Journal of Number Theory 131 (2011) 1734–1742

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On a divisor problem related to the Epstein zeta-function, II ☆

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ARTICLE INFO

Article history:
Received 6 November 2010
Revised 2 March 2011
Accepted 11 March 2011
Available online 14 May 2011
Communicated by K. Soundararajan

MSC:
11F30
11F11
11F66

Keywords:
Epstein zeta-function
Divisor problem
Modular form

ABSTRACT

Recently by using the theory of modular forms and the Riemann zeta-function, Lü improved the estimates for the error term in a divisor problem related to the Epstein zeta-function established by Sankaranarayanan. In this short note, we are able to further sharpen some results of Sankaranarayanan and of Lü, and to establish corresponding Ω-estimates.

1. Introduction

For a positive definite quadratic form $Q(\mathbf{y}) = Q(y_1, \dots, y_\ell)$ in $\ell \geqslant 2$ variables with integral coefficients, we can write it in Siegel's notation as

☆ The work of Guangshi Lü is supported in part by key project of the National Natural Science Foundation of China (Grant No. 11031004), Shandong Province Natural Science Foundation (Grant No. ZR2009AM007), and NCET. Wenguang Zhai is supported by the National Natural Science Foundation of China (Grant No. 10771127) and National Science Foundation of Beijing (Grant No. 1112010).

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.03.003

$$Q(\mathbf{y}) = \frac{1}{2}\mathbf{A}[\mathbf{y}] = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y} = \sum_{i<j} a_{ij} y_i y_j + \frac{1}{2}\sum_i a_{ii} y_i^2,$$

where $\mathbf{y}^{\mathrm{t}}$ is the transpose of $\mathbf{y}$, and the matrix $\mathbf{A} = (a_{ij})$ has integral entries which are even on the diagonal, i.e., $a_{ii} \equiv 0 \pmod 2$ for $0 \leqslant i \leqslant \ell$. Then the corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$Z_Q(s) := \sum_{\substack{y_1\in\mathbb{Z} \\ (y_1,\ldots,y_\ell)\neq(0,\ldots,0)}} \cdots \sum_{y_\ell\in\mathbb{Z}} Q(y_1,\ldots,y_\ell)^{-s} \tag{1.1}$$

for $\Re e\, s > \ell/2$. We can also rewrite it, in the same region, as

$$Z_Q(s) = \sum_{n\geqslant 1} a_n n^{-s},$$

where a_n is the number of the solutions of the equation $Q(\mathbf{y}) = n$ with $\mathbf{y} \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies the functional equation of Riemann type

$$\left(d^{1/\ell}/2\pi\right)^s \Gamma(s) Z_Q(s) = \left(d^{1-1/\ell}/2\pi\right)^{\ell/2-s} \Gamma(\ell/2 - s) Z_{\overline{Q}}(\ell/2 - s) \quad (s \in \mathbb{C}),$$

where d is the discriminant of Q and $\overline{Q}(\mathbf{y}) := \frac{1}{2}\mathbf{y}^{\mathrm{t}}(d\mathbf{A}^{-1})\mathbf{y}$ (cf. [9]).

If we write for any integer $k \geqslant 1$,

$$Z_Q(s)^k = \sum_{n\geqslant 1} a_k(n) n^{-s},$$

then

$$a_k(n) = \sum_{n_1\cdots n_k = n} a_{n_1}\cdots a_{n_k}.$$

In particular $a_1(n) = a_n$. It seems interesting to study the asymptotic behavior of the sum $\sum_{n\leqslant x} a_k(n)$. It is easy to show that its main term is

$$\operatorname*{Res}_{s=\ell/2}\left(Z_Q(s)^k x^s s^{-1}\right) = x^{\ell/2} P_k(\log x),$$

where $P_k(t)$ is a polynomial in t of degree $k-1$. Then the real hard work is to study the error term

$$\Delta_k^*(Q, x) := \sum_{n\leqslant x} a_k(n) - x^{\ell/2} P_k(\log x). \tag{1.2}$$

In 1912, Landau [7] proved that for $\ell = 2$, $\Delta_1^*(Q, x) \ll x^{1/3+\varepsilon}$, where and throughout this paper ε denotes an arbitrarily small positive constant. Landau's method can also be applied to treat the general case. In fact his method implies that for $k \geqslant 1$ and $\ell \geqslant 2$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2 - \ell/(k\ell+1)+\varepsilon}.$$

Later Chandrasekharan and Narasimhan [1] were able to delete the ε in the exponent of x. In [9], Sankaranarayanan improved these classical results by showing that for $k \geqslant 2$ and $\ell \geqslant 3$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon}. \tag{1.3}$$

Recently inspired by Iwaniec's book [5], Lü [8] was able to improve (1.3) for the quadratic forms of level one (see [5, Chapter 11]). These quadratic forms are defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{A}[\mathbf{y}]$ with $\operatorname{diag}(\mathbf{A}) = \operatorname{diag}(\mathbf{A}^{-1}) \equiv 0 \pmod 2$, where $\operatorname{diag}(\mathbf{A})$ denotes the set of entries on the diagonal of the matrix $\mathbf{A}$. Moreover we have that $\det(\mathbf{A}) = 1$, $\mathbf{A}$ is equivalent to $\mathbf{A}^{-1}$, and the number of variables satisfies $\ell \equiv 0 \pmod 8$. Denote by $\mathcal{Q}_\ell$ the set of quadratic forms of level one with ℓ variables. For $Q \in \mathcal{Q}_\ell$, we have (see [5, (11.32)] or [8, Lemma 2.1])

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geqslant 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \qquad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the nth Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Thus

$$Z_Q(s) = A_\ell \zeta(s - \ell/2 + 1)\zeta(s) + L(s, f) \quad (\Re e\, s > \ell/2), \tag{1.4}$$

where $L(s, f)$ is the Hecke L-function associated with $f(z, Q)$. According to the well known Deligne's work [2], we have

$$\left|a_f(n, Q)\right| \leqslant n^{(\ell/2-1)/2}\tau(n), \tag{1.5}$$

where $\tau(n)$ is the divisor function. With the help of these properties, Lü proved, by complex integration method, a better estimate than Sankaranarayanan's (1.3) for all $k \geqslant 3$ and $8 \mid \ell$. For $r \geqslant 0$, the r-dimensional divisor function $\tau_r(n)$ is defined by

$$\zeta(s)^r = \sum_{n \geqslant 1} \tau_r(n) n^{-s} \quad (\Re e\, s > 1).$$

The r-dimensional divisor problem concerns the estimate of the error term

$$\Delta_r(x) := \sum_{n \leqslant x} \tau_r(n) - \operatorname*{Res}_{s=1}\left(\zeta(s)^r x^s s^{-1}\right) = \sum_{n \leqslant x} \tau_r(n) - x G_r(\log x), \tag{1.6}$$

where $G_r(t)$ is a polynomial of degree $r-1$ if $r \geqslant 1$ and $G_0(t) \equiv 0$. It is known that

$$\Delta_r(x) \ll x^{\theta_r+\varepsilon} \quad (x \geqslant 2) \tag{1.7}$$

where

$$\theta_0 = 0, \qquad \theta_1 = 0, \qquad \theta_2 = 131/416, \qquad \theta_3 = 43/96 \tag{1.8}$$

and

$$\theta_r = \begin{cases} (3r-4)/(4r), & \text{if } 4 \leqslant r \leqslant 8, \\ 35/54, & \text{if } r = 9, \\ 41/60, & \text{if } r = 10, \\ 7/10, & \text{if } r = 11, \\ (r-2)/(r+2), & \text{if } 12 \leqslant r \leqslant 25, \\ (r-1)/(r+4), & \text{if } 26 \leqslant r \leqslant 50, \\ (31r-98)/(32r), & \text{if } 51 \leqslant r \leqslant 57, \\ (7r-34)/(7r), & \text{if } r \geqslant 58. \end{cases} \tag{1.9}$$

(The case of $r = 0, 1$ is trivial. See [3] for $r = 2$, [6] for $r = 3$ and [4, Theorem 13.2] for $r \geqslant 4$.) Lü's result (see [8, Theorem 1.2]) can be stated as follows

$$\Delta_k^*(Q, x) \ll \begin{cases} x^{\ell/2-1/2+\varepsilon}, & \text{if } k = 3, \\ x^{\ell/2-1+\theta_k+\varepsilon}, & \text{if } k \geqslant 4. \end{cases} \tag{1.10}$$

In this short note, we can further improve Sankaranarayanan's (1.3) with $k = 2$ and Lü's (1.10) with $k = 3$.

Theorem 1. *Let $k \geqslant 2$ and $8 \mid \ell$. Then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where θ_k is the exponent in (1.7).

For comparison, we note that

$$\ell/2 - 1 + \theta_k = \begin{cases} \ell/2 - 1/2 - 0.185\ldots & \text{if } k = 2, \\ \ell/2 - 1/2 - 0.052\ldots & \text{if } k = 3, \end{cases}$$

which are better than (1.3) with $k = 2$ and (1.10), respectively.

For $k = 2$ or 3, we also can establish Ω-type result.

Theorem 2. *Let $2 \leqslant k \leqslant 8$ and $8 \mid \ell$. If there is a positive constant δ such that*

$$\theta_r \leqslant (k-1)/(2k) - \delta \quad (0 \leqslant r \leqslant k-1), \tag{1.11}$$

then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ and $\varepsilon > 0$, we have

$$\Delta_k^*(Q, x) = \Omega\left(x^{\ell/2-1+(k-1)/(2k)} (\log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k-\varepsilon}\right) \tag{1.12}$$

where $\beta_k := (k^{(2k)/(k+1)} - 1)(k+1)/(2k)$ and $\gamma_k := (3k-1)/(4k)$.

In particular (1.12) *holds unconditionally for $k = 2$ or* 3.

Our method is different from [8]. First we shall establish relations between $\Delta_k(x)$ and $\Delta_k^*(Q, x)$ and then deduce Theorems 1 and 2 from known O-type and Ω-type estimates for $\Delta_k(x)$.

2. Preliminary lemmas

This section is devoted to establish three preliminary lemmas, which will be needed in the proof of Theorems 1 and 2.

Lemma 2.1. *For any $\varepsilon > 0$, we have*

$$\int_1^x \Delta_r(t)\,\mathrm{d}t \ll_{r,\varepsilon} x^{1+\delta_r+\varepsilon} \quad (x \geqslant 1),$$

where

$$\delta_r := \begin{cases} 1/2 - 1/r, & \text{if } r = 2, 4, 6, 8, \\ 1/2 - 1/(r+1), & \text{if } r = 1, 3, 5, 7. \end{cases} \tag{2.1}$$

Proof. By Perron's formula [11, Theorem II.2.3], we obtain, with $b := 1 + 1/\log x$,

$$\int_0^x \Delta_r(u)\,\mathrm{d}u = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s - \int_0^x uG_r(\log u)\,\mathrm{d}u, \tag{2.2}$$

where $b := 1 + 1/\log x$ and $F_r(s;x) := \zeta(s)^r x^{s+1}/\{s(s+1)\}$.

Let $\max\{1 - 6/r, 0\} < a < 1$. By using the classical estimate

$$\zeta(s) \ll \big(|t| + 2\big)^{\max\{(1-\sigma)/3,0\}} \log\big(|t| + 2\big),$$

we deduce that for all $\varepsilon > 0$ and $T > 0$,

$$\int_{a\leqslant\sigma\leqslant b,\ |\tau|=T} \big|F_r(s;x)\big||\mathrm{d}s| \ll \big(x^2T^{-2} + x^{1+a}T^{-2+\max\{(1-a)r/3,0\}}\big)(\log T)^r,$$

and

$$\int_{\sigma=b,\ |t|\geqslant T} \big|F_r(s;x)\big||\mathrm{d}s| \ll x^2T^{-1}(\log T)^r.$$

Using the preceding estimates and shifting the line of integration from $\sigma = b$ to $\sigma = a$, the residue theorem implies that

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s &= \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} F_r(s;x)\,\mathrm{d}s + O\left(\frac{x^2}{T}(\log T)^r\right) \\
&= \int_0^x uG_r(u)\,\mathrm{d}u + \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}T}^{a+\mathrm{i}T} F_r(s;x)\,\mathrm{d}s \\
&\quad + O\left(\frac{x^2}{T}(\log T)^r + \frac{x^{1+a}}{T^{2-\max\{(1-a)r/3,0\}}}(\log T)^r\right),
\end{aligned}$$

where we have used the relation

$$\operatorname*{Res}_{s=1}\big(F_r(s;x)\big) = \int_0^x uG_r(u)\,\mathrm{d}u.$$

Making $T \to \infty$ and inserting the obtained formula into (2.2), we find that

$$\begin{aligned}\int_1^x \Delta_r(u)\,\mathrm{d}u &= \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s \\ &\ll x^{1+a}\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^r}{(|t|+1)^2}\,\mathrm{d}t.\end{aligned}$$

When $r = 2, 4, 6, 8$, the last integral is convergent for any $a > 1/2 - 1/r \geqslant \max\{1 - 6/r, 0\}$ (see [4, Lemma 13.1 and Theorem 13.4]). For $r = 2k - 1$ $(1 \leqslant k \leqslant 4)$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^r}{(|t|+1)^2}\,\mathrm{d}t &\leqslant \left\{\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^{2(k-1)}}{(|t|+1)^2}\,\mathrm{d}t\right\}^{1/2}\left\{\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^{2k}}{(|t|+1)^2}\,\mathrm{d}t\right\}^{1/2} \\ &< \infty\end{aligned}$$

provided $a > 1/2 - 1/(2k) = 1/2 - 1/(r+1) \geqslant \max\{1 - 6/r, 0\}$. This completes the proof. □

Lemma 2.2. *For $r \geqslant 0$, we have*

$$\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} = x^{\ell/2}G_r^*(\log x) + x^{\ell/2-1}\Delta_r(x) + O\big(x^{\ell/2-1+\delta_r}\big), \tag{2.3}$$

where $G_r^*(t)$ is a polynomial of degree $r-1$ with the convention that $G_0^*(t) \equiv 0$ and the constant $\delta_r \geqslant 0$ is given by (2.1)*. In particular*

$$\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} = x^{\ell/2}G_r^*(\log x) + O\big(x^{\ell/2-1+\theta_r}\big). \tag{2.4}$$

Proof. With the help of (1.6) and Lemma 2.1, a simple partial summation yields

$$\begin{aligned}\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} &= \int_1^x t^{\ell/2-1}\big(tG_r(\log t)\big)'\,\mathrm{d}t + \int_{1-}^x t^{\ell/2-1}\,\mathrm{d}\Delta_r(t) \\ &= x^{\ell/2}G_r^*(\log x) + x^{\ell/2-1}\Delta_r(x) + O\big(x^{\ell/2-1+\delta_r}\big).\end{aligned}$$

This completes the proof. □

In order to state our third lemma, it is necessary to introduce some notation.

By (1.4), we can write, for $\Re e\, s > \ell/2$,

$$Z_Q(s)^k = \sum_{0\leqslant r\leqslant k} A_\ell^r C_k^r \zeta(s)^r L(s,f)^{k-r}\zeta(s-\ell/2+1)^r,$$

$$\zeta(s-\ell/2+1)^k = \sum_{0\leqslant r\leqslant k} A_\ell^{-k} C_k^r (-1)^{k-r}\zeta(s)^{-k} L(s,f)^{k-r} Z_Q(s)^r.$$

These imply that

$$a_k(n) = \sum_{0\leqslant r\leqslant k} A_\ell^r C_k^r \sum_{dm=n} b_{k,r}(d)\tau_r(m)m^{\ell/2-1}, \tag{2.5}$$

$$\tau_k(n)n^{\ell/2-1} = \sum_{0\leqslant r\leqslant k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{dm=n} c_{k,r}(d)a_r(m), \tag{2.6}$$

where $b_{k,r}$ and $c_{k,r}$ are defined by the relations

$$\zeta(s)^r L(s,f)^{k-r} = \sum_{n\geqslant 1} b_{k,r}(n)n^{-s}, \qquad \zeta(s)^{-k} L(s,f)^{k-r} = \sum_{n\geqslant 1} c_{k,r}(n)n^{-s},$$

for $\Re e\, s > \ell/2$.

Lemma 2.3. *Let $j\geqslant 0$, $k\geqslant 2$, $0\leqslant r\leqslant k$, $8 \mid \ell$ and $\theta > (\ell+2)/4$. Then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ and $d_{k,r} = b_{k,r}$ or $c_{k,r}$, we have*

$$\sum_{n\leqslant x} \frac{|d_{k,r}(n)|}{n^\theta} \ll_{j,\ell,\theta} 1 \quad (x\geqslant 2), \tag{2.7}$$

$$\sum_{n\leqslant x} \frac{d_{k,r}(n)(\log n)^j}{n^\theta} = C_f(j,k,r,\theta) + O\left(x^{-\theta+(\ell+2)/4+\varepsilon}\right) \quad (x\geqslant 2), \tag{2.8}$$

where $C_f(j,k,r,\theta)$ is a constant.

Proof. By the definition of $b_{k,r}$ and $c_{k,r}$, we have

$$b_{k,r}(n) = \sum_{d_1\cdots d_r m_1\cdots m_{k-r}=n} a_f(Q,m_1)\cdots a_f(Q,m_{k-r}),$$

$$c_{k,r}(n) = \sum_{d_1\cdots d_k m_1\cdots m_{k-r}=n} \mu(d_1)\cdots\mu(d_k)a_f(Q,m_1)\cdots a_f(Q,m_{k-r}).$$

We treat only the case of $b_{k,r}$ and the latter is completely similar. With the help of the Deligne inequality (1.5), we have

$$\begin{aligned}\sum_{n\leqslant x}\left|b_{k,r}(n)\right| &\leqslant \sum_{d\leqslant x}\tau_r(d)\sum_{m\leqslant x/d}\tau_{2(k-r)}(m)m^{(\ell-2)/4}\\ &\leqslant \sum_{d\leqslant x}\tau_r(d)(x/d)^{\ell/4+1/2}(\log x)^{2k-2r-1}\\ &\ll_{j,\ell} x^{(\ell+2)/4}(\log x)^{2k-2r-1} \quad (0\leqslant r\leqslant k).\end{aligned}$$

From this, a simple partial integration allows us to deduce (2.7) and (2.8). □

3. Proof of Theorem 1

By (2.5), (2.4) of Lemma 2.2 and (2.7) of Lemma 2.3, it follows that

$$\sum_{n\leqslant x} a_k(n) = x^{\ell/2} \sum_{0\leqslant r\leqslant k} C_k^r A_\ell^r \sum_{d\leqslant x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*\big(\log(x/d)\big) + O\big(x^{\ell/2-1+\theta_k+\varepsilon}\big).$$

Since $\ell/2 > (\ell+2)/4$, (2.8) of Lemma 2.3 implies that

$$\sum_{0\leqslant r\leqslant k} C_k^r A_\ell^r \sum_{d\leqslant x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*\big(\log(x/d)\big) = P_k(\log x) + O\big(x^{1/2-\ell/4+\varepsilon}\big).$$

Inserting it into the preceding formula, we get the required result. $\square$

4. Proof of Theorem 2

From (2.6) and (1.2), we can deduce that

$$\sum_{n\leqslant x} \tau_k(n) n^{\ell/2-1} = x^{\ell/2} G_k^*(\log x) + O\big(x^{(\ell+2)/4+\varepsilon}\big) + \sum_{0\leqslant r\leqslant k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d) \Delta_r^*(Q, x/d),$$

where we have used the following estimate

$$\sum_{0\leqslant r\leqslant k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d)(x/d)^{\ell/2} P_r\big(\log(x/d)\big) = x^{\ell/2} G_k^*(\log x) + O\big(x^{(\ell+2)/4+\varepsilon}\big).$$

Comparing with (2.3) of Lemma 2.2 yields

$$x^{\ell/2-1}\Delta_k(x) = \sum_{0\leqslant r\leqslant k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d)\Delta_r^*(Q, x/d) + O\big(x^{\ell/2-1+\delta_k}\big).$$

Under hypothesis (1.11), by (2.5), (2.3) of Lemma 2.2 and (2.7) of Lemma 2.3 we have

$$\begin{aligned}\Delta_r^*(Q, x) &\ll x^{\ell/2-1+r/\{2(r+1)\}-\delta+\varepsilon} \\ &\ll x^{\ell/2-1+(k-1)/(2k)-\delta/2}\end{aligned}$$

for $0 \leqslant r \leqslant k-1$. Inserting into the preceding formula and using (2.7), we can deduce

$$x^{\ell/2-1}\Delta_k(x) = A_\ell^{-k} \sum_{d\leqslant x} c_{k,k}(d)\Delta_k^*(Q, x/d) + O\big(x^{\ell/2-1+(k-1)/(2k)-\delta/2}\big). \tag{4.1}$$

On the other hand, according to Soundararajan [10], we have, for any $k \geqslant 2$,

$$\Delta_k(x) = \Omega\big((x\log x)^{(k-1)/(2k)}(\log_2 x)^{\beta_k}(\log_3 x)^{-\gamma_k}\big). \tag{4.2}$$

Now on noting (2.7) of Lemma 2.3, the first assertion of Theorem 2 follows from (4.1) and (4.2).

Finally in view of (1.8), it is easy check that the hypothesis (1.11) is satisfied when $k = 2$ or 3. Therefore (1.12) holds unconditionally for these two values of k.

This completes the proof of Theorem 2. □

Acknowledgment

The authors are grateful to the referee for carefully reading the manuscript and detailed comments.

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