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On the cohomology of Witt vectors of p-adic integers and a conjecture of Hesselholt

Amit Hogadi, Supriya Pisolkar *

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

A R T I C L E I N F O

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A B S T R A C T

Let K be a complete discrete valued field of characteristic zero with residue field k_K of characteristic $p > 0$. Let L/K be a finite Galois extension with Galois group G such that the induced extension of residue fields k_L/k_K is separable. Hesselholt (2004) [2] conjectured that the pro-abelian group $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ is zero, where $\mathcal{O}_L$ is the ring of integers of L and $W(\mathcal{O}_L)$ is the ring of Witt vectors in $\mathcal{O}_L$ w.r.t. the prime p. He partially proved this conjecture for a large class of extensions. In this paper, we prove Hesselholt's conjecture for all Galois extensions.

1. Introduction

Let p be a prime number and K be a complete discrete valued field of characteristic zero with residue field k_K of characteristic p. Let L/K be a finite Galois extension with Galois group G, such that the induced extension of residue fields k_K/k_L is separable. Let $W_n(\mathcal{O}_L)$ denote the ring of Witt vectors of length n in $\mathcal{O}_L$. In [2], Hesselholt conjectured that the pro-abelian group $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ vanishes, which means that for every integer n, there exists $m > n$ such that the map $H^1(G, W_m(\mathcal{O}_L)) \to H^1(G, W_n(\mathcal{O}_L))$ is zero. As explained in [2], this can be viewed as an analogue of Hilbert theorem 90 for the ring of Witt vectors $W(\mathcal{O}_L)$.

In order to prove the above conjecture, one easily reduces to the case where L/K is a totally ramified Galois extension of degree p (see Lemma 3.1). For such an extension, let $s = s(L/K)$ be the ramification break (see [4, IV, Remark 1]) in the ramification filtration of G. This is the largest integer such that G acts trivially on $\mathcal{O}_L/\mathfrak{P}_L^{s+1}$, where $\mathfrak{P}_L$ is the maximal ideal of $\mathcal{O}_L$. Hesselholt proved his

* Corresponding author.
E-mail addresses: amit@math.tifr.res.in (A. Hogadi), supriya@math.tifr.res.in (S. Pisolkar).

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conjecture for extensions with $s > e_K/p - 1$. The following theorem, which is the main result of this paper, proves Hesselholt's conjecture for all Galois extensions.

Theorem 1.1. *Let L/K be a finite Galois extension of complete discrete valued fields of mixed characteristic with Galois group G, such that the induced extension of residue fields k_L/k_K is separable. Then the pro-abelian group $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ is zero.*

As a corollary of the above result we will show the following.

Corollary 1.2. *Let L/K be as in Theorem* 1.1. *Then*

$$H^1\big(G, W(\mathcal{O}_L)\big) = \varprojlim H^1\big(G, W_n(\mathcal{O}_L)\big) = 0$$

We thank the referee for pointing out to us the difference between the vanishing of the pro-abelian group $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ and the vanishing of $\varprojlim H^1(G, W_n(\mathcal{O}_L))$. The former means that for every $n \in \mathbb{N}$, there exists an integer $m > n$ such that the map

$$H^1\big(G, W_m(\mathcal{O}_L)\big) \to H^1\big(G, W_n(\mathcal{O}_L)\big)$$

is zero and is stronger than saying that $\varprojlim H^1(G, W_n(\mathcal{O}_L))$ vanishes. For generalities on pro-abelian groups, we refer the reader to [3, Section 1].

Although the proof of Hesselholt's result does not generalize (for instance due to use of [2, 2.2]), our proof, which is based on an observation on addition in the ring of Witt vectors (see Lemma 2.2) relies on several ideas developed in [2]. One of these ideas which we use is the following.

Lemma 1.3. *(See [2, 1.1].) Let L/K be as in Theorem* 1.1. *Let $m \geqslant 1$ be an integer and suppose that the induced map*

$$H^1\big(G, W_{m+n}(\mathcal{O}_L)\big) \to H^1\big(G, W_n(\mathcal{O}_L)\big)$$

is zero for $n = 1$. Then the same is true, for all $n \geqslant 1$. In particular the pro-abelian group $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ vanishes.

Thus, in view of the above lemma and Lemma 3.1, to prove Theorem 1.1, it is enough to prove the following.

Theorem 1.4. *Let K be as above and L/K be degree-p totally ramified cyclic extension with Galois group G. Then there exists a positive integer $m \in \mathbb{N}$ such that the homomorphism $H^1(G, W_m(\mathcal{O}_L)) \to H^1(G, \mathcal{O}_L)$ is equal to zero.*

2. Remarks on addition of Witt vectors

The main observation of this section is Lemma 2.2, which lies at the heart of the proof of Theorem 1.1. We first recall from [4, II] how addition of Witt vectors is defined. For every positive integer n, define ghost polynomials $w_n \in \mathbb{Z}[X_0, \ldots, X_n]$ by

$$w_n(X_0, \ldots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + p^2X_2^{p^{n-2}} + \cdots + p^nX_n$$

One now defines addition of Witt vectors (thanks to Theorem 2.1) in such a way that if

$$(X_0, \ldots, X_n) + (Y_0, \ldots, Y_n) = (Z_0, \ldots, Z_n) \tag{1}$$

then

$$w_i(X_0, \dots, X_i) + w_i(Y_0, \dots, Y_i) = w_i(Z_0, \dots, Z_i) \quad \forall 0 \leqslant i \leqslant n$$

Theorem 2.1. *(See [4, II, §6].) For every positive integer n, there exists a unique $\phi_n \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ such that*

$$w_n(X_0, \dots, X_n) + w_n(Y_0, \dots, Y_n) = w_n(\phi_1, \dots, \phi_n)$$

In other words, in Eq. (1) above

$$Z_i = \phi_i(X_0, \dots, X_i, Y_0, \dots, Y_i)$$

Since ϕ_i's are polynomials with integral coefficients, the expression makes sense in all characteristics. We now consider addition of p-many Witt vectors. Let

$$(x_{10}, \dots, x_{1n}) + \dots + (x_{p0}, \dots, x_{pn}) = (z_0, \dots, z_n)$$

By the above discussion, for every $0 \leqslant i \leqslant n$, there exist polynomials in $p(i+1)$ variables, $g_i \in \mathbb{Z}[X_{10}, \dots, X_{1i}, \dots, X_{p0}, \dots, X_{pi}]$ such that

$$z_i = g_i(x_{10}, \dots, x_{1i}, \dots, x_{p0}, \dots, x_{pi})$$

The following observation is about the nature of these polynomials. In order to state this observations, without loss of generality we work over the polynomial ring

$$R_n = \mathbb{Z}\big[\{x_{ij} \mid 1 \leqslant i \leqslant p,\ 0 \leqslant j \leqslant n-1\}\big].$$

Lemma 2.2. *Let p be a prime number, and let R_n be the polynomial ring as above. For every $i \leqslant n$, the polynomial ring R_i is a subring of R_n. Let $\underline{x}_i = (x_{i0}, \dots, x_{i(n-1)}) \in W_n(R_n)$ for $1 \leqslant i \leqslant p$.*

$$(z_0, z_2, \dots, z_{(n-1)}) := \sum_{i=1}^{p} (x_{i0}, x_{i1}, \dots, x_{i(n-1)})$$

(1) *For all $0 \leqslant \ell \leqslant n-1$ there exists a polynomial $f_\ell \in R_\ell$ such that*

$$z_\ell = \sum_{i=1}^{p} x_{i\ell} + f_\ell$$

where each monomial of f_ℓ has degree $\geqslant p$. We set $f_0 = 0$.

(2) *For $\ell \geqslant 2$, there exists a polynomial $h_{\ell-2} \in R_{\ell-1}$ such that*

$$f_\ell = \frac{\sum_{i=1}^{p} x_{i,\ell-1}^p - (\sum_{i=1}^{p} x_{i,\ell-1})^p}{p} - \frac{1}{p}\left[\sum_{j=1}^{p-1} \binom{p}{j} \left(\sum_{i=1}^{p} x_{i,\ell-1}\right)^{p-j} f_{\ell-1}^j\right] + h_{\ell-2}$$

and each monomial appearing in $h_{\ell-2}$ has degree $\geqslant p^2$. For $\ell = 1$, the above expression remains valid by setting $h_{-1} = 0$.

Proof. (1) By the definition of addition of Witt vectors by using ghost polynomials we have, for $0 \leqslant \ell \leqslant (n-1)$,

$$\sum_{i=1}^{p} w_\ell(x_{i0}, \ldots, x_{i\ell}) = w_\ell(z_0, \ldots, z_\ell)$$

Using the expression for the polynomials w_ℓ and rearranging, we get

$$z_\ell = \sum_{i=1}^{p} x_{i\ell} + f_\ell$$

where

$$f_\ell = \frac{1}{p^\ell}\left(\sum_{i=1}^{p} x_{i0}^{p^\ell} - z_0^{p^\ell}\right) + \cdots + \frac{1}{p}\left(\sum_{i=1}^{p} x_{i(\ell-1)}^{p} - z_{\ell-1}^{p}\right)$$

The claim that f_ℓ has integral coefficients follows from Theorem 2.1. All the terms above involving the variables x_{ij}, $1 \leqslant i \leqslant p$, $0 \leqslant j \leqslant \ell - 1$ will have monomials of degree $\geqslant p$. This shows that every monomial appearing in the expression of f_ℓ has degree $\geqslant p$.

(2) Substitute $z_{\ell-1} = \sum_{i=1}^{p} x_{i(\ell-1)} + f_{\ell-1}$ in the expression of f_ℓ and rewrite f_ℓ as

$$f_\ell = \frac{(\sum_{i=1}^{p} x_{i(\ell-1)}^{p}) - (\sum_{i=1}^{p} x_{i(\ell-1)})^p}{p} - \frac{1}{p}\sum_{j=1}^{p-1}\binom{p}{j}\left(\sum_{i=1}^{p} x_{i(\ell-1)}\right)^{p-j} \cdot f_{\ell-1}^{j} + h_{\ell-2}$$

where

$$\begin{aligned} h_{\ell-2} = {} & -\frac{1}{p} f_{\ell-1}^{p} + \frac{1}{p^2}\left(x_{1(\ell-2)}^{p^2} + x_{2(\ell-2)}^{p^2} + \cdots + x_{p(\ell-2)}^{p^2} - z_{\ell-2}^{p^2}\right) + \cdots \\ & + \frac{1}{p^\ell}\left(x_{10}^{p^\ell} + x_{20}^{p^\ell} + \cdots + x_{p0}^{p^\ell} - z_0^{p^\ell}\right) \end{aligned}$$

As p is a prime number, every binomial coefficient $\binom{p}{j}$ with $1 \leqslant j < p$ is divisible by p. Thus the first two terms in the above expressions of f_ℓ have integral coefficients. Since we know that f_ℓ has integral coefficients, it follows that $h_{\ell-2}$ has integral coefficients too. Moreover, since all monomials appearing in $f_{\ell-1}$ have degree $\geqslant p$, all monomials appearing in $f_{\ell-1}^{p}$ have degree $\geqslant p^2$. Since each z_t for $0 \leqslant t \leqslant \ell - 2$, is again a polynomial without a constant term in the variables x_{ij}, all monomials appearing in the polynomial $z_t^{p^{\ell-t}}$ will have at least p^2. This shows that all monomials appearing in the expression of $h_{\ell-2}$ have degree $\geqslant p^2$. $\square$

3. Proof of the main theorem

In this section, we prove Theorem 1.4. As mentioned before, the main theorem, Theorem 1.1, follows immediately from Theorem 1.4 and Lemma 1.3. Throughout this section v_K (resp. v_L) will denote normalized valuation on K (resp. on L) so that their values at the respective uniformizers are equal to 1.

Lemma 3.1. *Let p be a prime number and L/K be a finite Galois extension of complete discrete fields with $G = \mathrm{Gal}(L/K)$. Suppose that k_L/k_K is separable. Then the following two statements are equivalent.*

(i) *The pro-abelian group* $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ *vanishes for all extensions* L/K *as above.*
(ii) *The pro-abelian group* $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ *vanishes for all* L/K *as above which are ramified and of degree* p.

Proof. (i) $\Longrightarrow$ (ii) is obvious. Now we prove (i) assuming (ii).

Let L/K be any Galois extension of complete discrete valued fields. Let L^t be the maximal subfield of L which is tamely ramified over K. The extension L^t/K is Galois and let $H = \mathrm{Gal}(L/L^t)$. Since L^t/K is tame, $\mathcal{O}_{L^t}$ is a projective $\mathcal{O}_K[G/H]$ module (see [1, I, Theorem 3]) which can be used to show the vanishing of the pro-abelian group $\{H^1(G/H, W_n(\mathcal{O}_{L^t}))\}_{n\in\mathbb{N}}$. Moreover, because of the following inflation–restriction exact sequence of pro-abelian groups

$$0 \to \bigl\{H^1\bigl(G/H, W_n(\mathcal{O}_{L^t})\bigr)\bigr\}_{n\in\mathbb{N}} \xrightarrow{inf} \bigl\{H^1\bigl(G, W_n(\mathcal{O}_L)\bigr)\bigr\}_{n\in\mathbb{N}} \xrightarrow{res} \bigl\{H^1\bigl(H, W_n(\mathcal{O}_L)\bigr)\bigr\}_{n\in\mathbb{N}}$$

vanishing of $\{H^1(G, W(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ is implied by that of $\{H^1(H, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$. Thus without loss of generality, we may replace K by L^t and assume that our extension L/K is totally wildly ramified Galois extension. Thus G is a p-group. Since any p-group has a normal subgroup of index p, again by induction and inflation–restriction exact sequence, we reduce ourselves to the case when L/K is of degree p. But in this case the vanishing of $\{H^1(G, W_n(\mathcal{O}_L))\}_{n\in\mathbb{N}}$ is guaranteed by (ii). This proves the lemma. $\square$

Let G be any finite cyclic group with a generator σ. Let M be a G-module. Then the cohomology group $H^i(G, M)$ is isomorphic to the ith cohomology group of the complex

$$M \xrightarrow{1-\sigma} M \xrightarrow{tr} M \xrightarrow{1-\sigma} M \xrightarrow{tr} M \to \cdots$$

where for $a \in M$, $tr(a) = \sum_{g\in G} ga$. Thus in the case at hand, where L/K is a cyclic Galois extension, we have a canonical isomorphism

$$H^1\bigl(G, W_m(\mathcal{O}_L)\bigr) \cong W_m(\mathcal{O}_L)^{tr=0}/(\sigma - 1)W_m(\mathcal{O}_L)$$

Henceforth, for K as before, we assume L/K is a totally ramified cyclic extension of degree p. For such an extension we will denote by s the ramification break. To prove Theorem 1.4 we need following lemmas and results from [2].

Lemma 3.2. *(See [2, 2.4].) Let* L/K *be as above. Suppose that* $x \in \mathcal{O}_L^{tr=0}$ *represents a non-zero class in* $H^1(G, \mathcal{O}_L)$. *Then* $v_L(x) \leqslant s - 1$.

Lemma 3.3. *(See [2, 2.1].) Let* L/K *be as above. For all* $a \in \mathcal{O}_L$,

$$v_K\bigl(tr(a)\bigr) \geqslant \bigl(v_L(a) + s(p-1)\bigr)/p$$

Lemma 3.4. *(See [2, 2.2].) Let* L/K *be as above. For all* $a \in \mathcal{O}_L$,

$$v_K\bigl(tr\bigl(a^p\bigr) - tr(a)^p\bigr) = e_K + v_L(a)$$

Lemma 3.5. *For* L/K *be as above, let* $\underline{x} = (x_0, x_1, \ldots, x_{n-1}) \in W_n(\mathcal{O}_L)^{tr=0}$. *Then* $tr(x_0) = 0$ *and for all* $1 \leqslant \ell \leqslant n-1$

$$-tr(x_\ell) = \frac{tr(x_{\ell-1}^p) - tr(x_{\ell-1})^p}{p} - C.tr(x_{\ell-1})^p + h_{\ell-2}$$

where C is the integer defined by

$$C = \frac{1}{p}\sum_{j=1}^{p-1}(-1)^j\binom{p}{j}$$

and $h_{\ell-2}$ is a polynomial in $x_0, \dots, x_{\ell-2}$ and its conjugates. Further each monomial appearing in $h_{\ell-2}$ is of degree $\geqslant p^2$.

Proof. Since $\underline{x} \in W_n(\mathcal{O}_L)^{tr=0}$ we have

$$\sum_{i=1}^{p}\left(\sigma^{i-1}x_0, \dots, \sigma^{i-1}x_{n-1}\right) = (0, \dots, 0)$$

Since $z_i = 0$ for all $0 \leqslant i \leqslant n-1$, the above claim follows directly from Lemma 2.2(2) by making the substitutions

$$x_{ij} = \sigma^{i-1}x_j \quad 1 \leqslant i \leqslant p,\ 0 \leqslant j \leqslant n-1 \quad \text{and} \quad f_i = -tr(x_i) \quad 0 \leqslant i \leqslant n-1 \qquad \square$$

Lemma 3.6. *Notation as in Lemma* 3.5. *For $\ell \geqslant 2$, $h_{\ell-2} \in \mathcal{O}_K$. Further*

$$v_K(h_{\ell-2}) \geqslant p \cdot \min\left\{v_L(x_i) \;\middle|\; 0 \leqslant i \leqslant \ell-2\right\}$$

Proof. The claim that $h_{\ell-2} \in \mathcal{O}_K$ follows from the following equation (see Lemma 3.5)

$$-tr(x_\ell) = \frac{tr(x_{\ell-1}^p) - tr(x_{\ell-1})^p}{p} - C.tr(x_{\ell-1})^p + h_{\ell-2}$$

and the fact that $tr(a) \in \mathcal{O}_K$ for any element $a \in \mathcal{O}_L$. Further since $h_{\ell-2}$ is a sum of monomials in $x_0, \dots, x_{\ell-2}$ and their conjugates, each of degree $\geqslant p^2$ (see Lemma 3.5), we have

$$v_L(h_{\ell-2}) \geqslant p^2 \cdot \min\left\{v_L(x_i) \;\middle|\; 0 \leqslant i \leqslant \ell-2\right\}$$

The lemma now follows from the fact that $v_L(h_{\ell-2}) = p \cdot v_K(h_{\ell-2})$. $\square$

Proof of Theorem 1.4. By Lemma 3.2, to prove Theorem 1.4 it is sufficient to find $M \in \mathbb{N}$ such that, for all $\underline{x} = (x_0, \dots, x_{M-1}) \in W_M(\mathcal{O}_L)^{tr=0}$, $v_L(x_0) \geqslant s$.

Step (1): Let n be a positive integer and $(x_0, \dots, x_{n-1}) \in W_n(\mathcal{O}_L)^{tr=0}$. We will prove by induction on ℓ that $v_L(x_\ell) \geqslant \frac{s(p-1)}{p}$ for $0 \leqslant \ell \leqslant n-2$.

By Lemma 3.5, and using the fact that $h_0 = 0$, $tr(x_0) = 0$ we have

$$-tr(x_1) = \frac{1}{p}\left(tr\left(x_0^p\right) - tr(x_0)^p\right)$$

But by Lemma 3.3, $v_K(tr(x_1)) \geqslant \frac{s(p-1)}{p}$. Thus

$$v_K\left(tr\left(x_0^p\right) - tr(x_0)^p\right) - e_K = v_K\left(tr(x_1)\right) \geqslant \frac{s(p-1)}{p}$$

By Lemma 3.4, $v_K(tr(x_0^p) - tr(x_0)^p) = v_L(x_0) + e_K$. Therefore $v_L(x_0) \geqslant \frac{s(p-1)}{p}$. This proves the claim for $\ell = 0$.

Now we assume that $\ell \geqslant 1$ and that for all $i \leqslant \ell - 1$, $v_L(x_i) \geqslant \frac{s(p-1)}{p}$. We will prove $v_L(x_\ell) \geqslant \frac{s(p-1)}{p}$. By Lemma 3.5, we have

$$-tr(x_{\ell+1}) = \frac{tr(x_\ell^p) - tr(x_\ell)^p}{p} - C \cdot tr(x_\ell)^p + h_{\ell-1}$$

Thus, using Lemma 3.4 we get

$$v_L(x_\ell) = v_K\left(\frac{tr(x_\ell^p) - tr(x_\ell)^p}{p}\right) \geqslant \inf\bigl\{v_K\bigl(tr(x_{\ell+1})\bigr), v_K\bigl(C \cdot tr(x_\ell)^p\bigr), v_K(h_{\ell-1})\bigr\}$$

Using Lemma 3.3, we have

$$v_K\bigl(tr(x_{\ell+1})\bigr) \geqslant s(p-1)/p$$

and

$$v_K\bigl(C \cdot tr(x_\ell)^p\bigr) \geqslant s(p-1)$$

By Lemma 3.6, and by induction hypothesis $v_K(h_{\ell-1}) \geqslant s(p-1)$. Combining the above, we get

$$v_L(x_\ell) \geqslant \frac{s(p-1)}{p}$$

Step (2): We will now see that the lower bound for $v_L(x_0)$ approaches s as the length of a Witt vector with first term x_0 goes to infinity. The argument that this implies the existence of an integer M such that for all $\underline{x} \in W_M(\mathcal{O}_L)$, $v_L(x_0) \geqslant s$ is at the end of the proof.

For any positive integer n and $(x_0, \dots, x_{n-1}) \in W_n(\mathcal{O}_L)^{tr=0}$, by Step (1) we have

$$v_L(x_i) \geqslant \frac{s(p-1)}{p} \quad \forall 0 \leqslant i \leqslant n-2$$

For a fixed n, and $2 \leqslant i \leqslant n$, we claim that

$$v_L(x_{n-i}) \geqslant \frac{s(p-1)}{p}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-2}}\right)$$

We prove this by induction on i. For $i = 2$, this is the claim that

$$v_L(x_{n-2}) \geqslant \frac{s(p-1)}{p}$$

which follows from Step (1). Now let i be an integer such that $2 \leqslant i \leqslant n-1$. Assuming the claim for i we will prove it for $i+1$. By induction hypothesis

$$v_L(x_{n-i}) \geqslant \frac{s(p-1)}{p}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-2}}\right)$$

Therefore by using Lemma 3.3 we get

$$v_K\big(tr(x_{n-i})\big) \geqslant \frac{v_L(x_{n-i})+s(p-1)}{p} \geqslant \frac{s(p-1)}{p}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{i-1}}\right)$$

By Lemma 3.5

$$-tr(x_{n-i}) = \frac{tr(x_{n-(i+1)}^p)-tr(x_{n-(i+1)})^p}{p} - C\cdot tr(x_{n-(i+1)})^p + h_{n-(i+2)}$$

By Lemma 3.3, $v_K(C\cdot tr(x_{n-(i+1)})^p) \geqslant s(p-1)$. By Step (1) and Lemma 3.6, $v_K(h_{n-(i+2)}) \geqslant s(p-1)$. Thus, using Lemma 3.4,

$$\begin{aligned}
v_L(x_{n-(i+1)}) &= v_K\left(\frac{tr(x_{n-(i+1)}^p)-tr(x_{n-(i+1)})^p}{p}\right)\\
&\geqslant \min\big\{v_K\big(tr(x_{n-i})\big), v_K\big(C\cdot tr(x_{n-(i+1)})^p\big), v_K\big(h_{n-(i+2)}\big)\big\}\\
&\geqslant \min\left\{\frac{s(p-1)}{p}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{i-1}}\right), s(p-1), s(p-1)\right\}\\
&= \frac{s(p-1)}{p}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{i-1}}\right)
\end{aligned}$$

This proves the claim. Hence

$$v_L(x_0) \geqslant \frac{s(p-1)}{p}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{n-2}}\right)$$

The right-hand side approaches s as n goes to ∞.

Step (3): There exists an integer M, such that

$$\frac{s(p-1)}{p}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{M-2}}\right) > s-1$$

Since v_L is a discrete valuation, for such M and for any $(x_0,\dots,x_{M-1}) \in W_M(\mathcal{O}_L)^{tr=0}$, we have shown that

$$v_L(x_0) \geqslant s \qquad \square$$

4. Proof of Corollary 1.2

In this section we prove Corollary 1.2. In view of Theorem 1.1, in order to prove this it is sufficient to show that $\varprojlim H^1(G, W_n(\mathcal{O}_L))$ coincides with $H^1(G, W(\mathcal{O}_L))$ for all Galois extensions L/K of complete discrete valued fields (see Corollary 4.2). Note that in general group cohomology does not commute with inverse limits.

Proposition 4.1. *Let G be a finite group and $\{A_i\}_{i\in\mathbb{N}}$ be an inverse system of G modules indexed by $\mathbb{N}$. For $j>i$, let $\phi_{ji}: A_j \to A_i$ denote the given maps. Then the following two statements hold.*

(i) *If ϕ_{ji} is surjective for all $j>i$ then*

$$H^1(G, \varprojlim A_i) \to \varprojlim H^1(G, A_i)$$

is surjective.

(ii) *If the induced maps $\phi_{ji}^G : A_j^G \to A_i^G$ are surjective for all $j > i$, then*

$$H^1(G, \varprojlim A_i) \to \varprojlim H^1(G, A_i)$$

is injective.

Corollary 4.2. *Let L/K be a finite Galois extension of complete discrete valued fields. Then the natural map*

$$\Phi : H^1\big(G, W(\mathcal{O}_L)\big) \to \varprojlim H^1\big(G, W_n(\mathcal{O}_L)\big)$$

is an isomorphism.

Proof. By construction of Witt vectors, the projection maps

$$W_{n+1}(\mathcal{O}_L) \to W_n(\mathcal{O}_L)$$

are surjective. Thus by the above proposition, Φ is surjective. In order to prove injectivity of Φ we need to prove surjectivity of

$$W_{n+1}(\mathcal{O}_L)^G \to W_n(\mathcal{O}_L)^G$$

This follows from the fact that $W_i(\mathcal{O}_L)^G = W_i(\mathcal{O}_K)$ for all i and from the surjectivity of the projection maps $W_{n+1}(\mathcal{O}_K) \to W_n(\mathcal{O}_K)$. $\square$

Proof of Proposition 4.1. (i) Suppose we are given an element $\alpha \in \varprojlim H^1(G, A_i)$. This is equivalent to giving datum $\alpha_i \in H^1(G, A_i)$ for all i such that $\alpha_{i+1} \mapsto \alpha_i$. We now inductively construct cocycles a_g^i representing the class α_i as follows. For $i = 1$, choose a_g^1 arbitrarily. Now, suppose a_g^n has been constructed. Then construct a_g^{n+1} as follows. First start with any cocycle b_g^{n+1} which represents α_{n+1}. For an element $b \in A_{n+1}$, let $\bar{b}$ denote its image in A_n. Thus $\bar{b}_g^{n+1}$ is a cocycle in A_n which represents the same class as that represented by a_g^n. Thus, there exists $c \in A_n$ such that

$$\bar{b}_g^{n+1} - a_g^n = gc - c$$

Since by assumption, $A_{n+1} \to A_n$ is surjective, there exists an element $d \in A_{n+1}$ such that $\bar{d} = c$. Now define

$$a_g^{n+1} = b_g^{n+1} - (gd - d)$$

This completes the inductive construction of the cocycles a_g^i. The cocycles have the property that for all i and g,

$$a_g^{i+1} \mapsto a_g^i$$

and thus they define a cocycles with values in $\varprojlim A_i$ whose class obviously maps to the element α we started with.

(ii) Suppose α is a class in $H^1(G, \varprojlim A_i)$ which maps to zero in $\varprojlim H^1(G, A_i)$, or equivalently maps to zero in $H^1(G, A_i)$ for each i. Under the given assumption we will show that $\alpha = 0$. Choose a cocycle a_g representing α. By abuse of notation, we will denote the image of a_g in A_n by $\bar{a}_g$. The n will be clear from context.

For each n, we will now inductively construct an element $b_n \in A_n$ such that

$$a_g = gb_n - b_n \quad \forall g \in G$$

and for all n, b_{n+1} maps to b_n. For $n = 1$, we know that the image of a_g in A_1 is a coboundary. Thus there exists an element $b_1 \in A_1$ such that

$$\bar{a}_g = gb_1 - b_1 \quad \forall g \in G$$

Now suppose we have defined b_n. To define b_{n+1} we first choose an element $c_{n+1} \in A_{n+1}$ such that

$$\bar{a}_g = gc_{n+1} - c_{n+1} \quad \forall g \in G$$

However the image of c_{n+1} in A_n, denoted by $\bar{c}_{n+1}$ satisfies

$$g\bar{c}_{n+1} - \bar{c}_{n+1} = gb_n - b_n$$

which means, there exists a $d \in A_n^G$ such that

$$b_n = \bar{c}_{n+1} + d$$

Since the map $A_{n+1}^G \to A_n^G$ is assumed to be surjective, we can lift d to an element $\tilde{d} \in A_{n+1}^G$. Now define

$$b_{n+1} = c_{n+1} + \tilde{d}$$

The elements b_n defined above are compatible elements and hence define an element b of $\varprojlim A_i$. Also, from the construction it is clear that

$$a_g = gb - b \quad \forall g \in G$$

holds, since it holds after taking image in A_i for all i. Thus the cocycle a_g is actually a coboundary and hence the class α we started with is trivial. $\square$

Proof of Corollary 1.2. In view of Theorem 1.1, the proof now follows immediately from Corollary 4.2. $\square$

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