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Sums of the triple divisor function over values of a ternary quadratic form

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A B S T R A C T

Let $\tau_3(n)$ be the triple divisor function which is the number of solutions of the equation $d_1d_2d_3 = n$ in natural numbers. It is shown that

$$\sum_{1\leq n_1,n_2,n_3\leq\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2) = c_1x^{\frac{3}{2}}(\log x)^2 + c_2x^{\frac{3}{2}}\log x + c_3x^{\frac{3}{2}} + O_\varepsilon(x^{\frac{11}{8}+\varepsilon})$$

for some constants c_1, c_2 and c_3.

Contents

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1. Introduction

The divisor functions

$$\tau_k(n) = \sum_{\substack{d_1 \cdots d_k = n \\ d_1, \ldots, d_k \in \mathbb{Z}^+}} 1,$$

are the basic arithmetic functions in number theory, with the generating Dirichlet series $\zeta^k(s)$ which are the simplest GL_k L-functions. While the Riemann zeta function $\zeta(s)$ has always been the most important and intensively studied L-function, the behavior of $\tau_k(n)$ is far less than perfectly understood even for $k = 2$. For example, Hooley [6] proved that

$$\sum_{n \leq x} \tau(n^2 + a) = c_1 x \log x + c_2 x + O\left(x^{\frac{8}{9}} (\log x)^3\right) \tag{1.1}$$

for any fixed $a \in \mathbb{Z}$ such that $-a$ is not a perfect square, where c_1 and c_2 are constants depending only on a. Here as usual $\tau(n) := \tau_2(n)$. However, so far there are no asymptotic formulas for the sum $\sum_{n \leq x} \tau(f(n))$ for $f(x)$ of degree $\deg f \geq 3$. For the average behavior of the divisor functions over values of quadratic forms, Yu [15] proved that, as $x \to \infty$,

$$\sum_{1 \leq n_1, n_2 \leq \sqrt{x}} \tau(n_1^2 + n_2^2) = c_3 x \log x + c_4 x + O_\varepsilon\left(x^{\frac{3}{4}+\varepsilon}\right), \tag{1.2}$$

where c_3 and c_4 are constants. Calderón and de Velasco [1] studied the average behavior of $\tau(n)$ over values of ternary quadratic form and established the asymptotic formula

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau(n_1^2 + n_2^2 + n_3^2) = \frac{4\zeta(3)}{5\zeta(5)} x^{\frac{3}{2}} \log x + O(x^{\frac{3}{2}}). \tag{1.3}$$

Recently, Guo and Zhai [4] improved (1.3) by showing that

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau(n_1^2 + n_2^2 + n_3^2) = \frac{4\zeta(3)}{5\zeta(5)} x^{\frac{3}{2}} \log x + c_5 x^{\frac{3}{2}} + O_\varepsilon(x^{\frac{4}{3}}), \tag{1.4}$$

where c_5 is a constant. The error term in (1.4) was further improved by Zhao [16] to $O(x \log x)$. Nothing of type (1.1), (1.2) or (1.3) is known for $\tau_k(n)$ with $k \geq 3$ and in fact the situation becomes even more difficult for $k \geq 3$ if one considers the sum

$$\sum_{n \leq x} a_n \tau_k(n)$$

for various sparse arithmetic sequences a_n. There are few results in this direction. For $\tau_3(n)$, Friedlander and Iwaniec [2] showed that, for $x \geq 3$,

$$\sum_{\substack{n_1^2+n_2^6 \leq x \\ (n_1,n_2)=1}} \tau_3(n_1^2 + n_2^6) = c x^{\frac{2}{3}} (\log x)^2 + O\left(x^{\frac{2}{3}} (\log x)^{\frac{7}{4}} (\log \log x)^{\frac{1}{2}} \right),$$

where c is a constant.

In this paper, we want to prove an asymptotic formula of type (1.4) for $\tau_3(n)$. Our main result is the following theorem.

Theorem 1.1. *For any $x \geq x_0$ (x_0 is a large absolute constant) and any $\varepsilon > 0$, we have*

$$\begin{aligned}\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) = {} & \frac{\mathcal{C}_0 \mathcal{J}_0}{4} x^{\frac{3}{2}} (\log x)^2 + \frac{1}{2} (\mathcal{C}_1 \mathcal{J}_0 + \mathcal{C}_0 \mathcal{J}_1) x^{\frac{3}{2}} \log x \\ & + \frac{1}{2} \left(\mathcal{C}_2 \mathcal{J}_0 + \mathcal{C}_1 \mathcal{J}_1 + \frac{1}{2} \mathcal{C}_0 \mathcal{J}_2 \right) x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon} \right),\end{aligned}$$

where for $\ell = 0, 1, 2,$

$$\mathcal{J}_\ell = \int_{-\infty}^{\infty} \left(\int_0^3 (\log u)^\ell e(-\beta u) \mathrm{d}u \right) \left(\int_0^1 e(\beta v^2) \mathrm{d}v \right)^3 \mathrm{d}\beta$$

and

$$\mathcal{C}_\ell = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_\ell(n, q) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} G(a, 0; q)^3 S(-\overline{a}, 0; q).$$

Here $\overline{a}$ denotes the multiplicative inverse of a mod q, $S(a, b; c)$ is the classical Kloosterman sum, $G(a, b; q)$ is the Gauss sum

$$G(a, b; q) = \sum_{d \bmod q} e\left(\frac{ad^2 + bd}{q} \right),$$

$P_0(n, q) = 1$ and $P_j(n, q)$ ($j = 1, 2$) are given by

$$P_1(n,q) = \frac{5}{3}\log n - 3\log q + 3\gamma - \frac{1}{3\tau(n)}\sum_{d|n}\log d,$$

$$P_2(n,q) = (\log n)^2 - 5\log q\log n + \frac{9}{2}(\log q)^2 + 3\gamma^2 - 3\gamma_1 + 7\gamma\log n - 9\gamma\log q$$
$$+ \frac{1}{\tau(n)}\left((\log n + \log q - 5\gamma)\sum_{d|n}\log d - \frac{3}{2}\sum_{d|n}(\log d)^2\right)$$

with $\gamma := \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right)$ *being the Euler constant and* $\gamma_1 := -\frac{\mathrm{d}}{\mathrm{d}s}\left(\zeta(s) - \frac{1}{s-1}\right)\Big|_{s=1}$ *being the Stieltjes constant.*

A similar asymptotic formula can be derived for the slightly modified sum

$$\sum_{\substack{1\le n_1^2+n_2^2+n_3^2\le x\\ (n_1,n_2,n_3)\in\mathbb{Z}^3}} \tau_3(n_1^2+n_2^2+n_3^2)$$

which is in some sense simpler than the sum in Theorem 1.1, since obviously we have

$$\sum_{1\le n_1^2+n_2^2+n_3^2\le x} \tau_3(n_1^2+n_2^2+n_3^2) = \sum_{1\le n\le x}\tau_3(n)r_3(n), \tag{1.5}$$

where

$$r_3(n) = \#\left\{(n_1,n_2,n_3)\in\mathbb{Z}^3 : n_1^2+n_2^2+n_3^2 = n\right\}.$$

Theorem 1.2. *For any* $x \ge x_0$ *(*x_0 *is the same as that in Theorem 1.1) and any* $\varepsilon > 0$*, we have*

$$\sum_{\substack{1\le n_1^2+n_2^2+n_3^2\le x\\ (n_1,n_2,n_3)\in\mathbb{Z}^3}} \tau_3(n_1^2+n_2^2+n_3^2) = 2\mathcal{C}_0\mathcal{K}_0 x^{\frac{3}{2}}(\log x)^2 + 4\left(\mathcal{C}_1\mathcal{K}_0 + \mathcal{C}_0\mathcal{K}_1\right)x^{\frac{3}{2}}\log x$$
$$+ 4\left(\mathcal{C}_2\mathcal{K}_0 + \mathcal{C}_1\mathcal{K}_1 + \frac{1}{2}\mathcal{C}_0\mathcal{K}_2\right)x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon}\right),$$

where C_ℓ*'s are as in Theorem 1.1 and*

$$\mathcal{K}_\ell = \int_{-\infty}^{\infty}\left(\int_0^1 (\log u)^\ell e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta.$$

Remark 1. In Theorems 1.1 and 1.2, x_0 is an absolute constant which can be explicitly computed. We can take $x_0 = 4^8$.

Remark 2. The proof of Theorem 1.2 is similar as that of Theorem 1.1 and we shall omit it for simplicity. In view of (1.5), the error term in Theorem 1.2 may be further improved by appealing to the analytic properties of the L-function

$$\sum_{n\geq 1}\frac{\tau_3(n)r_3(n)}{n^s}.$$

However, we will not take up this issue in this paper.

Now let us end the introduction by an outline of the proof of Theorem 1.1. We first reduce the long sum into (smooth) sums over dyadic intervals in Section 2. Then the smooth sums can be studied similarly as that in Sun [13] and Zhao [16]. That is, in Section 4, we apply the circle method to the smooth sum $\mathscr{S}(X)$, and use the Voronoi formula for $\tau_3(n)$ and an asymptotic formula for the exponential sum $\sum_{n\leq\sqrt{x}} e(\alpha n^2)$ to express $\mathscr{S}(X)$ as 16 sums which contain the character sum $\mathscr{C}(b_1,b_2,b_3,n,m,v;q)$ (see (4.28)). We then estimate $\mathscr{C}(b_1,b_2,b_3,n,m,v;q)$ in Section 9 and the main saving comes from square-root cancelation of a two dimensional twisted character sum (see Lemma 9.1) which benefits from the theorem of Fu [3]. After stripping the character sum we are led to bound a twisted average for $\sigma_{0,0}(k,l)$ (see (3.1) and Proposition 5.2) and this is completed in Section 8.

Notation. Throughout the paper, the letters q, m and n, with or without subscript, denote integers. The letter ε is an arbitrarily small positive constant, not necessarily the same at different occurrences. The symbol $\ll_{a,b,c}$ denotes that the implied constant depends at most on a, b and c.

2. Derivation of Theorem 1.1

Let $\mathcal{V}$ denote the set $[1,\sqrt{x}]\cap\mathbb{Z}$ and $r_3^*(n)=\sum_{\substack{n_1^2+n_2^2+n_3^2=n\\(n_1,n_2,n_3)\in\mathcal{V}^3}}1$. By dyadic subdivision, we decompose the aimed sum into partial sums

$$\sum_{1\leq n_1,n_2,n_3\leq\sqrt{x}}\tau_3(n_1^2+n_2^2+n_3^2)=\sum_{j\geq 1}\sum_{3x/2^j<n\leq 3x/2^{j-1}}\tau_3(n)r_3^*(n). \tag{2.1}$$

Notice that the inner sum in (2.1) vanishes for $j>\log 6x/\log 2$. So we are treating $O(\log x)$ sums of the form

$$\sum_{X_j/2<n\leq X_j}\tau_3(n)r_3^*(n),$$

where $X_j=3x/2^{j-1}$, $j\geq 1$. Let $\phi(y)$ be a smooth function supported on $[1/2,1]$, identically equal 1 on $[1/2+M^{-1},1-M^{-1}]$ with $M>4$, and satisfy $\phi^{(j)}(y)\ll_j M^j$ for any integer $j\geq 0$. Then we have

$$\sum_{X_j/2<n\le X_j} \tau_3(n) r_3^*(n) = \sum_{n\ge 1} \tau_3(n) r_3^*(n) \phi\left(\frac{n}{X_j}\right) + O_\varepsilon(X_j^{\frac{3}{2}+\varepsilon} M^{-1}). \tag{2.2}$$

Here the O-term comes from the bounds $\tau_3(n) \ll_\varepsilon n^\varepsilon$ and

$$r_3^*(n) \le r_3(n) = \sum_{\substack{n_1^2+n_2^2+n_3^2=n \\ (n_1,n_2,n_3)\in\mathbb{Z}^3}} 1 \ll_\varepsilon n^{\frac{1}{2}+\varepsilon}.$$

Thus it remains to study the smoothed sum

$$\mathscr{S}(X) = \sum_{n\ge 1} \tau_3(n) r_3^*(n) \phi\left(\frac{n}{X}\right).$$

We are going to prove the following asymptotic formula for $\mathscr{S}(X)$.

Theorem 2.1. *For any* $1 < X \le 3x$ *and any* $\varepsilon > 0$, *we have*

$$\mathscr{S}(X) = \frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}M + x^{\frac{3}{2}+\varepsilon}M^{-1}\right),$$

where

$$\mathcal{C}_\ell = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n\tau(n) P_\ell(n,q) \sideset{}{^*}\sum_{a=1}^{q} G(a,0;q)^3 S\left(-\overline{a}, 0; \frac{q}{n}\right), \tag{2.3}$$

and

$$\mathcal{I}_\ell(X) = \int_{-\infty}^{\infty} \left(\int_{X/2}^{X} e(-\beta u)(\log u)^\ell \mathrm{d}u\right) \left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta.$$

Remark 3. For $X \ll x^{\frac{7}{8}}$, we have

$$\begin{aligned}
\mathcal{I}_\ell(X) &\ll X(\log X)^\ell \int_{-\infty}^{\infty} \left(\frac{1}{1+|\beta|x}\right)^{\frac{3}{2}} \mathrm{d}\beta \\
&\ll X(\log X)^\ell \left\{ \int_{|\beta|<x^{-1}} \mathrm{d}\beta + x^{-\frac{3}{2}} \int_{|\beta|>x^{-1}} |\beta|^{-\frac{3}{2}} \mathrm{d}\beta \right\} \\
&\ll X(\log X)^\ell x^{-1} \\
&\ll_\ell x^{-\frac{1}{8}} (\log x)^\ell
\end{aligned}$$

and on taking $M = x^{\frac{1}{8}}$ the term $\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0 x^{\frac{3}{2}}$ is then absorbed into the O-terms. Thus the asymptotic formula in Theorem 2.1 is meaningful only when $X \gg x^{\frac{7}{8}}$.

We set the proof of Theorem 2.1 aside and continue the derivation of Theorem 1.1. Applying Theorem 2.1 to the sum on the right side of (2.2) and taking $M = x^{\frac{1}{8}}$ we get

$$\sum_{X_j/2<n\le X_j} \tau_3(n) r_3^*(n) = \frac{1}{2}\mathcal{I}_0(X_j)\mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2}\mathcal{I}_1(X_j)\mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4}\mathcal{I}_2(X_j)\mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon}\right). \tag{2.4}$$

Set $j_0 := j_0(x) = [\log x/\log 2]$. Plugging (2.4) into (2.1) we obtain

$$\begin{aligned}
&\sum_{1\le n_1,n_2,n_3\le\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2) \\
&= \sum_{1\le j\le j_0} \sum_{3x/2^j<n\le 3x/2^{j-1}} \tau_3(n) r_3^*(n) + O(1) \\
&= \frac{1}{2}\mathcal{C}_2 x^{\frac{3}{2}} \sum_{1\le j\le j_0} \mathcal{I}_0\left(\frac{3x}{2^{j-1}}\right) + \frac{1}{2}\mathcal{C}_1 x^{\frac{3}{2}} \sum_{1\le j\le j_0} \mathcal{I}_1\left(\frac{3x}{2^{j-1}}\right) + \frac{1}{4}\mathcal{C}_0 x^{\frac{3}{2}} \sum_{1\le j\le j_0} \mathcal{I}_2\left(\frac{3x}{2^{j-1}}\right) \\
&\quad + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon}\right),
\end{aligned} \tag{2.5}$$

where for $\ell = 0, 1, 2,$

$$\begin{aligned}
\sum_{1\le j\le j_0} \mathcal{I}_\ell\left(\frac{3x}{2^{j-1}}\right) &= \int_{-\infty}^{\infty} \left(\sum_{1\le j\le j_0} \int_{\frac{3x}{2^j}}^{\frac{3x}{2^{j-1}}} (\log u)^\ell e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta \\
&= x\int_{-\infty}^{\infty} \left(\int_{\frac{3}{2^{j_0}}}^{3} (\log ux)^\ell e(-\beta xu)\mathrm{d}u\right)\left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta \\
&= \sum_{i=0}^{\ell} C_\ell^i (\log x)^{\ell-i} \int_{-\infty}^{\infty} \left(\int_0^3 (\log u)^i e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta \\
&\quad - \mathcal{R}_\ell(x),
\end{aligned} \tag{2.6}$$

and for $x \ge 6$,

$$\mathcal{R}_\ell(x) = \sum_{i=0}^{\ell} C_\ell^i (\log x)^{\ell-i} \int_{-\infty}^{\infty} \left(\int_0^{\frac{3}{2^{j_0}}} (\log u)^i e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta v^2)\mathrm{d}\beta\right)^3 \mathrm{d}\beta$$

$$\ll_\ell (\log x)^\ell \int_0^{\frac{3}{2\mathcal{J}_0}} (-\log u)^\ell \mathrm{d}u$$
$$\ll_\ell x^{-1}(\log x)^4. \tag{2.7}$$

By (2.5)–(2.7) we obtain

$$\sum_{1\le n_1,n_2,n_3\le\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2) = \frac{\mathcal{C}_0\mathcal{J}_0}{4}x^{\frac{3}{2}}(\log x)^2 + \frac{1}{2}\left(\mathcal{C}_1\mathcal{J}_0+\mathcal{C}_0\mathcal{J}_1\right)x^{\frac{3}{2}}\log x$$
$$+\frac{1}{2}\left(\mathcal{C}_2\mathcal{J}_0+\mathcal{C}_1\mathcal{J}_1+\frac{1}{2}\mathcal{C}_0\mathcal{J}_2\right)x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon}\right),$$

where for $\mathcal{C}_\ell$ is defined in (2.3) and

$$\mathcal{J}_\ell = \int_{-\infty}^{\infty}\left(\int_0^3 (\log u)^\ell e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta.$$

This finishes the proof of Theorem 1.1. The following sections are devoted to the proof of Theorem 2.1.

3. Voronoi formula for the triple divisor function

The Voronoi formula for $\tau_3(n)$ was first proved by Ivić [8] and later in [11], Li derived a more explicit formula. To adopt Li's result, set

$$\sigma_{0,0}(k,l) = \sum_{\substack{d_1|l\\ d_1>0}} \sum_{\substack{d_2|\frac{l}{d_1}\\ d_2>0\\ (d_2,k)=1}} 1. \tag{3.1}$$

Let $\zeta(s)$ be the Riemann zeta function, $\gamma := \lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)$ be the Euler constant and $\gamma_1 := -\frac{\mathrm{d}}{\mathrm{d}s}\left(\zeta(s)-\frac{1}{s-1}\right)\Big|_{s=1}$ be the Stieltjes constant. For $\phi(y)\in C_c(0,\infty)$, $k=0,1$ and $\sigma > -1-2k$, set

$$\Phi_k(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} \left(\pi^3 y\right)^{-s}\frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3}\widetilde{\phi}(-s-k)\mathrm{d}s$$

with $\widetilde{\phi}(s) = \int_0^\infty \phi(u)u^{s-1}\mathrm{d}u$ the Mellin transform of ϕ, and

$$\Phi^{\pm}(y) = \Phi_0(y) \pm \frac{1}{i\pi^3 y}\Phi_1(y).$$

Lemma 3.1. *For $\phi(y) \in C_c^\infty(0,\infty)$, $a, \overline{a}, q \in \mathbb{Z}^+$ with $a\overline{a} \equiv 1 \pmod q$, we have*

$$\begin{aligned}
&\sum_{n\geq 1} \tau_3(n) e\left(\frac{an}{q}\right)\phi(n) \\
&= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{m\geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S\left(\pm m, \overline{a}; \frac{q}{n}\right) \Phi^{\pm}\left(\frac{mn^2}{q^3}\right) \\
&\quad + \frac{1}{2q^2}\widetilde{\phi}(1) \sum_{n|q} n\tau(n) P_2(n,q) S\left(0, \overline{a}; \frac{q}{n}\right) \\
&\quad + \frac{1}{2q^2}\widetilde{\phi}'(1) \sum_{n|q} n\tau(n) P_1(n,q) S\left(0, \overline{a}; \frac{q}{n}\right) \\
&\quad + \frac{1}{4q^2}\widetilde{\phi}''(1) \sum_{n|q} n\tau(n) S\left(0, \overline{a}; \frac{q}{n}\right),
\end{aligned}$$

where $S(a,b;c)$ is the classical Kloosterman sum,

$$P_1(n,q) = \frac{5}{3}\log n - 3\log q + 3\gamma - \frac{1}{3\tau(n)} \sum_{d|n} \log d, \tag{3.2}$$

and

$$\begin{aligned}
P_2(n,q) &= (\log n)^2 - 5\log q \log n + \frac{9}{2}(\log q)^2 + 3\gamma^2 - 3\gamma_1 + 7\gamma \log n - 9\gamma \log q \\
&\quad + \frac{1}{\tau(n)}\left((\log n + \log q - 5\gamma)\sum_{d|n} \log d - \frac{3}{2}\sum_{d|n} (\log d)^2\right).
\end{aligned} \tag{3.3}$$

The functions $\Phi^{\pm}(y)$ have the following properties (see Sun [13]).

Lemma 3.2. *Suppose that $\phi(y)$ is a smooth function of compact support in $[AX, BX]$, where $X > 0$ and $B > A > 0$, satisfying $\phi^{(j)}(y) \ll_{A,B,j} P^j$ for any integer $j \geq 0$. Then for $y > 0$ and any integer $\ell \geq 0$, we have*

$$\Phi^{\pm}(y) \ll_{A,B,\ell,\varepsilon} (yX)^{-\varepsilon}(PX)^3 \left(\frac{y}{P^3X^2}\right)^{-\ell}.$$

By Lemma 3.2, for any fixed $\varepsilon > 0$ and $yX \geq X^{\varepsilon}(PX)^3$, $\Phi^{\pm}(y)$ are negligibly small. Moreover, for $yX \gg X^{\varepsilon}$, we have an asymptotic formula for $\Phi_k(y)$ (see [8,10,12]).

Lemma 3.3. *Suppose that $\phi(y)$ is a smooth function of compact support on $[AX, BX]$, where $X > 0$ and $B > A > 0$. Then for $y > 0$, $yX \gg 1$, $\ell \geq 2$ and $k = 0, 1$, we have*

$$\Phi_k(y) = (\pi^3 y)^{k+1} \sum_{j=1}^{\ell} \int_0^\infty \phi(u) \left(a_k(j) e\left(3(yu)^{\frac{1}{3}}\right) + b_k(j) e\left(-3(yu)^{\frac{1}{3}}\right)\right) \frac{\mathrm{d}u}{(\pi^3 yu)^{\frac{j}{3}}}$$
$$+ O_{A,B,\varepsilon,\ell}\left((\pi^3 y)^k (\pi^3 yX)^{-\frac{\ell}{3}+\frac{1}{2}+\varepsilon}\right),$$

where $a_k(j)$, $b_k(j)$ are constants with

$$a_0(1) = -\frac{2\sqrt{3\pi}}{6\pi i}, \quad b_0(1) = \frac{2\sqrt{3\pi}}{6\pi i}, \quad a_1(1) = b_1(1) = -\frac{2\sqrt{3\pi}}{6\pi}.$$

4. Transformation of $\mathscr{S}(X)$

Applying the circle method, we have

$$\mathscr{S}(X) = \int_0^1 \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)\mathrm{d}\alpha,$$

where

$$\mathscr{F}(\alpha) = \sum_{n\in\mathcal{V}} e(\alpha n^2),$$

and

$$\mathscr{G}(\alpha) = \sum_{n\geq 1} \tau_3(n) e(-\alpha n)\phi\left(\frac{n}{X}\right). \tag{4.1}$$

Note that $\mathscr{F}^3(\alpha)\mathscr{G}(\alpha)$ is a periodic function of period 1. We have

$$\mathscr{S}(X) = \int_{-1/(Q+1)}^{Q/(Q+1)} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)\mathrm{d}\alpha,$$

where $Q = [5\sqrt{x}]$. Then we can evaluate $\mathscr{S}(X)$ by dissecting the interval $(-1/(Q+1), Q/(Q+1)]$ with Farey's points of order Q (see for example Iwaniec [9]). Let $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ be adjacent points, which are determined by the conditions

$$Q < q+q', q+q'' \leq q+Q, \quad aq' \equiv 1 \pmod{q}, \quad aq'' \equiv -1 \pmod{q}.$$

Then

$$\left(\frac{-1}{Q+1}, \frac{Q}{Q+1}\right] = \bigcup_{\substack{0\leq a<q\leq Q\\(a,q)=1}}\bigcup \left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right].$$

It follows that

$$\mathscr{S}(X) = \sum_{q \le Q} \sideset{}{^*}\sum_{a=1}^{q} \int\limits_{\mathscr{M}(a,q)} \mathscr{F}^3\left(\frac{a}{q}+\beta\right)\mathscr{G}\left(\frac{a}{q}+\beta\right)\mathrm{d}\beta,$$

where $*$ denotes the condition $(a, q) = 1$ and

$$\mathscr{M}(a,q) = \left(-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')}\right].$$

Exchanging the order of the summation over a and the integration over β as in Heath-Brown [5], we have

$$\mathscr{S}(X) = \sum_{q \le Q} \int\limits_{|\beta| \le \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v,q,\beta) \sideset{}{^*}\sum_{a=1}^{q} e\left(-\frac{\overline{a}v}{q}\right)\mathscr{F}^3\left(\frac{a}{q}+\beta\right)\mathscr{G}\left(\frac{a}{q}+\beta\right)\mathrm{d}\beta, \tag{4.2}$$

where $\varrho(v, q, \beta)$ satisfies

$$\varrho(v,q,\beta) \ll \frac{1}{1+|v|}. \tag{4.3}$$

For an asymptotic formula of $\mathscr{F}\left(\frac{a}{q}+\beta\right)$, we quote the following result (see Theorem 4.1 in [14] or Lemma 4.1 in [16]).

Lemma 4.1. *Let $Q = [5\sqrt{x}]$. Suppose that $(a, q) = 1$, $q \le Q$ and $|\beta| \le 1/(qQ)$. We have*

$$\mathscr{F}\left(\frac{a}{q}+\beta\right) = \frac{G(a,0;q)}{q}\Psi_0(\beta) + \sum_{-\frac{3q}{2} < b \le \frac{3q}{2}} G(a,b;q)\Psi(b,q,\beta), \tag{4.4}$$

where $G(a, b; q)$ is the Gauss sum

$$G(a,b;q) = \sum_{d \bmod q} e\left(\frac{ad^2+bd}{q}\right), \tag{4.5}$$

$\Psi_0(\beta)$ is the integral

$$\Psi_0(\beta) = \int_0^{\sqrt{x}} e(\beta u^2)\mathrm{d}u, \tag{4.6}$$

and $\Psi(b, q, \beta)$ satisfies

$$\sum_{-\frac{3q}{2} < b \le \frac{3q}{2}} |\Psi(b,q,\beta)| \ll \log(q+2). \tag{4.7}$$

For $\mathscr{G}(\alpha)$ in (4.1), we apply Lemma 3.1 with $\phi_\beta(y) = \phi\left(\frac{y}{X}\right) e(-\beta y)$ getting

$$\begin{aligned}\mathscr{G}\left(\frac{a}{q}+\beta\right) &= \sum_{n\geq 1} \tau_3(n) e\left(-\frac{an}{q}\right)\phi_\beta(n) \\ &= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) \\ &\quad \times S\left(\pm m, -\overline{a}; \frac{q}{n}\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right) \\ &\quad + \frac{1}{2q^2}\widetilde{\phi}_\beta(1)\sum_{n|q} n\tau(n)P_2(n,q)S\left(0,-\overline{a};\frac{q}{n}\right) \\ &\quad + \frac{1}{2q^2}\widetilde{\phi}'_\beta(1)\sum_{n|q} n\tau(n)P_1(n,q)S\left(0,-\overline{a};\frac{q}{n}\right) \\ &\quad + \frac{1}{4q^2}\widetilde{\phi}''_\beta(1)\sum_{n|q} n\tau(n)S\left(0,-\overline{a};\frac{q}{n}\right),\end{aligned} \tag{4.8}$$

where

$$\Phi_\beta^{\pm}(y) = \Phi_0(y,\beta) \pm \frac{1}{i\pi^3 y}\Phi_1(y,\beta) \tag{4.9}$$

with

$$\Phi_k(y,\beta) = \frac{1}{2\pi i}\int\limits_{\mathrm{Re}(s)=\sigma} \left(\pi^3 y\right)^{-s}\frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3}\widetilde{\phi}_\beta(-s-k)\mathrm{d}s, \tag{4.10}$$

and $P_j(n,q)$ $(j=1,2)$ defined in (3.2) and (3.3).

By (4.4) and (4.8), we have

$$\sum_{a=1}^{q}{}^{*} e\left(-\frac{\overline{a}v}{q}\right)\mathscr{F}^3\left(\frac{a}{q}+\beta\right)\mathscr{G}\left(\frac{a}{q}+\beta\right) = \sum_{j=1}^{16}\mathscr{B}_j(v,q,\beta), \tag{4.11}$$

where

$$\begin{aligned}\mathscr{B}_1(v,q,\beta) &= \frac{q}{2\pi^{\frac{3}{2}}}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right) \\ &\quad \times \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2} \\ 1\leq j\leq 3}} \Psi(b_1,q,\beta)\Psi(b_2,q,\beta)\Psi(b_3,q,\beta)\mathscr{C}(b_1,b_2,b_3,n,\pm m,v;q),\end{aligned} \tag{4.12}$$

$$\mathscr{B}_2(v,q,\beta) = \frac{3}{2\pi^{\frac{3}{2}}}\Psi_0(\beta)\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$
$$\times\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)\mathscr{C}(0,b_1,b_2,n,\pm m,v;q), \tag{4.13}$$

$$\mathscr{B}_3(v,q,\beta) = \frac{3}{2\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^2}{q}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$
$$\times\sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}}\Psi(b,q,\beta)\mathscr{C}(0,0,b,n,\pm m,v;q), \tag{4.14}$$

$$\mathscr{B}_4(v,q,\beta) = \frac{1}{2\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^3}{q^2}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$
$$\times\mathscr{C}(0,0,0,n,\pm m,v;q), \tag{4.15}$$

$$\mathscr{B}_5(v,q,\beta) = \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)}{q^2}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)\Psi(b_3,q,\beta)$$
$$\times\mathscr{C}(b_1,b_2,b_3,n,0,v;q), \tag{4.16}$$

$$\mathscr{B}_6(v,q,\beta) = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)}{q^2}\sum_{n|q}n\tau(n)P_1(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)\Psi(b_3,q,\beta)$$
$$\times\mathscr{C}(b_1,b_2,b_3,n,0,v;q), \tag{4.17}$$

$$\mathscr{B}_7(v,q,\beta) = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)}{q^2}\sum_{n|q}n\tau(n)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)\Psi(b_3,q,\beta)$$
$$\times\mathscr{C}(b_1,b_2,b_3,n,0,v;q), \tag{4.18}$$

$$\mathscr{B}_8(v,q,\beta) = \frac{3}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)$$
$$\times\mathscr{C}(0,b_1,b_2,n,0,v;q), \tag{4.19}$$

$$\mathscr{B}_9(v,q,\beta) = \frac{3}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q}n\tau(n)P_1(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}}\Psi(b_1,q,\beta)\Psi(b_2,q,\beta)$$
$$\times\mathscr{C}(0,b_1,b_2,n,0,v;q), \tag{4.20}$$

$$\mathscr{B}_{10}(v,q,\beta) = \frac{3}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q} n\tau(n) \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}} \Psi(b_1,q,\beta)\Psi(b_2,q,\beta) \times \mathscr{C}(0,b_1,b_2,n,0,v;q), \tag{4.21}$$

$$\mathscr{B}_{11}(v,q,\beta) = \frac{3}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_2(n,q) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \Psi(b,q,\beta) \times \mathscr{C}(0,0,b,n,0,v;q), \tag{4.22}$$

$$\mathscr{B}_{12}(v,q,\beta) = \frac{3}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \Psi(b,q,\beta) \times \mathscr{C}(0,0,b,n,0,v;q), \tag{4.23}$$

$$\mathscr{B}_{13}(v,q,\beta) = \frac{3}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \Psi(b,q,\beta) \times \mathscr{C}(0,0,b,n,0,v;q), \tag{4.24}$$

$$\mathscr{B}_{14}(v,q,\beta) = \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,v;q), \tag{4.25}$$

$$\mathscr{B}_{15}(v,q,\beta) = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_1(n,q)\mathscr{C}(0,0,0,n,0,v;q), \tag{4.26}$$

$$\mathscr{B}_{16}(v,q,\beta) = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)\mathscr{C}(0,0,0,n,0,v;q) \tag{4.27}$$

with

$$\mathscr{C}(b_1,b_2,b_3,n,m,v;q) = \sideset{}{^*}\sum_{a=1}^{q} e\left(\frac{-\overline{a}v}{q}\right) G(a,b_1;q)G(a,b_2;q)G(a,b_3;q)S\left(-\overline{a},m;\frac{q}{n}\right). \tag{4.28}$$

We will show that $\mathscr{B}_j$, $1\leq j\leq 13$, contribute the remainder terms, and $\mathscr{B}_j$, $14\leq j\leq 16$, contribute the main terms.

5. Contribution of $\mathscr{B}_j$, $1\leq j\leq 4$

The estimation of $\mathscr{B}_j$, $1\leq j\leq 4$, are similar as the arguments in [13]. Since the cancelation from the character sums $\mathscr{C}(b_1,b_2,b_3,n,m,v;q)$ is the main saving for our final result, we first have the following proposition.

Proposition 5.1. *Let q_1 be the largest factor of q such that $q_1|n$ and $\left(q_1, \frac{q}{q_1}\right) = 1$. Let q_2 be the largest factor of q/q_1 such that $q_2|n^\infty$ and $\left(q_2, \frac{q}{q_1, q_2}\right) = 1$. Let $q = q_1q_2q_3'q_3''$, $(q_3', 2q_3'') = 1$, q_3' square-free and $4q_3''$ square-full. For any $\varepsilon > 0$, we have*

$$\mathscr{C}(b_1, b_2, b_3, n, m, v; q) \ll_\varepsilon \frac{(q_1q_2q_3'')^{3+\varepsilon}{q_3'}^{\frac{5}{2}+\varepsilon}}{\sqrt{n}}. \tag{5.1}$$

Next, we need the following result which will be proved in Section 8.

Proposition 5.2. *For any $\varepsilon > 0$, we have*

$$\sum_{\pm}\sum_{m\geq 1}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2}, m\right)\left|\Phi_\beta^\pm\left(\frac{mn^2}{q^3}\right)\right| \ll_\varepsilon X^\varepsilon n^\varepsilon(M + |\beta|^2X^2).$$

By the second derivative test and the trivial estimation, $\Psi_0(\beta)$ in (4.6) is bounded by

$$\Psi_0(\beta) \ll \left(\frac{x}{1+|\beta|x}\right)^{\frac{1}{2}}. \tag{5.2}$$

Let q be as in Proposition 5.1. Denote $q_0 = q_2q_3''$. Then q_0 is square-full. By (4.7), (4.12) and Propositions 5.1–5.2, we have

$$\begin{aligned}
\mathscr{B}_1(v, q, \beta) &\ll q\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2}, m\right)\left|\Phi_\beta^\pm\left(\frac{mn^2}{q^3}\right)\right| \\
&\quad\times \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}|\Psi(b_1, q, \beta)||\Psi(b_2, q, \beta)||\Psi(b_3, q, \beta)||\mathscr{C}(b_1, b_2, b_3, n, \pm m, v; q)| \\
&\ll_\varepsilon X^\varepsilon\sum_{n\leq Q}n^{-\frac{3}{2}}q_1^4q_0^4{q_3'}^{\frac{7}{2}}\sum_{\pm}\sum_{m\geq 1}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2}, m\right)\left|\Phi_\beta^\pm\left(\frac{mn^2}{q^3}\right)\right| \\
&\ll_\varepsilon X^\varepsilon(M + |\beta|^2X^2)\sum_{n\leq Q}n^{-\frac{3}{2}+\varepsilon}q_1^4q_0^4{q_3'}^{\frac{7}{2}}.
\end{aligned}$$

Then by (4.3), we have

$$\begin{aligned}
&\sum_{q\leq Q}\int_{|\beta|\leq\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v, q, \beta)\mathscr{B}_1(v, q, \beta)\mathrm{d}\beta \\
&\ll_\varepsilon X^\varepsilon\sum_{n\leq Q}n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\leq Q/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\leq Q/(q_1q_3')\\ 4q_0 \text{ square-full}}}q_1^4q_0^4{q_3'}^{\frac{7}{2}}\int_{|\beta|\leq\frac{1}{q_1q_0q_3'Q}}(M + |\beta|^2X^2)\mathrm{d}\beta
\end{aligned}$$

$$\ll_\varepsilon X^\varepsilon \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} \sum_{\substack{q_0\le Q/(q_1q_3') \\ 4q_0 \text{ square-full}}} q_1^4 q_0^4 {q_3'}^{\frac{7}{2}} \left(\frac{M}{q_1q_0q_3'Q} + \frac{X^2}{(q_1q_0q_3'Q)^3}\right)$$

$$\ll_\varepsilon \frac{X^\varepsilon M}{Q} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^3 \sum_{q_3'\le Q/q_1} {q_3'}^{\frac{5}{2}} \left(\frac{Q}{q_1q_3'}\right)^{\frac{7}{2}}$$

$$+ \frac{X^{2+\varepsilon}}{Q^3} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1 \sum_{q_3'\le Q/q_1} {q_3'}^{\frac{1}{2}} \left(\frac{Q}{q_1q_3'}\right)^{\frac{3}{2}}$$

$$\ll_\varepsilon X^\varepsilon M Q^{\frac{5}{2}} + \frac{X^{2+\varepsilon}}{Q^{\frac{3}{2}}}$$

$$\ll_\varepsilon M x^{\frac{5}{4}+\varepsilon}. \tag{5.3}$$

Moreover, by (4.7), (4.13) and Propositions 5.1–5.2, we have

$$\mathscr{B}_2(v,q,\beta) \ll_\varepsilon x^\varepsilon \left(\frac{x}{1+|\beta|x}\right)^{\frac{1}{2}} \sum_{n\le Q} n^{-\frac{3}{2}} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}} X^\varepsilon n^\varepsilon (M + |\beta|^2 X^2)$$

$$\ll_\varepsilon x^{\frac{1}{2}+\varepsilon} \left((1+|\beta|x)^{\frac{3}{2}} + \frac{M}{\sqrt{1+|\beta|x}}\right) \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}}.$$

It follows that

$$\sum_{q\le Q} \int_{|\beta|\le \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v,q,\beta)\mathscr{B}_2(v,q,\beta)\mathrm{d}\beta$$

$$\ll_\varepsilon x^{\frac{1}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q_3'\le Q/q_1 \\ q_3' \text{ square-free}}} \sum_{\substack{q_0\le Q/(q_1q_3') \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}} \int_{|\beta|\le \frac{1}{q_1q_0q_3'Q}} (1+|\beta|x)^{\frac{3}{2}}\mathrm{d}\beta$$

$$+ x^\varepsilon M \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q_3'\le Q/q_1 \\ q_3' \text{ square-free}}} \sum_{\substack{q_0\le Q/(q_1q_3') \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}} \int_{|\beta|\le \frac{1}{q_1q_0q_3'Q}} \frac{1}{\sqrt{x^{-1}+|\beta|}}\mathrm{d}\beta$$

$$\ll_\varepsilon x^{\frac{1}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} \sum_{\substack{q_0\le Q/(q_1q_3') \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}} \left(\frac{1}{q_1q_0q_3'Q} + \frac{x^{\frac{3}{2}}}{(q_1q_0q_3'Q)^{\frac{5}{2}}}\right)$$

$$+ x^\varepsilon M \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} \sum_{\substack{q_0\le Q/(q_1q_3') \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 {q_3'}^{\frac{5}{2}} (q_1q_0q_3'Q)^{-\frac{1}{2}}$$

$$\ll_\varepsilon \frac{x^{\frac{1}{2}+\varepsilon}}{Q} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^2 \sum_{q_3'\le Q/q_1} {q_3'}^{\frac{3}{2}} \left(\frac{Q}{q_1q_3'}\right)^{\frac{5}{2}}$$

$$
\begin{aligned}
&+\frac{x^{2+\varepsilon}}{Q^{\frac{5}{2}}}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1^{\frac{1}{2}}\sum_{q_3'\le Q/q_1}\frac{Q}{q_1q_3'}\\
&+\frac{x^{\varepsilon}M}{Q^{\frac{1}{2}}}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1^{\frac{5}{2}}\sum_{q_3'\le Q/q_1}{q_3'}^2\left(\frac{Q}{q_1q_3'}\right)^3\\
&\ll_\varepsilon x^{\frac{1}{2}+\varepsilon}Q^{\frac{3}{2}}+\frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}}+x^{\varepsilon}MQ^{\frac{5}{2}}\\
&\ll_\varepsilon Mx^{\frac{5}{4}+\varepsilon}.
\end{aligned} \tag{5.4}
$$

Further, by (4.7), (4.14) and Propositions 5.1–5.2, we have

$$
\begin{aligned}
\mathscr{B}_3(v,q,\beta) &\ll_\varepsilon \frac{x^{1+\varepsilon}}{1+|\beta|x}\sum_{n\le Q} n^{-\frac{3}{2}}q_1^2q_0^2{q_3'}^{\frac{3}{2}}X^{\varepsilon}n^{\varepsilon}(M+|\beta|^2X^2)\\
&\ll_\varepsilon x^{1+\varepsilon}\left(1+|\beta|x+\frac{M}{1+|\beta|x}\right)\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}q_1^2q_0^2{q_3'}^{\frac{3}{2}}.
\end{aligned}
$$

It follows from this estimate and (4.3) that

$$
\begin{aligned}
&\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_3(v,q,\beta)\mathrm{d}\beta\\
&\ll_\varepsilon x^{1+\varepsilon}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le Q/q_1\\ q_3'\text{ square-free}}}\sum_{\substack{q_0\le Q/(q_1q_3')\\ 4q_0\text{ square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(1+|\beta|x)\mathrm{d}\beta\\
&\quad+x^{\varepsilon}M\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le Q/q_1\\ q_3'\text{ square-free}}}\sum_{\substack{q_0\le Q/(q_1q_3')\\ 4q_0\text{ square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(x^{-1}+|\beta|)^{-1}\mathrm{d}\beta\\
&\ll_\varepsilon x^{1+\varepsilon}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le Q/q_1}\sum_{\substack{q_0\le Q/(q_1q_3')\\ 4q_0\text{ square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\left(\frac{1}{q_1q_0q_3'Q}+\frac{x}{(q_1q_0q_3'Q)^2}\right)\\
&\quad+x^{\varepsilon}M\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le Q/q_1}\sum_{\substack{q_0\le Q/(q_1q_3')\\ 4q_0\text{ square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\\
&\ll_\varepsilon \frac{x^{1+\varepsilon}}{Q}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1\sum_{q_3'\le Q/q_1}{q_3'}^{\frac{1}{2}}\left(\frac{Q}{q_1q_3'}\right)^{\frac{3}{2}}\\
&\quad+\frac{x^{2+\varepsilon}}{Q^2}\sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le Q/q_1}{q_3'}^{-\frac{1}{2}}\left(\frac{Q}{q_1q_3'}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{\varepsilon} M \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^2 \sum_{q_3'\le Q/q_1} {q_3'}^{\frac{3}{2}} \left(\frac{Q}{q_1 q_3'}\right)^{\frac{5}{2}} \\
&\ll_{\varepsilon} x^{1+\varepsilon} Q^{\frac{1}{2}} + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} + x^{\varepsilon} M Q^{\frac{5}{2}} \\
&\ll_{\varepsilon} M x^{\frac{5}{4}+\varepsilon}.
\end{aligned} \tag{5.5}
$$

Lastly, by (4.7), (4.15) and Propositions 5.1–5.2, we have

$$
\begin{aligned}
\mathscr{B}_4(v,q,\beta) &\ll_{\varepsilon} x^{\varepsilon} \left(\frac{x}{1+|\beta|x}\right)^{\frac{3}{2}} \sum_{n\le Q} n^{-\frac{3}{2}} q_1 q_0 {q_3'}^{\frac{1}{2}} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2) \\
&\ll_{\varepsilon} \left(x^{\frac{3}{2}+\varepsilon} (1+|\beta|x)^{\frac{1}{2}} + \frac{x^{\varepsilon} M}{(x^{-1}+|\beta|)^{\frac{3}{2}}} \right) \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} q_1 q_0 {q_3'}^{\frac{1}{2}}.
\end{aligned}
$$

By (4.3) and the estimate as above, we have

$$
\begin{aligned}
&\sum_{q\le Q} \int_{|\beta|\le \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v,q,\beta) \mathscr{B}_4(v,q,\beta) \mathrm{d}\beta \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q_3'\le Q/q_1 \\ q_3' \text{ square-free}}} \sum_{\substack{q_0\le Q/(q_1 q_3') \\ 4q_0 \text{ square-full}}} q_1 q_0 {q_3'}^{\frac{1}{2}} \int_{|\beta|\le \frac{1}{q_1 q_0 q_3' Q}} (1+|\beta|x)^{\frac{1}{2}} \mathrm{d}\beta \\
&\quad + x^{\varepsilon} M \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q_3'\le Q/q_1 \\ q_3' \text{ square-free}}} \sum_{\substack{q_0\le Q/(q_1 q_3') \\ 4q_0 \text{ square-full}}} q_1 q_0 {q_3'}^{\frac{1}{2}} \int_{|\beta|\le \frac{1}{q_1 q_0 q_3' Q}} (x^{-1}+|\beta|)^{-\frac{3}{2}} \mathrm{d}\beta \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} \sum_{\substack{q_0\le Q/(q_1 q_3') \\ 4q_0 \text{ square-full}}} q_1 q_0 {q_3'}^{\frac{1}{2}} \left(\frac{1}{q_1 q_0 q_3' Q} + \frac{x^{\frac{1}{2}}}{(q_1 q_0 q_3' Q)^{\frac{3}{2}}} \right) \\
&\quad + M x^{\frac{1}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} \sum_{\substack{q_0\le Q/(q_1 q_3') \\ 4q_0 \text{ square-full}}} q_1 q_0 {q_3'}^{\frac{1}{2}} \\
&\ll_{\varepsilon} \frac{x^{\frac{3}{2}+\varepsilon}}{Q} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q_3'\le Q/q_1} {q_3'}^{-\frac{1}{2}} \left(\frac{Q}{q_1 q_3'}\right)^{\frac{1}{2}} \\
&\quad + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^{-\frac{1}{2}} \sum_{q_3'\le Q/q_1} {q_3'}^{-1} \\
&\quad + M x^{\frac{1}{2}+\varepsilon} \sum_{n\le Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1 \sum_{q_3'\le Q/q_1} {q_3'}^{\frac{1}{2}} \left(\frac{Q}{q_1 q_3'}\right)^{\frac{3}{2}}
\end{aligned}
$$

$$\ll_\varepsilon \frac{x^{\frac{3}{2}+\varepsilon}}{Q^{\frac{1}{2}}} + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} + Mx^{\frac{1}{2}+\varepsilon}Q^{\frac{3}{2}}$$
$$\ll_\varepsilon Mx^{\frac{5}{4}+\varepsilon}. \tag{5.6}$$

By (4.2), (4.11) and (5.3)–(5.6), the contribution from $\mathscr{B}_j$, $j = 1, 2, 3, 4$, is $O_\varepsilon(Mx^{\frac{5}{4}+\varepsilon})$.

6. Contribution of $\mathscr{B}_j$, $5 \le j \le 13$

First, we note that (recall (3.2) and (3.3))

$$P_j(n, q) \ll (\log(n + 2)(q + 2))^j, \qquad j = 1, 2 \tag{6.1}$$

and

$$\widetilde{\phi_\beta}^{(j)}(1) = \int_0^\infty \phi\left(\frac{u}{X}\right) e(-\beta u)(\log u)^j \mathrm{d}u \ll \frac{x(\log x)^j}{1 + |\beta|x}, \qquad j = 0, 1, 2. \tag{6.2}$$

Next, bounding the character sum $\mathscr{C}(b_1, b_2, b_3, n, m, v; q)$ by Weil's bound for Kloosterman sums, we have

$$\mathscr{C}(b_1, b_2, b_3, n, m, v; q) \ll q^{\frac{5}{2}} \left(\frac{q}{n}\right)^{\frac{1}{2}} \tau\left(\frac{q}{n}\right) \ll_\varepsilon q^{3+\varepsilon} n^{-\frac{1}{2}}. \tag{6.3}$$

By (4.7), (4.16)–(4.18) and (6.1)–(6.3), we have, for $j = 5, 6, 7$,

$$\mathscr{B}_j(v, q, \beta) \ll_\varepsilon \frac{1}{q^2} \frac{x(\log x)^2}{1 + |\beta|x} \sum_{n|q} n\tau(n) q^{3+\varepsilon} n^{-\frac{1}{2}} (\log(q + 2))^5 \ll_\varepsilon \frac{x^\varepsilon q^{\frac{3}{2}+\varepsilon}}{x^{-1} + |\beta|}. \tag{6.4}$$

By (4.3) and (6.4), we obtain, for $j = 5, 6, 7$,

$$\sum_{q \le Q} \int_{|\beta| \le \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta)\mathscr{B}_j(v, q, \beta)\mathrm{d}\beta$$
$$\ll_\varepsilon x^\varepsilon \sum_{q \le Q} q^{\frac{3}{2}+\varepsilon} \int_{|\beta| \le \frac{1}{qQ}} \frac{1}{x^{-1} + |\beta|} \mathrm{d}\beta$$
$$\ll_\varepsilon x^\varepsilon Q^{\frac{5}{2}+\varepsilon}. \tag{6.5}$$

By (4.7), (4.19)–(4.21) and (6.1)–(6.3), we have, for $j = 8, 9, 10$,

$$\mathscr{B}_j(v, q, \beta) \ll_\varepsilon \frac{1}{q^3} \frac{x(\log x)^2}{1 + |\beta|x} \left(\frac{x}{1 + |\beta|x}\right)^{\frac{1}{2}} \sum_{n|q} n\tau(n) q^{3+\varepsilon} n^{-\frac{1}{2}} (\log(q + 2))^4$$
$$\ll_\varepsilon x^\varepsilon q^{\frac{1}{2}+\varepsilon} \left(\frac{1}{x^{-1} + |\beta|}\right)^{\frac{3}{2}}. \tag{6.6}$$

By (4.3) and (6.6), we obtain, for $j = 8, 9, 10$,

$$\begin{aligned}\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_j(v,q,\beta)\mathrm{d}\beta \\ \ll_\varepsilon x^\varepsilon\sum_{q\le Q}q^{\frac12+\varepsilon}\int\limits_{|\beta|\le\frac{1}{qQ}}\left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac32}\mathrm{d}\beta \\ \ll_\varepsilon x^{\frac12+\varepsilon}Q^{\frac32+\varepsilon}.\end{aligned} \tag{6.7}$$

By (4.7), (4.22)–(4.24) and (6.1)–(6.3), we have, for $j = 11, 12, 13$,

$$\begin{aligned}\mathscr{B}_j(v,q,\beta) &\ll_\varepsilon \frac{1}{q^4}\frac{x(\log x)^2}{1+|\beta|x}\frac{x}{1+|\beta|x}\sum_{n|q}n\tau(n)q^{3+\varepsilon}n^{-\frac12}(\log(q+2))^3 \\ &\ll_\varepsilon x^\varepsilon q^{-\frac12+\varepsilon}\left(\frac{1}{x^{-1}+|\beta|}\right)^2.\end{aligned} \tag{6.8}$$

By (4.3) and (6.8), we obtain, for $j = 11, 12, 13$,

$$\begin{aligned}\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_j(v,q,\beta)\mathrm{d}\beta \\ \ll_\varepsilon x^\varepsilon\sum_{q\le Q}q^{-\frac12+\varepsilon}\int\limits_{|\beta|\le\frac{1}{qQ}}\left(\frac{1}{x^{-1}+|\beta|}\right)^{2}\mathrm{d}\beta \\ \ll_\varepsilon x^{1+\varepsilon}Q^{\frac12+\varepsilon}.\end{aligned} \tag{6.9}$$

By (4.2), (4.11), (6.5), (6.7) and (6.9), the contribution from $\mathscr{B}_j$, $5 \le j \le 13$, is $O_\varepsilon(x^{\frac54+\varepsilon})$.

7. Computation of the main terms

The three sums $\mathscr{B}_{14}(v,q,\beta)$, $\mathscr{B}_{15}(v,q,\beta)$ and $\mathscr{B}_{16}(v,q,\beta)$ in (4.25)–(4.27) contribute the main terms. Replacing $\widetilde{\phi_\beta}^{(j)}(1)$ by

$$\vartheta^{\flat,j}(\beta) = \int\limits_{X/2}^{X} e(-\beta u)(\log u)^j\mathrm{d}u, \qquad j = 0, 1, 2, \tag{7.1}$$

we need to estimate the remainder terms from

$$\vartheta^{\sharp,j}(\beta) = \widetilde{\phi_\beta}^{(j)}(1) - \vartheta^{\flat,j}(\beta).$$

Write correspondingly

$$\mathscr{B}_j^{\sharp}(v,q,\beta) = \mathscr{B}_j(v,q,\beta) - \mathscr{B}_j^{\flat}(v,q,\beta), \qquad j = 14, 15, 16.$$

First, we evaluate the remainder terms from $\mathscr{B}_j^{\sharp}(v,q,\beta)$, $j = 14, 15, 16$. Notice that

$$\vartheta^{\sharp,j}(\beta) = \int\limits_{X/2}^{X} \left(\phi\left(\frac{u}{X}\right) - 1\right) e(-\beta u)(\log u)^j \mathrm{d}u \ll XM^{-1}(\log X)^j.$$

Hence

$$\begin{aligned}
&\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q} \varrho(v,q,\beta)\mathscr{B}_{14}^{\sharp}(v,q,\beta)\mathrm{d}\beta \\
&= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\sharp,0}(\beta)\Psi_0(\beta)^3 \\
&\quad\times\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,v;q)\mathrm{d}\beta \\
&\ll_\varepsilon X^{1+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac12}\tau(n)|P_2(n,q)|\int\limits_{|\beta|\le\frac{1}{qQ}}\left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac32}\mathrm{d}\beta \\
&\ll_\varepsilon x^{\frac32+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac12}\tau(n)|P_2(n,q)| \\
&\ll_\varepsilon x^{\frac32+\varepsilon}M^{-1}.
\end{aligned} \tag{7.2}$$

Here we have used (4.3), (5.2) and (6.3). Similarly,

$$\begin{aligned}
&\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q} \varrho(v,q,\beta)\mathscr{B}_{15}^{\sharp}(v,q,\beta)\mathrm{d}\beta \\
&= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\sharp,1}(\beta)\Psi_0(\beta)^3 \\
&\quad\times\sum_{n|q} n\tau(n)P_1(n,q)\mathscr{C}(0,0,0,n,0,v;q)\mathrm{d}\beta \\
&\ll_\varepsilon X^{1+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac12}\tau(n)|P_1(n,q)|\int\limits_{|\beta|\le\frac{1}{qQ}}\left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac32}\mathrm{d}\beta \\
&\ll_\varepsilon x^{\frac32+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac12}\tau(n)|P_1(n,q)| \\
&\ll_\varepsilon x^{\frac32+\varepsilon}M^{-1},
\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}
&\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}^{\sharp}_{16}(v,q,\beta)\mathrm{d}\beta\\
&\quad=\frac{1}{4}\sum_{q\le Q}\frac{1}{q^5}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\sharp,2}(\beta)\Psi_0(\beta)^3\sum_{n|q}n\tau(n)\mathscr{C}(0,0,0,n,0,v;q)\mathrm{d}\beta\\
&\quad\ll_\varepsilon X^{1+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac{1}{2}}\tau(n)\int\limits_{|\beta|\le\frac{1}{qQ}}\left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac{3}{2}}\mathrm{d}\beta\\
&\quad\ll_\varepsilon x^{\frac{3}{2}+\varepsilon}M^{-1}\sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac{1}{2}}\tau(n)\\
&\quad\ll_\varepsilon x^{\frac{3}{2}+\varepsilon}M^{-1}.
\end{aligned}\tag{7.4}$$

Next, we want to compute the contributions from $\mathscr{B}^{\flat}_j(v,q,\beta)$ which constitute the main terms. Interchanging the order of summation over a and the integration over β, we have

$$\begin{aligned}
&\sum_{q\le Q}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}^{\flat}_{14}(v,q,\beta)\mathrm{d}\beta\\
&\quad=\frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\int\limits_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\sum_{n|q}n\tau(n)P_2(n,q)\\
&\qquad\times\sideset{}{^*}\sum_{a=1}^{q}e\left(\frac{-\overline{a}v}{q}\right)G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\mathrm{d}\beta\\
&\quad=\frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\int\limits_{\mathscr{M}(a,q)}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta.
\end{aligned}$$

Note that

$$\left[-\frac{1}{2qQ},\frac{1}{2qQ}\right]\subseteq\mathscr{M}(a,q)\subseteq\left[-\frac{1}{qQ},\frac{1}{qQ}\right].$$

As in [16], we write $\mathscr{M}(a,q)$ as

$$\mathscr{M}(a,q)=\mathscr{M}(a,q)\backslash\left[-\frac{1}{2qQ},\frac{1}{2qQ}\right]\bigcup\left[-\frac{1}{2qQ},\frac{1}{2qQ}\right].$$

Accordingly,

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_{14}^{\flat}(v,q,\beta)\mathrm{d}\beta$$
$$=\frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\int_{-\infty}^{\infty}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta$$
$$+\mathscr{B}_{14}^{*}-\mathscr{B}_{14}^{**}, \tag{7.5}$$

where

$$\mathscr{B}_{14}^{*}=\frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)$$
$$\times\int_{\mathscr{M}(a,q)\setminus\left[-\frac{1}{2qQ},\frac{1}{2qQ}\right]}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta$$
$$\mathscr{B}_{14}^{**}=\frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\int_{|\beta|>\frac{1}{2qQ}}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta.$$

Interchanging the order of summation over a and the integration over β again, we have

$$\begin{aligned}\mathscr{B}_{14}^{*} &= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\int_{\frac{1}{2qQ}\le|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\\ &\quad\sideset{}{^*}\sum_{a=1}^{q}e\left(\frac{-\overline{a}v}{q}\right)G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\mathrm{d}\beta\\ &\ll_\varepsilon \sum_{q\le Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac{1}{2}}\tau(n)|P_2(n,q)|\int_{\frac{1}{2qQ}\le|\beta|\le\frac{1}{qQ}}|\beta|^{-\frac{5}{2}}\mathrm{d}\beta\\ &\ll_\varepsilon Q^{\frac{5}{2}+\varepsilon},\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}\mathscr{B}_{14}^{**} &= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\int_{|\beta|>\frac{1}{2qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\\ &\quad\sideset{}{^*}\sum_{a=1}^{q}e\left(\frac{-\overline{a}v}{q}\right)G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right)\mathrm{d}\beta\end{aligned}$$

$$\ll_\varepsilon \sum_{q\le Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}}\tau(n)|P_2(n,q)| \int\limits_{|\beta|>\frac{1}{2qQ}} |\beta|^{-\frac{5}{2}}\mathrm{d}\beta$$

$$\ll_\varepsilon Q^{\frac{5}{2}+\varepsilon}. \tag{7.7}$$

Here we have used (4.3), (5.2) and (6.1)–(6.3). By (7.5)–(7.7), we obtain

$$\sum_{q\le Q} \int\limits_{|\beta|\le\frac{1}{qQ}} \sum_{v \bmod q} \varrho(v,q,\beta)\mathscr{B}^\flat_{14}(v,q,\beta)\mathrm{d}\beta$$

$$= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q} n\tau(n)P_2(n,q)\sum_{a=1}^{q}{}^{*} G(a,0;q)^3 S\left(-\overline{a},0;\frac{q}{n}\right)$$

$$\times \int\limits_{-\infty}^{\infty} \vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta + O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}\right). \tag{7.8}$$

Moreover, by (4.6),

$$\Psi_0(\beta) = \int\limits_0^{\sqrt{x}} e(\beta v^2)\mathrm{d}v = x^{\frac{1}{2}}\int\limits_0^1 e(\beta x v^2)\mathrm{d}v.$$

It follows that

$$\int\limits_{-\infty}^{\infty} \vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta = x^{\frac{3}{2}}\int\limits_{-\infty}^{\infty}\left(\int\limits_{X/2}^{X} e(-\beta u)\mathrm{d}u\right)\left(\int\limits_0^1 e(\beta x v^2)\mathrm{d}v\right)^3\mathrm{d}\beta := x^{\frac{3}{2}}\mathcal{I}_0(X),$$

where

$$\mathcal{I}_0(X) = \int\limits_{-\infty}^{\infty}\left(\int\limits_{X/2}^{X} e(-\beta u)\mathrm{d}u\right)\left(\int\limits_0^1 e(\beta x v^2)\mathrm{d}v\right)^3\mathrm{d}\beta. \tag{7.9}$$

Substituting in (7.8) and by (7.2), we have

$$\sum_{q\le Q} \int\limits_{|\beta|\le\frac{1}{qQ}} \sum_{v \bmod q} \varrho(v,q,\beta)\mathscr{B}_{14}(v,q,\beta)\mathrm{d}\beta$$

$$= \frac{1}{2}\mathcal{I}_0(X)x^{\frac{3}{2}}\sum_{q\le Q}\frac{1}{q^5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,0;q) + O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}M^{-1}\right). \tag{7.10}$$

Further, by (6.1) and (6.3), we have

$$\begin{aligned}
&\sum_{q>Q}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,0;q)\\
&\quad\ll_\varepsilon \sum_{q>Q}q^{-2+\varepsilon}\sum_{n|q}n^{\frac{1}{2}}\tau(n)|P_2(n,q)|\\
&\quad\ll_\varepsilon Q^{-\frac{1}{2}+\varepsilon}. \qquad (7.11)
\end{aligned}$$

By (7.10) and (7.11), we conclude that

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_{14}(v,q,\beta)\mathrm{d}\beta=\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right), \qquad (7.12)$$

where

$$\mathcal{C}_2=\sum_{q=1}^{\infty}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_2(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right). \qquad (7.13)$$

Similarly, we have

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}^{\flat}_{15}(v,q,\beta)\mathrm{d}\beta=\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}\right), \qquad (7.14)$$

where

$$\mathcal{I}_1(X)=\int_{-\infty}^{\infty}\left(\int_{X/2}^{X}e(-\beta u)(\log u)\mathrm{d}u\right)\left(\int_0^1 e(\beta xv^2)\mathrm{d}v\right)^3\mathrm{d}\beta, \qquad (7.15)$$

and

$$\mathcal{C}_1=\sum_{q=1}^{\infty}\frac{1}{q^5}\sum_{n|q}n\tau(n)P_1(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right). \qquad (7.16)$$

By (7.3) and (7.14), we have

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_{15}(v,q,\beta)\mathrm{d}\beta=\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right). \qquad (7.17)$$

Finally, we have

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_{16}^{\flat}(v,q,\beta)\mathrm{d}\beta=\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}\right), \quad (7.18)$$

where

$$\mathcal{I}_2(X)=\int_{-\infty}^{\infty}\left(\int_{X/2}^{X}e(-\beta u)(\log u)^2\mathrm{d}u\right)\left(\int_0^1 e(\beta xv^2)\mathrm{d}v\right)^3\mathrm{d}\beta, \quad (7.19)$$

and

$$\mathcal{C}_0=\sum_{q=1}^{\infty}\frac{1}{q^5}\sum_{n|q}n\tau(n)\sum_{a=1}^{q}{}^{*}G(a,0;q)^3S\left(-\overline{a},0;\frac{q}{n}\right). \quad (7.20)$$

By (7.4) and (7.18), we obtain

$$\sum_{q\le Q}\int_{|\beta|\le\frac{1}{qQ}}\sum_{v \bmod q}\varrho(v,q,\beta)\mathscr{B}_{16}(v,q,\beta)\mathrm{d}\beta=\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0X^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right). \quad (7.21)$$

By (4.2), (4.11), (7.12), (7.17) and (7.21), the contribution from $\mathscr{B}_j$, $14\le j\le 16$, is

$$\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right),$$

where $\mathcal{I}_j$ $(0\le j\le 2)$ and $\mathcal{C}_j$ $(0\le j\le 2)$ are defined in (7.9), (7.15), (7.19) and (7.13), (7.16), (7.20), respectively.

8. Proof of Proposition 5.2

Recall $\Phi_\beta^{\pm}(y)$ in (4.9) which we relabel as

$$\Phi_\beta^{\pm}(y)=\Phi_0(y,\beta)\pm\frac{1}{i\pi^3y}\Phi_1(y,\beta), \quad (8.1)$$

where for $\sigma>-1-k$,

$$\Phi_k(y,\beta)=(\pi^3y)^k\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma}(\pi^3y)^{-s}\frac{\Gamma\left(\frac{1+s+k}{2}\right)^3}{\Gamma\left(\frac{-s+k}{2}\right)^3}\widetilde{\phi_\beta}(-s)\mathrm{d}s \quad (8.2)$$

with $\phi_\beta(y) = \phi\left(\frac{y}{X}\right) e(-\beta y)$. Note that

$$\phi_\beta^{(j)}(y) \ll_j \left(\frac{M+|\beta|X}{X}\right)^j .$$

By Lemma 3.2, we have

$$\begin{aligned}
&\sum_{\pm}\sum_{m\geq 1}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right| \\
&= \sum_{\pm}\sum_{\frac{mn^2}{q^3}X<X^\varepsilon(M+|\beta|X)^3}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right| + O_\varepsilon(1). \qquad (8.3)
\end{aligned}$$

Moreover, by (3.1), trivially, we have

$$\sum_{m\leq L}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right) \ll \sum_{m\leq L}\tau_3(n)\tau_3(m) \ll_\varepsilon n^\varepsilon L^{1+\varepsilon}. \qquad (8.4)$$

For $yX \ll X^\varepsilon$, we move the line of integration in (8.2) to $\sigma = -1+\varepsilon$ to obtain

$$\begin{aligned}
\Phi_k(y,\beta) &= (\pi^3 y)^k \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-1+\varepsilon}(\pi^3 y)^{-s}\frac{\Gamma\left(\frac{1+s+k}{2}\right)^3}{\Gamma\left(\frac{-s+k}{2}\right)^3}\widetilde{\phi_\beta}(-s)\mathrm{d}s \\
&\ll y^k (yX)^{1-\varepsilon}\int_{-\infty}^{\infty}(1+|t|)^{-\frac{3}{2}+3\varepsilon}\mathrm{d}t \\
&\ll_\varepsilon y^k X^\varepsilon. \qquad (8.5)
\end{aligned}$$

Here we have used Stirling's formula and the estimate $\widetilde{\phi_\beta}(-s) = \int_0^\infty \phi_\beta(u)u^{-s-1}\mathrm{d}u \ll X^{-\sigma}$. By (8.1), (8.4) and (8.5), we have

$$\begin{aligned}
&\sum_{\pm}\sum_{\frac{mn^2}{q^3}X\ll X^\varepsilon}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right| \\
&\ll_\varepsilon X^\varepsilon \max_{1\leq L\ll\frac{q^3X^\varepsilon}{n^2X}}\frac{1}{L}\sum_{L<m\leq 2L}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right) \\
&\ll_\varepsilon X^\varepsilon \max_{1\leq L\ll\frac{q^3X^\varepsilon}{n^2X}}\frac{1}{L}n^\varepsilon L^{1+\varepsilon} \\
&\ll_\varepsilon X^\varepsilon n^\varepsilon. \qquad (8.6)
\end{aligned}$$

For $yX > X^{\varepsilon}$, by Lemma 3.3, we have

$$\Phi_k(y,\beta) = (\pi^3 y)^{k+1} \sum_{j=1}^{\ell} \int_0^{\infty} \phi\left(\frac{u}{X}\right) e(-\beta u) \left(a_k(j) e\left(3(yu)^{\frac{1}{3}}\right) + b_k(j) e\left(-3(yu)^{\frac{1}{3}}\right)\right) \times \frac{\mathrm{d}u}{(\pi^3 yu)^{\frac{j}{3}}} + O_{\varepsilon,\ell}\left((\pi^3 y)^k (\pi^3 yX)^{-\frac{\ell}{3}+\frac{1}{2}+\varepsilon}\right),$$

where $a_k(j)$ and $b_k(j)$ are constants. Then by (8.1),

$$\Phi_\beta^{\pm}(y) \ll_{\varepsilon,\ell} y \sum_{j=1}^{\ell} y^{-\frac{j}{3}} \left(|\mathscr{I}_j(y,\beta)| + |\mathscr{J}_j(y,\beta)|\right) + (yX)^{-\frac{\ell}{3}+\frac{1}{2}+\varepsilon}, \tag{8.7}$$

with

$$\mathscr{I}_j(y,\beta) = \int_0^{\infty} u^{-\frac{j}{3}} \phi\left(\frac{u}{X}\right) e(-\beta u) e\left(3(yu)^{\frac{1}{3}}\right) \mathrm{d}u,$$

$$\mathscr{J}_j(y,\beta) = \int_0^{\infty} u^{-\frac{j}{3}} \phi\left(\frac{u}{X}\right) e(-\beta u) e\left(-3(yu)^{\frac{1}{3}}\right) \mathrm{d}u.$$

By partial integration twice, we have

$$\mathscr{I}_j(y,\beta), \mathscr{J}_j(y,\beta) \ll (yX)^{-\frac{2}{3}} X^{1-\frac{j}{3}} \left(M + |\beta|^2 X^2\right). \tag{8.8}$$

Taking $\ell = 3$. By (8.7) and (8.8), we have

$$\Phi_\beta^{\pm}(y) \ll_{\varepsilon} (yX)^{\frac{1}{3}} (M + |\beta|^2 X^2) \sum_{j=1}^{3} (yX)^{-\frac{j}{3}} + (yX)^{-\frac{1}{2}+\varepsilon} \ll_{\varepsilon} M + |\beta|^2 X^2. \tag{8.9}$$

By (8.4) and (8.9), we have

$$\begin{aligned} &\sum_{\pm} \sum_{X^{\varepsilon} < \frac{n^2 m}{q^3} X < X^{\varepsilon}(M+|\beta|X)^3} \frac{1}{m} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) \left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right| \\ &\ll_{\varepsilon} (M + |\beta|^2 X^2)(\log X) \max_{\frac{q^3 X^{\varepsilon}}{n^2 X} < L < \frac{q^3 X^{\varepsilon}(M+|\beta|X)^3}{n^2 X}} \frac{1}{L} \sum_{L < m \le 2L} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) \\ &\ll_{\varepsilon} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2). \end{aligned} \tag{8.10}$$

Then Proposition 5.2 follows from (8.3), (8.6) and (8.10).

9. Estimation of the character sum $\mathscr{C}(b_1, b_2, b_3, n, m, v; q)$

In this section, we shall prove Proposition 5.1. Let $b_1, b_2, b_3, n, m, v \in \mathbb{Z}$ and $n|q$. We recall $\mathscr{C}(b_1, b_2, b_3, n, m, v; q)$ in (4.28) which we relabel as

$$\mathscr{C}(b_1, b_2, b_3, n, m, v; q) = \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{-\overline{a}v}{q}\right) G(a, b_1; q)G(a, b_2; q)G(a, b_3; q)S\left(-\overline{a}, m; \frac{q}{n}\right), \tag{9.1}$$

where $G(a, b; q)$ is the Gauss sum defined in (4.5).

We first list some well-known results for $G(a, b; q)$ (see for example Lemma 5.4.5 in [7]). For $(a, q) = 1$, we have

$$G(a, b; q) \ll \sqrt{q}. \tag{9.2}$$

For $(2a, q) = 1$, we have

$$G(a, b; q) = e\left(-\frac{\overline{4}\overline{a}b^2}{q}\right) G(a, 0; q) \tag{9.3}$$

and

$$G(a, 0; q) = \left(\frac{a}{q}\right) \epsilon_q \sqrt{q}, \tag{9.4}$$

where $\epsilon_q = \begin{cases} 1, & \text{if } q \equiv 1 \pmod 4, \\ i, & \text{if } q \equiv -1 \pmod 4. \end{cases}$

As in [13], we factor q as $q = q_1q_2q_3$, where q_1 is the largest factor of q such that $q_1|n$ and $\left(q_1, \frac{q}{q_1}\right) = 1$, and q_2 is the largest factor of q/q_1 such that $q_2|n^\infty$ and $\left(q_2, \frac{q}{q_1,q_2}\right) = 1$. Then q_2 is square-full and $n|q_1q_2$. Denote temporarily $q' = q_1q_2$ and $\widehat{q} = \frac{q'}{n}$. Then we have

$$\begin{aligned}
&\mathscr{C}(b_1, b_2, b_3, n, m, v; q) \\
&\quad = \sideset{}{^*}\sum_{a_1 \bmod q'} e\left(\frac{-\overline{a_1}v}{q'}\right) G(a_1, b_1; q')G(a_1, b_2; q')G(a_1, b_3; q')S(-\overline{a_1}q_3, m\overline{q_3}^2; \widehat{q}) \\
&\qquad \times \sideset{}{^*}\sum_{a_2 \bmod q_3} e\left(\frac{-\overline{a}_2v}{q_3}\right) G(a_2, b_1; q_3)G(a_2, b_2; q_3)G(a_2, b_3; q_3)S(-\overline{a_2}q', m\overline{\widehat{q}}^2; q_3) \\
&\quad := \mathscr{C}^*(b_1, b_2, b_3, n, m, v; q')\mathscr{C}^{**}(b_1, b_2, b_3, n, m, v; q_3)
\end{aligned} \tag{9.5}$$

say.

By (9.2) and Weil's bound for Kloosterman sum we have

$$\mathscr{C}^*(b_1,b_2,b_3,n,m,v;q') \ll q'^{\frac{5}{2}}\left(\overline{a_1}q_3, m\overline{q_3}^2, \frac{q'}{n}\right)^{\frac{1}{2}}\left(\frac{q'}{n}\right)^{\frac{1}{2}}\tau\left(\frac{q'}{n}\right) \ll \frac{q_1^3q_2^3\tau(q_1q_2)}{\sqrt{n}}. \tag{9.6}$$

To estimate $\mathscr{C}^{**}(b_1,b_2,b_3,n,m,v;q_3)$, we further factor q_3 as $q_3 = q_3'q_3''$ with $(q_3', 2q_3'') = 1$, q_3' square-free and $4q_3''$ square-full. Then

$$\mathscr{C}^{**}(b_1,b_2,b_3,n,m,v;q_3) = \mathscr{C}_1^{**}(b_1,b_2,b_3,n,m,v;q_3')\mathscr{C}_2^{**}(b_1,b_2,b_3,n,m,v;q_3''), \tag{9.7}$$

where

$$\begin{aligned}\mathscr{C}_1^{**}(b_1,b_2,b_3,n,m,v;q_3') = &\sideset{}{^*}\sum_{\gamma \bmod q_3'} e\left(\frac{-\overline{\gamma}v}{q_3'}\right)G(\gamma,b_1;q_3')G(\gamma,b_2;q_3')G(\gamma,b_3;q_3') \\ &\times S(-\overline{\gamma}q_3''q', m\overline{\overline{q}}^2\,\overline{q_3''}^2; q_3'),\end{aligned}$$

and

$$\begin{aligned}\mathscr{C}_2^{**}(b_1,b_2,b_3,n,m,v;q_3'') = &\sideset{}{^*}\sum_{\gamma \bmod q_3''} e\left(\frac{-\overline{\gamma}v}{q_3''}\right)G(\gamma,b_1;q_3'')G(\gamma,b_2;q_3'')G(\gamma,b_3;q_3'') \\ &\times S(-\overline{\gamma}q_3'q', m\overline{\overline{q}}^2\,\overline{q_3'}^2; q_3'').\end{aligned}$$

We estimate $\mathscr{C}_2^{**}(b_1,b_2,b_3,n,m,v;q_3'')$ similarly as $\mathscr{C}^*(b_1,b_2,b_3,n,m,v;q')$ getting

$$\mathscr{C}_2^{**}(b_1,b_2,b_3,n,m,v;q_3'') \ll q''^3\tau(q_3''). \tag{9.8}$$

To estimate $\mathscr{C}_1^{**} := \mathscr{C}_1^{**}(b_1,b_2,b_3,n,m,v;q_3')$, we factor q_3' as $q_3' = p_1p_2\cdots p_s$, p_i prime, and correspondingly,

$$\mathscr{C}_1^{**} = \prod_{i=1}^{s}\mathscr{T}(b_1,b_2,b_3,q'q_3''p_i', m\overline{\overline{q}}^2\,\overline{q_3''}^2\,\overline{p_i'}^2; p_i), \tag{9.9}$$

where $p_i' = q_3'/p_i$ and

$$\begin{aligned}&\mathscr{T}(b_1,b_2,b_3,r_1,r_2m;p) \\ &= \sideset{}{^*}\sum_{z \bmod p} e\left(\frac{-v\overline{z}}{p}\right)G(z,b_1;p)G(z,b_2;p)G(z,b_3;p)S(-r_1\overline{z}, r_2m;p)\end{aligned}$$

with $(p, 2r_1r_2) = 1$.

By (9.3) and (9.4), we write

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p)$$
$$= \epsilon_p^3 p^{\frac{3}{2}} \sideset{}{^*}\sum_{z \bmod p} \left(\frac{z}{p}\right) e\left(\frac{-\overline{4}(4v + b_1^2 + b_2^2 + b_3^2)\overline{z}}{p}\right) S(-r_1\overline{z}, r_2m; p).$$

By (9.5)–(9.9), Proposition 5.1 follows from the following lemma.

Lemma 9.1. *We have*

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p) \ll p^{\frac{5}{2}}.$$

Proof. If $p|m$, then $S(-r_1\overline{z}, r_2m; p) = -1$ and trivially,

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p) = -\epsilon_p^3 p^{\frac{3}{2}} \sideset{}{^*}\sum_{z \bmod p} \left(\frac{z}{p}\right) e\left(\frac{-\overline{4}(4v + b_1^2 + b_2^2 + b_3^2)\overline{z}}{p}\right) \ll p^{\frac{5}{2}}.$$

If $p \nmid m$, we open the Kloosterman sum to obtain

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p) = \epsilon_p^3 p^{\frac{3}{2}} \sum_{y,z \in \mathbb{F}_p^\times} \left(\frac{z}{p}\right) e\left(\frac{-\overline{4}(4v + b_1^2 + b_2^2 + b_3^2)\overline{z} - r_1 y\overline{z} + r_2 m\overline{y}}{p}\right).$$

If $p \nmid m$, $p|4v + b_1^2 + b_2^2 + b_3^2$, changing variable $y\overline{z} \to z$, we have

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p) = \epsilon_p^3 p^{\frac{3}{2}} \sum_{y \in \mathbb{F}_p^\times} \left(\frac{y}{p}\right) e\left(\frac{r_2 m\overline{y}}{p}\right) \sum_{z \in \mathbb{F}_p^\times} \left(\frac{\overline{z}}{p}\right) e\left(\frac{-r_1 z}{p}\right)$$
$$= \epsilon_p^3 p^{\frac{3}{2}} \left(\frac{\overline{r_2 m}}{p}\right) \left(\frac{-\overline{r_1}}{p}\right) \tau\left(\left(\frac{\cdot}{p}\right)\right)^2,$$

where $\tau\left(\left(\frac{\cdot}{p}\right)\right)$ is the Gauss sum associated with the quadratic residue $\left(\frac{\cdot}{p}\right)$, i.e.

$$\tau\left(\left(\frac{\cdot}{p}\right)\right) = \sum_{\gamma \bmod p} \left(\frac{\gamma}{p}\right) e\left(\frac{\gamma}{p}\right) = \epsilon_p \sqrt{p}.$$

Therefore,

$$\mathscr{T}(b_1, b_2, b_3, r_1, r_2m; p) \ll p^{\frac{5}{2}}.$$

If $p \nmid m$, $p \nmid 4v + b_1^2 + b_2^2 + b_3^2$, we denote $r_0 = -\overline{4}(4v + b_1^2 + b_2^2 + b_3^2)$ and let $f(y,z) = r_0 z^{-1} - r_1 z^{-1} y + r_2 m y^{-1} \in \mathbb{F}_p^\times[y, z, (yz)^{-1}]$. The Newton polyhedron $\Delta(f)$ of f is the quadrangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(-1,0)$, $(-1,1)$ and $(0,-1)$. Thus $\dim \Delta(f) = 2$. Moreover, for each of the following five polynomials

$$f_\sigma(y,z) = r_0 z^{-1}, -r_1 z^{-1} y, r_2 m y^{-1}, r_0 z^{-1} - r_1 z^{-1} y, r_0 z^{-1} + r_2 m y^{-1},$$

corresponding to the faces of $\Delta(f)$ not containing $(0,0)$, the locus of

$$\frac{\partial f_\sigma}{\partial y} = \frac{\partial f_\sigma}{\partial z} = 0$$

is empty in $\left(\overline{F_p}^\times\right)^2$. In other words f is non-degenerate with respect to $\Delta(f)$. By Corollary 0.3 in Fu [3], we have

$$\sum_{y,z\in\mathbb{F}_p^\times} \left(\frac{z}{p}\right) e\left(\frac{r_0\overline{z} - r_1\overline{z}y + r_2 m\overline{y}}{p}\right) \ll p.$$

This completes the proof of Lemma 9.1. □

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