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Note on the mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble

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ABSTRACT

In this paper we obtain an asymptotic formula for the first moment of quadratic Dirichlet L-functions $L(s, \chi_D)$ over the rational function field at the central point $s = \frac{1}{2}$, where D runs over all monic square-free polynomials of even degree.

1. Introduction and statement of results

It is an important problem in analytic number theory to understand the m-th moments of the family of quadratic Dirichlet L-functions $L(s, \chi_d)$ at $s = \frac{1}{2}$. Jutila [2] was the first to obtain the asymptotic formulas for the cases $m = 1, 2$, and Soundararajan [4] succeeded in the cubic case. In [5], Soundararajan and Young studied the second moment of quadratic twists of modular L-functions and claimed that they may adapt the technique described here to obtain an asymptotic formula for the fourth moment of quadratic Dirichlet L-functions under the G.R.H.

In a recent paper [1], Andrade and Keating obtained an asymptotic formula for the first moment of quadratic Dirichlet L-functions over function fields at the central point $s = \frac{1}{2}$. Their results are the function field analogues of those obtained previously by Jutila in the number-field setting and are

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consistent with recent general conjectures for the moments of L-functions motivated by Random Matrix Theory. To describe the results of Andrade and Keating more precisely, we need to introduce some notations at first. Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$ be the polynomial ring. Let $|N| = q^{\deg N}$ for any $N \in \mathbb{A}$, and $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$ be the zeta function associated to $\mathbb{A}$. Let $\mathrm{P}(s) = \prod_P (1 - (1 + |P|)^{-1}|P|^{-s})$, where P runs over all monic irreducible polynomials in $\mathbb{A}$. We assume that q is odd. For a nonnegative integer n, let $\mathcal{H}_n$ be the set of monic square-free polynomials of degree n in $\mathbb{A}$. For $D \in \mathcal{H}_n$, let χ_D be the Dirichlet character modulo D defined by the Jacobi symbol $\chi_D(N) = (\frac{D}{N})$ and $L(s, \chi_D)$ be the quadratic Dirichlet L-function associated to χ_D. Andrade and Keating gave the following formula of mean value of quadratic Dirichlet L-functions at $s = \frac{1}{2}$.

Theorem 1.1. *(See [1, Theorem 2.1].) Assume that $q \equiv 1 \bmod 4$. Then*

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right) = \frac{\mathrm{P}(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left\{\log_q |D| + 1 + \frac{4}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1)\right\} + O\left(|D|^{\frac{3}{4}+\frac{1}{2}\log_q 2}\right). \tag{1.1}$$

As a corollary, they obtained the following asymptotic mean value [1, Corollary 2.2]:

$$\frac{1}{\sharp\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right) \sim \frac{1}{2}\mathrm{P}(1)(2g+1) \quad (\text{as } g \to \infty). \tag{1.2}$$

We remark that they assumed that $q \equiv 1 \bmod 4$ for simplicity, but their results hold true for any odd $q > 3$. For any $D \in \mathcal{H}_{2g+1}$, the infinite place $\infty = (1/T)$ of k ramifies in $k(\sqrt{D})$, that is, $k(\sqrt{D})$ is a ramified imaginary quadratic extension of k. In this paper we study the values of the summation of $L(\frac{1}{2}, \chi_D)$ with $D \in \mathcal{H}_{2g+2}$ and its asymptotic mean value as $g \to \infty$. For any $D \in \mathcal{H}_{2g+2}$, the infinite place ∞ of k splits in $k(\sqrt{D})$, that is, $k(\sqrt{D})$ is a real quadratic extension of k. Our main result is the following theorem.

Theorem 1.2. *Assume that q is odd and greater than 3. Then we have*

$$\sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) = \frac{\mathrm{P}(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left\{\log_q |D| + \frac{4}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1) - 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right\} + O\left(|D|^{\frac{3}{4}+\frac{1}{2}\log_q 2}\right). \tag{1.3}$$

Comparing (2.2) with Theorem 1.2, we obtain the following asymptotic mean value.

Corollary 1.3. *Under the same assumption of Theorem 1.2, we have*

$$\frac{1}{\sharp\mathcal{H}_{2g+2}} \sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) \sim \frac{1}{2}\mathrm{P}(1)\left(\log_q |D|\right) = \mathrm{P}(1)(g+1) \quad (\text{as } g \to \infty). \tag{1.4}$$

2. Quadratic Dirichlet L-functions and their functional equations

2.1. Basic facts about $\mathbb{A} = \mathbb{F}_q[T]$

For $0 \neq N \in \mathbb{A}$, set $|N| = q^{\deg N}$ and if $N = 0$, set $|N| = 0$. Let $\mathbb{A}^+$ be the subset of $\mathbb{A}$ consisting of all monic polynomials of $\mathbb{A}$. For an integer $n \geqslant 0$, write $\mathbb{A}_n^+ = \{N \in \mathbb{A}^+ \colon \deg N = n\}$. The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) = \sum_{N \in \mathbb{A}^+} \frac{1}{|N|^s} = \prod_{\substack{P \in \mathbb{A}^+ \\ \text{irreducible}}} \left(1 - |P|^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$

which is

$$\zeta_{\mathbb{A}}(s) = \frac{1}{1 - q^{1-s}}.$$

The Möbius function $\mu(N)$ and the Euler totient function $\Phi(N)$ for $\mathbb{A}$ are defined as follows:

$$\mu(N) = \begin{cases} (-1)^t, & N = \alpha P_1 \cdots P_t, \\ 0, & \text{otherwise}, \end{cases}$$

where each P_i is distinct monic irreducible, and

$$\Phi(N) = \sum_{\substack{M \in \mathbb{A} \\ \deg M < \deg N \\ (N,M)=1}} 1.$$

It is well known [3, Proposition 2.7]:

$$\sum_{N \in \mathbb{A}_n^+} \Phi(N) = \left(1 - q^{-1}\right) q^{2n}. \tag{2.1}$$

2.2. Quadratic Dirichlet L-function

For an integer $n \geqslant 0$, write $\mathcal{H}_n$ for the set of monic square-free polynomials of degree n in $\mathbb{A}$. The cardinality of $\mathcal{H}_n$ is

$$\sharp\mathcal{H}_n = \begin{cases} q, & \text{if } n = 1, \\ (1 - q^{-1}) q^d, & \text{if } n \geqslant 2. \end{cases}$$

In particular, we have

$$\sharp\mathcal{H}_{2g+2} = (q-1) q^{2g+1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}. \tag{2.2}$$

For $D \in \mathcal{H}_d$, let χ_D be the Dirichlet character modulo D defined by the Jacobi symbol $\chi_D(N) = (\frac{D}{N})$ and $L(s, \chi_D)$ be the quadratic Dirichlet L-function associated to χ_D:

$$L(s, \chi_D) = \sum_{N \in \mathbb{A}^+} \frac{\chi_D(N)}{|N|^s} = \prod_{\substack{P \in \mathbb{A}^+ \\ \text{irreducible}}} \left(1 - \frac{\chi_D(P)}{|P|^s}\right)^{-1}.$$

We can write

$$L(s, \chi_D) = \sum_{n=0}^{\infty} \sigma_n(D) q^{-ns} \quad \text{with } \sigma_n(D) = \sum_{N \in \mathbb{A}_n^+} \chi_D(N).$$

Since $\sigma_n(D) = 0$ for $n \geqslant d = \deg D$, $L(s, \chi_D)$ is a polynomial in q^{-s} of degree $\leqslant d-1$. Putting $u = q^{-s}$, write

$$\mathcal{L}(u, \chi_D) = \sum_{n=0}^{d-1} \sigma_n(D) u^n = L(s, \chi_D).$$

We now assume that $D \in \mathcal{H}_{2g+2}$. Then $\mathcal{L}(u, \chi_D)$ has a "trivial" zero at $u = 1$. The "complete" L-function $\tilde{\mathcal{L}}(u, \chi_D)$ is defined by $\tilde{\mathcal{L}}(u, \chi_D) = (1-u)^{-1}\mathcal{L}(u, \chi_D)$. It is a polynomial of even degree $2g$ and satisfies the functional equation

$$\tilde{\mathcal{L}}(u, \chi_D) = \left(qu^2\right)^g \tilde{\mathcal{L}}\left((qu)^{-1}, \chi_D\right). \tag{2.3}$$

Lemma 2.1. *Let χ_D be a quadratic character, where $D \in \mathcal{H}_{2g+2}$. Then*

$$\begin{aligned}\left(1 - q^{-\frac{1}{2}}\right)\tilde{\mathcal{L}}\left(q^{-\frac{1}{2}}, \chi_D\right) = {} & \sum_{n=0}^{g} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) q^{-\frac{n}{2}} + q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) \\ & + \sum_{n=0}^{g-1} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) q^{-\frac{n}{2}} + q^{-\frac{g}{2}} \sum_{n=0}^{g-1} \sum_{N \in \mathbb{A}_n^+} \chi_D(N).\end{aligned}$$

Proof. Write

$$\tilde{\mathcal{L}}(u, \chi_D) = \sum_{n=0}^{2g} \tilde{\sigma}_n(D) u^n.$$

Since $\mathcal{L}(u, \chi_D) = (1-u)\tilde{\mathcal{L}}(u, \chi_D)$, we have

$$\sigma_n(D) = \begin{cases} \tilde{\sigma}_0(D), & \text{if } n = 0, \\ \tilde{\sigma}_n(D) - \tilde{\sigma}_{n-1}(D), & \text{if } 1 \leqslant n \leqslant 2g, \\ -\tilde{\sigma}_{2g}(D), & \text{if } n = 2g+1, \end{cases}$$

or

$$\tilde{\sigma}_n(D) = \sum_{i=0}^{n} \sigma_i(D) \quad (0 \leqslant n \leqslant 2g). \tag{2.4}$$

By substituting $\tilde{\mathcal{L}}(u, \chi_D) = \sum_{n=0}^{2g} \tilde{\sigma}_n(D) u^n$ into the functional equation (2.3),

$$\sum_{n=0}^{2g} \tilde{\sigma}_n(D) u^n = \sum_{n=0}^{2g} \tilde{\sigma}_n(D) q^{g-n} u^{2g-n} = \sum_{n=0}^{2g} \tilde{\sigma}_{2g-n}(D) q^{-g+n} u^n.$$

By equating coefficients, we have that

$$\tilde{\sigma}_n(D) = \tilde{\sigma}_{2g-n}(D) q^{-g+n} \quad \text{or} \quad \tilde{\sigma}_{2g-n}(D) = \tilde{\sigma}_n(D) q^{g-n}$$

and we can write $\tilde{\mathcal{L}}(u,\chi_D)$ as

$$\tilde{\mathcal{L}}(u,\chi_D)=\sum_{n=0}^{g}\tilde{\sigma}_n(D)u^n+q^g u^{2g}\sum_{n=0}^{g-1}\tilde{\sigma}_n(D)q^{-n}u^{-n}.$$

In particular, we have

$$\tilde{\mathcal{L}}\big(q^{-\frac{1}{2}},\chi_D\big)=\sum_{n=0}^{g}\tilde{\sigma}_n(D)q^{-\frac{n}{2}}+\sum_{n=0}^{g-1}\tilde{\sigma}_n(D)q^{-\frac{n}{2}}. \tag{2.5}$$

By substituting (2.4) into (2.5), we have

$$\begin{aligned}\tilde{\mathcal{L}}\big(q^{-\frac{1}{2}},\chi_D\big)&=\sum_{n=0}^{g}\sum_{i=n}^{g}q^{-\frac{i}{2}}\sigma_n(D)+\sum_{n=0}^{g-1}\sum_{i=n}^{g-1}q^{-\frac{i}{2}}\sigma_n(D)\\&=\sum_{n=0}^{g}\sigma_n(D)\left(\frac{q^{-\frac{n}{2}}-q^{-\frac{(g+1)}{2}}}{1-q^{-\frac{1}{2}}}\right)+\sum_{n=0}^{g-1}\sigma_n(D)\left(\frac{q^{-\frac{n}{2}}-q^{-\frac{g}{2}}}{1-q^{-\frac{1}{2}}}\right).\qquad\square\end{aligned}$$

3. Some preparations

In this section we give several results which will be used to prove the main theorem in Section 4. The following three lemmas are due to Andrade and Keating [1].

Lemma 3.1. *(See [1, Proposition 5.2].) Let $L\in\mathbb{A}^+$ with $\deg L\leqslant 2g+1$. Then*

$$\sum_{\substack{D\in\mathcal{H}_{2g+2}\\(D,L)=1}}1=\frac{|D|}{\zeta_{\mathbb{A}}(2)\prod_{P|L}(1+|P|^{-1})}+O\left(\sqrt{|D|}\frac{\Phi(L)}{|L|}\right).$$

Lemma 3.2. *(See [1, Lemma 5.7].) We have*

$$\sum_{L\in\mathbb{A}_m^+}\prod_{P|L}\big(1+|P|^{-1}\big)^{-1}=q^m\sum_{\substack{M\in\mathbb{A}^+\\\deg M\leqslant m}}\frac{\mu(M)}{|M|}\prod_{P|M}\frac{1}{1+|P|}.$$

Let

$$\mathrm{P}(s)=\prod_{\substack{P\in\mathbb{A}^+\\\text{irreducible}}}\left(1-\frac{1}{(1+|P|)|P|^s}\right).$$

Then we have

$$\frac{\mathrm{P}'(s)}{\mathrm{P}(s)}=\log q\sum_{P}\frac{\deg P}{(1+|P|)|P|^s-1}.$$

In particular

$$\sum_{P} \frac{\deg P}{(1+|P|)|P|-1} = \frac{1}{\log q} \frac{\mathrm{P}'}{\mathrm{P}}(1).$$

Lemma 3.3. *(See [1, Lemma 5.9 and Lemma 5.10].) We have*

$$\sum_{\substack{M \in \mathbb{A}^+ \\ \deg M \leqslant [\frac{g}{2}]}} \frac{\mu(M)}{|M|} \prod_{P|M} \frac{1}{1+|P|} = \mathrm{P}(1) + O\left(q^{-\frac{g}{2}}\right).$$

Lemma 3.4. *We have*

$$\sum_{\substack{M \in \mathbb{A}^+ \\ \deg M \leqslant [\frac{g}{2}]}} \mu(M) \prod_{P|M} \frac{1}{1+|P|} \leqslant \left[\frac{g}{2}\right] + 1.$$

Proof.

$$\sum_{\substack{M \in \mathbb{A}^+ \\ \deg M \leqslant [\frac{g}{2}]}} \mu(M) \prod_{P|M} \frac{1}{1+|P|} \leqslant \sum_{\substack{M \in \mathbb{A}^+ \\ \deg M \leqslant [\frac{g}{2}]}} \mu^2(M) \prod_{P|M} \frac{1}{|P|} \leqslant \sum_{\substack{M \in \mathbb{A}^+ \\ \deg M \leqslant [\frac{g}{2}]}} |M|^{-1} = \left[\frac{g}{2}\right] + 1. \qquad \square$$

Proposition 3.5. *We have*

$$\sum_{m=0}^{[\frac{g}{2}]} q^{-m} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 = \frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)} |D| \left\{ \left[\frac{g}{2}\right] + 1 + \frac{1}{\log q} \frac{\mathrm{P}'}{\mathrm{P}}(1) \right\} + O\left(g q^{\frac{3}{2}g+2}\right) \tag{3.1}$$

and

$$q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 = \frac{q^{-\frac{(g-1)}{2} + [\frac{g}{2}]}}{(q-1)\zeta_{\mathbb{A}}(2)} \mathrm{P}(1)|D| + O\left(g q^{\frac{3}{2}g+\frac{3}{2}}\right). \tag{3.2}$$

Proof. The first part is a mild modification of Proposition 5.1 in [1]. We only give the proof of second part. By Lemma 3.1, we have

$$q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 = \frac{q^{-\frac{(g+1)}{2}}}{\zeta_{\mathbb{A}}(2)} |D| \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \prod_{P|L} \left(1 + |P|^{-1}\right)^{-1}$$
$$+ O\left(q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \sqrt{|D|} \frac{\Phi(L)}{|L|} \right).$$

By using (2.1), we have

$$\begin{aligned} q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L\in\mathbb{A}_m^+} \sqrt{|D|}\frac{\Phi(L)}{|L|} &= q^{-\frac{(g+1)}{2}} \sqrt{|D|} \sum_{m=0}^{[\frac{g}{2}]} q^{-m} \sum_{L\in\mathbb{A}_m^+} \Phi(L) \\ &= q^{-\frac{(g+1)}{2}} \sqrt{|D|}\big(1-q^{-1}\big) \sum_{m=0}^{[\frac{g}{2}]} q^m \\ &= q^{\frac{(g+1)}{2}}\big(1-q^{-1}\big)\frac{q^{[\frac{g}{2}]+1}-1}{q-1} \ll q^{g+\frac{1}{2}}. \end{aligned}$$

Using Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned} &\frac{q^{-\frac{(g+1)}{2}}}{\zeta_{\mathbb{A}}(2)}|D| \sum_{m=0}^{[\frac{g}{2}]} \sum_{L\in\mathbb{A}_m^+} \prod_{P|L}\big(1+|P|^{-1}\big)^{-1} \\ &= \frac{q^{-\frac{(g+1)}{2}}}{\zeta_{\mathbb{A}}(2)}|D| \sum_{m=0}^{[\frac{g}{2}]} q^m \sum_{\substack{M\in\mathbb{A}^+ \\ \deg M\leqslant m}} \frac{\mu(M)}{|M|} \prod_{P|M}\frac{1}{1+|P|} \\ &= \frac{q^{-\frac{(g+1)}{2}}}{\zeta_{\mathbb{A}}(2)}|D| \sum_{m=0}^{[\frac{g}{2}]} \sum_{M\in\mathbb{A}_m^+} \frac{\mu(M)}{|M|} \prod_{P|M}\frac{1}{1+|P|}\left(\frac{q^{[\frac{g}{2}]+1}-q^m}{q-1}\right) \\ &= \frac{q^{-\frac{(g-1)}{2}+[\frac{g}{2}]}}{(q-1)\zeta_{\mathbb{A}}(2)}|D| \sum_{\substack{M\in\mathbb{A}^+ \\ \deg M\leqslant [\frac{g}{2}]}} \frac{\mu(M)}{|M|} \prod_{P|M}\frac{1}{1+|P|} - q^{-\frac{(g+3)}{2}}|D| \sum_{\substack{M\in\mathbb{A}^+ \\ \deg M\leqslant [\frac{g}{2}]}} \mu(M) \prod_{P|M}\frac{1}{1+|P|} \\ &= \frac{q^{-\frac{(g-1)}{2}+[\frac{g}{2}]}}{(q-1)\zeta_{\mathbb{A}}(2)}|D|\mathrm{P}(1) + O\big(gq^{\frac{3}{2}g+\frac{3}{2}}\big). \qquad \square \end{aligned}$$

Proposition 3.6. *We have*

$$\sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) q^{-\frac{n}{2}} = O\big(2^{g+1} q^{\frac{3}{2}g+\frac{3}{2}}\big) \tag{3.3}$$

and

$$q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) = O\big(2^{g+1} q^{\frac{3}{2}g+\frac{3}{2}}\big). \tag{3.4}$$

Proof. The first part is a mild modification of Proposition 6.1 in [1]. We only give the proof of second part. As in [1, Lemma 6.4], for any non-square $N\in\mathbb{A}^+$, we have

$$\left|\sum_{D\in\mathcal{H}_{2g+2}} \chi_D(N)\right| \ll q^{g+1} 2^{\deg N-1}.$$

Hence we have

$$q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)\leqslant q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{N\in\mathbb{A}_n^+}q^{g+1}2^{\deg N-1}$$

$$\leqslant q^{\frac{(g+1)}{2}}\sum_{n=0}^{g}(2q)^n\ll 2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}. \qquad \square$$

4. Proof of main theorem

In this section we give the proof of Theorem 1.2. Since $L(\frac{1}{2},\chi_D)=(1-q^{-\frac{1}{2}})\tilde{\mathcal{L}}(q^{-\frac{1}{2}},\chi_D)$, by Lemma 2.1, we have

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)q^{-\frac{n}{2}}+q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)$$
$$+\sum_{n=0}^{g-1}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)q^{-\frac{n}{2}}+q^{-\frac{g}{2}}\sum_{n=0}^{g-1}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N). \tag{4.1}$$

By (3.1) and (3.3), we have

$$\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)q^{-\frac{n}{2}}=\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N=\square}}\chi_D(N)q^{-\frac{n}{2}}+\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)q^{-\frac{n}{2}}$$
$$=\sum_{m=0}^{[\frac{g}{2}]}q^{-m}\sum_{L\in\mathbb{A}_m^+}\sum_{\substack{D\in\mathcal{H}_{2g+2}\\ (L,D)=1}}1+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right)$$
$$=\frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)}|D|\left\{\left[\frac{g}{2}\right]+1+\frac{1}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1)\right\}+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right) \tag{4.2}$$

and, by (3.2) and (3.4), we have

$$q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)=q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N=\square}}\chi_D(N)$$
$$+q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)$$
$$=q^{-\frac{(g+1)}{2}}\sum_{m=0}^{[\frac{g}{2}]}\sum_{L\in\mathbb{A}_m^+}\sum_{\substack{D\in\mathcal{H}_{2g+2}\\ (L,D)=1}}1+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right)$$
$$=\frac{q^{-\frac{(g-1)}{2}+[\frac{g}{2}]}}{(q-1)\zeta_{\mathbb{A}}(2)}\mathrm{P}(1)|D|+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right). \tag{4.3}$$

Similarly, we have

$$\sum_{n=0}^{g-1}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)q^{-\frac{n}{2}} = \frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)}|D|\left\{\left[\frac{g-1}{2}\right]+1+\frac{1}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1)\right\}+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right) \tag{4.4}$$

and

$$q^{-\frac{g}{2}}\sum_{n=0}^{g-1}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{N\in\mathbb{A}_n^+}\chi_D(N) = \frac{q^{-\frac{g}{2}+[\frac{g-1}{2}]+1}}{(q-1)\zeta_{\mathbb{A}}(2)}\mathrm{P}(1)|D|+O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right). \tag{4.5}$$

It can be easily shown that

$$\left[\frac{g}{2}\right]+1+\left[\frac{g-1}{2}\right]+1=\frac{1}{2}\log_q|D|, \qquad q^{-\frac{(g-1)}{2}+[\frac{g}{2}]}+q^{-\frac{g}{2}+[\frac{g-1}{2}]+1}=1+\sqrt{q}$$

and

$$2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}=|D|^{\frac{3}{4}+\frac{\log_q 2}{2}}.$$

By inserting (4.2), (4.3), (4.4) and (4.5) into (4.1), we have

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=\frac{\mathrm{P}(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left\{\log_q|D|+\frac{4}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1)-2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right\}+O\left(|D|^{\frac{3}{4}+\frac{1}{2}\log_q 2}\right).$$

Corollary 1.3 follows immediately from Theorem 1.2 by using (2.2) and computing the limit as $g\to\infty$.

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References

[1] J.C. Andrade, J.P. Keating, The mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble, J. Number Theory 132 (12) (2012) 2793–2816.
[2] M. Jutila, On the mean value of $L(\frac{1}{2},\chi)$ for real characters, Analysis 1 (1981) 149–196.
[3] M. Rosen, Number Theory in Function Fields, Grad. Texts in Math., vol. 210, Springer-Verlag, New York, 2002.
[4] K. Soundararajan, Nonvanishing of quadratic Dirichlet L-functions at $s=\frac{1}{2}$, Ann. of Math. 152 (2000) 447–488.
[5] K. Soundararajan, M. Young, The second moment of quadratic twists of modular L-functions, J. Eur. Math. Soc. 12 (5) (2010) 1097–1116.