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General Section

Hilbert modularity of some double octic Calabi–Yau threefolds [☆]

Sławomir Cynk [a,*], Matthias Schütt [b,c], Duco van Straten [d]

[a] Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland

[b] Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

[c] Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany

[d] Institut für Mathematik, FB 08 – Physik, Mathematik und Informatik, Johannes Gutenberg-Universität, Staudingerweg 9, 55128 Mainz, Germany

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ABSTRACT

We exhibit three double octic Calabi–Yau threefolds, a non-rigid threefold defined over $\mathbb{Q}$ and two rigid threefolds over the quadratic fields $\mathbb{Q}[\sqrt{5}], \mathbb{Q}[\sqrt{-3}]$, and prove their modularity. The non-rigid threefold has two conjugate Hilbert modular forms for the field $\mathbb{Q}[\sqrt{2}]$ of weight $[4,2]$ and $[2,4]$ attached while the two rigid threefolds correspond to a Hilbert modular form of weight $[4,4]$ and to the twist of the restriction of a classical modular form of weight 4.

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* Corresponding author.
E-mail addresses: slawomir.cynk@im.uj.edu.pl (S. Cynk), schuett@math.uni-hannover.de (M. Schütt), straten@mathematik.uni-mainz.de (D. van Straten).

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1. Introduction

After the modularity of elliptic curves over $\mathbb{Q}$ was proven, much emphasis has been put on Calabi–Yau threefolds. Indeed, for rigid Calabi–Yau threefolds over $\mathbb{Q}$, the modularity has been established in the meantime independently in [8], [11] as a consequence of Serre's conjecture [15], [16]. Over number fields other than $\mathbb{Q}$, however (where Serre's conjecture is completely open), there seems to be only one (Hilbert) modular example to date (outside the CM case), the Consani–Scholten quintic from [3], [10].

This paper will provide three more examples of modular Calabi-Yau threefolds, each of which is defined over some quadratic field K. In detail, we will exhibit

(1) a non-rigid Calabi–Yau threefold X over $\mathbb{Q}$ (with $b_3(X) = 4$) admitting a rational self-map over $\mathbb{Q}[\sqrt{2}]$ which can be used to split $H^3(X)$ into two 2-dimensional eigenspaces; the corresponding two-dimensional Galois representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ are proved to correspond to a Hilbert modular form over $\mathbb{Q}[\sqrt{2}]$ of weight $[4, 2]$ and its Galois conjugate (Theorem 4.1, Remark 4.2);
(2) a rigid Calabi–Yau threefold Y over $\mathbb{Q}[\sqrt{5}]$ such that the Galois representation on $H^3(Y)$ corresponds to a Hilbert modular form over $\mathbb{Q}[\sqrt{5}]$ of weight $[4, 4]$ (Theorem 5.1);
(3) a rigid Calabi–Yau threefold Z over $\mathbb{Q}[\sqrt{-3}]$ such that the Galois representation on $H^3(Z)$ corresponds to a twist of the restriction of a classical modular form of weight 4 and level 72 (Theorem 6.2).

The Calabi–Yau threefolds will be constructed as crepant resolutions of certain double octics. More precisely, we will choose the branch locus to consist of 8 distinct planes. The construction, following [19], will be reviewed in Section 3.

Calabi–Yau varieties come with the benefit that the Hodge diamond is relatively simple. It follows that all but the middle cohomology is spanned by algebraic cycles, so that the Galois group acts on the even cohomology through a finite group after a Tate twist (and this can be determined explicitly from the geometry, see Section 4 e.g.). In our cases, the automorphy of all cohomology thus only depends on the middle cohomology.

The proof of the modularity of the non-rigid threefold X relies on a detailed geometric study. We can write down explicitly its deformation family, the Picard-Fuchs operator of this family is symmetric and the considered threefold corresponds to a fixed point of this symmetry. Using a presentation of X as a Kummer fibration associated to a pair of rational elliptic surfaces, we are able to construct a rational, generically two-to-one correspondence between the symmetric threefolds. This map gives a rational self-map Ψ of X that acts as multiplication by $\sqrt{2}$ on $H^{3,0} \oplus H^{0,3}$ and as multiplication by $-\sqrt{2}$ on $H^{2,1} \oplus H^{1,2}$. Consequently it decomposes the restriction of the Galois action on $H^3(X)$ to the absolute Galois group G_K of the number field $K = \mathbb{Q}[\sqrt{2}]$ into two 2-dimensional pieces $H^3_+ \oplus H^3_-$. Equivalently, the Galois representations satisfy $H^3(X) = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} H^3_+$.

With these preparations, the proofs of the mentioned results amount to comparing the two-dimensional Galois representations attached to the motive of the third cohomology of the Calabi–Yau threefolds (or the given submotives H^3_+, H^3_- in the first case) on the one hand and to the Hilbert or classical modular forms in question on the other. In practice, this can be achieved by working with the 2-adic (or $\sqrt{2}$-adic) representations and applying a method going back to Faltings and Serre and worked out in detail by Livné in [17]. This technique will be explained, with a view towards the given base fields, in Section 2. In essence, it reduces the proof of modularity to comparing a suitable number of traces and determinants of the Galois representations at certain Frobenius elements; these, in turn, can be obtained from extensive point counting using the Lefschetz fixed point formula (and the rational self-map Ψ), and from Hilbert modular forms calculations as incorporated in MAGMA.

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2. Faltings–Serre–Livné method

In this section we study special cases of [17, Thm. 4.3]; they will be instrumental in establishing the modularity of the three two-dimensional Galois representations attached to the Calabi–Yau threefolds X, Y, Z to be introduced in the next sections.

Throughout this section, the set-up comprises continuous two-dimensional 2-adic Galois representations of the absolute Galois group of a specified number field K which are unramified outside a given finite set S of prime ideals in the integer ring $\mathcal{O}_K$ of K. For simplicity, we list prime ideals just by a single generator. The norms will be included in the tables to follow.

Proposition 2.1. *Let $K = \mathbb{Q}[\sqrt{2}]$ and $E = \mathbb{Q}_2[\sqrt{2}]$ and let $\mathcal{P} := \sqrt{2}\,\mathbb{Z}_2[\sqrt{2}]$ be the maximal ideal of the ring of integers of E. Let $S := \{\sqrt{2}, 3\}$ and*

$$\begin{aligned} T = \{&5, 11, \sqrt{2}+3,\ \sqrt{2}-3,\ 3\sqrt{2}-1,\ \sqrt{2}+5,\ \sqrt{2}-5,\ 4\sqrt{2}-1,\ 4\sqrt{2}+1,\ 5\sqrt{2}-3,\\ &\sqrt{2}-7,\ \sqrt{2}+7,\ 4\sqrt{2}-11,\ 1-7\sqrt{2}\}\\ U = \{&5, 11, 13,\ \sqrt{2}-3,\ 3\sqrt{2}-1,\ \sqrt{2}-5,\ 4\sqrt{2}-1,\ 5\sqrt{2}-3\} \end{aligned}$$

be two sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_2(E)$ are continuous Galois representations unramified outside S and satisfying

1. $\mathrm{Tr}(\rho_1(Frob_{\mathfrak{p}})) \equiv \mathrm{Tr}(\rho_2(Frob_{\mathfrak{p}})) \equiv 0 \pmod{\mathcal{P}}$ *for* $\mathfrak{p} \in U$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,

3. $\mathrm{Tr}(\rho_1(Frob_{\mathfrak{p}})) = \mathrm{Tr}(\rho_2(Frob_{\mathfrak{p}}))$ *and* $\det(\rho_1(Frob_{\mathfrak{p}})) = \det(\rho_2(Frob_{\mathfrak{p}}))$ *for* $\mathfrak{p} \in T$.

Then ρ_1 *and* ρ_2 *have isomorphic semisimplifications.*

Proof. Following the arguments of [17] we first verify that assumption 1. implies that $\mathrm{Tr}(\rho_1) \equiv \mathrm{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$. Indeed, suppose that $\mathrm{Tr}(\rho_i) \not\equiv 0 \pmod{\mathcal{P}}$ and denote by L/K the Galois extension cut out by the kernel $\mathrm{Ker}\,\bar{\rho}_i$ of the reduction $\bar{\rho}_i$ of ρ_i modulo $\mathcal{P}$. By inspection, we have $\mathrm{im}(\bar{\rho}_i) \subseteq \mathrm{GL}_2(\mathbb{F}_2)$ where the elements of odd trace are exactly those of order 3 (which by assumption will correspond to some Frobenius elements). Hence, the Galois group of the extension L/K is isomorphic to S_3 or C_3, so it is the Galois closure of a degree 3 extension M/K.

Then M is a degree 6 extension of $\mathbb{Q}$ unramified outside $\{2, 3\}$. The database [14] lists 398 such fields presented by a monic degree 6 polynomial with rational coefficients. The assumption that M contains the subfield $K = \mathbb{Q}[\sqrt{2}]$ implies that the minimal polynomial of any primitive element of the extension $M/\mathbb{Q}$ factors over $\mathbb{Q}[\sqrt{2}]$. We check that exactly 25 of the 398 polynomials from [14] satisfy this condition. For each of these 25 degree 6 polynomials, we determine a prime integer p such that the reduction of the degree 3 polynomial over K modulo a prime $\mathfrak{p}$ in $\mathcal{O}_K$ over p stays irreducible over $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_{\mathfrak{p}}$. We list these data below.

$$x^6 - 2x^3 - 1 = \left(x^3 + \sqrt{2} - 1\right) \times \left(x^3 - \sqrt{2} - 1\right), \ p = 5$$

$$x^6 - 12x^4 + 36x^2 - 8 = \left(x^3 - 6x - 2\sqrt{2}\right) \times \left(x^3 - 6x + 2\sqrt{2}\right), \ p = 5$$

$$x^6 - 2 = \left(x^3 + \sqrt{2}\right) \times \left(x^3 - \sqrt{2}\right), \ p = 7$$

$$x^6 - 4x^3 + 2 = \left(x^3 + \sqrt{2} - 2\right) \times \left(x^3 - \sqrt{2} - 2\right), \ p = 5$$

$$x^6 + 6x^4 + 9x^2 - 8 = \left(x^3 + 3x + 2\sqrt{2}\right) \times \left(x^3 + 3x - 2\sqrt{2}\right), \ p = 11$$

$$x^6 + 6x^4 + 9x^2 - 2 = \left(x^3 + 3x - \sqrt{2}\right) \times \left(x^3 + 3x + \sqrt{2}\right), \ p = 5$$

$$\begin{aligned} x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 &= \left(x^3 + 3\sqrt{2}x^2 + 6x + 2\sqrt{2} - 3\right) \\ &\quad \times \left(x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} - 3\right), \ p = 7 \end{aligned}$$

$$x^6 - 18 = \left(x^3 - 3\sqrt{2}\right) \times \left(x^3 + 3\sqrt{2}\right), \ p = 7$$

$$\begin{aligned} x^6 - 6x^4 - 12x^3 + 12x^2 - 72x + 28 &= \left(x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} - 6\right) \\ &\quad \times \left(x^3 + 3\sqrt{2}x^2 + 6x + 2\sqrt{2} - 6\right), \ p = 13 \end{aligned}$$

$$x^6 - 6x^3 - 9 = \left(x^3 - 3\sqrt{2} - 3\right) \times \left(x^3 + 3\sqrt{2} - 3\right), \ p = 5$$

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$$x^6 - 6\,x^4 - 4\,x^3 + 9\,x^2 + 12\,x - 14 = \left(x^3 - 3\,x - 3\,\sqrt{2} - 2\right) \times \left(x^3 - 3\,x + 3\,\sqrt{2} - 2\right),\ p = 5$$

$$x^6 - 18\,x^4 - 12\,x^3 + 81\,x^2 + 108\,x + 18 = \left(x^3 - 9\,x + 3\,\sqrt{2} - 6\right) \times \left(x^3 - 9\,x - 3\,\sqrt{2} - 6\right),\ p = 5$$

$$x^6 + 6\,x^4 - 4\,x^3 - 9\,x^2 + 12\,x - 4 = \left(x^3 + 3\,\sqrt{2}x + 3\,x - 2\,\sqrt{2} - 2\right) \times \left(x^3 - 3\,\sqrt{2}x + 3\,x + 2\,\sqrt{2} - 2\right),\ p = 31$$

$$x^6 + 6\,x^4 - 4\,x^3 + 9\,x^2 - 12\,x - 4 = \left(x^3 + 3\,x + 2\,\sqrt{2} - 2\right) \times \left(x^3 + 3\,x - 2\,\sqrt{2} - 2\right),\ p = 23$$

$$x^6 - 6\,x^4 - 4\,x^3 + 9\,x^2 + 12\,x - 4 = \left(x^3 - 3\,x - 2\,\sqrt{2} - 2\right) \times \left(x^3 - 3\,x + 2\,\sqrt{2} - 2\right),\ p = 5$$

$$x^6 - 12\,x^3 + 18 = \left(x^3 + 3\,\sqrt{2} - 6\right) \times \left(x^3 - 3\,\sqrt{2} - 6\right),\ p = 5$$

$$x^6 - 12\,x^3 - 36 = \left(x^3 + 6\,\sqrt{2} - 6\right) \times \left(x^3 - 6\,\sqrt{2} - 6\right),\ p = 5$$

$$x^6 - 6\,x^4 - 4\,x^3 - 9\,x^2 - 12\,x - 4 = \left(x^3 - 3\,\sqrt{2}x - 3\,x - 2\,\sqrt{2} - 2\right) \times \left(x^3 + 3\,\sqrt{2}x - 3\,x + 2\,\sqrt{2} - 2\right),\ p = 41$$

$$x^6 - 8\,x^3 - 18\,x^2 - 48\,x - 16 = \left(x^3 - 3\,\sqrt{2}x - 4\,\sqrt{2} - 4\right) \times \left(x^3 + 3\,\sqrt{2}x + 4\,\sqrt{2} - 4\right),\ p = 7$$

$$x^6 + 6\,x^4 - 12\,x^3 + 9\,x^2 - 36\,x + 28 = \left(x^3 + 3\,x - 2\,\sqrt{2} - 6\right) \times \left(x^3 + 3\,x + 2\,\sqrt{2} - 6\right),\ p = 17$$

$$x^6 - 8\,x^3 - 18\,x^2 + 24\,x + 8 = \left(x^3 - 3\,\sqrt{2}x + 2\,\sqrt{2} - 4\right) \times \left(x^3 + 3\,\sqrt{2}x - 2\,\sqrt{2} - 4\right),\ p = 13$$

$$x^6 - 16\,x^3 - 18\,x^2 + 48\,x + 32 = \left(x^3 + 3\,\sqrt{2}x - 4\,\sqrt{2} - 8\right) \times \left(x^3 - 3\,\sqrt{2}x + 4\,\sqrt{2} - 8\right),\ p = 11$$

$$x^6-18\,x^4-36\,x^3-81\,x^2-108\,x+36=\left(x^3-9\sqrt{2}x-9\,x-12\sqrt{2}-18\right)\times\left(x^3+9\sqrt{2}x-9\,x+12\sqrt{2}-18\right),\ p=11$$

$$x^6-18\,x^4-12\,x^3+81\,x^2+108\,x-36=\left(x^3-9\,x-6\sqrt{2}-6\right)\times\left(x^3-9\,x+6\sqrt{2}-6\right),\ p=11$$

$$x^6-18\,x^4-36\,x^3+81\,x^2+324\,x+252=\left(x^3-9\,x-6\sqrt{2}-18\right)\times\left(x^3-9\,x+6\sqrt{2}-18\right),\ p=11$$

Given M and $\mathfrak{p}$ as above, it follows that any element in the conjugacy class of $\mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{Gal}(L/K)$ has order 3; consequently $\mathrm{Tr}(\rho_i(\mathrm{Frob}_{\mathfrak{p}}))\equiv 1 \pmod{\mathcal{P}}$, contradicting our assumptions. Thus we see that the set U was indeed chosen in such a way that condition 1. implies that

$$\mathrm{Tr}(\rho_1)\equiv\mathrm{Tr}(\rho_2)\equiv 0 \pmod{\mathcal{P}}.$$

The traces being even is the key ingredient to apply [17, Thm. 4.3]. To this end, let K_S be the compositum of all quadratic extensions of K unramified outside S. Since the ring $\mathcal{O}_K$ is a unique factorization domain, the compositum K_S is obtained by extracting square roots of generators of $\mathcal{O}_K^*$ and of prime elements $\alpha\in\mathcal{O}_K$ with norm $N_K(\alpha)$ divisible only by elements of S. Presently, generators of K_S/K can be taken as

$$\sqrt{-1},\sqrt[4]{2},\sqrt{\sqrt{2}-1},\sqrt{3}.$$

One computes the table of quadratic characters $\mathrm{Gal}(K_S/K)\longrightarrow(\mathbb{Z}/2)^4$ at the primes from T as follows:

$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{2}$	3	-1	$\sqrt{2}-1$	$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{2}$	3	-1	$\sqrt{2}-1$
5	25	1	0	0	0	$4\sqrt{2}-1$	31	0	1	1	0
11	121	0	0	0	1	$4\sqrt{2}+1$	31	1	1	1	1
$\sqrt{2}+3$	7	0	1	1	1	$5\sqrt{2}-3$	41	1	1	0	0
$\sqrt{2}-3$	7	1	1	1	0	$\sqrt{2}-7$	47	0	0	1	0
$3\sqrt{2}-1$	17	1	1	0	1	$\sqrt{2}+7$	47	1	0	1	1
$\sqrt{2}+5$	23	0	0	1	1	$4\sqrt{2}-11$	89	0	1	0	1
$\sqrt{2}-5$	23	1	0	1	0	$1-7\sqrt{2}$	97	1	0	0	1

From the table we infer that the image of the Frobenius elements $\mathrm{Frob}_t, t\in T$, contains 14 different non-zero elements, hence it is non–cubic in the terminology of [17, Def. 4.1] (see e.g. [22, p. 53]). Thus the assertion that the Galois representations ρ_1,ρ_2 have isomorphic semisimplifications follows from [17, Thm. 4.3]. □

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Remark 2.2. Since the image of $\bar{\rho}_i$ lies in the solvable group $\mathrm{GL}_2(\mathbb{F}_2)$, one can also compute the set U using class field theory. This approach was used in [1], [9], and [20].

Note in particular that Proposition 2.1 implies that ρ_1, ρ_2 have the same L-series, a feature which will be centrally used in the proof of the modularity results stated in the introduction. The next two propositions concern the same kind of problem for different base fields and adjusted ramification sets. Since the arguments are very similar, we give only the essential ingredients.

Proposition 2.3. *Let $K = \mathbb{Q}[\sqrt{5}]$, $E = \mathbb{Q}_2$ and let $\mathcal{P} := 2\mathbb{Z}_2$ be the maximal ideal of the ring of integers of E. Let $S := \{2\}$ and*

$$T = \{3, 13, \sqrt{5}+4, 2\sqrt{5}+7, \sqrt{5}+6, \sqrt{5}-6, 2\sqrt{5}+9\}$$

be two sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_2(E)$ are continuous Galois representations unramified outside S and satisfying

1. $\mathrm{Tr}(\rho_1(Frob_3)) \equiv \mathrm{Tr}(\rho_2(Frob_3)) \equiv 0 \pmod{\mathcal{P}}$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,
3. $\mathrm{Tr}(\rho_1(Frob_{\mathfrak{p}})) = \mathrm{Tr}(\rho_2(Frob_{\mathfrak{p}}))$ *and* $\det(\rho_1(Frob_{\mathfrak{p}})) = \det(\rho_2(Frob_{\mathfrak{p}}))$ *for* $\mathfrak{p} \in T$.

Then ρ_1 and ρ_2 have isomorphic semisimplifications.

Proof. By [14] there are 106 degree 6 extensions of $\mathbb{Q}$ unramified outside $\{2, 5\}$. Only one of them contains $\mathbb{Q}[\sqrt{5}]$; it is defined by the following polynomial:

$$\begin{aligned} x^6 - 2\,x^5 - 2\,x - 1 = &\left(x^3 - x^2 + \tfrac{1}{2}(-1+\sqrt{5})x + \tfrac{1}{2}(-1+\sqrt{5})\right) \\ &\times \left(x^3 - x^2 - \tfrac{1}{2}(1+\sqrt{5})x - \tfrac{1}{2}(1+\sqrt{5})\right) \end{aligned}$$

As both cubic polynomials over $\mathbb{Z}[\sqrt{5}]$ are irreducible modulo 3, assumption 1. implies that $\mathrm{Tr}(\rho_1) \equiv \mathrm{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$ as required.

The compositum K_S of quadratic extensions of K unramified outside S is obtained from the three quadratic extensions $K[\sqrt{2}], K[\sqrt{-1}], K[\sqrt{\frac{1}{2}(\sqrt{5}-1)}]$. The table of characters is computed as follows:

$\mathfrak{p}$	$N(\mathfrak{p})$	2	-1	$\frac{1}{2}(\sqrt{5}-1)$	$\mathfrak{p}$	$N(\mathfrak{p})$	2	-1	$\frac{1}{2}(\sqrt{5}-1)$
3	9	1	1	0	$\sqrt{5}+6$	31	1	0	0
13	169	1	1	1	$\sqrt{5}-6$	31	1	0	1
$\sqrt{5}+4$	11	0	0	1	$2\sqrt{5}+9$	61	0	1	0
$2\sqrt{5}+7$	29	0	1	1					

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From the table, we infer that the image of the Frobenius elements $\mathrm{Frob}_t, t \in T$ equals $(\mathbb{Z}/2)^3 \setminus \{0\}$; in particular, it is non-cubic, and the Proposition follows from [17, Thm. 4.3] as before. □

Proposition 2.4. *Let $K = \mathbb{Q}[\sqrt{-3}]$, $E = \mathbb{Q}_2$ and let $\mathcal{P} := 2\mathbb{Z}_2$ be the maximal ideal of the ring of integers of E. Let $S := \{2, \sqrt{-3}\}$ and*

$$T = \{\sqrt{-3} - 2,\ \sqrt{-3} + 2,\ 1 + 2\sqrt{-3},\ \sqrt{-3} - 4,\ \sqrt{-3} + 4, 5 + 2\sqrt{-3},\ 5 + 4\sqrt{-3}\}$$
$$U = \{11, \sqrt{-3} - 2, 1 + 2\sqrt{-3},\ \sqrt{-3} - 4\}$$

be three sets of primes in $\mathcal{O}_K$. Suppose that $\rho_1, \rho_2 : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_2(E)$ are continuous Galois representations unramified outside S and satisfying

1. $\mathrm{Tr}(\rho_1(Frob_{\mathfrak{p}})) \equiv \mathrm{Tr}(\rho_2(Frob_{\mathfrak{p}})) \equiv 0 \pmod{\mathcal{P}}$ *for* $\mathfrak{p} \in U$,
2. $\det(\rho_1) \equiv \det(\rho_2) \pmod{\mathcal{P}}$,
3. $\mathrm{Tr}(\rho_1(Frob_{\mathfrak{p}})) = \mathrm{Tr}(\rho_2(Frob_{\mathfrak{p}}))$ *and* $\det(\rho_1(Frob_{\mathfrak{p}})) = \det(\rho_2(Frob_{\mathfrak{p}}))$ *for* $\mathfrak{p} \in T$.

Then ρ_1 and ρ_2 have isomorphic semisimplifications.

Proof. We claim that condition 1. implies $\mathrm{Tr}(\rho_1) \equiv \mathrm{Tr}(\rho_2) \equiv 0 \pmod{\mathcal{P}}$. To prove this we have to determine all cubic extensions of $\mathbb{Q}[\sqrt{-3}]$, which are unramified outside S; they give degree 6 extensions of $\mathbb{Q}$ unramified outside $\{2, 3\}$ as in the proof of Proposition 2.1. Out of the 398 extensions 16 contain $\mathbb{Q}[\sqrt{-3}]$; they are given by the following degree six polynomials

$$x^6 - x^3 + 1,\quad x^6 - 3x^5 + 5x^3 - 3x + 1,\quad x^6 + 3,\quad x^6 - 3x^5 + 3x^3 + 6x^2 - 9x + 3,$$
$$x^6 - 3x^4 - 2x^3 + 9x^2 + 12x + 4,\quad x^6 - 2x^3 + 4,\quad x^6 - 3x^3 + 3,\quad x^6 + 12,$$
$$x^6 + 48,\quad x^6 - 3x^4 + 9x^2 - 18x + 12,\quad x^6 - 6x^3 + 12,\quad x^6 - 6x^3 + 36,$$
$$x^6 + 18x^4 + 81x^2 + 12,\quad x^6 + 3x^4 - 2x^3 + 9x^2 - 12x + 4,\quad x^6 + 27x^2 - 36x + 12,$$
$$x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 52$$

each of which factors into a product of degree 3 polynomials over $\mathbb{Q}[\sqrt{-3}]$. One readily verifies that each degree 3 polynomial has irreducible reduction modulo at least one prime from U. The evenness of the traces follows.

The compositum K_S of all quadratic extensions of $\mathbb{Q}[\sqrt{-3}]$ unramified outside $\{2, \sqrt{-3}\}$ equals the compositum of $\mathbb{Q}[\sqrt{-3}, \sqrt{2}]$, $\mathbb{Q}[\sqrt[4]{-3}]$, $\mathbb{Q}[\sqrt{\frac{1}{2}(\sqrt{-3} + 1)}]$. From the table of quadratic characters below, we read off that the elements from the Galois group $\mathrm{Gal}(K_S/\mathbb{Q}[\sqrt{-3}])$ corresponding to Frobenius elements at the primes from T form a non-cubic set. Now the proposition follows from [17, Thm. 4.3].

$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{-3}$	2	$\frac{1}{2}(\sqrt{-3}+1)$	$\mathfrak{p}$	$N(\mathfrak{p})$	$\sqrt{-3}$	2	$\frac{1}{2}(\sqrt{-3}+1)$
$\sqrt{-3}-2$	7	0	0	1	$\sqrt{-3}+4$	19	1	1	1
$\sqrt{-3}+2$	7	1	0	1	$5+2\sqrt{-3}$	37	0	1	0
$1+2\sqrt{-3}$	13	1	1	0	$5+4\sqrt{-3}$	73	1	0	0
$\sqrt{-3}-4$	19	0	1	1					

□

3. Double octics

In this paper we shall study the modularity of three Calabi-Yau threefolds constructed as crepant resolution of a double cover of the projective space $\mathbb{P}^3$ branched along an arrangement of eight planes $S = P_1 \cup \cdots \cup P_8$.

If the planes satisfy the following two conditions:

$$\text{no six planes intersect, no four planes contain a common line,} \tag{3.1}$$

then the double cover admits a projective crepant resolution of singularities. One calls the resulting Calabi-Yau threefold a *double octic*. It is sometimes useful to note that the crepant resolution can be arranged in such a way that it exhibits the double cover as a double cover of a blow-up of the projective space.

One of the key features of double octics is that one can control their invariants, in particular their Hodge numbers (where only $h^{1,1}, h^{1,2}$ are essential for Calabi–Yau threefolds). In particular, this is instrumental for constructing rigid double octics or one-dimensional families (accounting for all the infinitesimal deformations of the smooth members, i.e. $h^{1,2} = 1$). For brevity, we omit the details here and refer the reader to the section 4.2 of C. Meyer's monograph [19].

4. Double octic with real multiplication by $\mathbb{Q}[\sqrt{2}]$

Let X be the double octic Calabi-Yau threefold constructed as a resolution of the double covering of $\mathbb{P}^3$ branched along the following 8 hyperplanes:

$$u^2 = x(x-z)(x-v)(x-z-v)y(y-z)(y-v)(y+2z+v).$$

By separating the variables x, y on the right-hand side, one realizes that the double octic X admits a fibration by Kummer surfaces (the fibration is induced by the map $(x, y, z, v, u) \mapsto (z, v)$). Following [21], this Kummer structure arises from the fiber product of rational elliptic surfaces with singular fibers I_4, I_4, I_2, I_2 and $I_2, I_2, I_2, I_2, I_2, I_2$ where the singular fibers are located as follows:

∞	0	1	-1	$-\frac{1}{2}$	$-\frac{1}{3}$
I_4	I_4	I_2	I_2		
I_2	I_2	I_2	I_2	I_2	I_2

The Calabi-Yau threefold X is isomorphic to the element corresponding to $t = -1/2$ of the one parameter family defined by the Arrangement No. 250 ([19]). In particular

$$h^{11}(X) = 37, \quad h^{12}(X) = 1, \tag{4.1}$$

and the only primes of bad reduction of X are 2 and 3. The Riemann-symbol of the Picard-Fuchs operator of the family of Calabi–Yau threefolds defined by the Arrangement No. 250 is

$$\left\{\begin{array}{cccccc} -2 & -1 & -1/2 & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\ 2 & 1 & 4 & 1 & 2 & 3/2 \end{array}\right\}$$

(for details see [6]). The Picard-Fuchs operator is symmetric with respect to the involution $t \mapsto -1 - t$ and its fixed point $t = -\frac{1}{2}$ is an apparent singularity. The family, however, does not seem to be symmetric in an obvious way. Our findings depend in an essential way on a correspondence between members of the family exchanged by the involution. Applied to the given Calabi–Yau threefold X, the correspondence induces a two-to-one rational map

$$\Psi : X \longrightarrow X$$

defined over $\mathbb{Q}[\sqrt{2}]$ by

$$\Psi : \begin{pmatrix} x \\ y \\ z \\ v \\ u \end{pmatrix} \mapsto \left(\begin{array}{c} x\,(x-v-z)\,(z-v)\,(3\,y+v) \\ \frac{1}{2}\,(3\,z+v)\left(v^2-2\,xv+zv+2\,x^2-2\,xz\right)(y-v) \\ \frac{1}{2}\,\left(v^2-2\,xv+zv+2\,x^2-2\,xz\right)(3\,y+v)\,(z+v) \\ \frac{1}{2}\,\left(v^2-2\,xv+zv+2\,x^2-2\,xz\right)(3\,y+v)\,(z-v) \\ \frac{\sqrt{2}}{2}\,(v-z)\,(v+3\,y)^2\,v^2\,(2\,x-v-z)\,(v+z)\,(3\,z+v) \\ \times\left(v^2-2\,xv+zv+2\,x^2-2\,xz\right)^2 u \end{array} \right)$$

The pullback by Ψ of a canonical form ω_X is $\Psi^*\omega_X = \sqrt{2}\omega_X$. In particular the map Ψ^* acts as multiplication by $\sqrt{2}$ on $H^{3,0}(X) \oplus H^{0,3}(X)$. On the other hand, the map Ψ^* acts as multiplication by (-1) on the infinitesimal deformation space $H^1(\mathcal{T}_X)$ and hence as multiplication by $(-\sqrt{2})$ on $H^{1,2}(X) \oplus H^{2,1}(X)$ (by Serre duality there is an isomorphism between vector spaces $H^{1,2}(X)$ and $(H^1(\mathcal{T}_X) \otimes H^{3,0})^*$ compatible with the action induced by Ψ). Consequently the map Ψ decomposes the motive $H^3(X)$ into a direct sum of two two-dimensional submotives

$$H^3(X) = H^3_+ \oplus H^3_- \tag{4.2}$$

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defined as $(\pm\sqrt{2})$–eigenspaces of Ψ^*. The restriction of the Galois action to the sub-group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ preserves the two submotives and hence decomposes $H^3(X)$ as the direct sum of two Galois-conjugate Galois representations

$$\rho, \bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}]) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_2[\sqrt{2}]).$$

We shall study these Galois representations using the Lefschetz fixed point formula (in order to eventually prove their modularity). To this end, we have to inspect the crepant resolution of the double octic more closely.

The Calabi-Yau threefold X is a double covering of a blow-up X' of the projective space $\mathbb{P}^3$, consequently there is an involution $i : X \longrightarrow X$ acting on X. This involution induces a decomposition

$$\mathrm{Pic}(X) = H^2(X, \mathbb{Z}) = H^2_{sym}(X, \mathbb{Z}) \oplus H^2_{skew}(X, \mathbb{Z})$$

of the Picard group of X into symmetric and skew-symmetric part. The symmetric part $H^2_{sym}(X, \mathbb{Z})$ is isomorphic to the cohomology group $H^2(X', \mathbb{Z})$. The octic arrangement defining the Calabi-Yau threefold X has 28 double lines and 8 fourfold points, consequently in the process of resolution of singularities of X we blow-up the doubly covered projective space 36 times and the rank of the cohomology group $H^2(X', \mathbb{Z})$ equals 37. It follows from (4.1) that the cohomology group $H^2(X, \mathbb{Z})$ is generated by classes of symmetric divisors defined over $\mathbb{Q}$. By the comparison theorem, for any prime $p \geq 5$ the Frobenius morphism Frob_p acts on the étale cohomology $H^2_{\mathrm{et}}(\bar{X}_p, \mathbb{Q}_l)$ by multiplication by p, where X_p is the reduction of X modulo p and $\bar{X}_p = X \otimes \bar{\mathbb{F}}_p$, and likewise for all powers $\mathrm{Frob}_q = \mathrm{Frob}_p^r$.

In order to compute the trace of the Frobenius morphism Frob_q using the Lefschetz fixed point formula, first we count the points N_q on the singular double octic over $\mathbb{F}_q$ using a computer program, then we add the correction terms for the crepant resolution of singularities. Over $\mathbb{Q}$, the exceptional locus of the blow-up of a fourfold point not contained in any triple line (type p_4^0 in the notation of [19]) is isomorphic to a surface

$$E = \{u^2 = \alpha xyz(x + y + z) \subset \mathbb{P}^3(1, 1, 1, 2)\}, \text{ where } \alpha \in \mathbb{Q}. \tag{4.3}$$

The number of points on E over $\mathbb{F}_q$ equals $q^2 + 2q + 1$ if $-\alpha$ is a square in $\mathbb{F}_q$ and $q^2 + 1$ otherwise (for details see [19, p. 56]). Presently the eight fourfold points are

$$\begin{aligned} &(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),\\ &(1, 0, 0, 1), (1, 0, 1, 0), (0, 1, -1, 1), (1, 1, 1, 1). \end{aligned} \tag{4.4}$$

The projective transformation $(x, y, z, v) \mapsto (-x + v, y - v, x - z, v)$ maps the point $(1, 1, 1, 1)$ to $(0, 0, 0, 1)$ and the octic arrangement $x(x - z)(x - v)(x - z - v)y(y - z)(y - v)(y + 2z + v)$ to $xyz(x + y + z)(x - v)(z - v)(-2x + y - 2z + 4v)$, so the coefficient α for $(1, 1, 1, 1)$ equals 4. In a similar way, the coefficients α for all fourfold points in the order of (4.4) can be computed as $8, -1, 2, 1, 1, 2, 8, 4$. Any other blow-up adds q^2+q points to X. Consequently

$$\#\bar{X}_p(\mathbb{F}_{p^2}) = N_{p^2} + 28(p^4+p^2) + 8(p^4+2p^2)$$

and

$$\#\bar{X}_p(\mathbb{F}_p) = N_p + 28(p^2+p) + (p^2+2p) + 3\left(p^2+p+\left(\tfrac{-1}{p}\right)p\right) + 4\left(p^2+p+\left(\tfrac{-2}{p}\right)p\right).$$

Since the Frobenius morphism Frob_q acts on $H^2(\bar{X}_p)$ by multiplication by q we have $\mathrm{Tr}(\mathrm{Frob}_q\,|H^2(\bar{X}_p)) = 37q$ and $\mathrm{Tr}(\mathrm{Frob}_q\,|H^4(\bar{X}_p)) = 37q^2$. By the Lefschetz fixed point formula, the trace of Frob_{p^2} on $H^3_{\mathrm{et}}(\bar{X}_p, \mathbb{Q}_l)$ thus equals

$$a_{p^2} := \mathrm{Tr}(\mathrm{Frob}_{p^2}\,|H^3(\bar{X}_p)) = -N_{p^2} + p^6 + p^4 + 9p^2 + 1$$

By (4.2), the Galois representation on $H^3(X)$ equals its tensor product with the Dirichlet character associated to the Legendre symbol $(\frac{2}{p})$. Hence, if p is an *inert* prime in $\mathbb{Q}[\sqrt{2}]$, the trace of Frob_p on $H^3(X)$ vanishes:

$$a_p := \mathrm{Tr}(\mathrm{Frob}_p\,|H^3(X)) = 0;$$

consequently the Frobenius polynomial equals

$$X^4 - \frac{a_{p^2}}{2}X^2 + p^6.$$

If $F_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ is a Frobenius element, then the trace of the value of ρ and $\bar{\rho}$ at F_p equals

$$\mathrm{Tr}(\rho(F_p)) = \mathrm{Tr}(\bar{\rho}(F_p)) = \frac{1}{2}a_{p^2}.$$

If p is a *split* prime, then the trace of Frob_{p^2} can be computed as before by a point count in $\mathbb{F}_{p^2}$. In order to compute the trace of Frob_p with a point-count, we have to take into account the contribution from the eight fourfold points of the arrangement. Using (4.3) we get in this situation

$$a_p := \mathrm{Tr}(\mathrm{Frob}_p\,|H^3(X)) = -N_p + p^3 + p^3 + p\left(2 + 3\left(\tfrac{-1}{p}\right) + 4\left(\tfrac{-2}{p}\right)\right) + 1.$$

The Frobenius polynomial equals

$$\chi(\mathrm{Frob}_p) = X^4 - a_pX^3 - \tfrac{1}{2}(a_p^2 + a_{p^2})X^2 - a_p p^3 X + p^6.$$

In the following table we collect Frobenius polynomials for the values of p that we will need to prove modularity.

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p	a_p	a_{p^2}	F_p
5	0	20	$X^4 - 10\,X^2 + 15625$
7	32	-796	$X^4 - 32\,X^3 + 910\,X^2 - 10976\,X + 117649$ $(X^2 + 4\sqrt{2}X - 16\,X + 343) \times (X^2 - 4\sqrt{2}X - 16\,X + 343)$
11	0	-1452	$X^4 + 726\,X^2 + 1771561$ $(X^2 - 44\,X + 1331) \times (X^2 + 44\,X + 1331)$
17	-124	-10940	$X^4 + 124\,X^3 + 13158\,X^2 + 609212\,X + 24137569$ $(X^2 + 16\sqrt{2}X + 62\,X + 4913) \times (X^2 - 16\sqrt{2}X + 62\,X + 4913)$
23	80	-45212	$X^4 - 80\,X^3 + 25806\,X^2 - 973360\,X + 148035889$ $(X^2 + 8\sqrt{2}X - 40\,X + 12167) \times (X^2 - 8\sqrt{2}X - 40\,X + 12167)$
31	272	-59068	$X^4 - 272\,X^3 + 66526\,X^2 - 8103152\,X + 887503681$ $(X^2 - 76\sqrt{2}X - 136\,X + 29791) \times (X^2 + 76\sqrt{2}X - 136\,X + 29791)$
41	84	-148252	$X^4 - 84\,X^3 + 77654\,X^2 - 5789364\,X + 4750104241$ $(X^2 - 176\sqrt{2}X - 42\,X + 68921) \times (X^2 + 176\sqrt{2}X - 42\,X + 68921)$
47	-64	-134460	$X^4 + 64\,X^3 + 69278\,X^2 + 6644672\,X + 10779215329$ $(X^2 + 264\sqrt{2}X + 32\,X + 103823) \times (X^2 - 264\sqrt{2}X + 32\,X + 103823)$
89	-2476	507556	$X^4 + 2476\,X^3 + 2811510\,X^2 + 1745503244\,X + 496981290961$ $(X^2 + 256\sqrt{2}X + 1238\,X + 704969) \times (X^2 - 256\sqrt{2}X + 1238\,X + 704969)$
97	1284	-2822268	$X^4 - 1284\,X^3 + 2235462\,X^2 - 1171872132\,X + 832972004929$ $(X^2 + 32\sqrt{2}X - 642\,X + 912673) \times (X^2 - 32\sqrt{2}X - 642\,X + 912673)$

To avoid working with four-dimensional Galois representations (as in [10]), we have to determine the precise traces of $\mathrm{Frob}_{\mathfrak{p}}$ on H^3_+ and H^3_- for a prime of $\mathcal{O}_K$ above p. To this end, we shall exploit the rational self-map Ψ; more precisely, we study the action of $\mathrm{Frob}_{\mathfrak{p}} \circ \Psi$ on $H^3(X)$. This map preserves the Kummer fibration and transforms the fiber at (z, v) into the fiber at $(z + v, z - v)$. This allows us to determine the number of fixed points of Ψ; indeed, we can restrict ourselves to the fibers at $(1 \pm \sqrt{2}, 1)$. At those points, the fiber is isomorphic to the Kummer surface of the product of the elliptic curves

$$u^2 = x^3 - 30x + 56 \quad \text{and} \quad u^2 = y^3 - y,$$

and the map Ψ is induced by the complex multiplications given

$$x \longmapsto -\frac{x^2 - 4x + 18}{2(x-4)} \quad \text{and} \quad y \longmapsto -\frac{y+1}{y-1}.$$

As the map Ψ acts on $H^{0,3} \oplus H^{3,0}$ as multiplication by $\sqrt{2}$ and on $H^{2,1} \oplus H^{1,2}$ as multiplication by $-\sqrt{2}$, we infer that the trace of the induced map on the third cohomology is zero: $\mathrm{tr}(\Psi^*|H^3) = 0$. Using Magma we computed that the map Ψ has Lefschetz number equal 12, so we get

$$\mathrm{tr}(\Psi^*|H^0) = 1, \;\; \mathrm{tr}(\Psi^*|H^2) + \mathrm{tr}(\Psi^*|H^4) = 9, \;\; \mathrm{tr}(\Psi^*|H^6) = 2.$$

In a similar way we computed the trace of the composition $\mathrm{Frob}_{\mathfrak{p}} \circ \Psi$ for split primes $p = 7, 17, 23, 31, 47, 89$. As the Picard group of X is defined by divisors defined over $\mathbb{Q}$, the Frobenius morphism $\mathrm{Frob}_{\mathfrak{p}}^*$ acts on H^{2k} as multiplication by p^k, $k = 0, 1, 2, 3$. By direct computations with Magma (using the code given in [7], the computations took about 30 hours), we found the Lefschetz numbers listed in the table below.

$\mathfrak{p}$	$3+\sqrt{2}$	$3-\sqrt{2}$	$3\sqrt{2}-1$	$5+\sqrt{2}$	$5-\sqrt{2}$	$4\sqrt{2}+1$
$N(\mathfrak{p})$	7	7	17	23	23	31
$\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}} \circ \Psi)$	944	976	11404	27104	27040	64208
$\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* \| H^3_+)$	$16+4\sqrt{2}$	$16-4\sqrt{2}$	$-62-16\sqrt{2}$	$40-8\sqrt{2}$	$40+8\sqrt{2}$	$136+76\sqrt{2}$
$\mathfrak{p}$	$4\sqrt{2}-1$	$5\sqrt{2}-3$	$\sqrt{2}+7$	$\sqrt{2}-7$	$4\sqrt{2}-11$	$1-7\sqrt{2}$
$N(\mathfrak{p})$	31	41	47	47	89	97
$\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}} \circ \Psi)$	64816	147116	219936	217824	1450924	1872652
$\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* \| H^3_+)$	$136-76\sqrt{2}$	$42-176\sqrt{2}$	$-32-264\sqrt{2}$	$-32+264\sqrt{2}$	$-1238-256\sqrt{2}$	$642+32\sqrt{2}$

The table also lists the traces of $\mathrm{Frob}_{\mathfrak{p}} \circ \Psi$ on $H^3(X)$. These can be determined as follows. Since we do not know which factor of the Frobenius polynomial F_p corresponds to the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$ on H^3_+ and which to H^3_-, we can determine the trace of $\mathrm{Frob}_{\mathfrak{p}} \circ \Psi$ a priori only up to a sign. From the table on page 7 we get the following values of traces of $\mathrm{Frob}_{\mathfrak{p}}^*$ on H^3_+/H^3_-

7	17	23	31	41	47	89	97
$16 \pm 4\sqrt{2}$	$-62 \pm 16\sqrt{2}$	$40 \pm 8\sqrt{2}$	$136 \pm 76\sqrt{2}$	$42 \pm 176\sqrt{2}$	$-32 \pm 264\sqrt{2}$	$-1238 \pm 256\sqrt{2}$	$642 \pm 32\sqrt{2}$

We have for any split prime $p \in \mathbb{Z}$ and any prime $\mathfrak{p} \in \mathbb{Z}[\sqrt{2}]$ over p

$$\begin{aligned}\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}}^* \circ \Psi) = 1 + p\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}(\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* |H^3_+) - \mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* |H^3_-))\\ + p^2\,\mathrm{tr}(\Psi^*|H^4) + 2p^3,\end{aligned}$$

and

$$\mathrm{tr}(\Psi^*|H^2) + \mathrm{tr}(\Psi^*|H^4) = 9.$$

In the case $p = 7$, $\mathfrak{p} = 3 - \sqrt{2}$ we get two possibilities

$$976 = 1 + 7\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}((16 - 4\sqrt{2}) - (16 + 4\sqrt{2})) + 49\,\mathrm{tr}(\Psi^*|H^4) + 686$$

or

$$976 = 1 + 7\,\mathrm{tr}(\Psi^*|H^2) - \sqrt{2}((16 + 4\sqrt{2}) - (16 - 4\sqrt{2})) + 49\,\mathrm{tr}(\Psi^*|H^4) + 686,$$

or equivalently,

$$273 = 7(\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4)) \qquad \text{and} \qquad 305 = 7(\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4))$$

As $7 \nmid 305$, the second option is impossible and consequently

$$\mathrm{tr}(\Psi^*|H^2) + 7\,\mathrm{tr}(\Psi^*|H^4) = 39.$$

Together with

$$\mathrm{tr}(\Psi^*|H^2) + \mathrm{tr}(\Psi^*|H^4) = 9,$$

this yields

$$\mathrm{tr}(\Psi^*|H^2) = 4, \qquad \mathrm{tr}(\Psi^*|H^4) = 5.$$

Now, we get

$$\sqrt{2}(\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* \,|H_+^3) - \mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}^* \,|H_-^3)) = -\mathcal{L}(\mathrm{Frob}_{\mathfrak{p}}^* \circ \Psi) + 1 + 4p + 5p^2 + 2p^3,$$

and we can compute the entries of the above table.

Using Magma one finds 3 Hilbert modular forms for $K = \mathbb{Q}[\sqrt{2}]$ of weight $[4,2]$ and level $6\sqrt{2}\mathcal{O}_K$. For one of them, let us denote it by h_1, the Hecke eigenvalues agree exactly with the traces of the action of Frobenius on H_+^3 computed in the above table. (For the reader's convenience, we provide a table of eigenvalues at [7].)

Theorem 4.1. *The Galois representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}])$ *on the motive* H_+^3 *is isomorphic to the Galois representation of the Hilbert modular form* h_1 *for* $K = \mathbb{Q}[\sqrt{2}]$ *of weight* $[4,2]$ *and level* $6\sqrt{2}\mathcal{O}_K$.

Proof. There exist continuous Galois representations

$$\rho_1, \rho_2 : \quad \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{2}]) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_2[\sqrt{2}])$$

defined by the motive H_+^3 and the Hilbert modular form h_1. We shall verify that the representations ρ_1 and ρ_2 satisfy the assumptions of Proposition 2.1. We have computed the traces of $\mathrm{Frob}_{\mathfrak{p}} \,|H_+^3$ for $\mathfrak{p} \in T$ and verified with MAGMA that they agree with Hecke eigenvalues of h_1; for $\mathfrak{p} \in U$ we check that both traces are even. Moreover for any $\mathfrak{p} \in T$ we check that $\det(\rho_1(\mathrm{Frob}_{\mathfrak{p}})) = p^3$. Since $\det(\rho_1(\mathrm{Frob}_{\mathfrak{p}}))|p^6$ for any $p \geq 5$ and any prime $\mathfrak{p}$ in $\mathbb{Q}[\sqrt{2}]$ over p, it follows that $\det(\rho_1(\mathrm{Frob}_{\mathfrak{p}}))$ is odd.

Finally, h_1 is a Hilbert modular form with trivial character, so $\det(\rho_2(\mathrm{Frob}_{\mathfrak{p}})) = N(\mathfrak{p})^3$ (which is odd). Thus the assumptions of Proposition 2.1 are satisfied, and applying the proposition concludes the proof. □

Remark 4.2. It follows that the Galois representation on the motive H_-^3 is isomorphic to the Galois representation of the Hilbert modular form $\bar{h}_1$ for $\mathbb{Q}[\sqrt{2}]$ of weight $[2,4]$ and level $6\sqrt{2}\mathcal{O}$. Observe the divisibility condition $\bar{a}_{\mathfrak{p}} \in \mathfrak{p}$ for all Hecke eigenvalues of $\bar{h}_1$ in the given range of primes (in agreement with [12, §3]). We believe that this could be proven geometrically using the Hodge type $(2,1)+(1,2)$ of H_-^3 along the lines of [18]. This could then also simplify the determination of the factor of the characteristic polynomial of Frobenius corresponding to H_+^3 at ordinary primes.

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Remark 4.3. It might be possible to approach the modularity of the above examples, and of those to come, by appealing to modularity lifting theorems, or by extending the parallel weight results from [2] to the non-parallel weight case. As it stands, this would, however, only give potential modularity and leave the precise determination of the corresponding modular form.

Remark 4.4. By [13, Main Theorem] there exists a Siegel modular form of degree 2, weight 3 and paramodular level $8^2 \times 72 = 4608 = 2^9 3^2$ with L-function equal to the L-function of X (cf. discussion in [13, p. 545] in the case of the Hilbert modular form for the Consani–Scholten quintic).

5. Hilbert modular rigid Calabi-Yau threefold over $\mathbb{Q}[\sqrt{5}]$

Let Y be the double octic defined as a crepant resolution of singularities of the hypersurface

$$\begin{aligned} u^2 &= xyzv\,(x+y+z)\,(\varphi y - z + v)\,(x+y+\varphi v)\,((1-\varphi)x + y - \varphi z + \varphi v) \\ &\quad \subset \mathbb{P}(1,1,1,1,4), \end{aligned}$$

where $\varphi = \frac{1}{2}(-1+\sqrt{5})$. Then Y is a rigid Calabi-Yau threefold with $h^{11} = 38$ by [5, Prop. 5.4], and one verifies as before that the Picard group is generated by divisors defined over $K = \mathbb{Q}[\sqrt{5}]$ while the only prime of bad reduction of Y is 2.

For prime numbers $p \equiv 1, 4 \pmod 5$ we computed the numbers n_p and n_{p^2} of points of the singular double covering over $\mathbb{F}_p$ and over $\mathbb{F}_{p^2}$. The resolution of singularities is blowing up 28 double lines and 9 fourfold points; for each blow-up of a line, the number of points over the field $\mathbb{F}_q$ increases by q^2+q, while a blow-up of a fourfold point increases the number of points by q^2+2q or q^2 as around (4.3). Consequently,

$$\#\bar{Y}_{\mathfrak{p}}(\mathbb{F}_p) = n_{\mathfrak{p}} + 28(p^2+p) + a(p^2+2p) + (9-a)p^2 = n_{\mathfrak{p}} + 37p^2 + (28+2a)p,$$

where $a \in \{0, \ldots, 9\}$ and $\#\bar{Y}_{\mathfrak{p}}(\mathbb{F}_{p^2}) = n_{\mathfrak{p}^2} + 37p^4 + 46p^2$. By the Lefschetz fixed point formula we get the following traces on $H^3(\bar{Y}_{\mathfrak{p}})$ (where we suppress the cohomology group to ease the notation):

$$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}^2}) = -n_{\mathfrak{p}^2} + p^6 + p^4 - 8p^2 + 1, \quad \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}) = -n_{\mathfrak{p}} + p^3 + p^2 + cp + 1,$$
$$c = 10 - 2a \in \{-8, \ldots, 10\},$$

in a similar way as for the Calabi-Yau threefold X. Moreover, comparing the actions of $\mathrm{Frob}_{\mathfrak{p}}$ and $\mathrm{Frob}_{\mathfrak{p}^2}$

$$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}^2}) = \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})^2 - 2p^3.$$

We observe that these equalities often suffice to determine $\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$. The next table collects the results of the computations for the split primes which we shall need in the proof of modularity.

p	$\mathfrak{p}$	φ	$n_{\mathfrak{p}}$	$n_{\mathfrak{p}^2}$	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$	$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}^2})$
11	$\sqrt{5}+4$	3	1459	1784297	60	938
	$\sqrt{5}-4$	7	1461	1786601	36	-1366
29	$2\sqrt{5}+7$	5	25217	595525129	-218	-1254
	$2\sqrt{5}-7$	23	25089	595564553	-90	-40678
31	$\sqrt{5}+6$	12	30685	888442233	192	-22718
	$\sqrt{5}-6$	18	31003	888475001	-64	-55486
61	$2\sqrt{5}-9$	17	230471	51534519081	354	-328646
	$2\sqrt{5}+9$	43	230215	51534272297	610	-81862

To apply Proposition 2.3, we also require information at two inert primes. For $p = 3$ we were able to compute the number of points of the singular double cover of $\mathbb{P}^3$ over $\mathbb{F}_{3^2}$ and $\mathbb{F}_{3^4}$, obtaining $n_9 = 815$, $n_{81} = 538617$. Similar computations as in the case of split primes give $\mathrm{Tr}(\mathrm{Frob}_9) = -1262$ and $\mathrm{Tr}(\mathrm{Frob}_3) = 14$. For $p = 13$ we computed that the number of points over $\mathbb{F}_{13^2}$ equals 4857691. To obtain the contribution for the exceptional divisors (4.3) over the fourfold points, notice that the values of α in question,

$$6-2\sqrt{5},\ \tfrac{1}{2}(-3+3\sqrt{5}),\ \tfrac{1}{2}(-3+3\sqrt{5}),\ 4\sqrt{5}-8,\ -6\sqrt{5}+14,\ \tfrac{1}{2}(3-1\sqrt{5}),$$
$$6\sqrt{5}-14,\ \sqrt{5}-3,\ -1+\sqrt{5},$$

are all squares in $\mathbb{F}_{13^2}$. From the Lefschetz fixed point formula we thus derive the trace of Frob_{13} on $H^3(\bar{Y}_{13})$ as

$$\mathrm{Tr}(\mathrm{Frob}_{13}) = -(4857961 + 9\cdot 13^2) + 1 + 13^2 + 13^4 + 13^6 = -3942.$$

Using Magma we found 24 Hilbert modular forms for $\mathbb{Q}[\sqrt{5}]$ of parallel weight $[4,4]$ and level $16\mathcal{O}$; one of them, denoted by h_2, has exactly the same Hecke eigenvalues as the Frobenius traces above. Proposition 2.3 thus guarantees the modularity of Y:

Theorem 5.1. *The Galois representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{5}])$ *on* $H^3_{et}(Y,\mathbb{Q}_l)$ *is Hilbert modular with corresponding Hilbert modular form* h_2.

6. Modular Calabi-Yau threefold over $\mathbb{Q}[\sqrt{-3}]$

Let Z be the double octic defined as a resolution of singularities of the hypersurface

$$u^2 = xyzv\,(x+y)\,(x+y+z-v)\,(\zeta\,x-y+\zeta\,z)\,(y-\zeta\,z-v) \subset \mathbb{P}(1,1,1,1,4), \quad (6.1)$$

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where $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ and, of course, $i^2 = -1$. Then Z is a rigid Calabi-Yau threefold with $h^{11} = 46$ as can be checked by considering it as a member of the one-dimensional family of double octics given by arrangement No. 262 in [19]. As before, one verifies that the Picard group is generated by divisors defined over $K = \mathbb{Q}[\sqrt{-3}]$, and that the only prime of bad reduction of Z is 2.

Proposition 6.1. *Z is birational to a Calabi-Yau threefold defined over $\mathbb{Q}[i]$.*

Proof. The standard crepant resolution of a double octic proceeds as follows: blow-up successively fivefold points, triple lines, fourfold points and double lines. The resolution depends on the order of blow-ups of double lines; to overcome this subtlety we modify the last step and blow-up the union of all double lines in the singular double cover (cf. [4]). Then the map

$$(x, y, z, v, u) \mapsto (\zeta\, x, -\zeta\, x - \zeta\, y, -(\zeta+1)\, x - y - (\zeta+1)\, z, -(\zeta+1)\,(x+y+z-t)\,, iu)$$

defines an isomorphism of Z and its Galois conjugate over K, hence Z isomorphic to a variety defined over $\mathbb{Q}[i]$ by the Weil Galois Descent Theorem ([23, Thm. 3]). □

We can count points over $\mathbb{F}_p$ only if $p \equiv 1 \pmod 6$, i.e. p is a split prime in K. Above a given split prime p there are two prime ideals $\mathfrak{p}$ in the ring of integers of $\mathbb{Q}[\sqrt{-3}]$; this corresponds to two choices for $\zeta \in \mathbb{F}_\mathfrak{p}$ and two possibilities for the trace of Frobenius on $H^3(\bar{Z}_\mathfrak{p})$ which we list in the next table.

p	ζ	$\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p})$	ζ	$\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p})$
7	4	−12	2	12
13	3	−58	9	−58
19	11	−136	7	136
31	25	20	5	−20
37	26	−18	10	−18
43	6	−200	36	200
61	47	−458	13	−458
67	29	−496	37	496
73	64	602	8	602
79	55	1108	23	−1108
97	61	−206	35	−206

The computed traces agree up to sign with the Fourier coefficients of a modular form f of weight 4 for $\Gamma_0(72)$ (72/1 in Meyer's notation in [19]):

p	7	13	19	31	37	43	61	67	73	79	97
a_p	−12	58	−136	20	−18	−200	−458	−496	−602	1108	206

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We compare the signs with characters from the table in the proof of Proposition 2.4 to notice that the sign changes are governed by the character corresponding to the extension $\mathbb{Q}[\sqrt[4]{-3}]/\mathbb{Q}[\sqrt{-3}]$.

Theorem 6.2. *Consider the Galois representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}[\sqrt{-3}])$ *on* $H^3_{et}(\bar{Z}, \mathbb{Q}_l)$ *and the one associated to the modular form* f *restricted to* $\mathbb{Q}[\sqrt{-3}]$ *and then twisted by the quadratic character associated to the extension* $\mathbb{Q}[\sqrt[4]{-3}]/\mathbb{Q}[\sqrt{-3}]$*. Then the Galois representations have isomorphic semi-simplifications.*

Proof. In view of Proposition 2.4, compared to the present data, it suffices to check the following two properties:

- $\sqrt{-3} = 2\zeta + 1$ is a square in $\mathbb{F}_p$ if and only if the corresponding choice of $\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}})$ matches the Fourier coefficient a_p of f;
- the Galois representations have even trace at $p = 11$.

The latter condition follows easily since $a_{11} = 64$ and any double octic Calabi–Yau threefold given by an arrangement of 8 planes satisfying condition (3.1) is checked to have an even number of points over any finite field of odd parity by an elementary combinatorial argument. □

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