mathfrak{s}^0_{k,p^\alpha}(n)}{n^{\frac{1}{2}-s-iy}}\,ds,$$

$$I_{2,p^\alpha}(X,Y) = \frac{i^k C_\chi}{2\pi i}\int_{(c_1)} \left(\frac{4\pi^2}{q^2p^\alpha}\right)^{s+iy} \frac{X^s\Gamma(1+\frac{s}{h})}{s} G_k\left(s+\frac{1}{2}+iy\right)\sum_{n\leqslant Y}\frac{\overline{\chi}(n)\mathfrak{s}^0_{k,p^\alpha}(n)}{n^{\frac{1}{2}-s-iy}}\,ds.$$

Here $h=(k+1)/2$ and $Y\geqslant 1$. We can move the path of integration in I_{2,p^α} from the line $\sigma = -k/2-\varepsilon$ to the line $\sigma=\varepsilon$. By using (13) and (14), we have

$$I_{2,p^\alpha}(X,Y) = \frac{i^k C_\chi}{2\pi i}\int_{(\varepsilon)} \left(\frac{4\pi^2}{q^2p^\alpha}\right)^{s+iy} \frac{X^s\Gamma(1+\frac{s}{h})}{s} G_k\left(s+\frac{1}{2}+iy\right)\sum_{n\leqslant Y}\frac{\overline{\chi}(n)\mathfrak{s}^0_{k,p^\alpha}(n)}{n^{\frac{1}{2}-s-iy}}\,ds.$$

From Lemma 3 we can see that

$$\begin{aligned} E_{p^\alpha}(X) &= \sum_{n=1}^{\infty} \frac{\chi(n)e^{-(\frac{n}{X})^h}}{n^{\frac{1}{2}+iy}} \mathfrak{s}^{\infty}_{k,p^\alpha}(n) \\ &= e^{-(\frac{1}{X})^h}\bigl(1-c_p(\alpha)\bigr) + \begin{cases} O(p^{-k+\frac{1}{2}}\sum_{n=1}^\infty e^{-(\frac{n}{X})^h} n^{\frac{k}{2}-1}), & \alpha=1, \\ O((\alpha+1)p^{-1-(\alpha-1)(k-\frac{1}{2})}\sum_{n=1}^\infty e^{-(\frac{n}{X})^h} n^{\frac{k}{2}-1}), & \alpha\geqslant 2. \end{cases} \end{aligned}$$

And we have

$$\sum_{n=1}^{\infty} e^{-(\frac{n}{X})^h} n^{\frac{k}{2}-1} = \sum_{n\leqslant X} e^{-(\frac{n}{X})^h} n^{\frac{k}{2}-1} + \sum_{n>X} e^{-(\frac{n}{X})^h} n^{\frac{k}{2}-1} \ll X^{\frac{k}{2}}.$$

Then we obtain

$$E_{p^\alpha}(X) = e^{-(\frac{1}{X})^h}\bigl(1 - c_p(\alpha)\bigr) + \begin{cases} O(p^{-k+\frac{1}{2}} X^{\frac{k}{2}}), & \alpha = 1, \\ O((\alpha+1) p^{-1-(\alpha-1)(k-\frac{1}{2})} X^{\frac{k}{2}}), & \alpha \geqslant 2. \end{cases} \tag{23}$$

We estimate I_{1,p^α} and I_{2,p^α} for $Y > 1$. From Lemma 3 we have

$$\begin{aligned} I_{1,p^\alpha}(X,Y) &\ll \left(\frac{qp^{\frac{\alpha}{2}}}{2\pi}\right)^{k+2\varepsilon} X^{-\frac{k}{2}-\varepsilon} \sum_{n>Y} \frac{1}{n^{\frac{1}{2}+\frac{k}{2}+\varepsilon}} \times \begin{cases} n^{\frac{k-1}{2}} p^{-\frac{k}{2}}, & \alpha = 1, \\ n^{\frac{k-1}{2}} p^{-2}, & \alpha = 2, \\ n^{\frac{k-1}{2}} p^{-\frac{k(\alpha-1)}{2}-1}, & \alpha \geqslant 3 \end{cases} \\ &\quad \times \int_{(c_1)} \bigl|G_k(s+1/2+iy)\bigr|\bigl|\Gamma(1+s/h)s^{-1}\bigr|\,|dt| \\ &\ll q^{k+2\varepsilon} p^{\alpha\varepsilon} X^{-\frac{k}{2}-\varepsilon} Y^{-\varepsilon} \times \begin{cases} 1, & \alpha = 1, \\ p^{k-2}, & \alpha = 2, \\ p^{k/2-1}, & \alpha \geqslant 3 \end{cases} \\ &\quad \times \int_{(c_1)} \bigl|G_k(s+1/2+iy)\bigr|\bigl|\Gamma(1+s/h)s^{-1}\bigr|\,|ds|, \\ I_{2,p^\alpha}(X,Y) &\ll \left(\frac{qp^{\frac{\alpha}{2}}}{2\pi}\right)^{2\varepsilon} X^{-\varepsilon} \sum_{n\leqslant Y} \frac{1}{n^{\frac{1}{2}+\varepsilon}} \times \begin{cases} n^{\frac{k-1}{2}} p^{-\frac{k}{2}}, & \alpha = 1, \\ n^{\frac{k-1}{2}} p^{-2}, & \alpha = 2, \\ n^{\frac{k-1}{2}} p^{-\frac{k(\alpha-1)}{2}-1}, & \alpha \geqslant 3 \end{cases} \\ &\quad \times \int_{(\varepsilon)} \bigl|G_k(s+1/2+iy)\bigr|\bigl|\Gamma(1+s/h)s^{-1}\bigr|\,|ds| \\ &\ll q^{2\varepsilon} p^{\alpha(-\frac{k}{2}+\varepsilon)} X^{-\varepsilon} Y^{\frac{k}{2}-\varepsilon} \times \begin{cases} 1, & \alpha = 1, \\ p^{k-2}, & \alpha = 2, \\ p^{k/2-1}, & \alpha \geqslant 3 \end{cases} \\ &\quad \times \int_{(\varepsilon)} \bigl|G_k(s+1/2+iy)\bigr|\bigl|\Gamma(1+s/h)s^{-1}\bigr|\,|ds|. \end{aligned}$$

From (14) we have

$$\begin{aligned} &\int_{(c_1)} \bigl|G_k(s+1/2+iy)\bigr|\bigl|\Gamma(1+s/h)s^{-1}\bigr|\,|ds| \\ &\quad \ll \int_{-\infty}^{\infty} \bigl(1+|t+y|\bigr)^{k+2\varepsilon} \frac{|\Gamma(\frac{1-2\varepsilon}{2h} + i\frac{t}{h})|}{|\frac{k}{2}+\varepsilon+i\frac{t}{h}|}\,|dt| \end{aligned}$$

$$\ll \int_{-\infty}^{\infty} \left(1+|t+y|\right)^{k+2\varepsilon} \left| \Gamma\left(1+\frac{1-2\varepsilon}{2h}+i\frac{t}{h}\right)\right| |dt|. \tag{24}$$

To estimate this integral, we use (13) and

$$\left|\log \Gamma(s)\right| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\left(1+O\left(|t|^{-1}\right)\right) \tag{25}$$

for $-\pi+\delta < \arg s < \pi-\delta$, $|\frac{\sigma}{it}|<1$ and $|\sigma|<C$, where C is an absolute constant. We have (25) by using Stirling's formula (13). On the right-hand side of (24), we apply (13) to the integral (24) for $|t| \leqslant 2h$ and (25) to the integral (24) for $|t|>2h$. Then we have

$$\int_{(c_1)} \left|G_k(s+1/2+iy)\right|\left|\Gamma(1+s/h)s^{-1}\right| |ds| \ll \left(1+|y|\right)^{k+2\varepsilon}.$$

By the same type of calculations we have

$$\begin{aligned}\int_{(\varepsilon)} \left|G_k(s+1/2+iy)\right|\left|\Gamma(1+s/h)s^{-1}\right| |ds| &\ll \int_{-\infty}^{\infty} \left(1+|t+y|\right)^{2\varepsilon}\left|\Gamma\left(1-\frac{\varepsilon}{h}+i\frac{t}{h}\right)\right| |dt| \\ &\ll \left(1+|y|\right)^{2\varepsilon}.\end{aligned}$$

Therefore we obtain

$$I_{1,p^\alpha}(X,Y) \ll q^{k+2\varepsilon}p^{\alpha\varepsilon}X^{-\frac{k}{2}-\varepsilon}Y^{-\varepsilon}\left(1+|y|\right)^{k+2\varepsilon} \times \begin{cases} 1, & \alpha=1, \\ p^{k-2}, & \alpha=2, \\ p^{k/2-1}, & \alpha \geqslant 3 \end{cases}$$

and

$$I_{2,p^\alpha}(X,Y) \ll q^{2\varepsilon}p^{\alpha(-\frac{k}{2}+\varepsilon)}X^{-\varepsilon}Y^{\frac{k}{2}-\varepsilon}\left(1+|y|\right)^{2\varepsilon} \times \begin{cases} 1, & \alpha=1, \\ p^{k-2}, & \alpha=2, \\ p^{k/2-1}, & \alpha \geqslant 3. \end{cases}$$

We put $Y = p^\alpha X^{-1}q^2(1+|y|)^2$ and obtain

$$I_{1,p^\alpha}(X,Y)+I_{2,p^\alpha}(X,Y) \ll X^{-\frac{k}{2}}q^k\left(1+|y|\right)^k \times \begin{cases} 1, & \alpha=1, \\ p^{k-2}, & \alpha=2, \\ p^{\frac{k}{2}-1}, & \alpha \geqslant 3. \end{cases} \tag{26}$$

From the estimates (23) and (26) we have

$$\begin{aligned}&E_{p^\alpha}(X)+I_{1,p^\alpha}(X,Y)+I_{2,p^\alpha}(X,Y) \\ &\quad = e^{-(\frac{1}{X})^h}\left(1-c_p(\alpha)\right) + \begin{cases} O(p^{-k+\frac{1}{2}}X^{\frac{k}{2}}+X^{-\frac{k}{2}}q^k(1+|y|)^k), & \alpha=1, \\ O(p^{-1/2-k}X^{\frac{k}{2}}+X^{-\frac{k}{2}}q^k(1+|y|)^kp^{k-2}), & \alpha=2, \\ O((\alpha+1)p^{-1-(\alpha-1)k+\frac{(\alpha-1)}{2}}X^{\frac{k}{2}}+X^{-\frac{k}{2}}q^k(1+|y|)^kp^{\frac{k}{2}-1}), & \alpha \geqslant 3. \end{cases}\end{aligned}$$

We put

$$X = \begin{cases} p^{1-\frac{1}{2k}} q(1+|y|), & \alpha = 1, \\ p^{2-\frac{3}{2k}} q(1+|y|), & \alpha = 2, \\ p^{\alpha-\frac{1}{2}-\frac{\alpha-1}{2k}} q(1+|y|), & \alpha \geqslant 3, \end{cases}$$

so that $Y > 1$, as needed, and we obtain

$$\begin{aligned} \mathfrak{S}_k\bigl(p^\alpha\bigr) &= E_{p^\alpha}(X) + I_{1,p^\alpha}(X,Y) + I_{2,p^\alpha}(X,Y) \\ &= e^{-(\frac{1}{X})^h}\bigl(1-c_p(\alpha)\bigr) + \begin{cases} O(p^{-\frac{k}{2}+\frac{1}{4}} q^{\frac{k}{2}} (1+|y|)^{\frac{k}{2}}), & \alpha = 1, \\ O(p^{-\frac{5}{4}} q^{\frac{k}{2}} (1+|y|)^{\frac{k}{2}}), & \alpha = 2, \\ O((\alpha+1) p^{-1-\frac{\alpha-1}{2}k+\frac{\alpha-1}{4}+\frac{k}{4}} q^{\frac{k}{2}} (1+|y|)^{\frac{k}{2}}), & \alpha \geqslant 3. \end{cases} \end{aligned}$$

From (12) we see

$$\begin{aligned} e^{-(\frac{1}{X})^h} &= \frac{1}{2\pi i} \int_{(1)} X^s \frac{\Gamma(1+\frac{s}{h})}{s}\, ds = 1 + \frac{1}{2\pi i} \int_{(-\frac{k}{2})} X^s \frac{\Gamma(1+\frac{s}{h})}{s}\, ds \\ &= 1 + O\bigl(X^{-\frac{k}{2}}\bigr) \end{aligned}$$

and we obtain Theorem 1.

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