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# $p$ -adic Eisenstein–Kronecker series and non-critical values of $p$ -adic Hecke $L$ -function of an imaginary quadratic field when the conductor is divisible by $p$

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## ABSTRACT

*Text.* We relate non-critical special values of  $p$ -adic  $L$ -functions associated to algebraic Hecke characters of an imaginary quadratic number field with class number one to  $p$ -adic Eisenstein–Kronecker series constructed as the Coleman function, when the conductors of the algebraic Hecke characters are divisible by  $p$ .

*Video.* For a video summary of this paper, please click [here](http://youtu.be/AZemqgfp5pQ) or visit <http://youtu.be/AZemqgfp5pQ>.

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## 0. Introduction

Let  $p$  be a rational prime number. The purpose of this article is to relate non-critical values of the  $p$ -adic  $L$ -functions associated to algebraic Hecke characters whose conductors are divisible by  $p$  of an imaginary quadratic field with class number 1 to  $p$ -adic Eisenstein–Kronecker series. We expect to use this result in the future to consider the  $p$ -adic Beilinson conjecture as in [2] for the corresponding Hecke character. The Beilinson conjectures about special values of  $L$ -functions are a vast generalization of the class number formula for Dedekind zeta function (see [5]), which state that non-critical values of  $L$ -functions of an algebraic variety can be expressed using invariants arising from the Beilinson regulator map. More generally, these conjectures can be formulated for motives. The  $p$ -adic Beilinson conjectures are the  $p$ -adic analogues of the Beilinson conjectures, which state that non-critical values of  $p$ -adic  $L$ -functions of an algebraic variety may be concretely expressed by invariants arising from the syntomic regulator. The  $p$ -adic Beilinson conjectures were formulated and proved by Gros in the case of Dirichlet motives [17,18] and were generalized to  $p$ -adic  $L$ -functions of motives by Perrin-Riou (see [25, §4.2]). For  $p$ -adic  $L$ -functions of Abelian Artin motives, Coleman related special values of these  $p$ -adic  $L$ -functions to  $p$ -adic polylogarithms defined by using his theory of  $p$ -adic integration [12]. The polylogarithms have a motivic interpreter defined by Beilinson and Deligne. In other words, the Beilinson conjectures and their  $p$ -adic analogues suggest that special values of  $L$ -functions and  $p$ -adic  $L$ -functions may be expressed by using motivic elements.

The  $p$ -adic  $L$ -functions of algebraic Hecke characters of our case was first constructed by Manin and Višik [24], N. Katz [21], R.I. Yager [31], and de Shalit [15]. Bannai and Kobayashi gave a different construction of these  $p$ -adic  $L$ -functions using the Kronecker theta functions associated to the Poincaré bundle (see [4, §2]). In addition, Bannai, Kobayashi, and Tsuji expressed the elliptic polylogarithms on an elliptic curve with complex multiplication in terms of the “Eisenstein–Kronecker series” (for details, see [32]), by using the Kronecker theta functions (see [3, Theorem 1.17]). Deuring’s theorem indicates that  $L$ -functions of an elliptic curve with complex multiplication by the integer ring of an imaginary quadratic field can be expressed as Hecke  $L$ -functions of an imaginary quadratic field. Then the  $p$ -adic  $L$ -functions of an elliptic curve with complex multiplication become the  $p$ -adic  $L$ -functions of an imaginary quadratic field associated to algebraic Hecke characters.

When the conductors of algebraic Hecke characters are not divisible by  $p$ , Bannai, Kobayashi, and Tsuji related non-critical values of  $p$ -adic  $L$ -functions of an imaginary

quadratic field with class number 1 associated to algebraic Hecke characters to  $p$ -adic Eisenstein–Kronecker series by the measure construction with the connection functions (see [4, Proposition 2.27]). In addition, they calculated the syntomic regulator in terms of  $p$ -adic Eisenstein–Kronecker series (see [4, §4.3]) in order to construct and explicitly calculate the  $p$ -adic elliptic polylogarithm, and expressed concretely non-critical values of  $p$ -adic  $L$ -functions associated to algebraic Hecke characters whose conductors are not divisible by  $p$  with  $p$ -adic elliptic polylogarithm class in the rigid syntomic cohomology (see [4, §5.3]). In our paper, we extend the result [4, Proposition 2.27] of Bannai, Kobayashi, and Tsuji to Hecke characters whose conductors are divisible by  $p$ . In future research, we hope to apply our result to consider the  $p$ -adic Beilinson conjecture for algebraic Hecke characters, extending the work of [2] to the case for characters whose conductors are divisible by  $p$ .

Let  $K$  be an imaginary quadratic field and  $E$  be an elliptic curve defined over  $K$  with complex multiplication by the integer ring  $\mathcal{O}_K$  of  $K$ . Then  $K$  has class number 1 since the class number of  $K$  equals to  $[K(j(E)) : K]$  for  $j$ -invariant  $j(E)$  of  $E$  by [26, II, §4, Theorem 4.3]. We fix a Weierstrass model  $E$  by

$$E: y^2 = 4x^3 - g_2x - g_3 \quad (g_2, g_3 \in \mathcal{O}_K)$$

defined over  $\mathcal{O}_K$ . We assume that  $E$  has a good ordinary reduction at a prime above  $p \geq 5$ . Let  $\Gamma$  be the period lattice corresponding to the invariant differential  $\omega = dx/y$  obtained by the uniformization theorem. Then we have  $E(\mathbb{C}) \cong \mathbb{C}/\Gamma$ . By [27, VI, Theorem 4.1(a)] we have  $\text{End}(E(\mathbb{C})) \cong \{\alpha \in \mathbb{C} \mid \alpha\Gamma \subset \Gamma\}$ . In addition, since the elliptic curve  $E$  has complex multiplication by  $\mathcal{O}_K$ , we have  $\mathcal{O}_K\Gamma = \Gamma$ . Hence there exists a **complex period**  $\Omega \in \mathbb{C}^\times$  satisfying  $\Gamma = \Omega\mathcal{O}_K$ . Let  $\psi := \psi_{E/K} : \mathbb{A}_K^\times/K^\times \rightarrow \mathbb{C}^\times$  be the Hecke character of  $K$  associated to  $E$ , where  $\mathbb{A}_K^\times$  is the idèle group of  $K$  [26, II, Theorem 9.2]. Let  $\mathfrak{p}$  be a prime ideal of  $\mathcal{O}_K$  above  $p$ . Since  $E$  has a good reduction at  $\mathfrak{p}$ , by [26, II, Theorem 9.2], the Hecke character  $\psi$  of  $K$  associated to  $E$  is unramified, i.e.  $\psi(\mathcal{O}_{K_{\mathfrak{p}}}^\times) = 1$ , where  $K_{\mathfrak{p}}$  is the completion at  $\mathfrak{p}$  and  $\mathcal{O}_{K_{\mathfrak{p}}}$  is its integer ring. Then we define  $\psi(\mathfrak{p})$  to be

$$\psi(\mathfrak{p}) := \psi\left(\dots, 1, 1, \underset{\substack{\uparrow \\ \mathfrak{p}\text{-th}}}{\pi}, 1, 1, \dots\right),$$

where  $\pi$  is a uniformizer at  $\mathfrak{p}$ . Since  $\psi$  is unramified at  $\mathfrak{p}$ ,  $\psi(\mathfrak{p})$  is well-defined independent of the choice of  $\mathfrak{p}$ . From the assumption that  $E$  is ordinary at a prime  $\mathfrak{p}$  above  $p$ ,  $p$  splits  $(p) = \mathfrak{p}\bar{\mathfrak{p}}$  by [22, Chapter 10, §4, Theorem 10], where  $\bar{\mathfrak{p}}$  is the complex conjugation of  $\mathfrak{p}$ . Since  $\psi(\mathfrak{p})$  is the value of  $\psi$  at an idèle with 1's in its archimedean components, we have  $\psi(\mathfrak{p}) \in \mathcal{O}_K$ . Then we have  $\pi = \psi(\mathfrak{p})$  and  $p = \pi\bar{\pi}$  ( $\bar{\pi}$  is the complex conjugation of  $\pi$ ). Now we take an immersion  $\bar{K} \hookrightarrow \mathbb{C}_p$  satisfying  $|\pi| < 1$  in  $\mathbb{C}_p$ .

For a natural number  $N$ , we let  $\mathfrak{g}' := \mathfrak{g}p^N$  be an integral ideal of  $\mathcal{O}_K$  which is divisible by the conductor  $\mathfrak{f}$  of  $\psi$ , where  $\mathfrak{g} = (g)$  is the integral ideal of  $\mathcal{O}_K$  prime to  $p$ .

Let  $I(\mathfrak{g}')$  be a group of fractional ideals of  $\mathcal{O}_K$  prime to  $\mathfrak{g}'$ . Let  $\varphi : I(\mathfrak{g}') \rightarrow \overline{K}^\times$  be an algebraic Hecke character of infinite type  $(m, n) \in \mathbb{Z}^2$  whose conductor divides  $\mathfrak{g}'$ : In other words,  $\varphi$  is the group homomorphism satisfying

$$\varphi((\alpha)) = \chi(\alpha)\alpha^m\bar{\alpha}^n \quad \text{for any } \alpha \in \mathcal{O}_K \text{ prime to } \mathfrak{g}'$$

for some finite character  $\chi : (\mathcal{O}_K/\mathfrak{g}')^\times \rightarrow \overline{K}^\times$ . The classical complex Hecke  $L$ -function of  $\varphi$  is defined by

$$L_{\mathfrak{g}'}(s, \varphi) := \sum_{(\mathfrak{a}, \mathfrak{g}')=1} \frac{\varphi(\mathfrak{a})}{N(\mathfrak{a})^s} \quad \text{for } \operatorname{Re}(s) \gg 0 \quad (1)$$

where the sum runs over all integral ideals  $\mathfrak{a}$  of  $\mathcal{O}_K$  prime to  $\mathfrak{g}'$ . By defining  $\varphi(\mathfrak{a})$  by 0 if  $\mathfrak{a}$  is not prime to  $\mathfrak{g}'$ , we can consider  $\varphi$  as a function on the group of all fractional ideals of  $\mathcal{O}_K$ . If  $\operatorname{Re}(s) > (m+n)/2 + 1$ , the Hecke  $L$ -function (1) converges absolutely. The analytic continuation and functional equation of  $L_{\mathfrak{g}'}(s, \varphi)$  is well known. If we put

$$\widehat{L}_{\mathfrak{g}'}(s, \varphi) := \frac{(d_K N(\mathfrak{g}'))^{s/2} \Gamma(s - \min\{m, n\}) L_{\mathfrak{g}'}(s, \varphi)}{(2\pi)^{s - \min\{m, n\}}}$$

for  $\pi = 3.1415\dots$  and discriminant  $d_K$  of  $K$ , we have

$$\widehat{L}_{\mathfrak{g}'}(s, \varphi) = W \cdot \widehat{L}_{\mathfrak{g}'}(1 + m + n - s, \bar{\varphi}), \quad (2)$$

where  $W$  is a constant of absolute value 1, called the Artin root number. Since  $\Gamma$ -functions of both sides of (2) have no poles on  $\{(m, n) \in \mathbb{Z}^2 \mid m < 0, n \geq 0\}$  or  $\{(m, n) \in \mathbb{Z}^2 \mid n < 0, m \geq 0\}$ , following Deligne [14, Définition 1.3], these two sets are **critical** domains. In addition, for integers  $m, n$  with  $m < 0$  and  $n \geq 0$  (resp.  $n < 0$  and  $m \geq 0$ ), by Damerell's theorem [3, Corollary 2.12], we have

$$\frac{L_{\mathfrak{g}'}(0, \varphi)}{\Omega^{n-m}} \in \overline{\mathbb{Q}} \quad \left( \text{resp. } \frac{L_{\mathfrak{g}'}(0, \bar{\varphi})}{\Omega^{m-n}} \in \overline{\mathbb{Q}} \right). \quad (3)$$

(3) asserts that Deligne's conjecture [14, Conjecture 1.8] holds with respect to  $L$ -functions associated to algebraic Hecke characters. The main theorem of this article is that the  $p$ -adic analogue of  $L_{\mathfrak{g}'}(0, \varphi)/\Omega^{n-m}$  can be expressed by using the  $p$ -adic Eisenstein–Kronecker series in the non-critical domain. In order to achieve our purpose, we use the  $p$ -adic Eisenstein–Kronecker series as Coleman functions constructed by [1], which we call the **Coleman Eisenstein–Kronecker series**.

By the conditions  $(g, p) = 1$  and  $(\pi, \bar{\pi}) = 1$ , by the Chinese remainder theorem, we have

$$(\mathcal{O}_K/\mathfrak{g}')^\times \cong (\mathcal{O}_K/\mathfrak{g})^\times \times (\mathcal{O}_K/(\pi^N))^\times \times (\mathcal{O}_K/(\bar{\pi}^N))^\times.$$

Therefore if we restrict the finite character  $\chi$  on  $(\mathcal{O}_K/\mathfrak{g}')^\times$  respectively to  $\chi_{\mathfrak{g}}: (\mathcal{O}_K/\mathfrak{g})^\times \rightarrow \overline{K}^\times$ ,  $\chi_1: (\mathcal{O}_K/(\pi^N))^\times \rightarrow \overline{K}^\times$ , and  $\chi_2: (\mathcal{O}_K/(\overline{\pi}^N))^\times \rightarrow \overline{K}^\times$ , we can decompose  $\chi$  by

$$\chi(\alpha) = \chi_{\mathfrak{g}}(\alpha)\chi_1(\alpha)\chi_2(\alpha).$$

We extend  $\chi_{\mathfrak{g}}, \chi_1, \chi_2$  respectively into characters with  $\mathbb{C}_p$ -values by using an inclusion map  $i: \overline{K}^\times \hookrightarrow \mathbb{C}_p^\times$ .

Now we put  $\mathbb{X} := \varprojlim_n (\mathcal{O}_K/\mathfrak{g}p^n \mathcal{O}_K)^\times$ . We define the *p-adic character*  $\phi_p: \mathbb{X} \rightarrow \mathbb{C}_p^\times$  by [15, Chapter II, §1, (5)]. Similarly to [15, Chapter II, §4.16, (49)] or [4, §2.4], for a measure  $\mu_g$  on  $\mathbb{X}$  (for details, see (27)), we define the value of the *p-adic L-function* at the *p-adic character*  $\phi_p: \mathbb{X} \rightarrow \mathbb{C}_p^\times$  by

$$L_p(\phi_p) := \int_{\mathbb{X}} \phi_p(\alpha) d\mu_g(\alpha).$$

Now for the *p-adic character*  $\varphi_p: \mathbb{X} \rightarrow \mathbb{C}_p^\times$  defined by  $\varphi_p(\alpha) := \varphi((\alpha))$  for any  $\alpha \in \mathcal{O}_K$  prime to  $\mathfrak{g}p$ , since

$$\mathbb{X} \cong (\mathcal{O}_K/\mathfrak{g})^\times \times (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times,$$

we have  $\varphi_p = \chi_{\mathfrak{g}}\chi_1\chi_2\kappa_1^m\kappa_2^n$  as *p-adic characters* on  $\mathbb{X}$ , where  $\kappa_1, \kappa_2$  are the projections to the first and second factors of the following isomorphism:

$$(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \mathcal{O}_{K_p}^\times \times \mathcal{O}_{K_{\overline{p}}}^\times \xrightarrow{\cong} \mathbb{Z}_p^\times \times \mathbb{Z}_{\overline{p}}^\times, \quad \alpha \mapsto (\kappa_1(\alpha), \kappa_2(\alpha)). \quad (4)$$

Note that  $\chi_1, \chi_2$  which are components of  $\varphi_p$  can be regarded as characters on  $\mathbb{X}$  by the following liftings of the natural projections:

$$\begin{aligned} \mathbb{X} &\twoheadrightarrow \varprojlim_n (\mathcal{O}_K/p^n \mathcal{O}_K)^\times \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \xrightarrow{\kappa_1} \mathcal{O}_{K_p}^\times \twoheadrightarrow (\mathcal{O}_K/(\pi^N))^\times \xrightarrow{\chi_1} \overline{K}^\times \xrightarrow{i} \mathbb{C}_p^\times, \\ \mathbb{X} &\twoheadrightarrow \varprojlim_n (\mathcal{O}_K/p^n \mathcal{O}_K)^\times \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \xrightarrow{\kappa_2} \mathcal{O}_{K_{\overline{p}}}^\times \twoheadrightarrow (\mathcal{O}_K/(\overline{\pi}^N))^\times \xrightarrow{\chi_2} \overline{K}^\times \xrightarrow{i} \mathbb{C}_p^\times. \end{aligned}$$

Let  $\Omega_p \in \mathcal{O}_{\mathbb{C}_p}^\times$  be the *p-adic period* obtained from the isomorphism between the formal group of the elliptic curve and the multiplicative formal group (for details, see (22)). By calculation, we know that for integers  $m, n$  with  $m < 0$  and  $n \geq 0$ ,

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = (\text{interpolation factor}) \times \frac{L_{\mathfrak{g}'}(0, \varphi)}{\Omega^{n-m}}$$

holds. Hence  $L_p(\varphi_p)/\Omega_p^{n-m}$  is the *p-adic analogue* of  $L_{\mathfrak{g}'}(0, \varphi)/\Omega^{n-m}$ .

K. Bannai, S. Kobayashi, and T. Tsuji related the non-critical values of the *p-adic L-function* and *p-adic Eisenstein–Kronecker series* constructed by the measure when the

conductor of algebraic Hecke character is *not divisible* by  $p$ . However when the conductor of the algebraic Hecke character is *divisible* by  $p$ , we *cannot express* special values of the  $p$ -adic  $L$ -function by using the  $p$ -adic Eisenstein–Kronecker series constructed by the measure. So we use the  $p$ -adic Eisenstein–Kronecker series constructed by the *Coleman integration* established in [1]. For any integers  $m, n$  with  $n \geq 0$ , let  $E_{m,n}^{\text{col}}(z)$  be the Coleman Eisenstein–Kronecker series on  $E(\mathbb{C}_p) \setminus [0]$ . We expressed non-critical values of the  $p$ -adic  $L$ -function by using the Coleman Eisenstein–Kronecker series as follows:

**Theorem 0.1** (= Theorem 3.11). *Let  $m, n$  be any integers with  $n \geq 0$ . For a primitive  $\mathfrak{g}'$ -torsion point  $\xi_{\mathfrak{g}'} := i_*(\Omega C/gp^N)$  that  $C$  is a special constant of  $z$  contained in the following equation, we have*

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = \frac{g^{-1}n!(-1)^{m+n+1}}{\tau(\overline{\chi_1})\overline{\pi}^N} \sum_{z \in (\mathcal{O}_K/\mathfrak{g}')^\times} \chi(z) E_{m+1,n+1}^{\text{col}}(\xi_{\mathfrak{g}'} z),$$

where  $\tau(\overline{\chi_1})$  is the Gauss sum defined by Lemma 3.3 and  $i_* : E(\overline{K}) \hookrightarrow E(\mathbb{C}_p)$  is the inclusion map induced by the inclusion map  $i : \overline{K} \hookrightarrow \mathbb{C}_p$ . Note that  $\Omega C/gp^N$  is the element in  $E(\overline{K})$  through the isomorphism  $\mathfrak{g}'^{-1}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}'] \cong E(\overline{K})[\mathfrak{g}']$ . Here  $E(\overline{K})[\mathfrak{g}']$  (resp.  $E(\mathbb{C})[\mathfrak{g}']$ ) is a subgroup of  $E(\overline{K})$  (resp.  $E(\mathbb{C})$ ), which consists of  $\mathfrak{g}'$ -torsion points.

## 1. Coleman integration theory and its applications

### 1.1. A brief review of Coleman integration theory

In this section, we give a brief review of Coleman integration theory. Coleman constructed  $p$ -adic integration theory by using rigid analysis which Tate introduced in [29] and defined  $p$ -adic polylogarithms in his studies of  $p$ -adic analogue of Bloch's results which related special values of  $L$ -functions of algebraic varieties to Quillen's  $K$ -groups with dilogarithms (see [12,9]). For Tate's rigid analysis, see ([10,16], or [6, §0]). The Coleman function is, roughly speaking, the class of  $p$ -adic analytic functions generalizing the rigid analytic functions and given as power series which converges on each open unit disk. For details, see [7]. In addition, Besser gave generalization of the Coleman integration on a smooth and proper algebraic variety which has a good reduction in [8].

In the Coleman integration theory, there are two important properties.

- (A) The **uniqueness principle** holds. Thus we can consider analytic continuation (for example, see [13, Corollary 2.4.5]).
- (B) We can locally integrate any differential forms and it is **unique** up to a constant by Frobenius invariance.

In Tate's rigid analysis, (A) holds (for example, see [6, Proposition 0.1.13]). But (B) does not hold because for example if  $t = 0$  is removed for a local parameter  $t$ , we cannot

integrate  $dt/t$  in the affinoid algebra. So we will extend into a bigger ring so that we can integrate any differential forms.

Let  $\mathcal{O}_{\mathbb{C}_p}$  be an integer ring of the completion  $\mathbb{C}_p$  of algebraic closure of  $\mathbb{Q}_p$ . Then a residue field of  $\mathbb{C}_p$  is  $\bar{\mathbb{F}}_p$ . Let  $X$  be a proper, smooth, and connected scheme of locally finite type of relative dimension 1 defined over  $\mathcal{O}_{\mathbb{C}_p}$  with a good reduction. Let  $X(\mathbb{C}_p)$  be a generic fiber and  $X(\bar{\mathbb{F}}_p)$  be a special fiber. According to [6, Proposition 0.3.5],  $X(\mathbb{C}_p)$  is isomorphic to a rigid analytic  $\mathbb{C}_p$ -space  $X^{\text{an}}$  obtained by a rigid analyzation of  $X$ , so we may also denote  $X(\mathbb{C}_p)$  by  $X(\mathbb{C}_p)^{\text{an}}$ .

Let  $Y \subset X$  be an open affine subscheme defined over  $\mathcal{O}_{\mathbb{C}_p}$  which is proper, smooth, connected and has a good reduction. Let  $Y(\bar{\mathbb{F}}_p)$  be a special fiber of  $Y$ . Then we can take finite points  $e_1, \dots, e_n$  such that  $X(\bar{\mathbb{F}}_p) \setminus Y(\bar{\mathbb{F}}_p) = \{e_1, \dots, e_n\}$ .

For an  $\bar{\mathbb{F}}_p$ -subscheme  $S \subset X(\bar{\mathbb{F}}_p)$ , let  $|S| := \text{sp}^{-1}(S) \subset X(\mathbb{C}_p)^{\text{an}}$  be a tube of  $S$ , where  $\text{sp}$  is the specialization map

$$\text{sp}: X(\mathbb{C}_p)^{\text{an}} \xrightarrow{\text{reduction}} X(\bar{\mathbb{F}}_p).$$

In particular, for a closed point  $x \in X(\bar{\mathbb{F}}_p)$ ,  $|x| := \text{sp}^{-1}(x)$  is an open unit disk by [6, Proposition 1.1.1]. In other words,  $|x| \cong \{z \in \mathbb{C}_p \mid |z| < 1\}$ . We denote a local parameter  $z$  around  $x$  by  $z_x$ . For  $0 < r \leq 1$ , we put  $U_r := X(\mathbb{C}_p)^{\text{an}} \setminus \bigcup_{i=1}^n D(\tilde{e}_i, r)^-$ . Here,  $\tilde{e}_i \in X(\mathbb{C}_p)$  is a lift of  $e_i$  and  $D(\tilde{e}_i, r)^-$  is a closed disk centered at  $\tilde{e}_i$  with radius  $r$ .

Let  $\mathcal{O}_{X^{\text{an}}}$  be a structure sheaf of the rigid analytic  $\mathbb{C}_p$ -space  $X^{\text{an}} = X(\mathbb{C}_p)^{\text{an}}$ . Let  $U := \text{sp}^{-1}(X(\bar{\mathbb{F}}_p)) = \varprojlim_{r \rightarrow 1} U_r$ . Let  $A(U)$  be a subset of a locally rigid analytic function  $\mathcal{L}(U)$  on  $U$  such that

$$A(U) := \{f \in \mathcal{L}(U) \mid f|_{X^{\text{an}}} \in \mathcal{O}_{X^{\text{an}}}(X^{\text{an}})\}$$

where  $\mathcal{O}_{X^{\text{an}}}(X^{\text{an}}) = \Gamma(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$  is an algebra of rigid analytic functions. Let  $\Omega^1(U)$  be a space of 1-forms. These are respectively rings of overconvergent functions and overconvergent 1-forms in the sense of Monsky–Washnitzer. In other words, we can regard  $A(U)$  as  $\Gamma(Y(\bar{\mathbb{F}}_p)[, j^\dagger \mathcal{O}_{Y(\bar{\mathbb{F}}_p)[}])$  and  $\Omega^1(U)$  as  $\Gamma(Y(\bar{\mathbb{F}}_p)[, j^\dagger \Omega^1_{Y(\bar{\mathbb{F}}_p)[}])$ , where for an open immersion  $j: S \hookrightarrow \bar{S}$  corresponding to a closed subscheme  $\bar{S} \subset X(\bar{\mathbb{F}}_p)$  and an open  $S$  of  $\bar{S}$ ,  $j^\dagger$  is a functor defined by [6, §2.1, (2.1.1.1)].

A **branch** of  $p$ -adic logarithms is any locally analytic homomorphism  $\log: \mathbb{C}_p^\times \rightarrow \mathbb{C}_p^+$  with the usual expansion for  $\log$  around 1. Such a function is determined by choosing  $\pi \in \mathbb{C}_p$  such that  $|\pi| < 1$  and declaring  $\log(\pi) = 0$ . Coleman's  $p$ -adic integration theory depends on the choice of the branch of the  $p$ -adic logarithms. We choose such a branch “log” of  $p$ -adic logarithms. We define

$$A_{\log}(|x|) := \begin{cases} A(|x|) & \text{if } x \in Y(\bar{\mathbb{F}}_p), \\ \lim_{r \rightarrow 1} A(|x| \cap U_r)[\log(z_x)] & \text{if } x \in X(\bar{\mathbb{F}}_p) \setminus Y(\bar{\mathbb{F}}_p), \end{cases}$$

$$\Omega^1_{\log}(|x|) := A_{\log}(|x|) dz_x.$$

Here, note that if  $x \in Y(\bar{\mathbb{F}}_p)$ , then  $A([x])$  is the ring  $\mathcal{O}_{[x]}([x])$  consisting of formal power series  $f(z_x) = \sum_{n=0}^{\infty} a_n z_x^n$  which converges on  $\{z_x \in \mathbb{C}_p \mid |z_x| < 1\}$ , and if  $x \in X(\bar{\mathbb{F}}_p) \setminus Y(\bar{\mathbb{F}}_p)$ , then formal power series  $f(z_x) = \sum_{n=-\infty}^{\infty} a_n z_x^n$  which converges on  $\{z_x \in \mathbb{C}_p \mid r < |z_x| < 1\}$  for some  $r < 1$ . We define rings of locally analytic functions and 1-forms on  $U$  by

$$A_{\text{loc}}(U) := \prod_{x \in X(\bar{\mathbb{F}}_p)} A_{\log}([x]), \quad \Omega_{\text{loc}}^1(U) := \prod_{x \in X(\bar{\mathbb{F}}_p)} \Omega_{\log}^1([x]).$$

These are independent of the choice of  $z_x$ . We can define a differential  $d : A_{\text{loc}}(U) \rightarrow \Omega_{\text{loc}}^1(U)$  in the natural way. Then  $d : A_{\text{loc}}(U) \rightarrow \Omega_{\text{loc}}^1(U)$  is surjective. The point is that we are able to integrate  $dz/z$  by adding logarithms. So we can integrate any elements in  $\Omega_{\text{loc}}^1(U)$  i.e. (B) holds. But since  $\text{Ker}(d) = \prod_{x \in X(\bar{\mathbb{F}}_p)} \mathbb{C}_p$ , we do not have the notion of analytic continuation yet i.e. (A) does not hold.

**Definition 1.1** (*Coleman function*). Coleman defined a subalgebra  $M(U) \subset A_{\text{loc}}(U)$  equipped with an integration map

$$\int : M(U) \otimes_{A(U)} \Omega^1(U) \rightarrow M(U)/\mathbb{C}_p, \quad \omega \mapsto F_\omega := \int \omega$$

which is one of  $\mathbb{C}_p$ -linear maps, in order to obtain the notion of analytic continuation, with the surjectivity of  $d$  keeping as follows. A map  $\int$  is characterized by three properties:

- i) (The existence of a primitive function)  $dF_\omega = \omega$ .
- ii) (Frobenius invariance) For a Frobenius automorphism  $\phi : U \rightarrow U$ , we have

$$\int (\phi^*(\omega)) = \phi^* \left( \int \omega \right).$$

- iii)  $\int dg = g + C$  for  $g \in A(U)$ , where  $C$  is a constant in  $\mathbb{C}_p$ .

We call  $M(U)$  a space of **Coleman functions** on  $U$ . As for the construction of such a space  $M(U)$ , see [7, §2].

In summary, when  $f$  is a function in  $A_{\text{loc}}(U)$  and  $P(x)$  is a polynomial with  $\mathbb{C}_p$ -coefficients whose roots do not contain the roots of 1, if  $df \in M(U) \otimes_{A(U)} \Omega_{\text{loc}}^1(U)$  and  $P(\phi^*)f \in M(U)$ , then we have  $f \in M(U)$ . Note that we extend the classes of integrable differential forms so that the integration is unique up to a constant in  $\mathbb{C}_p$ , not in  $\prod_{x \in X(\bar{\mathbb{F}}_p)} \mathbb{C}_p$ . In other words, we have an exact sequence

$$0 \longrightarrow \mathbb{C}_p \longrightarrow M(U) \xrightarrow{d} M(U) \otimes_{A(U)} \Omega^1(U) \longrightarrow 0.$$



The entire theory turns out to be independent of the choice of  $\phi$ . The important idea is to extend the classes of the integrable differential forms from  $d(A(U))$  by using Frobenius invariance.

## 1.2. Applications of Coleman integrations

Put  $U := \mathbb{P}^1(\mathbb{C}_p) \setminus \{0, 1, \infty\}$ . Coleman defined  $p$ -adic polylogarithms recursively as follows:

**Definition 1.2** (*p-adic polylogarithm*). Let  $k$  be an integer. If  $k \geq 0$ , we define a locally analytic function

$$\ell_k \in M(U) \quad (k \geq 0)$$

satisfying

- i)  $\ell_0(z) = \frac{z}{1-z}$ .
- ii)  $d\ell_k(z) = \ell_{k-1}(z) \frac{dz}{z}$ .
- iii)  $\lim_{z \rightarrow 0} \ell_k(z) = 0$ .

$\ell_k(z)$  is an analytic function  $\ell_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$  on  $|z| < 1$ . The existence and uniqueness of  $\ell_k$  is insured by [11, Corollaire 2.2.2.1]. If  $k \leq 0$ , we take  $\ell_k \in A(U)$  satisfying i), ii), iii). This  $\ell_k$  is the **p-adic polylogarithm** defined by Coleman [12].

The important application of the  $p$ -adic polylogarithm is that Kubota–Leopoldt  $p$ -adic  $L$ -function  $L_p(s, \chi)$  associated to a non-trivial Dirichlet character  $\chi$  can be written as the sum of the  $p$ -adic polylogarithms at positive integers (i.e. non-critical values). It is well known that Kubota–Leopoldt  $p$ -adic  $L$ -function  $L_p(s, \chi)$  is obtained by interpolating at negative integers (i.e. critical points) of the complex Dirichlet  $L$ -function (see [20, §3, Theorem 3 ii])).

**Theorem 1.3.** (See Coleman [12, §7].) Let  $p$  be an odd prime. Let  $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \rightarrow \mathbb{C}_p^\times$  be a Dirichlet character with a conductor  $d > 1$  such that  $d$  is prime to  $p$ , and let  $\omega : \mathbb{Z}_p^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$  be a Teichmüller character. For all integers  $k \geq 1$ , we have

$$L_p(k, \chi\omega^{1-k}) = \left(1 - \frac{\chi(p)}{p^k}\right) \frac{g(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \overline{\chi(a)} \ell_k(\zeta^{-a}),$$

where  $\zeta$  is a primitive  $d$ -th root of 1 and  $g(\chi, \zeta)$  is the Gauss sum defined by  $g(\chi, \zeta) = \sum_{a=0}^{d-1} \chi(a) \zeta^a$ .

Substituted for  $k = 1$ , Theorem 1.3 is reduced to [20, §5, Theorem 3]. Note that Theorem 1.3 is the  $p$ -adic analogue of the classical formula

$$L(k, \chi) = \frac{g(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \overline{\chi(a)} \ell_k(\zeta^{-a}).$$

We can show this formula by using two properties of the Gauss sum

$$\sum_{a=0}^{d-1} \overline{\chi(a)} \zeta^{-an} = \chi(n) \overline{g(\chi, \zeta)}$$

and

$$g(\chi, \zeta) \overline{g(\chi, \zeta)} = d.$$

The main theorem of this article is the elliptic analogue of [Theorem 1.3](#).

## 2. Review of the classical Eisenstein–Kronecker series and its $p$ -adic analogue

In this section, we review the definition of the classical Eisenstein–Kronecker series by A. Weil [\[30\]](#) and of the  $p$ -adic Eisenstein–Kronecker series as the Coleman function by K. Bannai, H. Furusho, and S. Kobayashi in [\[1\]](#). The classical Eisenstein–Kronecker series is the elliptic analogue of the classical complex polylogarithm by using the Bloch–Wigner–Ramakrishnan polylogarithm which is invariant under the map  $z \mapsto qz$  on an elliptic curve  $\mathbb{C}^\times/q^\mathbb{Z}$  where  $q = e^{2\pi i\tau}$  for  $\text{Im}(\tau) > 0$  (see [\[32, Theorem 1\]](#)). The  $p$ -adic Eisenstein–Kronecker series as the Coleman function i.e. Coleman Eisenstein–Kronecker series is defined by constructing with generating functions appeared in the Laurent coefficients of the Kronecker theta function.

### 2.1. Review of the classical Eisenstein–Kronecker series

Recall the definition of the classical Eisenstein–Kronecker series [\[30, VIII, §12\]](#).

Let  $\Gamma \subset \mathbb{C}$  be a lattice,  $\varpi = 3.1415\dots$ ,  $A(\Gamma) = (\text{Area of } \mathbb{C}/\Gamma)/\varpi$ , and  $\chi_w(z)_\Gamma := \exp((z\bar{w} - w\bar{z})/A(\Gamma))$  for any  $z, w \in \mathbb{C}$ .

In particular, if we can write  $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$  with  $\text{Im}(\omega_2/\omega_1) > 0$ , note that

$$A(\Gamma) = \frac{1}{\varpi} \text{Im}(\omega_2/\omega_1) = \frac{1}{2\varpi i} (\omega_2\bar{\omega}_1 - \omega_1\bar{\omega}_2).$$

In addition, by direct calculations, we know the following properties.

- i)  $\chi_w(z)_\Gamma = \chi_z(-w)_\Gamma = \chi_z(w)_\Gamma^{-1}$ .
- ii)  $\chi_w(az)_\Gamma = \chi_{\bar{a}w}(z)_\Gamma$  for any  $a \in \mathbb{C}$ .
- iii)  $z \in \Gamma \Leftrightarrow \chi_\gamma(z)_\Gamma = 1$  for any  $\gamma \in \Gamma$ .

**Definition 2.1** (*Eisenstein–Kronecker–Lerch series*). Let  $a$  be an integer and  $z_0, w_0 \in \mathbb{C}$  be complex numbers. The **Eisenstein–Kronecker–Lerch series** is defined by

$$K_a^*(z_0, w_0, s; \Gamma) := \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\bar{z}_0 + \bar{\gamma})^a}{|z_0 + \gamma|^{2s}} \chi_{w_0}(\gamma) \quad (s \in \mathbb{C}).$$

This series converges absolutely for  $\operatorname{Re}(s) > a/2 + 1$ .

Hereafter, by abuse of notations, we omit “ $\Gamma$ ” except the case where we want to express the lattice clearly.  $K_a^*(z_0, w_0, s)$  has the following important properties.

**Proposition 2.2.** *Let  $a$  be an integer and  $z_0, w_0 \in \mathbb{C}$  be complex numbers.*

- i)  $K_a^*(z_0, w_0, s)$  can be continued meromorphically on  $\mathbb{C}$  as a function of  $s$ . Moreover, if  $a = 0$  and  $w_0 \in \Gamma$ ,  $K_a^*(z_0, w_0, s)$  has a simple pole at  $s = 1$ .
- ii)  $K_a^*(z_0, w_0, s)$  has a functional equation:

$$\Gamma(s) K_a^*(z_0, w_0, s) = A^{a+1-2s} \Gamma(a+1-s) K_a^*(w_0, z_0, a+1-s) \chi_{z_0}(w_0),$$

where  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$  ( $\operatorname{Re}(s) > 0$ ) is a Gamma function.

**Proof.** If  $a \geq 0$ , see [30, VIII, §13]. If  $a \leq 0$ , see [4, Proposition 2.4].  $\square$

**Definition 2.3** (*Eisenstein–Kronecker number*). Let  $z_0, w_0 \in \mathbb{C}$  and we take  $a, b \in \mathbb{Z}$  as  $(a, b) \neq (1, -1)$  if  $w_0 \in \Gamma$ . The **Eisenstein–Kronecker number**  $e_{a,b}^*(z_0, w_0)$  is defined by

$$e_{a,b}^*(z_0, w_0) := K_{a+b}^*(z_0, w_0, b) = \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\bar{z}_0 + \bar{\gamma})^a}{(z_0 + \gamma)^b} \chi_{w_0}(\gamma).$$

For  $(a, b) = (0, 0)$ , we have  $e_{0,0}^*(z_0, w_0) := K_0^*(z_0, w_0, 0) = -\chi_{z_0}(w_0)$ .

We define the Kronecker theta function. The Kronecker theta function was defined by using the reduced theta function associated to the divisor  $[0]$  (i.e. the holomorphic pseudo-periodic function with the Appell–Humbert data) by using that a group of isomorphism classes of invertible sheaves on the torus  $\mathbb{C}/\Gamma$  is classified by Appell–Humbert’s theorem. For details, see [3, Example 1.9].

**Definition 2.4** (*Kronecker theta function*). Let  $\theta(z)$  be a reduced theta function associated to the divisor  $[0]$  defined by [3, Example 1.9].  $\theta(z)$  is characterized by  $\theta'(0) = 1$ .

By using this  $\theta(z)$ , the Kronecker theta function is defined as follows. For any  $z, w \in \mathbb{C}$ , we define the **Kronecker theta function**  $\Theta(z, w)$  by

$$\Theta(z, w) := \frac{\theta(z+w)}{\theta(z)\theta(w)}.$$

In addition, for  $z_0, w_0 \in \mathbb{C}$ , we define

$$\Theta_{z_0, w_0}(z, w) := \exp\left(-\frac{z_0 \bar{w}_0}{A}\right) \exp\left(-\frac{z \bar{w}_0 + w \bar{z}_0}{A}\right) \Theta(z + z_0, w + w_0). \quad (5)$$

According to [3, Proposition 1.16],  $\Theta_{z_0, w_0}(z, w)$  has the following distribution relation: Let  $c, c' \in \Gamma$ ,  $n$  be a natural number, and  $z_0, w_0 \in \mathbb{C}$ . Then we have

$$\sum_{w_n \in \pi^{-n} \Gamma / \Gamma} \chi_{w_n}(c) \Theta_{z_0, w_0 + w_n}(z, w) = \pi^n \chi_c(w_0) \Theta_{(z_0 - c)/\bar{\pi}^n, \pi^n w_0}(z/\bar{\pi}^n, \pi^n w), \quad (6)$$

$$\sum_{z_m \in \pi^{-m} \Gamma / \Gamma} \chi_{c'}(z_m) \Theta_{z_0 + z_m, w_0}(z, w) = \pi^m \Theta_{\pi^m z_0, c'/\bar{\pi}^m}(\pi^m z, w/\bar{\pi}^m). \quad (7)$$

$\Theta_{z_0, w_0}(z, w)$  can be expanded to the Laurent series of  $z, w$  as the generating function of the following Eisenstein–Kronecker number.

**Theorem 2.5.**  $\Theta_{z_0, w_0}(z, w)$  has a Laurent expansion in the neighborhood of  $(z, w) = (0, 0)$ , that is,

$$\Theta_{z_0, w_0}(z, w) = \chi_{z_0}(w_0) \frac{\delta_{z_0}}{z} + \frac{\delta_{w_0}}{w} + \sum_{a, b \geq 0} (-1)^{a+b} \frac{e_{a, b+1}^*(z_0, w_0)}{a! A^a} z^b w^a, \quad (8)$$

where  $\delta_x$  is defined by

$$\delta_x = \begin{cases} 1 & (x \in \Gamma), \\ 0 & (\text{otherwise}). \end{cases}$$

**Proof.** See [3, §1.14, Theorem 1.17].  $\square$

Substituting  $w_0 = 0$  for the formula (8), we define a function  $F_{z_0, b}(z)$  as follows.

**Definition 2.6.** For any  $z_0 \in \mathbb{C}$ , we define  $F_{z_0, b}$  by a function satisfying

$$\Theta_{z_0, 0}(z, w) = \sum_{b \geq 0} F_{z_0, b}(z) w^{b-1}. \quad (9)$$

If  $z_0 = 0$ , we define  $F_b(z) := F_{0, b}(z)$ . We have  $F_0(z) = 1$  by observing coefficient of  $w^{-1}$  in the formula (8). We observe  $F_1(z)$ . Noting that  $\Theta_{0, 0}(z, w) = \Theta(z, w) := \theta(z + w)/\theta(z)\theta(w)$  and observing coefficients of  $w^0$  in the formulas (8) and (9), we find that  $F_1(z)$  satisfies

$$F_1(z) = \lim_{w \rightarrow 0} (\Theta(z, w) - w^{-1}) = \frac{\theta'(z)}{\theta(z)}.$$

$F_{z_0,b}(z)$  is dependent only on a choice of  $z_0$  modulo  $\Gamma$  because we have

$$\Theta_{z_0+\gamma,0}(z,w) = \exp\left[-\frac{w(\bar{z}_0 + \bar{\gamma})}{A}\right] \Theta(z+z_0+\gamma,w) = \Theta_{z_0,0}(z,w).$$

As we define later, the  $p$ -adic analogue of  $F_{z_0,b}$  for a variable  $z_0$  is constructed as the Coleman function by gluing together each open unit disk. By using the  $p$ -adic analogue of  $F_{z_0,b}$ , we construct the  $p$ -adic analogue of the Eisenstein–Kronecker series  $E_{m,n}$ . We have the Laurent expansion of  $F_{z_0,b}(z)$  from [Theorem 2.5](#).

**Corollary 2.7** (*Generating function*). *For any  $b \geq 0$ , the Laurent series of  $F_{z_0,b}(z)$  at  $z = 0$  can be written as*

$$F_{z_0,b}(z) = \frac{\delta_{z_0,b}}{z} + \sum_{a \geq 0} (-1)^{a+b-1} \frac{e_{a,b}^*(0, z_0)}{a! A^a} z^a,$$

where

$$\delta_{x,b} = \begin{cases} 1 & (b = 0 \text{ and } x \in \Gamma), \\ 0 & (\text{otherwise}). \end{cases}$$

**Proof.** See [\[4, Corollary 2.11\]](#).  $\square$

When we define the  $p$ -adic Eisenstein–Kronecker series later, we use the connection function  $L_n(z)$  of  $F_b(z)$  for  $b \geq 0$ . We define the connection function  $L_n(z)$  by

$$\Xi(z,w) := \exp(-F_1(z)w) \Theta(z,w) = \sum_{n \geq 0} L_n(z) w^{n-1}.$$

Since  $F_b(z) := F_{0,b}(z)$ ,  $\Theta(z,w) = \Theta_{0,0}(z,w) = \sum_{b \geq 0} F_b(z) w^{b-1}$ , and by giving “exp” Taylor expansion, we have

$$L_n(z) = \sum_{\substack{a+b=n \\ a \geq 0, b \geq 0}} \frac{(-F_1(z))^a F_b(z)}{a!} = \sum_{b=0}^n \frac{(-F_1(z))^{n-b}}{(n-b)!} F_b(z). \quad (10)$$

By the translation by  $\gamma \in \Gamma$  of  $\Xi(z,w)$ ,  $L_n(z)$  is the periodic function on  $\mathbb{C}/\Gamma$ , i.e. the elliptic function and the holomorphic function on  $\mathbb{C} \setminus \Gamma$ . If  $n = 0, 1$ , by the formula [\(10\)](#), we have

$$L_0(z) = F_0(z) = 1, \quad L_1(z) = -F_1(z)F_0(z) + F_1(z) = 0.$$

Since  $\Theta_{z_0,0}(z,w) = \exp(F_{z_0,1}(z)w) \Xi(z+z_0,w)$ , we have a relation

$$F_{z_0,b}(z) = \sum_{n=0}^b \frac{F_{z_0,1}(z)^{b-n}}{(b-n)!} L_n(z+z_0) \quad (11)$$

between  $F_{z_0,b}(z)$  and  $L_n(z)$ .

Now we assume that a complex torus has an algebraic model. Let  $K$  be an imaginary quadratic field and we fix an immersion  $K \hookrightarrow \mathbb{C}$ . We define an elliptic curve  $E$  over  $K$  by a Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in K,$$

and its invariant differential  $\omega$  by  $\omega = dx/y$ . By the uniformization theorem, there exists a period lattice  $\Gamma \subset \mathbb{C}$  of  $E$  satisfying an isomorphism

$$\xi : \mathbb{C}/\Gamma \xrightarrow{\sim} E(\mathbb{C}), \quad z \mapsto (\wp(z), \wp'(z)),$$

where  $\wp(z)$  is the Weierstrass  $\wp$ -function. Then we have  $\omega = d\wp(z)/\wp'(z) = dz$ . According to [4, Proposition 1.12], the connection function  $L_n(z)$  is *algebraic* since we have  $L_n(z) \in K[\wp(z), \wp'(z)]$  as the rational function on  $E$  over  $K$ . In addition, we assume that the elliptic curve  $E$  has complex multiplication by the integer ring  $\mathcal{O}_K$  of  $K$ . Then by Damerell's theorem,  $\Theta_{z_0, w_0}(z, w)$  and  $F_{z_0, b}(z)$  are *algebraic* in the following sense:

- If  $z_0, w_0$  correspond to torsion points in  $\mathbb{C}/\Gamma \cong E(\mathbb{C})$ , then

$$\Theta_{z_0, w_0}(z, w) - \chi_{z_0}(w_0) \frac{\delta_{z_0}}{z} - \frac{\delta_{w_0}}{w} \in \overline{\mathbb{Q}}[[z, w]], \quad (12)$$

where if  $x \in \Gamma$  then  $\delta_x = 1$ , and otherwise  $\delta_x = 0$  (see [1, Theorem 2.13]).

- If  $z_0 \in \mathbb{C}$  corresponds to a torsion point in  $\mathbb{C}/\Gamma \cong E(\mathbb{C})$ , then

$$F_{z_0, b}(z) - \frac{\delta_{z_0, b}}{z} \in \overline{\mathbb{Q}}[[z]], \quad (13)$$

where if  $b = 1$  and  $x \in \Gamma$  then  $\delta_{x, b} = 1$ , and otherwise  $\delta_{x, b} = 0$  (see [1, Corollary 2.14]).

These algebraicities allow us to view this value as an element in  $\mathbb{C}_p$  through the immersion  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ . In addition, since  $E$  has complex multiplication by the integer ring  $\mathcal{O}_K$ , by [26, II, §1, Proposition 1.1], there is a unique isomorphism

$$[\cdot] : \mathcal{O}_K \xrightarrow{\sim} \text{End}(E)$$

such that for any invariant differential  $\omega = dx/y$  on  $E$ ,

$$[\alpha]^*(\omega) = \alpha\omega \quad \text{for all } \alpha \in \mathcal{O}_K.$$

For any  $0 \neq \alpha \in \mathcal{O}_K$ , let  $E[\alpha]$  be a subgroup of  $E(\overline{\mathbb{Q}})$  such that  $E[\alpha] := \{P \in E(\overline{\mathbb{Q}}) \mid [\alpha]P = 0\}$ . According to [1, Proposition 2.15], the function  $F_{z_0,b}(z)$  is known to satisfy the following distribution relation with respect to  $E[\alpha]$ :

$$\sum_{z_\alpha \in E[\alpha]} F_{z_0+z_\alpha,b}(z) = \alpha \bar{\alpha}^{1-b} F_{\alpha z_0,b}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K. \quad (14)$$

## 2.2. Review of the $p$ -adic analogue of the Eisenstein–Kronecker series

For an integer  $b \geq 0$ , we review that the  $p$ -adic analogue of  $F_{z_0,b}$  is constructed as the Coleman function on an elliptic curve along [1].

Let  $E$  be an elliptic curve in Section 0. Let  $t := -2x/y$  be a formal parameter of  $E$  at the origin. Let  $\hat{E}$  be a formal group of  $E$  for  $t$  equipped with a maximal ideal of a complete local ring of  $\mathcal{O}_K$  and group operations  $\oplus$ . Let  $\lambda : \hat{E} \xrightarrow{\sim} \hat{\mathbb{G}}_a$  be a normalized formal logarithm for an additive formal group  $\hat{\mathbb{G}}_a$ . For a torsion point  $z_0 \in E(\overline{\mathbb{Q}})_{\text{tors}}$ , we define  $\hat{F}_{z_0,b}(t)$  by

$$\hat{F}_{z_0,b}(t) := F_{z_0,b}(z)|_{z=\lambda(t)} = F_{z_0,b}(\lambda(t)).$$

By formula (11) and  $\lambda(z_0) = 0$ , we have

$$\hat{F}_{z_0,b}(t) = \sum_{n=0}^b \frac{\hat{F}_{z_0,1}(t)^{b-n}}{(b-n)!} \hat{L}_{z_0,n}(t) \quad (15)$$

where  $\hat{L}_{z_0,n}(t) := L_n(z + z_0)|_{z=\lambda(t)}$ .

By formula (13), we consider  $\hat{F}_{z_0,b}(t)$  as power series with  $\mathbb{C}_p$ -coefficients through the immersion  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ . In other words, if  $z_0 \in E(\overline{\mathbb{Q}})$  is a torsion point with an order prime to  $\mathfrak{p}$ , according to [4, Proposition 2.16], the following series

$$\hat{F}_{z_0,b}(t) - \frac{\delta_{z_0,b}}{t} = \sum_{a \geq 0} (-1)^{a+b-1} \frac{e_{a,b}^*(0, z_0)}{a! A^a} z^a \Big|_{z=\lambda(t)} \in \mathbb{C}_p[[t]] \quad (16)$$

converges on  $B(0, 1) := \{t \in \mathbb{C}_p \mid |t| < 1\}$  if  $b \neq 1$  or  $z_0 \neq 0$ . In particular, this series is a rigid analytic function on  $B(0, 1)$ . Moreover  $\hat{F}_1(t) := \hat{F}_{0,1}(t)$  converges on  $\{t \in \mathbb{C}_p \mid 0 < |t| < 1\}$ .

In addition, we have a formula for translation by  $\pi^n$ -torsion points.

**Proposition 2.8** (Translation). *Let  $z_0 \in E(\overline{\mathbb{Q}})$  be a torsion point with an order prime to  $\mathfrak{p}$ . Then we have*

$$\hat{F}_{z_0,b}(t \oplus t_n) = \hat{F}_{z_0+z_n,b}(t),$$

where  $t_n \in \widehat{E}[\pi^n]$  is a  $\pi^n$ -torsion point and  $z_n \in E(\overline{\mathbb{Q}})_{\text{tors}}$  is the image of  $t_n$  through an inclusion map  $\widehat{E}(\mathfrak{m}_{\mathbb{C}_p})_{\text{tors}} \hookrightarrow E(\overline{\mathbb{Q}})_{\text{tors}} \hookrightarrow \mathbb{C}/\Gamma$ . Here  $\mathfrak{m}_{\mathbb{C}_p}$  is a maximal ideal of an integer ring  $\mathcal{O}_{\mathbb{C}_p}$  of  $\mathbb{C}_p$ .

**Proof.** See [1, Lemma 2.17].  $\square$

Let  $F \subset \mathbb{C}_p$  be a finite extension field of  $K_{\mathfrak{p}}$ . By abuse of notations, we denote the extension of  $E$  into an integer ring  $\mathcal{O}_F$  of  $F$  by again  $E$ . For  $\pi := \psi(\mathfrak{p})$ , let

$$\phi : E \rightarrow E$$

be a Frobenius automorphism induced by a multiplication by  $[\pi]$ .

Let  $E(\mathbb{C}_p) := E(\mathbb{C}_p)^{\text{an}}$  be an extension into  $\mathbb{C}_p$  of a rigid analytic  $F$ -space  $E(F)^{\text{an}}$ , and we fix a variable  $z$  on  $E(\mathbb{C}_p)$ .

Each residue disk of  $E(\mathbb{C}_p)$  contains a Teichmüller representative, where a Teichmüller representative is a unique element in the residue disk fixed by some power of proper Frobenius. By the choice of Frobenius morphism  $\phi$ , a Teichmüller representative is a torsion point  $z_0$  with an order prime to  $\mathfrak{p}$ .

For  $t = -2x/y$ , an open unit disk  $\{t \in \mathbb{C}_p \mid |t| < 1\}$  expresses the residue disk  $]0[ := \text{sp}^{-1}(0) \subset E(\mathbb{C}_p)$  containing a unit element with respect to the group operations of the elliptic curve  $E$ , where  $\text{sp} : E(\mathbb{C}_p)^{\text{an}} \xrightarrow{\text{reduction}} E(\overline{\mathbb{F}}_p)$  is a specialization map.

Let  $z_0 \in E(\mathbb{C}_p)$  be a torsion point with an order prime to  $\mathfrak{p}$  and  $\tau_{z_0} : E \rightarrow E$  be  $\tau_{z_0}(z) := z + z_0$ . For this  $\tau_{z_0}$ , we define  $]z_0[$  by

$$]z_0[ := \tau_{z_0}(]0[). \quad (17)$$

Then  $]z_0[$  is a residue disk containing  $z_0$ .

Let  $U := E(\mathbb{C}_p) \setminus [0]$ , where  $[0]$  is a unit element in group laws of the elliptic curve. If  $t_{z_0}$  is a local parameter of  $E$  at a point  $z_0$ , by the formula (16),  $\widehat{F}_{z_0,b}(t)$  defines an element in  $A(]z_0[)$  via  $]z_0[ \cong \{t_{z_0} \in \mathbb{C}_p \mid |t_{z_0}| < 1\}$ .

**Lemma 2.9.** We define  $F_1^{\text{col}} \in A_{\text{loc}}(U)$  by

$$F_1^{\text{col}}(z)|_{]z_0[} := \widehat{F}_{z_0,1}(t) \in A(]z_0[) \subset A_{\text{log}}(]z_0[)$$

on each residue disk  $]z_0[$ , where  $z_0 \in E(\overline{\mathbb{Q}})$  is a torsion point with an order prime to  $\mathfrak{p}$ . Then  $F_1^{\text{col}}$  is a Coleman function on  $U$ .

**Proof.** See [1, Lemma 3.4].  $\square$

The formula (10) indicates that  $L_n$  is a rational function on  $E$  with poles only at  $[0]$  in  $E$ , hence is in particular a Coleman function on  $U$ . The set of Coleman functions is a ring, and we define  $F_b^{\text{col}}$  as follows.



For  $b \geq 1$ , we define  $F_b^{\text{col}}$  by if  $b = 1$ ,  $F_1^{\text{col}}(z)|_{]z_0[} := \widehat{F}_{z_0,1}(t)$ , and if  $b > 1$ ,

$$F_b^{\text{col}} := \sum_{n=0}^b \frac{(F_1^{\text{col}})^{b-n}}{(b-n)!} L_n.$$

According to [1, Proposition 3.6],  $F_b^{\text{col}}$  interpolates  $F_{z_0,b}(z)$  on each open unit disk as follows: For a torsion point  $z_0 \in E(\overline{\mathbb{Q}})_{\text{tors}}$  with an order prime to  $\mathfrak{p}$ , we have

$$F_b^{\text{col}}(z)|_{]z_0[} = \widehat{F}_{z_0,b}(t) \in A_{\log}(]z_0[).$$

In addition,  $F_b^{\text{col}}$  has the following distribution relation by [1, Proposition 3.7]:

$$\sum_{z_\alpha \in E[\alpha]} F_b^{\text{col}}(z + z_\alpha) = \alpha \bar{\alpha}^{1-b} F_b^{\text{col}}(\alpha z) \quad \text{for all } 0 \neq \alpha \in \mathcal{O}_K. \quad (18)$$

In [1, §3.3], K. Bannai, H. Furusho, and S. Kobayashi constructed the  $p$ -adic Eisenstein–Kronecker series as the Coleman function by using  $F_b^{\text{col}}$  whose constant term is chosen to satisfy the distribution relation. The distribution relation plays an important role to resolve the ambiguity of integration constants.

**Definition 2.10** (*Coleman Eisenstein–Kronecker series*). Let  $m, b$  be integers with  $b \geq 0$ . We define the **Coleman Eisenstein–Kronecker series**  $E_{m,b}^{\text{col}}$  on  $U := E(\mathbb{C}_p) \setminus [0]$  recursively as follows:

i)  $E_{0,b}^{\text{col}} := (-1)^{b-1} F_b^{\text{col}}.$

This function satisfies the distribution relation by the formula (18).

ii) If  $m > 0$ , we define  $E_{m,b}^{\text{col}}$  by the Coleman function

$$E_{m,b}^{\text{col}} := - \int E_{m-1,b}^{\text{col}} \omega$$

with a constant term normalized by satisfying the distribution relation

$$\sum_{z_\alpha \in E[\alpha]} E_{m,b}^{\text{col}}(z + z_\alpha) = \alpha^{1-m} \bar{\alpha}^{1-b} E_{m,b}^{\text{col}}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K. \quad (19)$$

iii) If  $m < 0$ , we define  $E_{m,b}^{\text{col}}$  by

$$dE_{m+1,b}^{\text{col}} := -E_{m,b}^{\text{col}} \omega,$$

where  $\omega$  is the invariant differential of the elliptic curve.

There exists uniquely such a Coleman function  $E_{m,b}^{\text{col}}$  on  $U$  defined by the iterated integration  $E_{m+1,b}^{\text{col}} := -\int E_{m,b}^{\text{col}} \omega$  satisfying the distribution relation

$$\sum_{z_\alpha \in E[\alpha]} E_{m+1,b}^{\text{col}}(z + z_\alpha) = \alpha^{-m} \bar{\alpha}^{1-b} E_{m+1,b}^{\text{col}}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.$$

The existence and uniqueness are insured by [1, Proposition 3.9].

**Remark 2.11.** The distribution relation (19) is the  $p$ -adic analogue of the distribution relation of the classical complex Eisenstein–Kronecker series

$$\sum_{z_\alpha \in E[\alpha]} E_{m,b}(z + z_\alpha) = \alpha^{1-m} \bar{\alpha}^{1-b} E_{m,b}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.$$

We can show this by using the following orthogonality of character

$$\sum_{z_\alpha \in E[\alpha]} \exp\left(\frac{z_\alpha \bar{\gamma} - \bar{z}_\alpha \gamma}{A}\right) = \begin{cases} N(\alpha) (= \alpha \bar{\alpha}) & \text{if } \gamma \in \bar{\alpha} \Gamma, \\ 0 & \text{if } \gamma \notin \bar{\alpha} \Gamma. \end{cases}$$

The construction of  $p$ -adic Eisenstein–Kronecker series in Definition 2.10 allows us to choose constant term when  $m > 0$ . By the convergence property of  $F_1$  in (16),  $E_{m,1}^{\text{col}}$  is defined at any point in  $U := E(\mathbb{C}_p) \setminus [0]$ , and if  $b > 1$  then  $E_{m,b}^{\text{col}}$  is defined on  $E(\mathbb{C}_p)$ . When  $b = 0$ , since  $F_0 = 1$  and the definition of  $E_{0,b}^{\text{col}}$ , we have  $E_{0,0}^{\text{col}} = -F_0^{\text{col}} = -1$ . This shows that we have  $E_{a,0}^{\text{col}} = 0$  for  $a < 0$ . Note that the values of  $E_{m,b}^{\text{col}}(z)$  are independent of the choice of the branch of the  $p$ -adic logarithm (see [1, Lemma 3.12]).

According to [1, Proposition 3.11], the  $p$ -adic Eisenstein–Kronecker series  $E_{m,b}^{\text{col}}$  interpolates the classical complex Eisenstein–Kronecker series  $E_{m,b}$  for  $m \leq 0$ .

For  $m, b$  being integers with  $b \geq 0$ , we define

$$E_{m,b}^{(p)}(z) := E_{m,b}^{\text{col}}(z) - \frac{1}{\pi^m \bar{\pi}^b} E_{m,b}^{\text{col}}(\pi z). \quad (20)$$

### 3. Main result

In previous results, the construction of the  $p$ -adic measure interpolating special values of algebraic Hecke characters was established by Manin and Višik [24], N. Katz [21], R.I. Yager [31], and de Shalit [15] etc. In [4], when the conductor of an algebraic Hecke character is prime to  $p$ , K. Bannai, S. Kobayashi, T. Tsuji related  $p$ -adic Eisenstein–Kronecker numbers and non-critical values of the  $p$ -adic  $L$ -function associated to algebraic Hecke characters under the another construction of the measure with the Kronecker theta function.

In this paper, by the method of  $p$ -adic analogue as Coleman functions, we related  $p$ -adic Eisenstein–Kronecker series and the non-critical values of  $p$ -adic  $L$ -function of an

imaginary quadratic field with class number 1 associated to the algebraic Hecke character whose conductor is divisible by  $p$ .

### 3.1. Preludes for main results

We keep the notation in Section 0. The  $p$ -adic  $L$ -function interpolates the classical  $L$ -function at critical points in the meaning of Deligne [14]. Let  $\varphi : I(\mathfrak{g}) \rightarrow \overline{K}^\times$  be an algebraic Hecke character of infinite type  $(m, n) \in \mathbb{Z}^2$  whose conductor divides  $\mathfrak{g}$  and let  $\varphi_p : \mathbb{X} \rightarrow \mathbb{C}_p^\times$  be a  $p$ -adic character. We fix a pair  $(\Omega, \Omega_p) \in \mathbb{C}^\times \times \mathcal{O}_{\mathbb{C}_p}^\times$  of a complex period and a  $p$ -adic period. There exists a  $p$ -adic function  $L_p(\varphi_p)$  such that

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = (-m-1)! \left( \frac{2\pi}{\sqrt{d_K}} \right)^n \left( 1 - \frac{\varphi^{-1}(\mathfrak{p})}{p} \right) (1 - \varphi(\bar{\mathfrak{p}})) \frac{L_{\mathfrak{g}}(0, \varphi)}{\Omega^{n-m}}$$

for  $m < 0$  and  $n \geq 0$ , both of which lie in  $\overline{\mathbb{Q}}$  (see [4, Proposition 2.26]). We call  $L_p(\varphi_p)$  the  **$p$ -adic  $L$ -function** at the  $p$ -adic character  $\varphi_p$ .

We give the construction of this  $p$ -adic  $L$ -function along [4, §2.4]. If  $\hat{E}$  is a formal group associated to  $E \otimes_{\mathcal{O}_K} \mathcal{O}_{K_p}$  for  $t = -2x/y$ ,  $\hat{E}$  is the Lubin–Tate formal group over  $\mathcal{O}_{K_p}$ . Let a formal group law of  $\hat{E}$  be  $\oplus$ . Let  $\lambda : \hat{E} \xrightarrow{\sim} \hat{\mathbb{G}}_a$  be a normalized formal logarithm by  $\lambda'(0) = 1$ . We have  $\mathcal{O}_{K_p}$ -linear isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\hat{E}, \hat{\mathbb{G}}_m) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(\hat{E}), T_p(\hat{\mathbb{G}}_m)) \xleftarrow{\cong} \mathcal{O}_{K_p}, \quad (21)$$

the first isomorphism holds by [28, §4.2, Corollary 1] and the second isomorphism holds since  $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(\hat{E}), T_p(\hat{\mathbb{G}}_m))$  is a free  $\mathcal{O}_{K_p}$ -module of rank 1 by [28, §4.2, Proposition 12], where  $T_p(\hat{E}) := \varprojlim_n \hat{E}[p^n]$  (resp.  $T_p(\hat{\mathbb{G}}_m) := \varprojlim_n \hat{\mathbb{G}}_m[p^n]$ ) is a Tate module and  $\hat{\mathbb{G}}_m$  is a multiplicative formal group.  $T_p(\hat{\mathbb{G}}_m)$  is a  $\mathbb{Z}_p$ -module of rank 1 (see [28, §2.1, Example (b)]). Then there exists a homomorphism  $\eta_p : \hat{E} \rightarrow \hat{\mathbb{G}}_m$  such that if  $t \in \hat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi]$  then  $1 + \eta_p(t)$  is a  $p$ -th power root of 1 and that the image of  $\hat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi]$  does not contain power roots of 1. If  $\eta_p(t) = \Omega_p^{-1}t + \cdots \in \mathcal{O}_{\mathbb{C}_p}[[t]]$ , by the uniqueness of the formal logarithm  $\lambda$ , we have the commutative diagram:

$$\begin{array}{ccc} & \hat{E} & \\ \eta_p \swarrow & \downarrow \lambda \cong & \\ \hat{\mathbb{G}}_m & \xrightarrow{w \mapsto \Omega_p \log(1+w)} & \hat{\mathbb{G}}_a \end{array}$$

where the isomorphism  $\hat{\mathbb{G}}_a \xrightarrow{\sim} \hat{\mathbb{G}}_m$  follows from the fact that a characteristic of  $\mathcal{O}_{\mathbb{C}_p}$  is 0.

Since  $E$  is ordinary,  $\widehat{E}$  has a height 1 [27, V, Theorem 3.1(b)]. Then  $\eta_{\mathfrak{p}} : \widehat{E} \rightarrow \widehat{\mathbb{G}}_m$  is isomorphism (see [23, §4, Corollary 4.3.3]). Then we can take the isomorphism  $\eta_{\mathfrak{p}} : \widehat{E} \xrightarrow{\cong} \widehat{\mathbb{G}}_m$  over  $\mathcal{O}_{\mathbb{C}_p}$  by

$$\eta_{\mathfrak{p}}(t) = \exp\left(\frac{\lambda(t)}{\Omega_{\mathfrak{p}}}\right) - 1. \quad (22)$$

We call  $\Omega_{\mathfrak{p}} \in \mathcal{O}_{\mathbb{C}_p}^{\times}$  a  **$p$ -adic period**, which is regarded as the  $p$ -adic analogue of the complex period  $\Omega$ . The second isomorphism of (21) is given by associating to any  $x \in \mathcal{O}_{K_p}$  the homomorphism of formal groups defined by  $\exp(x\lambda(s)/\Omega_{\mathfrak{p}})$ , and depends on the choice of  $\Omega_{\mathfrak{p}}$ . In addition, since  $E$  is ordinary, we have  $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ . Therefore we have  $\mathcal{O}_{K_p} \cong \mathbb{Z}_p$ .

Now we fix a natural number  $N$ . Let  $s_N \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N]$  be any  $\pi^N$ -torsion point. Let  $z_N$  be the image of  $s_N$  by an inclusion map  $\widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N] \hookrightarrow E(\bar{\mathbb{Q}}) \hookrightarrow E(\mathbb{C}) \cong \mathbb{C}/\Gamma$ . Then  $z_N$  is a  $\pi^N$ -torsion point in  $E(\mathbb{C})[\pi^N]$ . Now we put  $z_N := \Omega/\pi^N \in \pi^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]$ . Then we can take an isomorphism  $\eta_{\mathfrak{p}} : \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N] \xrightarrow{\sim} \widehat{\mathbb{G}}_m(\mathfrak{m}_{\mathbb{C}_p})[p^N]$  of formal groups over  $\mathcal{O}_{\mathbb{C}_p}$  satisfying

$$1 + \eta_{\mathfrak{p}}(s_N) = \chi_{z_N}(\Omega) := \exp\left(\frac{\Omega\bar{z}_N - \bar{\Omega}z_N}{A(\Gamma)}\right). \quad (23)$$

This  $\eta_{\mathfrak{p}}$  is induced by  $\eta_{\mathfrak{p}}(t) = \exp(\lambda(t)/\Omega_{\mathfrak{p}}) - 1$ .

Let  $W$  be the integer ring of the completion field at  $p$  of a maximal unramified extension of  $\mathbb{Q}_p$ . According to [3, Corollary 2.18], the Kronecker theta function has the property of  $p$ -adic integrality as follows: For torsion points  $z_0, w_0 \in E(\bar{K})_{\text{tors}}$  with orders prime to  $\mathfrak{p}$ , we have

$$\widehat{\Theta}_{z_0, w_0}(s, t) := \Theta_{z_0, w_0}(z, w)|_{z=\lambda(s), w=\lambda(t)} \in W[s^{-1}, t^{-1}][[s, t]].$$

In particular,

$$\widehat{\Theta}_{z_0, w_0}^*(s, t) := \widehat{\Theta}_{z_0, w_0}(s, t) - \chi_{z_0}(w_0)\delta_{z_0}s^{-1} - \delta_{w_0}t^{-1} \in W[[s, t]],$$

where  $\delta_x = 1$  if  $x \in \Gamma$  and  $\delta_x = 0$  otherwise. Therefore for non-zero torsion points  $z_0, w_0 \in E(\bar{K})_{\text{tors}}$  with orders prime to  $\mathfrak{p}$ , we can characterize a  $p$ -adic measure  $\mu_{z_0, w_0} : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow W$  by

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1+s)^x (1+t)^y d\mu_{z_0, w_0}(x, y) = \widehat{\Theta}_{z_0, w_0}^*(s, t). \quad (24)$$

In particular, if  $z_0, w_0 \notin \Gamma$ , we have  $\widehat{\Theta}_{z_0, w_0}^*(s, t) = \widehat{\Theta}_{z_0, w_0}(s, t)$ . The existence and uniqueness of this  $p$ -adic measure  $\mu_{z_0, w_0} : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow W$  follows from  $\widehat{\Theta}_{z_0, w_0}^*(s, t) \in W[[s, t]]$  (see [31, §6]). By the isomorphism  $\eta_{\mathfrak{p}} : \widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$ , we can rewrite the formula (24) as

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp\left(\frac{x\lambda(s)}{\Omega_p}\right) \exp\left(\frac{y\lambda(t)}{\Omega_p}\right) d\mu_{z_0, w_0}(x, y) = \widehat{\Theta}_{z_0, w_0}^*(s, t). \quad (25)$$

For a non-zero torsion point  $z_0 \in E(\overline{K})_{\text{tors}}$  with an order prime to  $p$  and  $g$  as above, we define a variant measure  $\mu_{z_0, 0}^{(g)}$  on  $\mathbb{Z}_p \times \mathbb{Z}_p$  of the measure  $\mu_{z_0, 0}$  in (25) by the formula

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp\left(\frac{x\lambda(s)}{\Omega_p}\right) \exp\left(\frac{y\lambda(t)}{\Omega_p}\right) d\mu_{z_0, 0}^{(g)}(x, y) = \widehat{\Theta}_{z_0, 0}^*([g^{-1}]s, [\bar{g}]t). \quad (26)$$

For  $\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times$ , let  $\alpha_0\Omega/g \in \mathfrak{g}^{-1}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}]$  be a primitive  $\mathfrak{g}$ -torsion point and  $\mu_{\alpha_0\Omega/g, 0}^{(g)}$  induce a measure on  $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$  through the isomorphism (4). Similarly to [4, §2.4], for a continuous function  $f: \mathbb{X} \rightarrow \mathbb{C}_p$ , we define the measure  $\mu_g$  on  $\mathbb{X}$  by

$$\int_{\mathbb{X}} f(\alpha) d\mu_g(\alpha) := g^{-1}\Omega_p \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \int_{(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times} f(\alpha_0\alpha) \kappa_1(\alpha) d\mu_{\alpha_0\Omega/g, 0}^{(g)}(\alpha). \quad (27)$$

**Definition 3.1** (*p-adic L-function*). Let  $\mathfrak{f}$  be an integral ideal of  $\mathcal{O}_K$  prime to  $p$ . The  $p$ -adic  $L$ -function of  $K$  is the function whose domain is the set of all  $p$ -adic continuous characters  $\phi_p: \mathbb{X} \rightarrow \mathbb{C}_p^\times$  on  $\mathbb{X}$ , and is defined at the character  $\phi_p$  by

$$L_p(\phi_p) := \int_{\mathbb{X}} \phi_p(\alpha) d\mu_g(\alpha). \quad (28)$$

Here the measure  $\mu_g$  corresponds to the measure denoted by  $\mu$  in [15, Theorem 4.14] through the canonical isomorphism  $\text{Gal}(K(\mathfrak{g}p^\infty)/K) \cong \mathbb{X}$ .

By using the formula (27), we can rewrite the  $p$ -adic  $L$ -function (28) at the  $p$ -adic character  $\varphi_p$  in Section 0 as

$$\begin{aligned} L_p(\varphi_p) &= g^{-1}\Omega_p \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_{\mathfrak{g}}(\alpha_0) \\ &\quad \times \int_{(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times} \chi_1(\alpha) \chi_2(\alpha) \kappa_1(\alpha)^{m+1} \kappa_2(\alpha)^n d\mu_{\alpha_0\Omega/g, 0}^{(g)}(\alpha), \end{aligned} \quad (29)$$

where  $\chi_1, \chi_2$  in (29) are characters in Section 0. In addition by the following isomorphism

$$\mathcal{O}_{K_p}^\times \times \mathcal{O}_{K_{\bar{p}}}^\times \xrightarrow{\cong} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, \quad (\alpha_1, \alpha_2) \mapsto (\alpha_1^{-1}, \alpha_2),$$

and putting  $x := \kappa_1(\alpha)^{-1}$  and  $y := \kappa_2(\alpha)$ , we rewrite the formula (29) as

$$\begin{aligned}
L_p(\varphi_p) &= g^{-1} \Omega_{\mathfrak{p}} \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_{\mathfrak{g}}(\alpha_0) \\
&\times \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \chi_1(x)^{-1} \chi_2(y) x^{-m-1} y^n d\mu_{\alpha_0 \Omega/g, 0}^{(g)}(x, y). \quad (30)
\end{aligned}$$

Since  $p = \pi \bar{\pi}$  and  $\chi_1$  (resp.  $\chi_2$ ) is defined by extending to 0 at all values not prime to  $\pi$  (resp.  $\bar{\pi}$ ),  $\chi_1$  (resp.  $\chi_2$ ) is 0 on  $p\mathbb{Z}_p$ . Therefore if we replace the integration domain of (30) by  $\mathbb{Z}_p \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ , or  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ , all integrations of the right side of the formula (30) have the same value on  $\mathbb{Z}_p \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ , or  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ .

**Definition 3.2** (*Gauss sum*).

a) For  $x \in \mathcal{O}_K/(\pi^N)$ , we define the **Gauss sum** on  $\mathcal{O}_K/(\pi^N)$  by

$$\begin{aligned}
\tau(\chi_1, x) &:= \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \overline{\chi_1(u)} \langle \Omega, xuz_N \rangle \\
&= \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \overline{\chi_1(u)} \exp\left(\frac{\Omega \overline{xuz_N} - \bar{\Omega} xuz_N}{A}\right),
\end{aligned}$$

where  $z_N := \Omega/\pi^N \in \pi^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]$  is a  $\pi^N$ -torsion point.

b) For  $y \in \mathcal{O}_K/(\bar{\pi}^N)$ , we define the Gauss sum on  $\mathcal{O}_K/(\bar{\pi}^N)$  by

$$\begin{aligned}
\tau(\chi_2, y) &:= \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \overline{\chi_2(v)} \langle \Omega, \bar{y}\bar{v}w_N \rangle \\
&= \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \overline{\chi_2(v)} \exp\left(\frac{\Omega \overline{\bar{y}\bar{v}w_N} - \bar{\Omega} \bar{y}\bar{v}w_N}{A}\right),
\end{aligned}$$

where  $w_N := \Omega/\pi^N \in \pi^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]$  is a  $\pi^N$ -torsion point.

**Lemma 3.3** (*Properties of Gauss sum*). Under the assumption of Definition 3.2, we have

- a)  $\tau(\chi_1, x) = \chi_1(x)\tau(\chi_1)$ , where  $\tau(\chi_1) := \tau(\chi_1, 1)$ . In addition, if  $x$  is not prime to  $\pi$ , we have  $\tau(\chi_1, x) = 0$ , and if  $x$  is prime to  $\pi$ , we have  $|\tau(\chi_1, x)| = \sqrt{N(\mathfrak{p}^N)} = \sqrt{p^N}$ .
- b)  $\tau(\chi_2, y) = \chi_2(y)\tau(\chi_2)$ , where  $\tau(\chi_2) := \tau(\chi_2, 1)$ . In addition, if  $y$  is not prime to  $\bar{\pi}$ , we have  $\tau(\chi_2, y) = 0$  and if  $y$  is prime to  $\bar{\pi}$ , we have  $|\tau(\chi_2, y)| = \sqrt{N(\bar{\mathfrak{p}}^N)} = \sqrt{p^N}$ .

**Proof.** For the proof of b), if we replace  $y$  by  $\bar{y}$  and  $v$  by  $\bar{v}$ , we can reduce to a). So we have only to prove a).

If  $x$  is prime to  $\pi$ , since  $(\mathcal{O}_K/(\pi^N))^\times \rightarrow (\mathcal{O}_K/(\pi^N))^\times, u \mapsto xu$  is bijective and  $\chi_1^{-1} = \overline{\chi_1}$  because  $\chi_1$  is a finite character, then we have

$$\tau(\chi_1, x) = \chi_1(x)\tau(\chi_1).$$

Next we consider when  $x$  is not prime to  $\pi$ . Let  $d \neq 1$  be a greatest common divisor of  $x$  and  $\pi$ . Since  $\chi_1$  is primitive, there exists  $a \in (\mathcal{O}_K/(\pi^N))^\times$  such that

$$\chi_1(a) \neq 1 \quad \text{and} \quad a \equiv 1 \pmod{\frac{\pi^N}{d}}.$$

So we have  $xa \equiv x \pmod{\pi^N}$ . Therefore we have

$$\overline{\chi_1(a)}\tau(\chi_1, x) = \tau(\chi_1, x).$$

Since  $\chi_1(a) \neq 1$ , we have  $\tau(\chi_1, x) = 0$ . On the other hand, since  $\chi_1(x) = 0$ , we have  $\chi_1(x)\tau(\chi_1) = 0$ . Hence  $\tau(\chi_1, x) = \chi_1(x)\tau(\chi_1)$ .

We show the latter part of a). Since  $\Gamma = \Omega\mathcal{O}_K$ , we have  $A := A(\Gamma) = \sqrt{d_K}\Omega\overline{\Omega}/2\varpi$ , where  $d_K$  is a discriminant of  $K$ . Then

$$\tau(\chi_1, x) = \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \overline{\chi_1(u)} \exp \left[ \frac{2\varpi}{\sqrt{d_K}} \left( \overline{\left( \frac{xu}{\pi^N} \right)} - \frac{xu}{\pi^N} \right) \right].$$

Now for  $z \in \mathcal{O}_K/(\pi^N)$ , we have

$$\begin{aligned} & \tau(\chi_1, z)\overline{\tau(\chi_1, z)} \\ &= \sum_{u, v \in (\mathcal{O}_K/(\pi^N))^\times} \overline{\chi_1(u)}\chi_1(v) \exp \left[ \frac{2\varpi}{\sqrt{d_K}} \left( \frac{\bar{z}}{\pi^N}(\bar{u} - \bar{v}) - \frac{z}{\pi^N}(u - v) \right) \right]. \end{aligned} \quad (31)$$

If we take the sum of both sides of (31) over  $z \in \mathcal{O}_K/(\pi^N)$ , since  $\tau(\chi_1, z) = 0$  holds if  $(z, \pi) \neq 1$ , so the sum over  $z \in \mathcal{O}_K/(\pi^N)$  of the left side of (31) and the sum over  $z \in (\mathcal{O}_K/(\pi^N))^\times$  have the same value. Then giving the sum over  $z \in (\mathcal{O}_K/(\pi^N))^\times$  to the left side of (31), we have

$$\begin{aligned} & \sum_{z \in (\mathcal{O}_K/(\pi^N))^\times} \tau(\chi_1, z)\overline{\tau(\chi_1, z)} \\ &= \sum_{u, v \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u)\overline{\chi_1(v)} \sum_{z \in (\mathcal{O}_K/(\pi^N))^\times} \langle \Omega, zu z_N \rangle \langle \overline{\Omega}, \bar{z}\bar{u}\bar{z}_N \rangle. \end{aligned}$$

Since  $z$  is prime to  $\pi$ ,  $(\mathcal{O}_K/(\pi^N))^\times \rightarrow (\mathcal{O}_K/(\pi^N))^\times, u \mapsto z^{-1}u, v \mapsto z^{-1}v$  is bijective, we have

$$\begin{aligned} \sum_{z \in (\mathcal{O}_K/(\pi^N))^\times} \tau(\chi_1, z) \overline{\tau(\chi_1, z)} &= \varepsilon(\pi^N) \sum_{u, v \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) \overline{\chi_1(v)} \langle \Omega, uz_N \rangle \langle \overline{\Omega}, \bar{u} \bar{z}_N \rangle \\ &= \varepsilon(\pi^N) |\tau(\chi_1)|^2 = \varepsilon(\pi^N) |\tau(\chi_1, x)|^2, \end{aligned}$$

where  $\varepsilon(\pi^N) := \#(\mathcal{O}_K/(\pi^N))^\times$  is an Euler function and we used  $\chi_1(z) \overline{\chi_1(z)} = |\chi_1(z)|^2 = 1$ . Next we give the sum over  $z \in (\mathcal{O}_K/(\pi^N))^\times$  to the right side of (31). Since

$$\begin{aligned} &\sum_{z \in (\mathcal{O}_K/(\pi^N))^\times} \exp \left[ \frac{2\varpi}{\sqrt{d_K}} \left( \frac{\bar{z}}{\bar{\pi}^N} (\bar{u} - \bar{v}) - \frac{z}{\pi^N} (u - v) \right) \right] \\ &= \begin{cases} \varepsilon(\pi^N) N(\mathfrak{p}^N) & \text{if } u \equiv v \pmod{\pi^N}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

the sum over  $z \in (\mathcal{O}_K/(\pi^N))^\times$  of the right side of (31) is  $\varepsilon(\pi^N) p^N$ . Therefore we have

$$|\tau(\chi_1, x)|^2 = N(\mathfrak{p}^N) = p^N. \quad \square$$

By Lemma 3.3 and using the formula (23), we can rewrite the  $p$ -adic  $L$ -function (30) at the  $p$ -adic character  $\varphi_p$  as

$$\begin{aligned} L_p(\varphi_p) &= \frac{g^{-1} \Omega_{\mathfrak{p}}}{\tau(\overline{\chi_1}) \tau(\chi_2)} \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_{\mathfrak{g}}(\alpha_0) \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_1(u) \overline{\chi_2(v)} \\ &\quad \times \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x \lambda([u] s_N)}{\Omega_{\mathfrak{p}}} \right) \exp \left( \frac{y \lambda([\bar{v}] t_N)}{\Omega_{\mathfrak{p}}} \right) x^{-m-1} y^n d\mu_{\alpha_0 \Omega/g, 0}^{(g)}(x, y), \quad (32) \end{aligned}$$

where  $x \in \mathbb{Z}_p$  (resp.  $y \in \mathbb{Z}_p$ ) in the formula (32) is an element obtained by the lifting via the natural projection  $\mathbb{Z}_p \cong \mathcal{O}_{K_p} \twoheadrightarrow \mathcal{O}_K/(\pi^N)$  (resp.  $\mathbb{Z}_p \cong \mathcal{O}_{K_{\bar{p}}} \twoheadrightarrow \mathcal{O}_K/(\bar{\pi}^N)$ ) and  $s_N \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N]$  (resp.  $t_N \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N]$ ) is the  $\pi^N$ -torsion point in the formal group corresponding to the  $\pi^N$ -torsion point  $z_N = \Omega/\pi^N \in \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]$  (resp.  $w_N = \Omega/\pi^N \in \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]$ ) through the inclusion map  $\widehat{E}(\mathfrak{m}_{\mathbb{C}_p})[\pi^N] \hookrightarrow E(\overline{\mathbb{Q}})[\pi^N] \hookrightarrow \mathbb{C}/\Gamma$ .

Since the  $p$ -adic  $L$ -function (32) is rewritten by using Lemma 3.3, similarly to (30), the  $p$ -adic  $L$ -function (32) has the same value on  $\mathbb{Z}_p \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ ,  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ , or  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ .

### 3.2. The non-critical values of the $p$ -adic Hecke $L$ -function whose conductor is divisible by $p$ and $p$ -adic Eisenstein–Kronecker series

We calculate the rewritten  $p$ -adic  $L$ -function (32), focusing on each summand of the  $p$ -adic  $L$ -function. The method of calculation of the  $p$ -adic  $L$ -function (32) is as follows: We first rewrite the  $p$ -adic  $L$ -function (32) constructed using the two-variable measure in terms of the one-variable measure (see Proposition 3.6). Then we express the integration



in Proposition 3.6 using the formula (20) (see Lemma 3.7). Finally, by using Lemma 3.10, we express the  $p$ -adic  $L$ -function (32) using the Coleman Eisenstein–Kronecker series (see Proposition 3.9).

**Lemma 3.4.** *For any  $s, t \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})$ , we have*

$$\begin{aligned} & \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \overline{\chi_2(v)} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp\left(\frac{x\lambda(s \oplus [u]s_N)}{\Omega_p}\right) \exp\left(\frac{y\lambda(t \oplus [\bar{v}]t_N)}{\Omega_p}\right) \mu_{\alpha_0\Omega/g,0}^{(g)}(x, y) \\ &= \frac{\pi^N}{\tau(\overline{\chi_2})} \sum_{\gamma \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(\gamma) \widehat{\Theta}_{(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N, 0}^*([g^{-1}\bar{\pi}^{-N}](s \oplus [u]s_N), [\pi^N\bar{g}]t). \end{aligned} \quad (33)$$

**Proof.**

$$\begin{aligned} (\text{Left side}) &= \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \overline{\chi_2(v)} \widehat{\Theta}_{\alpha_0\Omega/g, \bar{g}\bar{v}w_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t) \\ &= \frac{1}{\tau(\overline{\chi_2})} \sum_{\gamma \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(\gamma) \\ &\quad \times \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \langle \Omega, \bar{v}\bar{\gamma}w_N \rangle \widehat{\Theta}_{\alpha_0\Omega/g, \bar{g}\bar{v}w_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t). \end{aligned}$$

Here the first equation can be calculated by the formula (26) and the  $p$ -adic translation of the Kronecker theta function (see [3, Corollary 2.22]) and the second equation can be calculated by  $\tau(\overline{\chi_2}, v) = \overline{\chi_2(v)}\tau(\overline{\chi_2})$  by Lemma 3.3 b). The result of this lemma can be showed by the following Lemma 3.5.  $\square$

**Lemma 3.5.**

$$\begin{aligned} & \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \langle \Omega, \bar{v}\bar{\gamma}w_N \rangle \widehat{\Theta}_{\alpha_0\Omega/g, \bar{g}\bar{v}w_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t) \\ &= \sum_{w'_N \in \pi^{-N}\Gamma/\Gamma} \langle \gamma g^{-1}\Omega, w''_N \rangle \widehat{\Theta}_{\alpha_0\Omega/g, w''_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t). \end{aligned}$$

**Proof.** Since the formula (33) is 0 when  $v$  is not prime to  $\bar{\pi}$ , we have only consider the above formula when  $v$  is prime to  $\bar{\pi}$ . Since  $\mathcal{O}_K/(\bar{\pi}^N) \xrightarrow{\cong} \pi^{-N}\Gamma/\Gamma$ ,  $v \xrightarrow{\sim} \bar{v}w_N$  is bijective, if we put  $w'_N := \bar{v}w_N$ , we have

$$\begin{aligned} & \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \langle \Omega, \bar{v}\bar{\gamma}w_N \rangle \widehat{\Theta}_{\alpha_0\Omega/g, \bar{g}\bar{v}w_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t) \\ &= \sum_{w'_N \in \pi^{-N}\Gamma/\Gamma} \langle \gamma\Omega, w'_N \rangle \widehat{\Theta}_{\alpha_0\Omega/g, \bar{g}w'_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t). \end{aligned}$$

Since  $g$  is prime to  $p$  by the assumption,  $g$  is prime to  $\bar{\pi}$ . Then  $\pi^{-N}\Gamma/\Gamma \rightarrow \pi^{-N}\Gamma/\Gamma$ ,  $w'_N \mapsto \bar{g}w'_N$  is bijective. If we put  $w''_N := \bar{g}w'_N$ , then we have

$$\begin{aligned} & \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \langle \Omega, \bar{v}\bar{\gamma}w_N \rangle \hat{\Theta}_{\alpha_0\Omega/g, \bar{g}\bar{v}w_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t) \\ &= \sum_{w''_N \in \pi^{-N}\Gamma/\Gamma} \langle \gamma g^{-1}\Omega, w''_N \rangle \hat{\Theta}_{\alpha_0\Omega/g, w''_N}^*([g^{-1}](s \oplus [u]s_N), [\bar{g}]t). \end{aligned}$$

Since  $g$  is prime to  $p$ ,  $g$  is prime to  $\bar{\pi}$ . Hence  $\gamma g^{-1} \in (\mathcal{O}_K/(\bar{\pi}^N))^\times$ . By the lifting  $\gamma g^{-1}$  into  $\mathcal{O}_K$ , we have  $\gamma g^{-1}\Omega \in \mathcal{O}_K\Omega = \Gamma$ . So we can use the distribution relation of the Kronecker theta function (6). Then we obtain this lemma.  $\square$

We show that the  $p$ -adic  $L$ -function (32) can be rewritten as follows:

**Proposition 3.6.** *Let  $m, n$  be any integers with  $n \geq 0$ . Then we have*

$$\begin{aligned} L_p(\varphi_p) &= \frac{g^{-1}\Omega_{\mathfrak{p}}\pi^N}{\tau(\bar{\chi}_1)\tau(\bar{\chi}_2)\tau(\bar{\chi}_2)} \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_{\mathfrak{g}}(\alpha_0) \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) \sum_{\gamma \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(\gamma) \\ &\quad \times \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_{\mathfrak{p}}}\right) x^{-m-1} y^n d\mu_{(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N, 0}(x, y). \end{aligned}$$

**Proof.** Since  $\alpha_0\Omega/g \in \mathfrak{g}^{-1}\Gamma/\Gamma$  and  $\gamma g^{-1}\Omega \in \Gamma$ , this value  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N \in \mathfrak{g}^{-1}\bar{\pi}^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}\bar{\pi}^N]$  is a  $g\bar{\pi}^N$ -torsion point. Since  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N$  is a torsion point with an order prime to  $\pi$ , by the formula (25), Lemma 3.4 becomes

$$\begin{aligned} & \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp\left(\frac{x\lambda(s)}{\Omega_{\mathfrak{p}}}\right) \exp\left(\frac{y\lambda(t)}{\Omega_{\mathfrak{p}}}\right) d\mu_1(x, y) \\ &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^{-N}]s)}{\Omega_{\mathfrak{p}}}\right) \exp\left(\frac{y\lambda([\pi^N\bar{g}]t)}{\Omega_{\mathfrak{p}}}\right) d\mu_2(x, y). \end{aligned} \quad (34)$$

Here for simplicity, we put

$$\begin{aligned} d\mu_1(x, y) &:= \sum_{v \in (\mathcal{O}_K/(\pi^N))^\times} \bar{\chi}_2(v) \exp\left(\frac{x\lambda([u]s_N)}{\Omega_{\mathfrak{p}}}\right) \exp\left(\frac{y\lambda([\bar{v}]t_N)}{\Omega_{\mathfrak{p}}}\right) d\mu_{\alpha_0\Omega/g, 0}^{(g)}(x, y), \\ d\mu_2(x, y) &:= \frac{\pi^N}{\tau(\bar{\chi}_2)} \\ &\quad \times \sum_{\gamma \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(\gamma) \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^{-N}u]s_N)}{\Omega_{\mathfrak{p}}}\right) d\mu_{(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N, 0}(x, y). \end{aligned}$$

By the isomorphism of formal groups  $\eta_{\mathfrak{p}} : \widehat{E}(\mathfrak{m}_{\mathbb{C}_p}) \rightarrow \widehat{\mathbb{G}}_m(\mathfrak{m}_{\mathbb{C}_p})$  the formula (34) is

$$\begin{aligned} & \int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1 + \eta_{\mathfrak{p}}(s))^x (1 + \eta_{\mathfrak{p}}(t))^y d\mu_1(x, y) \\ &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1 + \eta_{\mathfrak{p}}([g^{-1}\bar{\pi}^{-N}]s))^x (1 + \eta_{\mathfrak{p}}([\pi^N \bar{g}]t))^y d\mu_2(x, y). \end{aligned}$$

By the binomial expansion, we have

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \eta_{\mathfrak{p}}(s)^m \eta_{\mathfrak{p}}(t)^n \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \binom{x}{m} \binom{y}{n} d\mu_1(x, y) \\ &= \sum_{m,n=0}^{\infty} \eta_{\mathfrak{p}}([g^{-1}\bar{\pi}^{-N}]s)^m \eta_{\mathfrak{p}}([\pi^N \bar{g}]t)^n \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \binom{x}{m} \binom{y}{n} d\mu_2(x, y). \end{aligned}$$

Now we put

$$\begin{aligned} f(x, y) &:= \sum_{m,n=0}^{\infty} \eta_{\mathfrak{p}}(s)^m \eta_{\mathfrak{p}}(t)^n \binom{x}{m} \binom{y}{n}, \\ g(x, y) &:= \sum_{m,n=0}^{\infty} \eta_{\mathfrak{p}}([g^{-1}\bar{\pi}^{-N}]s)^m \eta_{\mathfrak{p}}([\pi^N \bar{g}]t)^n \binom{x}{m} \binom{y}{n}. \end{aligned}$$

Since  $|\eta_{\mathfrak{p}}(s)| < 1$ ,  $|\eta_{\mathfrak{p}}(t)| < 1$  by  $s, t \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})$ , then  $\lim_{m,n \rightarrow \infty} \eta_{\mathfrak{p}}(s)^m \eta_{\mathfrak{p}}(t)^n = 0$ . By Mahler's theorem,  $f(x, y)$  is a continuous function on  $\mathbb{Z}_p \times \mathbb{Z}_p$ . On the other hand, since  $g$  and  $\bar{\pi}$  are prime to  $\pi$ ,  $[g^{-1}\bar{\pi}^{-N}]s \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})$ . Then  $\lim_{m,n \rightarrow \infty} \eta_{\mathfrak{p}}([g^{-1}\bar{\pi}^{-N}]s)^m \times \eta_{\mathfrak{p}}([\pi^N \bar{g}]t)^n = 0$ . By Mahler's theorem,  $g(x, y)$  is a continuous function on  $\mathbb{Z}_p \times \mathbb{Z}_p$ . By the arbitrariness of  $s, t \in \widehat{E}(\mathfrak{m}_{\mathbb{C}_p})$ ,  $f(x, y)$  and  $g(x, y)$  are arbitrary continuous functions on  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Since we pick up a part of the  $p$ -adic  $L$ -function (32), we can regard  $f(x, y)$  and  $g(x, y)$  as continuous functions of  $(x, y)$  on  $\mathbb{Z}_p^{\times} \times \mathbb{Z}_p$ . Therefore we can replace  $f(x, y)$  by  $x^{-m-1}y^n f(x, y)$  and  $g(x, y)$  by  $x^{-m-1}y^n g(x, y)$ . Then we obtain this proposition.  $\square$

Next we show that

$$\int_{\mathbb{Z}_p^{\times} \times \mathbb{Z}_p} \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_{\mathfrak{p}}}\right) x^{-m-1}y^n d\mu_{(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N, 0}(x, y) \quad (35)$$

can be expressed as a Coleman function on  $E(\mathbb{C}_p)$ . Note that the value  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N$  is an algebraic element since  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N \in \mathfrak{g}^{-1}\bar{\pi}^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}\bar{\pi}^N] \cong E(\bar{K})[\mathfrak{g}\bar{\pi}^N]$ , so that  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N$  can be embedded into  $E(\mathbb{C}_p)$  by

an inclusion  $i : \bar{K} \hookrightarrow \mathbb{C}_p$ . For an inclusion map  $i_* : E(\bar{K}) \hookrightarrow E(\mathbb{C}_p)$  induced by the inclusion map  $i$  through the lifting  $E(\bar{K})[\mathfrak{g}p^N] \xrightarrow{\times \pi^N} E(\bar{K})[\mathfrak{g}\bar{\pi}^N]$ , we put  $\widetilde{\alpha}_0 := i_*(\alpha_0\Omega/gp^N)$  and  $\widetilde{\gamma} := i_*(\gamma\Omega/gp^N)$ .

**Lemma 3.7.** *The formula (35) can be expressed as a Coleman function on  $E(\mathbb{C}_p)$  as follows:*

$$\begin{aligned} & \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_p}\right) x^{-m-1} y^n d\mu_{(\alpha_0\Omega/g-\gamma\Omega/g)/\bar{\pi}^N,0}(x,y) \\ &= n!(-1)^{m+n+1} \Omega_p^{n-m-1} E_{m+1,n+1}^{(p)}(\lambda([g^{-1}\bar{\pi}^N u]s_N) + \pi^N(\widetilde{\alpha}_0 - \widetilde{\gamma})). \end{aligned}$$

Before proving Lemma 3.7, we introduce a new definition.

**Definition 3.8** (= [4, Definition 2.18]). For a non-zero torsion point  $z_0 \in E(\bar{\mathbb{Q}})_{\text{tors}}$  with an order to prime  $\mathfrak{p}$  and any integer  $b \geq 0$ , we define the  $p$ -adic measure  $\mu_{z_0,b}$  on the set  $\mathcal{C}^{\text{an}}(\mathcal{O}_{K_p}, \mathbb{C}_p)$  consisting of locally  $K_p$ -analytic functions on  $\mathcal{O}_{K_p}$ :

$$\int_{\mathcal{O}_{K_p}} \exp\left(\frac{x\lambda(s)}{\Omega_p}\right) d\mu_{z_0,b}(x) := \widehat{F}_{z_0,b}(s).$$

Then according to [4, §2.4], for any integer  $a, b$  with  $b \geq 0$ , we have

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{(\alpha_0\Omega/g-\gamma\Omega/g)/\bar{\pi}^N,0}(x,y) = b! \Omega_p^b \int_{\mathcal{O}_{K_p}^\times} x^a d\mu_{(\alpha_0\Omega/g-\gamma\Omega/g)/\bar{\pi}^N,b+1}(x).$$

Since  $x^a$  is a continuous function on  $\mathbb{Z}_p^\times \cong \mathcal{O}_{K_p}^\times$  and  $a$  is arbitrary,  $x^a$  is arbitrary continuous function on  $\mathbb{Z}_p^\times \cong \mathcal{O}_{K_p}^\times$ . So we can replace it by

$$x^a \mapsto x^a \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_p}\right).$$

If we substitute  $a = -m-1, b = n$ , then we have

$$\begin{aligned} & \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^{-m-1} y^n \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_p}\right) d\mu_{(\alpha_0\Omega/g-\gamma\Omega/g)/\bar{\pi}^N,0}(x,y) \\ &= n! \Omega_p^n \int_{\mathcal{O}_{K_p}^\times} x^{-m-1} \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^N u]s_N)}{\Omega_p}\right) d\mu_{(\alpha_0\Omega/g-\gamma\Omega/g)/\bar{\pi}^N,n+1}(x). \end{aligned} \quad (36)$$

**Proof of Lemma 3.7.** For a torsion point  $z_0 \in E(\overline{K})_{\text{tors}} \xrightarrow{i_*} E(\mathbb{C}_p)_{\text{tors}}$  with a prime to  $\pi$  and any integer  $b \geq 0$ , by using the induction on  $m$  and the distribution relation of the  $p$ -adic Eisenstein–Kronecker series (19), we can show the following relation (for details, see [1, Remark 3.17]):

$$E_{m,b}^{(p)}(z)|_{i_*(z_0)} = (-1)^{b-1} (-\Omega_p)^m \int_{\mathcal{O}_{K_p}^\times} x^{-m} \exp\left(\frac{x\lambda(t)}{\Omega_p}\right) d\mu_{z_0,b}(x) \quad (37)$$

where  $t$  and  $z$  are variables related by  $z = \lambda(t)$  with the normalized formal logarithm  $\lambda: \widehat{E} \xrightarrow{\cong} \widehat{\mathbb{G}}_a$ .

Now since  $(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N$  has an order prime to  $\pi$ , we can use (37). Then (36) becomes

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^{-m-1} y^n \exp\left(\frac{x\lambda([g^{-1}\bar{\pi}^{-N}u]s_N)}{\Omega_p}\right) d\mu_{(\alpha_0\Omega/g - \gamma\Omega/g)/\bar{\pi}^N, 0}(x, y)$$

by using (37), we have

$$\begin{aligned} &= n!(-1)^{m+n+1} \Omega_p^{n-m-1} E_{m+1,n+1}^{(p)}(\lambda([g^{-1}\bar{\pi}^{-N}u]s_N))|_{\pi^N(\widetilde{\alpha}_0 - \widetilde{\gamma})} \\ &= n!(-1)^{m+n+1} \Omega_p^{n-m-1} E_{m+1,n+1}^{(p)}(z)|_{\pi^N(\widetilde{\alpha}_0 - \widetilde{\gamma})}|_{z=\lambda([g^{-1}\bar{\pi}^{-N}u]s_N)} \end{aligned}$$

by the translation of the residue disk (17), we have

$$= n!(-1)^{m+n+1} \Omega_p^{n-m-1} E_{m+1,n+1}^{(p)}(z + \pi^N(\widetilde{\alpha}_0 - \widetilde{\gamma}))|_{0}|_{z=\lambda([g^{-1}\bar{\pi}^{-N}u]s_N)}$$

since  $z = \lambda([g^{-1}\bar{\pi}^{-N}u]s_N) \in ]0[$ , we have

$$= n!(-1)^{m+n+1} \Omega_p^{n-m-1} E_{m+1,n+1}^{(p)}(\lambda([g^{-1}\bar{\pi}^{-N}u]s_N) + \pi^N(\widetilde{\alpha}_0 - \widetilde{\gamma})).$$

Then we obtain this lemma.  $\square$

Here for  $u \in (\mathcal{O}_K/(\pi^N))^\times$ , the value  $\lambda([g^{-1}\bar{\pi}^{-N}u]s_N) = g^{-1}\bar{\pi}^{-N}uz_N = u\Omega/gp^N$  defines a primitive  $\mathfrak{g}p^N$ -torsion point. Since  $u\Omega/gp^N$  is the element in  $E(\overline{K})[\mathfrak{g}p^N]$  through the isomorphism  $\mathfrak{g}^{-1}p^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}p^N] \cong E(\overline{K})[\mathfrak{g}p^N]$ , the value  $u\Omega/gp^N$  can be embedded in  $E(\mathbb{C}_p)$ . For an inclusion map  $i_*: E(\overline{K}) \hookrightarrow E(\mathbb{C}_p)$  induced by the inclusion map  $i: \overline{K} \hookrightarrow \mathbb{C}_p$ , we put  $\tilde{u} := i_*(u\Omega/gp^N)$ . Then the value of the  $p$ -adic  $L$ -function (32) can be calculated as follows:

**Proposition 3.9.** *Let  $m, n$  be any integers with  $n \geq 0$ . Then we have*

$$\frac{L_p(\varphi_p)}{\Omega_{\mathfrak{p}}^{n-m}} = \frac{g^{-1}n!(-1)^{m+n+1}}{\tau(\overline{\chi_1})\overline{\pi}^N} \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_{\mathfrak{g}}(\alpha_0) \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) \sum_{\gamma \in (\mathcal{O}_K/(\overline{\pi}^N))^\times} \chi_2(\gamma) \\ \times E_{m+1, n+1}^{\text{col}}(\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma})).$$

**Proof.** First, we combine Lemma 3.6 with Lemma 3.7. Second, by Lemma 3.3 b), we have  $\tau(\chi_2)\tau(\overline{\chi_2}) = \chi_2(-1)p^N$ . Third, we use the fact that  $\chi_2(-1)^{-1} = \chi_2(-1)$  holds since  $\chi_2(-1) = \pm 1$  and that  $\gamma \mapsto -\gamma$  is bijective on  $(\mathcal{O}_K/(\overline{\pi}^N))^\times$ . Finally, if we recall (20) and use the following Lemma 3.10, we obtain this proposition.  $\square$

**Lemma 3.10.**

$$\sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) E_{m+1, n+1}^{\text{col}}(\pi(\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma}))) = 0.$$

**Proof.** Since  $(\pi, \pi^N) = \pi$  and  $\chi_1$  is a primitive character on  $(\mathcal{O}_K/(\pi^N))^\times$ , there exists  $a \in (\mathcal{O}_K/(\pi^N))^\times$  such that

$$\chi_1(a) \neq 1 \quad \text{and} \quad a \equiv 1 \pmod{\pi^{N-1}}.$$

Therefore

$$\chi_1(a) \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) E_{m+1, n+1}^{\text{col}}(\pi(\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma}))) \\ = \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(au) E_{m+1, n+1}^{\text{col}}(\pi(\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma})))$$

since  $a$  is prime to  $\pi$ ,  $u \mapsto a^{-1}u$  is bijective on  $(\mathcal{O}_K/(\pi^N))^\times$ , then we have

$$= \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) E_{m+1, n+1}^{\text{col}}(\pi(a^{-1}\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma})))$$

since  $a \equiv 1 \pmod{\pi^{N-1}}$  and the value  $a^{-1}\tilde{u} = i_*(\alpha^{-1}u\Omega/gp^N)$  is the image of  $gp^N$ -torsion point in  $\mathfrak{g}^{-1}p^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[gp^N]$ , we have

$$= \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) E_{m+1, n+1}^{\text{col}}(\pi(\tilde{u} + \pi^N(\widetilde{\alpha_0} + \tilde{\gamma}))).$$

Since  $\chi_1(a) \neq 1$ , this lemma holds.  $\square$

We order Proposition 3.9 by gathering toward a character on  $(\mathcal{O}_K/\mathfrak{g}')^\times$ . First we gather  $\chi_{\mathfrak{g}}, \chi_1, \chi_2$  by solving the following simultaneous congruences of first degree:

$$z \equiv \alpha_0 \pmod{\mathfrak{g}}, \quad z \equiv u \pmod{\pi^N}, \quad z \equiv \gamma \pmod{\bar{\pi}^N}.$$

We assume that  $z = s_1 \in \mathcal{O}_K$  is a solution of  $p^N z = \pi^N \bar{\pi}^N z \equiv 1 \pmod{\mathfrak{g}}$ . The existence of this solution follows from the fact that  $\mathfrak{g}$  is prime to  $p$ . Similarly, we assume that  $z = s_2 \in \mathcal{O}_K$  (resp.  $z = s_3 \in \mathcal{O}_K$ ) is a solution of  $g\bar{\pi}^N z \equiv 1 \pmod{\pi^N}$  (resp.  $g\pi^N z \equiv 1 \pmod{\bar{\pi}^N}$ ). If we put  $z = \alpha_0 p^N s_1 + ug\bar{\pi}^N s_2 + \gamma g\pi^N s_3$ ,  $z$  satisfies simultaneously  $z \equiv \alpha_0 \pmod{\mathfrak{g}}$ ,  $z \equiv u \pmod{\pi^N}$ ,  $z \equiv \gamma \pmod{\bar{\pi}^N}$ . Therefore the solution of the given simultaneous congruences of first degree is

$$z \equiv \alpha_0 p^N s_1 + ug\bar{\pi}^N s_2 + \gamma g\pi^N s_3 \pmod{\mathfrak{g}p^N}.$$

Recalling  $\chi = \chi_{\mathfrak{g}}\chi_1\chi_2$  and  $\mathfrak{g}' = \mathfrak{g}p^N$ , Proposition 3.9 is

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = \frac{g^{-1}n!(-1)^{m+n+1}}{\tau(\bar{\chi}_1)\bar{\pi}^N} \sum_{z \in (\mathcal{O}_K/\mathfrak{g}')^\times} \chi(z) E_{m+1, n+1}^{\text{col}}(\tilde{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma})).$$

Next, we express the contents  $\tilde{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma})$  of  $E_{m+1, n+1}^{\text{col}}$  by the formula of  $z$ . We find a constant  $C$  satisfying

$$u + \pi^N(\alpha_0 + \gamma) = Cz = C(\alpha_0 p^N s_1 + ug\bar{\pi}^N s_2 + \gamma g\pi^N s_3)$$

in  $\mathcal{O}_K/\mathfrak{g}'$ . Since  $u$ ,  $\alpha_0$ , and  $\gamma$  are arbitrary, comparing respectively coefficients of  $u$ ,  $\alpha_0$ , and  $\gamma$ , we have

$$1 = Cg\bar{\pi}^N s_2, \quad \pi^N = Cp^N s_1, \quad \pi^N = Cg\pi^N s_3$$

in  $\mathcal{O}_K/\mathfrak{g}'$ . Since  $s_1$ ,  $s_2$ , and  $s_3$  respectively satisfies  $p^N s_1 \equiv 1 \pmod{\mathfrak{g}}$ ,  $g\bar{\pi}^N s_2 \equiv 1 \pmod{\pi^N}$ , and  $g\pi^N s_3 \equiv 1 \pmod{\bar{\pi}^N}$ , we have the following simultaneous congruences of first degree:

$$C \equiv 1 \pmod{\pi^N}, \quad C \equiv \pi^N \pmod{\mathfrak{g}}, \quad C \equiv \pi^N \pmod{\bar{\pi}^N}.$$

Solving these simultaneous congruences of first degree, for the above  $s_1$ ,  $s_2$ , and  $s_3$ , we have

$$C \equiv \pi^N p^N s_1 + g\bar{\pi}^N s_2 + g\pi^{2N} s_3 \pmod{\mathfrak{g}p^N}.$$

Now if we put  $\xi_{\mathfrak{g}'} := i_*(\Omega(\pi^N p^N s_1 + g\bar{\pi}^N s_2 + g\pi^{2N} s_3)/g p^N) = i_*(\Omega C/g p^N)$ ,  $\xi_{\mathfrak{g}'}$  is a primitive  $\mathfrak{g}'$ -torsion point.  $\xi_{\mathfrak{g}'}$  is the primitive  $\mathfrak{g}'$ -torsion point in  $E(\mathbb{C}_p)$ , which is the special case when we substitute  $\alpha_0 = \pi^N$ ,  $u = 1$ , and  $\gamma = \pi^N$  for  $z \equiv \alpha_0 p^N s_1 + ug\bar{\pi}^N s_2 + \gamma g\pi^N s_3 \pmod{\mathfrak{g}p^N}$ . Then we have

**Theorem 3.11** (Main theorem). *Let  $m, n$  be any integers with  $n \geq 0$ . For the above symbols, we have*

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = \frac{g^{-1}n!(-1)^{m+n+1}}{\tau(\overline{\chi_1})\overline{\pi}^N} \sum_{z \in (\mathcal{O}_K/\mathfrak{g}')^\times} \chi(z) E_{m+1, n+1}^{\text{col}}(\xi_{\mathfrak{g}'} z),$$

where  $\tau(\overline{\chi_1})$  is the Gauss sum defined by Lemma 3.3.

The main theorem is the  $p$ -adic analogue of the relation of the complex Hecke  $L$ -function and the classical Eisenstein–Kronecker series

$$L_{\mathfrak{g}'}(s, \varphi) = \frac{1}{w_{\mathfrak{g}'}} K_{|m-n|}^*(\alpha, 0, s - \min\{m, n\}; ((\alpha)^{-1}\mathfrak{g}')^\delta), \quad (38)$$

where  $w_{\mathfrak{g}'}$  is the number of roots of 1 in  $\mathcal{O}_K^\times$  congruent to 1 modulo  $\mathfrak{g}'$ ,  $\alpha$  is any element of  $\mathfrak{a}^{-1}$  in (1) such that  $\alpha \equiv 1 \pmod{\mathfrak{g}'}$  and  $\delta \in \text{Gal}(\mathbb{C}/\mathbb{R})$  is an element satisfying that  $\delta$  is trivial if and only if  $m - n > 0$  (see [3, Proposition 1.6]).

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## Appendix A. Supplementary material

The online version of this article contains additional supplementary material. Please visit <http://dx.doi.org/10.1016/j.jnt.2013.07.019>.

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