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Congruences concerning Legendre polynomials II

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Let $p>3$ be a prime, and let m be an integer with $p\nmid m$. In the paper we solve some conjectures of Z.W. Sun concerning $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \pmod{p^2}$, $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}/m^k \pmod p$ and $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/m^k \pmod{p^2}$. In particular, we show that $\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\equiv 0 \pmod{p^2}$ for $p\equiv 3,5,6 \pmod 7$. Let $\{P_n(x)\}$ be the Legendre polynomials. In the paper we also show that $P_{[\frac p4]}(t)\equiv -(\frac 6p)\sum_{x=0}^{p-1}(\frac{x^3-\frac32(3t+5)x+9t+7}{p}) \pmod p$, where t is a rational p-adic integer, $[x]$ is the greatest integer not exceeding x and $(\frac ap)$ is the Legendre symbol. As consequences we determine $P_{[\frac p4]}(t) \pmod p$ in the cases $t=-\frac53,-\frac79,-\frac{65}{63}$ and confirm many conjectures of Z.W. Sun.

1. Introduction

Let p be an odd prime. Following [A] we define

$$A(p,\lambda)=\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}^2\binom{\frac{p-1}{2}+k}{k}\lambda^{kp}.$$

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It is easily seen that (see [S2, Lemmas 2.2, 2.4 and 2.5]) for $k \in \{0, 1, \dots, \frac{p-1}{2}\}$

$$(-1)^k \binom{\frac{p-1}{2}+k}{k} \equiv \binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod p,$$
$$\binom{\frac{p-1}{2}+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}. \tag{1.1}$$

Thus, by Fermat's little theorem, for any rational p-adic integer λ,

$$A(p,\lambda) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \pmod p. \tag{1.2}$$

For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly say that $n = ax^2 + by^2$. In 1987, Beukers [B] conjectured a congruence for $A(p,1) \pmod{p^2}$ equivalent to

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4. \end{cases} \tag{1.3}$$

This congruence was proved by several authors including Ishikawa [Is] ($p \equiv 1 \pmod 4$), van Hamme [H] ($p \equiv 3 \pmod 4$) and Ahlgren [A]. In 1998, by using the hypergeometric series ${}_3F_2(\lambda)_p$ over the finite field $\mathbb{F}_p$, Ono [O] obtained congruences for $A(p,\lambda) \pmod p$ in the cases $\lambda = -1, -8, -\frac{1}{8}, 4, \frac{1}{4}, 64, \frac{1}{64}$, see also [A]. Hence from (1.2) we deduce congruences for $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{m^k}\binom{2k}{k}^3 \pmod p$ in the cases $m = 1, -8, 16, -64, 256, -512, 4096$. Recently the author's brother Zhi-Wei Sun [Su1,Su2,Su3] conjectured corresponding congruences modulo p^2. In particular, he conjectured that

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7. \end{cases} \tag{1.4}$$

We note that $p \mid \binom{2k}{k}$ for $\frac{p+1}{2} \leqslant k \leqslant p-1$. In the paper, by using the Legendre polynomials and the work of Coster and van Hamme [CH] we completely solve Zhi-Wei Sun's such conjectures.

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \geqslant 1).$$

It is well known that (see [MOS, pp. 228–232], [G, (3.132)–(3.133)])

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\left(x^2 - 1\right)^n, \tag{1.5}$$

where $[x]$ is the greatest integer not exceeding x. From (1.5) we see that

$$P_n(-x) = (-1)^n P_n(x), \qquad P_{2m+1}(0) = 0 \quad \text{and} \quad P_{2m}(0) = \frac{(-1)^m}{2^{2m}} \binom{2m}{m}. \tag{1.6}$$

The first few Legendre polynomials are as follows:

$$P_0(x)=1,\qquad P_1(x)=x,\qquad P_2(x)=\frac{1}{2}\big(3x^2-1\big),\qquad P_3(x)=\frac{1}{2}\big(5x^3-3x\big),$$
$$P_4(x)=\frac{1}{8}\big(35x^4-30x^2+3\big),\qquad P_5(x)=\frac{1}{8}\big(63x^5-70x^3+15x\big).$$

Let $\mathbb{Z}$ be the set of integers. For a prime p, let $\mathbb{Z}_p$ be the set of those rational numbers whose denominator is not divisible by p, let $\mathbb{Q}_p$ denote the field of p-adic numbers, and let $(\frac{a}{p})$ be the Legendre symbol. On the basis of the work of Ono [O] (see also [A, Theorem 2] and [LR]), in [S2, Theorem 2.11] the author showed that for any prime $p>3$ and $t\in\mathbb{Z}_p$,

$$P_{\frac{p-1}{2}}(t)\equiv-\left(\frac{-6}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-3(t^2+3)x+2t(t^2-9)}{p}\right)\pmod p. \tag{1.7}$$

In this paper, by using elementary arguments we prove that for any prime $p>3$ and $t\in\mathbb{Z}_p$,

$$P_{[\frac{p}{4}]}(t)\equiv-\left(\frac{6}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-\frac{3}{2}(3t+5)x+9t+7}{p}\right)\pmod p. \tag{1.8}$$

From (1.8) we deduce (1.7) immediately. As consequences of (1.8), we determine $P_{[\frac{p}{4}]}(-\frac{5}{3})$, $P_{[\frac{p}{4}]}(-\frac{7}{9})$, $P_{[\frac{p}{4}]}(-\frac{65}{63})\pmod p$ and use them to solve Z.W. Sun's conjectures [Su1] on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{m^k}\pmod p\quad\text{and}\quad\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\ \big(\mathrm{mod}\ p^2\big).$$

For instance, for any prime $p>7$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}\equiv\begin{cases}4C^2\pmod p & \text{if } p\equiv 1,2,4\pmod 7\text{ and so } p=C^2+7D^2,\\ 0\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7.\end{cases} \tag{1.9}$$

For any odd prime p and $m\in\mathbb{Z}_p$ we also establish the following general congruences:

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k}\equiv P_{\frac{p-1}{2}}\left(\sqrt{1-\frac{64}{m}}\right)^2\ \big(\mathrm{mod}\ p^2\big)\quad\text{for } m\not\equiv 0\pmod p, \tag{1.10}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\big(x(1-64x)\big)^k\equiv\left(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\right)^2\ \big(\mathrm{mod}\ p^2\big). \tag{1.11}$$

2. Congruences for $P_{[\frac{p}{4}]}(t) \pmod p$

Lemma 2.1. *Let p be an odd prime and $k \in \{0, 1, \dots, [\frac{p}{4}]\}$. Then*

(i)
$$\binom{\frac{p-1}{2}-k}{k} \equiv \frac{1}{(-4)^k}\binom{4k}{2k} \pmod p,$$

(ii)
$$\binom{[\frac{p}{4}]+k}{2k} \equiv \frac{1}{(-64)^k}\binom{4k}{2k} \pmod p,$$

(iii)
$$\binom{p-1-2k}{\frac{p-1}{2}} \equiv \frac{(-1)^{\frac{p-1}{2}}}{16^k}\binom{4k}{2k} \pmod p.$$

Proof. It is clear that

$$\frac{\binom{\frac{p-1}{2}-k}{k}}{\binom{\frac{p-1}{2}}{2k}} = \frac{(2k)!}{\frac{p-1}{2}\cdot\frac{p-3}{2}\cdots(\frac{p-1}{2}-k+1)\cdot k!} \equiv \frac{(-2)^k\cdot(2k)!}{1\cdot 3\cdots(2k-1)\cdot k!} = (-4)^k \pmod p.$$

This together with (1.1) yields (i).

Suppose $r = 1$ or 3 according as $4 \mid p-1$ or $4 \mid p-3$. Then clearly

$$\begin{aligned}\binom{[\frac{p}{4}]+k}{2k} &= \frac{(\frac{p-r}{4}+k)(\frac{p-r}{4}+k-1)\cdots(\frac{p-r}{4}-k+1)}{(2k)!} \\ &\equiv (-1)^k\frac{(4k-r)(4k-r-4)\cdots(4-r)\cdot r(r+4)\cdots(4k+r-4)}{4^{2k}\cdot(2k)!} \\ &= \frac{(-1)^k\cdot(4k)!}{2^{2k}\cdot(2k)!\cdot 4^{2k}\cdot(2k)!} = \frac{\binom{4k}{2k}}{(-64)^k} \pmod p\end{aligned}$$

and

$$\begin{aligned}&(-1)^{\frac{p-1}{2}}\binom{p-1-2k}{\frac{p-1}{2}} \\ &= (-1)^{\frac{p-1}{2}}\frac{(p-1-2k)(p-2-2k)\cdots(p-(\frac{p-1}{2}+2k))}{\frac{p-1}{2}!} \\ &\equiv \frac{(2k+1)(2k+2)\cdots(\frac{p-1}{2}+2k)}{\frac{p-1}{2}!} = \frac{(\frac{p-1}{2}+2k)(\frac{p-1}{2}+2k-1)\cdots(\frac{p-1}{2}+1)}{(2k)!} \\ &\equiv \frac{(4k-1)(4k-3)\cdots 3\cdot 1}{2^{2k}\cdot(2k)!} = \frac{(4k)!}{2^{2k}\cdot(2k)!\cdot 2^{2k}\cdot(2k)!} = \frac{1}{2^{4k}}\binom{4k}{2k} \pmod p.\end{aligned}$$

Thus (ii) and (iii) are true and the proof is complete. □

Lemma 2.2. *Let p be an odd prime and let t be a variable. Then*

$$P_{[\frac{p}{4}]}(t) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k}\binom{2k}{k}\left(\frac{1-t}{128}\right)^k \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k}\binom{2k}{k}\left(\frac{1+t}{128}\right)^k \pmod p.$$

Proof. It is known that [G, (3.135)]

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{t-1}{2}\right)^k. \tag{2.1}$$

Observe that $\binom{n}{k}\binom{n+k}{k} = \binom{n+k}{2k}\binom{2k}{k}$. By (2.1) and Lemma 2.1(ii) we have

$$P_{[\frac{p}{4}]}(t) = \sum_{k=0}^{[p/4]} \binom{[\frac{p}{4}]+k}{2k}\binom{2k}{k}\left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}\left(\frac{t-1}{2}\right)^k \pmod p.$$

By (1.6), $P_{[\frac{p}{4}]}(t) = (-1)^{[\frac{p}{4}]}P_{[\frac{p}{4}]}(-t)$. Thus the result follows. $\square$

Theorem 2.1. *Let p be a prime greater than 3 and let t be a variable. Then*

$$P_{[\frac{p}{4}]}(t) \equiv -\sum_{x=0}^{p-1}\left(x^3+4x^2+2(1-t)x\right)^{\frac{p-1}{2}} \equiv -\left(\frac{6}{p}\right)\sum_{x=0}^{p-1}\left(x^3-\frac{3}{2}(3t+5)x+9t+7\right)^{\frac{p-1}{2}} \pmod p.$$

Proof. For any positive integer k it is well known that (see [IR, Lemma 2, p. 235])

$$\sum_{x=0}^{p-1} x^k \equiv \begin{cases} p-1 \pmod p & \text{if } p-1 \mid k,\\ 0 \pmod p & \text{if } p-1 \nmid k.\end{cases}$$

Suppose that $(x^3+4x^2+2(1-t)x)^{\frac{p-1}{2}} = \sum_{k=1}^{3(p-1)/2} a_k x^k$. From the above we deduce that

$$a_{p-1} \equiv -\sum_{k=1}^{3(p-1)/2} a_k \sum_{x=0}^{p-1} x^k = -\sum_{x=0}^{p-1}\left(x^3+4x^2+2(1-t)x\right)^{\frac{p-1}{2}} \pmod p.$$

On the other hand,

$$\begin{aligned}\left(x^3+4x^2+2(1-t)x\right)^{\frac{p-1}{2}} &= x^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}\left(x^2+4x\right)^{\frac{p-1}{2}-k}\left(2(1-t)\right)^k \\ &= x^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \sum_{r=0}^{\frac{p-1}{2}-k} \binom{\frac{p-1}{2}-k}{r} x^{2r}(4x)^{\frac{p-1}{2}-k-r}\left(2(1-t)\right)^k.\end{aligned}$$

Hence, applying (1.1) and Lemmas 2.1–2.2 we obtain

$$\begin{aligned} a_{p-1} &= \sum_{k=0}^{[p/4]} \binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}-k}{k} 4^{\frac{p-1}{2}-2k}\bigl(2(1-t)\bigr)^k \\ &\equiv \sum_{k=0}^{[p/4]} \frac{1}{(-4)^k}\binom{2k}{k}\frac{1}{(-4)^k}\binom{4k}{2k}4^{-2k}\cdot 2^k(1-t)^k \\ &= \sum_{k=0}^{[p/4]} \binom{4k}{2k}\binom{2k}{k}\left(\frac{1-t}{128}\right)^k \equiv P_{[\frac{p}{4}]}(t) \pmod p. \end{aligned}$$

Therefore,

$$\begin{aligned} P_{[\frac{p}{4}]}(t) \equiv a_{p-1} &\equiv -\sum_{x=0}^{p-1}\bigl(x^3+4x^2+2(1-t)x\bigr)^{\frac{p-1}{2}} \\ &\equiv -\sum_{x=0}^{p-1}\left(\left(x-\frac{4}{3}\right)^3+4\left(x-\frac{4}{3}\right)^2+2(1-t)\left(x-\frac{4}{3}\right)\right)^{\frac{p-1}{2}} \\ &= -\sum_{x=0}^{p-1}\left(x^3-\frac{2}{3}(3t+5)x+\frac{8}{27}(9t+7)\right)^{\frac{p-1}{2}} \\ &= -\sum_{x=0}^{p-1}\left(\left(\frac{2x}{3}\right)^3-\frac{2}{3}(3t+5)\cdot\frac{2x}{3}+\frac{8}{27}(9t+7)\right)^{\frac{p-1}{2}} \\ &= -\left(\frac{8}{27}\right)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\left(x^3-\frac{3}{2}(3t+5)x+9t+7\right)^{\frac{p-1}{2}} \pmod p. \end{aligned}$$

This yields the result. □

Corollary 2.1. *Let $p>3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\sum_{x=0}^{p-1}\left(\frac{x^3-\frac{3(3t+5)}{2}x+9t+7}{p}\right) = \left(\frac{2}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3+\frac{3(3t-5)}{2}x+9t-7}{p}\right).$$

Proof. Since $P_{[\frac{p}{4}]}(t)=(-1)^{[\frac{p}{4}]}P_{[\frac{p}{4}]}(-t)$, by Theorem 2.1 we obtain

$$\begin{aligned} \sum_{x=0}^{p-1}\left(\frac{x^3-\frac{3(3t+5)}{2}x+9t+7}{p}\right) &\equiv (-1)^{[\frac{p}{4}]}\sum_{x=0}^{p-1}\left(\frac{x^3-\frac{3(-3t+5)}{2}x-9t+7}{p}\right) \\ &= \left(\frac{-2}{p}\right)\sum_{x=0}^{p-1}\left(\frac{(-x)^3+\frac{3(3t-5)}{2}(-x)-9t+7}{p}\right) \\ &= \left(\frac{2}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3+\frac{3(3t-5)}{2}x+9t-7}{p}\right) \pmod p. \end{aligned}$$

For $p \in \{5, 11, 13\}$ it is easy to check that the result is true for $t = 0, 1, \dots, p-1$. Now suppose $p \geqslant 17$. By Hasse's estimate [C, Theorem 14.12, p. 315],

$$\left|\sum_{x=0}^{p-1}\left(\frac{x^3 - \frac{3(\pm 3t+5)}{2}x + 9t \pm 7}{p}\right)\right| \leqslant 2\sqrt{p}.$$

Since $4\sqrt{p} < p$, from the above we deduce the result for $p \geqslant 17$. $\square$

Theorem 2.2. *Let p be a prime greater than 3. Then*

$$P_{[\frac{p}{4}]}\left(-\frac{7}{9}\right) \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{72^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{576^k}$$
$$\equiv \begin{cases} (-1)^{\frac{p-1}{4}}(\frac{p}{3})2a \pmod p & \text{if } 4 \mid p-1,\ p = a^2+b^2 \text{ and } 4 \mid a-1, \\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. From [BEW, Theorem 6.2.9] or [S1, (2.15)–(2.16)] we have

$$\sum_{x=0}^{p-1}\left(\frac{x^3-4x}{p}\right) = \begin{cases} -2a & \text{if } p \equiv 1 \pmod 4,\ p = a^2+b^2 \text{ and } a \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

Thus taking $t = -7/9$ in Lemma 2.2 and Theorem 2.1 we deduce the result. $\square$

Theorem 2.3. *Let p be a prime greater than 3. Then*

$$P_{[\frac{p}{4}]}\left(-\frac{5}{3}\right) \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{48^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}$$
$$\equiv \begin{cases} 2A \pmod p & \text{if } 3 \mid p-1,\ p = A^2+3B^2 \text{ and } 3 \mid A-1, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. It is known that (see for example [S1, (2.7)–(2.9)] or [BEW, pp. 195–196])

$$\sum_{x=0}^{p-1}\left(\frac{x^3-8}{p}\right) = \begin{cases} -2A(\frac{-2}{p}) & \text{if } 3 \mid p-1,\ p = A^2+3B^2 \text{ and } 3 \mid A-1, \\ 0 & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{2.3}$$

Thus, putting $t = -5/3$ in Lemma 2.2 and Theorem 2.1 we deduce the result. $\square$

Theorem 2.4. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$P_{[\frac{p}{4}]}\left(-\frac{65}{63}\right) \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{63^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}$$
$$\equiv \begin{cases} 2C(\frac{p}{3})(\frac{C}{7}) \pmod p & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = C^2+7D^2, \\ 0 \pmod p & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Proof. Putting $t=-65/63$ in Lemma 2.2 we get

$$P_{[\frac p4]}\Big(-\frac{65}{63}\Big)\equiv\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}\binom{2k}{k}}{63^k}\equiv(-1)^{[\frac p4]}\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}\pmod p.$$

Taking $t=-65/63$ in Theorem 2.1 we see that

$$\begin{aligned}P_{[\frac p4]}\Big(-\frac{65}{63}\Big)&\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3+4x^2+\frac{256}{63}x}{p}\Big)=-\sum_{x=0}^{p-1}\Big(\frac{(\frac4{21}x)^3+4(\frac4{21}x)^2+\frac{256}{63}\cdot\frac4{21}x}{p}\Big)\\&=-\sum_{x=0}^{p-1}\Big(\frac{(\frac4{21})^3(x^3+21x^2+112x)}{p}\Big)\pmod p.\end{aligned}$$

From [R1,R2] we have

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+21x^2+112x}{p}\Big)=\begin{cases}-2C(\frac C7)&\text{if } p=C^2+7D^2\equiv1,2,4\ (\mathrm{mod}\ 7),\\0&\text{if } p\equiv3,5,6\ (\mathrm{mod}\ 7).\end{cases}\tag{2.4}$$

Thus the result follows. □

For related conjectures concerning Theorems 2.2–2.4, see [Su1, Conjectures A47–A49].

Lemma 2.3. *Let p be an odd prime and let u be a variable. Then*

$$(\sqrt u)^{\frac{p-1}2}P_{\frac{p-1}2}(\sqrt u)\equiv(-1)^{[\frac p4]}\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}u^{\frac{p-1}2-k}\pmod p.$$

Proof. From (1.5) we see that

$$(\sqrt u)^{\frac{p-1}2}P_{\frac{p-1}2}(\sqrt u)=\frac1{2^{\frac{p-1}2}}\sum_{k=0}^{[p/4]}\binom{\frac{p-1}2}{k}\binom{p-1-2k}{\frac{p-1}2}(-1)^ku^{\frac{p-1}2-k}.$$

Thus applying (1.1) and Lemma 2.1 we obtain

$$(\sqrt u)^{\frac{p-1}2}P_{\frac{p-1}2}(\sqrt u)\equiv\Big(\frac{-2}{p}\Big)\sum_{k=0}^{[p/4]}\frac1{(-4)^k}\binom{2k}{k}\cdot\frac1{2^{4k}}\binom{4k}{2k}(-1)^ku^{\frac{p-1}2-k}\pmod p.$$

Noting that $(\frac{-2}p)=(-1)^{[\frac p4]}$ we then obtain the result. □

Theorem 2.5. *Let $p>3$ be a prime, $u\in\mathbb Z_p$ and $u\not\equiv0\pmod p$. Then*

$$\begin{aligned}(\sqrt u)^{\frac{p-1}2}P_{\frac{p-1}2}(\sqrt u)&\equiv\Big(\frac up\Big)P_{[\frac p4]}\Big(\frac2u-1\Big)\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3-2ux^2+ux}{p}\Big)\\&\equiv-\Big(\frac6p\Big)\sum_{x=0}^{p-1}\Big(\frac{ux^3-(3u+9)x+18-2u}{p}\Big)\pmod p.\end{aligned}$$

Proof. By (1.6), Lemmas 2.2 and 2.3 we have

$$P_{[\frac p4]}\Big(\frac 2u-1\Big)=\Big(\frac{-2}{p}\Big)P_{[\frac p4]}\Big(1-\frac 2u\Big)\equiv(-1)^{[\frac p4]}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\binom{2k}{k}\frac{1}{(64u)^k}$$
$$\equiv\Big(\frac up\Big)(\sqrt u)^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt u)\pmod p.$$

By Theorem 2.1, we have

$$P_{[\frac p4]}(1-2/u)\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3+4x^2+\frac 4ux}{p}\Big)=-\sum_{x=0}^{p-1}\Big(\frac{(-\frac{2x}{u})^3+4(-\frac{2x}{u})^2+\frac 4u(-\frac{2x}{u})}{p}\Big)$$
$$=-\sum_{x=0}^{p-1}\Big(\frac{(-\frac 2u)^3(x^3-2ux^2+ux)}{p}\Big)=-\Big(\frac{-2u}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-2ux^2+ux}{p}\Big)\pmod p$$

and

$$P_{[\frac p4]}\Big(\frac 2u-1\Big)\equiv-\Big(\frac 6p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{3(u+3)}{u}x+\frac{18-2u}{u}}{p}\Big)$$
$$=-\Big(\frac{6u}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{ux^3-3(u+3)x+18-2u}{p}\Big)\pmod p.$$

Thus the theorem is proved. □

3. Congruences for $\sum_{k=0}^{\frac{p-1}{2}}\frac{1}{m^k}\binom{2k}{k}^3 \pmod{p^2}$

Lemma 3.1. *For any nonnegative integers m and n we have*

$$\sum_{k=0}^{m}\binom{2k}{k}^2\binom{n+k}{2k}\binom{k}{m-k}=\sum_{k=0}^{m}\binom{2k}{k}\binom{n+k}{2k}\binom{2m-2k}{m-k}\binom{n+m-k}{2m-2k}.$$

Proof. Let m and n be nonnegative integers. For $k\in\{0,1,\dots,m\}$ set

$$F_1(m,k)=\binom{2k}{k}\binom{n+k}{2k}\binom{2k}{k}\binom{k}{m-k},$$
$$F_2(m,k)=\binom{2k}{k}\binom{n+k}{2k}\binom{2(m-k)}{m-k}\binom{n+m-k}{2(m-k)}.$$

For $k\in\{0,1,\dots,m+1\}$ set

$$G_1(m,k)=-(m+2)k^2\binom{2k}{k}^2\binom{n+k}{2k}\binom{k}{m+2-k},$$
$$G_2(m,k)=-\frac{k^2(3mn^2-2n^2k+m^2+3mn-mk+6n^2-2nk+4m+6n-2k+4)}{(m+2-k)^2}$$

$$\times \binom{2k}{k}\binom{n+k}{2k}\binom{2(m+1-k)}{m+1-k}\binom{n+m+1-k}{2(m+1-k)}.$$

For $i=1,2$ and $k\in\{0,1,\dots,m\}$, using Maple it is easy to check that

$$\begin{aligned}&(m+2)^3F_i(m+2,k)+(2m+3)\big(m^2-2n^2+3m-2n+2\big)F_i(m+1,k)\\&\quad+(m+1)(m+2n+2)(m-2n)F_i(m,k)=G_i(m,k+1)-G_i(m,k).\end{aligned}\tag{3.1}$$

Set $S_i(r)=\sum_{k=0}^r F_i(r,k)$ for $r=0,1,2,\dots$. Then

$$\begin{aligned}&(m+2)^3\big(S_i(m+2)-F_i(m+2,m+2)-F_i(m+2,m+1)\big)\\&\quad+(2m+3)\big(m^2-2n^2+3m-2n+2\big)\big(S_i(m+1)-F_i(m+1,m+1)\big)\\&\quad+(m+1)(m+2n+2)(m-2n)S_i(m)\\&=(m+2)^3\sum_{k=0}^m F_i(m+2,k)+(2m+3)\big(m^2-2n^2+3m-2n+2\big)\sum_{k=0}^m F_i(m+1,k)\\&\quad+(m+1)(m+2n+2)(m-2n)\sum_{k=0}^m F_i(m,k)\\&=\sum_{k=0}^m\big(G_i(m,k+1)-G_i(m,k)\big)=G_i(m,m+1)-G_i(m,0)=G_i(m,m+1).\end{aligned}$$

From the above we deduce that for $i=1,2$ and $m=0,1,2,\dots,$

$$\begin{aligned}&(m+2)^3S_i(m+2)+(2m+3)\big(m^2-2n^2+3m-2n+2\big)S_i(m+1)\\&\quad+(m+1)(m+2n+2)(m-2n)S_i(m)\\&=G_i(m,m+1)+(m+2)^3\big(F_i(m+2,m+2)+F_i(m+2,m+1)\big)\\&\quad+(2m+3)\big(m^2-2n^2+3m-2n+2\big)F_i(m+1,m+1)=0.\end{aligned}\tag{3.2}$$

Since $S_1(0)=1=S_2(0)$ and $S_1(1)=2n(n+1)=S_2(1)$, from (3.2) we deduce $S_1(r)=S_2(r)$ for all $r=0,1,2,\dots$. This completes the proof. □

Remark 3.1. We actually find (3.1) and prove Lemma 3.1 by using WZ method and Maple. For the WZ method, see [PWZ].

Definition 3.1. Note that $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$. For any nonnegative integer n we define

$$S_n(x)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}x^k=\sum_{k=0}^n\binom{2k}{k}^2\binom{n+k}{2k}x^k.$$

Lemma 3.2. *Let n be a nonnegative integer. Then*

$$S_n(x)=P_n(\sqrt{1+4x})^2\quad\text{and}\quad P_n(x)^2=S_n\Big(\frac{x^2-1}{4}\Big).$$

Proof. From (2.1) we have

$$P_n(x)^2 = \left(\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}\left(\frac{x-1}{2}\right)^k\right)\left(\sum_{r=0}^{n}\binom{2r}{r}\binom{n+r}{2r}\left(\frac{x-1}{2}\right)^r\right)$$
$$= \sum_{m=0}^{2n}\left(\frac{x-1}{2}\right)^m\sum_{k=0}^{m}\binom{2k}{k}\binom{n+k}{2k}\binom{2m-2k}{m-k}\binom{n+m-k}{2m-2k}.$$

On the other hand,

$$S_n\left(\frac{x^2-1}{4}\right) = \sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\left(\frac{x-1}{2}\right)^k\left(1+\frac{x-1}{2}\right)^k$$
$$= \sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\left(\frac{x-1}{2}\right)^k\sum_{r=0}^{k}\binom{k}{r}\left(\frac{x-1}{2}\right)^r$$
$$= \sum_{m=0}^{2n}\left(\frac{x-1}{2}\right)^m\sum_{k=0}^{m}\binom{2k}{k}^2\binom{n+k}{2k}\binom{k}{m-k}.$$

Hence, from the above and Lemma 3.1 we deduce

$$P_n(x)^2 = S_n\left(\frac{x^2-1}{4}\right) \quad \text{and so} \quad P_n(\sqrt{1+4x})^2 = S_n(x).$$

This proves the lemma. $\square$

Remark 3.2. Since $S_n(-\frac{1}{4}) = P_n(0)^2$ by Lemma 3.2, using (1.6) we deduce that

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{1}{(-4)^k} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{2^{2n}}\binom{n}{n/2}^2 & \text{if } n \text{ is even.} \end{cases}$$

This is an identity due to Bell [G, (6.35)].

Theorem 3.1. *Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k} \equiv P_{\frac{p-1}{2}}\left(\sqrt{1-\frac{64}{m}}\right)^2 \pmod{p^2}.$$

Proof. By Definition 3.1 and (1.1) we have

$$S_{\frac{p-1}{2}}\left(-\frac{16}{m}\right) = \sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\binom{\frac{p-1}{2}+k}{2k}\left(-\frac{16}{m}\right)^k \equiv \sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k} \pmod{p^2}.$$

On the other hand, by Lemma 3.2 we get

$$S_{\frac{p-1}{2}}\left(-\frac{16}{m}\right)=P_{\frac{p-1}{2}}\left(\sqrt{1+4\left(-\frac{16}{m}\right)}\right)^2=P_{\frac{p-1}{2}}\left(\sqrt{\frac{m-64}{m}}\right)^2.$$

Thus the result follows. $\square$

Theorem 3.2. *Let p be an odd prime, $m\in\mathbb{Z}_p$ and $m\not\equiv 0,64 \pmod p$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k}\equiv\left(\frac{m(m-64)}{p}\right)P_{[\frac{p}{4}]}\left(\frac{m+64}{m-64}\right)^2 \pmod p.$$

Moreover,

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k}\equiv 0 \pmod{p^2} \quad\Leftrightarrow\quad P_{[\frac{p}{4}]}\left(\frac{m+64}{m-64}\right)\equiv 0 \pmod p.$$

Proof. Set $u=1-64/m$. From Theorem 2.5 we know that

$$(\sqrt{u})^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt{u})\equiv\left(\frac{u}{p}\right)P_{[\frac{p}{4}]}\left(\frac{2}{u}-1\right) \pmod p.$$

Thus,

$$\left(\frac{u}{p}\right)P_{\frac{p-1}{2}}(\sqrt{u})^2\equiv u^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt{u})^2\equiv P_{[\frac{p}{4}]}\left(\frac{2}{u}-1\right)^2 \pmod p.$$

It then follows from Theorem 3.1 that

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k}\equiv P_{\frac{p-1}{2}}(\sqrt{u})^2\equiv\left(\frac{u}{p}\right)P_{[\frac{p}{4}]}\left(\frac{2}{u}-1\right)^2=\left(\frac{m(m-64)}{p}\right)P_{[\frac{p}{4}]}\left(\frac{m+64}{m-64}\right)^2 \pmod p.$$

From (1.5) we see that $(\sqrt{u})^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt{u})\in\mathbb{Z}_p$. Thus, by Theorems 3.1 and 2.5,

$$\begin{aligned}&\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{m^k}\equiv 0 \pmod{p^2}\\ &\quad\Leftrightarrow\quad P_{\frac{p-1}{2}}(\sqrt{u})^2\equiv 0 \pmod{p^2}\quad\Leftrightarrow\quad\left((\sqrt{u})^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt{u})\right)^2\equiv 0 \pmod{p^2}\\ &\quad\Leftrightarrow\quad(\sqrt{u})^{\frac{p-1}{2}}P_{\frac{p-1}{2}}(\sqrt{u})\equiv 0 \pmod p\quad\Leftrightarrow\quad P_{[\frac{p}{4}]}\left(\frac{2}{u}-1\right)\equiv 0 \pmod p\\ &\quad\Leftrightarrow\quad P_{[\frac{p}{4}]}\left(\frac{m+64}{m-64}\right)\equiv 0 \pmod p.\end{aligned}$$

Thus the theorem is proved. $\square$

Theorem 3.3. *Let p be an odd prime. Then*

(i) $$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{4096^k}\equiv 0 \pmod{p^2}\quad \text{for } p\equiv 3,5,6 \pmod 7,$$

(ii) $$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv 0 \pmod{p^2}\quad \text{for } p\equiv 3 \pmod 4,$$

(iii) $$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{16^k}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{256^k}\equiv 0 \pmod{p^2}\quad \text{for } p\equiv 2 \pmod 3,$$

(iv) $$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv 0 \pmod{p^2}\quad \text{for } p\equiv 5,7 \pmod 8.$$

Proof. Taking $m=1,4096,-8,-512,16,256,-64$ in Theorem 3.2 and then applying Theorems 2.2–2.4 and (1.6) we obtain the result. □

Theorem 3.4. *Let p be an odd prime.*

(i) *If $p\equiv 1,2,4 \pmod 7$ and so $p=C^2+7D^2$ with $C,D\in\mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^3\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{4096^k}\equiv 4C^2-2p \pmod{p^2}.$$

(ii) *If $p\equiv 1 \pmod 4$ and so $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $2\nmid a$, then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv(-1)^{\frac{p-1}{4}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv 4a^2-2p \pmod{p^2}.$$

(iii) *If $p\equiv 1 \pmod 3$ and so $p=A^2+3B^2$ with $A,B\in\mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{16^k}\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{256^k}\equiv 4A^2-2p \pmod{p^2}.$$

(iv) *If $p\equiv 1,3 \pmod 8$ and so $p=c^2+2d^2$ with $c,d\in\mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv(-1)^{\frac{p-1}{2}}\left(4c^2-2p\right) \pmod{p^2}.$$

Proof. From [CH, (48), Tables II and III] we know that

$$P_{\frac{p-1}{2}}(3\sqrt{-7}) \equiv x - y\sqrt{-7} \pmod{p^2} \quad \text{for } p = x^2+7y^2 \text{ with } 4 \mid x+y-1,$$

$$P_{\frac{p-1}{2}}\left(\frac{3\sqrt{7}}{8}\right) \equiv \left(\frac{\sqrt{-7}}{\sqrt{7}}\right)^{\frac{p-1}{2}}(x - y\sqrt{-7}) \pmod{p^2} \quad \text{for } p = x^2+7y^2 \text{ with } 4 \mid x+y-1,$$

$$P_{\frac{p-1}{2}}(3) \equiv (-1)^{\frac{p-1}{4}}(x - y\sqrt{-1}) \pmod{p^2} \quad \text{for } p = x^2+y^2 \equiv 1 \pmod 4 \text{ with } 4 \mid x-1,$$

$$P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right) \equiv (\sqrt{-1})^{\frac{y}{2}}(x - y\sqrt{-1}) \pmod{p^2} \quad \text{for } p = x^2+y^2 \equiv 1 \pmod 4 \text{ with } 4 \mid x-1,$$

$$P_{\frac{p-1}{2}}(\sqrt{-3}) \equiv (-1)^{\frac{p-1}{2}}(x - y\sqrt{-3}) \pmod{p^2} \quad \text{for } p = x^2+3y^2 \equiv 1 \pmod 3 \text{ with } 4 \mid x+y-1,$$

$$P_{\frac{p-1}{2}}\left(\frac{\sqrt{3}}{2}\right) \equiv \left(-\frac{\sqrt{-3}}{\sqrt{3}}\right)^{\frac{p-1}{2}}(x - y\sqrt{-3}) \pmod{p^2}$$
$$\text{for } p = x^2+3y^2 \equiv 1 \pmod 3 \text{ with } 4 \mid x+y-1,$$

$$P_{\frac{p-1}{2}}(\sqrt{2}) \equiv \left(\frac{\sqrt{-2}}{\sqrt{2}}\right)^{-y}(x - y\sqrt{-2}) \pmod{p^2}$$
$$\text{for } p = x^2+2y^2 \equiv 1, 3 \pmod 8 \text{ with } 4 \mid x-1,$$

where $\sqrt{-d} \in \mathbb{Q}_p$ and $|x - y\sqrt{-d}|_p = 1$ ($|\cdot|_p$ is the usual valuation on $\mathbb{Q}_p$).

If $(\frac{-d}{p}) = 1$, $p = x^2+dy^2$ and $A \in \mathbb{Q}_p$ with $A \equiv x - y\sqrt{-d} \pmod{p^2}$ and $|x - y\sqrt{-d}|_p = 1$, then

$$(A-x)^2 \equiv -dy^2 = x^2 - p \pmod{p^2} \quad \text{and so} \quad A^2 - 2xA \equiv -p \pmod{p^2}.$$

Hence $A \equiv 2x \pmod p$ and so $2xA \equiv (A-2x)^2 + 2xA = A^2 - 2xA + 4x^2 \equiv 4x^2 - p \pmod{p^2}$. Therefore,

$$A \equiv 2x - \frac{p}{2x} \pmod{p^2} \quad \text{and so} \quad A^2 \equiv 4x^2 - 2p \pmod{p^2}.$$

From all the above we deduce that

$$P_{\frac{p-1}{2}}(3\sqrt{-7})^2 \equiv (-1)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{7}}{8}\right)^2 \equiv 4x^2 - 2p \pmod{p^2} \quad \text{for } p = x^2+7y^2,$$

$$P_{\frac{p-1}{2}}(3)^2 \equiv (-1)^{\frac{p-1}{4}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right)^2 \equiv 4x^2 - 2p \pmod{p^2}$$
$$\text{for } p = x^2+y^2 \equiv 1 \pmod 4 \text{ with } 2 \nmid x \ \big(\text{see also [S2, Corollary 2.3 and Theorem 2.9]}\big),$$

$$P_{\frac{p-1}{2}}(\sqrt{-3})^2 \equiv (-1)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{\sqrt{3}}{2}\right)^2 \equiv 4x^2 - 2p \pmod{p^2}$$
$$\text{for } p = x^2+3y^2 \equiv 1 \pmod 3,$$

$$P_{\frac{p-1}{2}}(\sqrt{2})^2 \equiv (-1)^y(4x^2-2p) = (-1)^{\frac{p-1}{2}}(4x^2-2p) \pmod{p^2}$$
$$\text{for } p = x^2+2y^2 \equiv 1, 3 \pmod 8.$$

Now taking $m = 1, 4096, -8, -512, 16, 256, -64$ in Theorem 3.1 and then applying the above we derive the result. □

Corollary 3.1. *Let p be a prime such that $p \ne 2, 3, 7$. Then*

$$\sum_{k=0}^{(p-1)/2} k\binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{21}(3p-4C^2) \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = C^2 + 7D^2, \\ \frac{8}{21}p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Proof. By [Su2, Theorem 1.3] we have

$$\sum_{k=0}^{(p-1)/2} (21k+8)\binom{2k}{k}^3 \equiv \sum_{k=0}^{p-1} (21k+8)\binom{2k}{k}^3 \equiv 8p \pmod{p^2}.$$

Thus applying Theorems 3.3(i) and 3.4(i) we conclude the result. □

Remark 3.3. Theorems 3.3, 3.4 and Corollary 3.1 were conjectured by Zhi-Wei Sun in [Su1,Su3].

Lemma 3.3. *For any positive integer n we have the following identities:*

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k}{(-4)^k} = \begin{cases} -\frac{n^2}{2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2 \nmid n, \\ \frac{n(n+1)}{2^{2n}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2 \mid n, \end{cases}$$

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k^2}{(-4)^k} = \begin{cases} -\frac{n^2(2n^2+2n-1)}{3\cdot 2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2 \nmid n, \\ \frac{n^2(n+1)^2}{3\cdot 2^{2n-1}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2 \mid n, \end{cases}$$

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k^3}{(-4)^k} = \begin{cases} -\frac{n^2(4n^2(n+1)^2-n(n+1)+1)}{15\cdot 2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2 \nmid n, \\ \frac{n^2(n+1)^2(2n+1)^2}{15\cdot 2^{2n}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2 \mid n. \end{cases}$$

Proof. We only prove the first identity. The other identities can be proved similarly. Let $f'(x) = \frac{d}{dx}f(x)$ be the derivative of $f(x)$. By Lemma 3.2, $S_n(x) = P_n(\sqrt{1+4x})^2$. If $2 \mid n$, then $P_n(\sqrt{1+4x})$ is a polynomial in $1+4x$. Thus

$$S_n'(x) = 2P_n(\sqrt{1+4x}) \cdot \frac{d}{dx}P_n(\sqrt{1+4x})$$

and so

$$S_n'\left(-\frac{1}{4}\right) = 2P_n(0)\frac{d}{dx}P_n(\sqrt{1+4x})\Big|_{x=-\frac{1}{4}}.$$

From (1.5) we see that

$$\frac{d}{dx}P_n(\sqrt{1+4x})=\frac{d}{dx}\frac{1}{2^n}\sum_{k=0}^{n/2}\binom{n}{k}(-1)^k\binom{2n-2k}{n}(1+4x)^{\frac{n}{2}-k}$$
$$=\frac{4}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}(-1)^k\binom{2n-2k}{n}\left(\frac{n}{2}-k\right)(1+4x)^{\frac{n}{2}-k-1}$$

and therefore

$$\frac{d}{dx}P_n(\sqrt{1+4x})\Big|_{x=-\frac14}=\frac{4}{2^n}\binom{n}{\frac{n}{2}-1}(-1)^{\frac{n}{2}-1}\binom{2n-2(\frac{n}{2}-1)}{n}$$
$$=(-1)^{\frac{n}{2}-1}\frac{n(n+1)}{2^{n-1}}\binom{n}{n/2}.$$

Combining the above with (1.6) we get

$$S_n'\left(-\frac14\right)=2\cdot\frac{(-1)^{n/2}}{2^n}\binom{n}{n/2}\cdot(-1)^{\frac{n}{2}-1}\frac{n(n+1)}{2^{n-1}}\binom{n}{n/2}=-\frac{n(n+1)}{2^{2n-2}}\binom{n}{n/2}^2.$$

By Definition 3.1,

$$S_n'\left(-\frac14\right)=\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k}{(-4)^{k-1}}. \tag{3.3}$$

Thus the first identity is true when n is even.

Now we assume $2\nmid n$. Set

$$A_n(x)=\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}(-1)^k\binom{2n-2k}{n}(1+4x)^{\frac{n-1}{2}-k}.$$

From Lemma 3.2 and (1.5) we have

$$S_n(x)=(1+4x)\left(\frac{P_n(\sqrt{1+4x})}{\sqrt{1+4x}}\right)^2=\frac{1}{2^{2n}}(1+4x)A_n(x)^2.$$

Thus,

$$S_n'(x)=\frac{1}{2^{2n}}\left(4A_n(x)^2+(1+4x)\cdot 2A_n(x)A_n'(x)\right)$$

and therefore

$$S_n'\left(-\frac14\right)=\frac{4}{2^{2n}}A_n\left(-\frac14\right)^2=\frac{4}{2^{2n}}\left(\binom{n}{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}}\binom{2n-(n-1)}{n}\right)^2$$
$$=\frac{4(n+1)^2}{2^{2n}}\binom{n}{\frac{n-1}{2}}^2=\frac{4n^2}{2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2.$$

This together with (3.3) yields the first identity in the case $2\nmid n$. The proof is now complete. □

We remark that Lemma 3.3 can also be proved by using WZ method.

Lemma 3.4. *Let p be a prime of the form $4k+1$ and so $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $2\nmid a$. Then*

$$\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\equiv 4a^2-2p \pmod{p^2}.$$

Proof. From [CDE] or [BEW] we know that for $a\equiv 1 \pmod 4$,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv\frac{2^{p-1}+1}{2}\left(2a-\frac{p}{2a}\right) \pmod{p^2}.$$

Set $q_p(2)=(2^{p-1}-1)/p$. We then have

$$\begin{aligned}\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2&\equiv\frac{1}{2^{p-1}}\cdot\frac{(2^{p-1}+1)^2}{4}\left(4a^2-2p\right)\\&=\frac14\cdot\frac{(2+pq_p(2))^2}{1+pq_p(2)}\left(4a^2-2p\right)\equiv 4a^2-2p \pmod{p^2}.\end{aligned}$$

This proves the lemma. $\square$

Theorem 3.5. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{k\binom{2k}{k}^3}{64^k}\equiv\begin{cases}(2p-2+2^{p-1})\binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ \frac{p}{2}-a^2 \pmod{p^2} & \text{if } p=a^2+4b^2\equiv 1 \pmod 4,\end{cases}$$

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{k^2\binom{2k}{k}^3}{64^k}\equiv\begin{cases}(1-p-2^{p-2})\binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ \frac{1}{12}(2a^2-p) \pmod{p^2} & \text{if } p=a^2+4b^2\equiv 1 \pmod 4,\end{cases}$$

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{k^3\binom{2k}{k}^3}{64^k}\equiv\begin{cases}\frac15(2^{p-2}-1+p)\binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4.\end{cases}$$

Proof. By (1.1) and Lemma 3.3,

$$\begin{aligned}\sum_{k=1}^{\frac{p-1}{2}}\frac{k\binom{2k}{k}^3}{64^k}&\equiv\sum_{k=1}^{\frac{p-1}{2}}\binom{2k}{k}^2\binom{\frac{p-1}{2}+k}{2k}\frac{k}{(-4)^k}\\&=\begin{cases}-\frac{(p-1)^2}{2^{p-1}}\binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\equiv-\frac14\cdot\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4.\end{cases}\end{aligned}$$

Set $q_p(2)=(2^{p-1}-1)/p$. For $p\equiv 3 \pmod 4$ we see that

$$-\frac{(p-1)^2}{2^{p-1}} \equiv \frac{2p-1}{1+pq_p(2)} \equiv (2p-1)\big(1-pq_p(2)\big)$$
$$\equiv -1+\big(2+q_p(2)\big)p = 2p-2+2^{p-1} \pmod{p^2}.$$

Now combining all the above with Lemma 3.4 we obtain the first congruence. The remaining congruences can be proved similarly. □

Remark 3.4. In [Su3, (1.10)], Zhi-Wei Sun obtained the congruence for $\sum_{k=1}^{\frac{p-1}{2}} \frac{k^3\binom{2k}{k}^3}{64^k} \pmod p$. He also conjectured that $\sum_{k=1}^{\frac{p-1}{2}} \frac{k^3\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^2}$ for any prime $p \equiv 1 \pmod 4$ with $p \neq 5$.

4. A general congruence modulo p^2

Lemma 4.1. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{n-k} (-64)^{n-k} = \sum_{k=0}^{n} \binom{2k}{k} \binom{4k}{2k} \binom{2(n-k)}{n-k} \binom{4(n-k)}{2(n-k)}.$$

Proof. Let m be a nonnegative integer. For $k \in \{0,1,\dots,m\}$ set

$$F_1(m,k) = \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k},$$
$$F_2(m,k) = \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)}.$$

For $k \in \{0,1,\dots,m+1\}$ set

$$G_1(m,k) = 64k^2(m+2)\binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m+2-k} (-64)^{m+1-k},$$
$$G_2(m,k) = \frac{4k^2(16m^2-16mk+55m-26k+46)}{(m+2-k)^2} \times \binom{2k}{k} \binom{4k}{2k} \binom{2(m+1-k)}{m+1-k} \binom{4(m+1-k)}{2(m+1-k)}.$$

For $i=1,2$ and $k \in \{0,1,\dots,m\}$, using Maple it is easy to check that

$$(m+2)^3 F_i(m+2,k) - 8(2m+3)\big(8m^2+24m+19\big)F_i(m+1,k) + 1024(m+1)(2m+1)(2m+3)F_i(m,k) = G_i(m,k+1) - G_i(m,k). \tag{4.1}$$

Set $S_i(n) = \sum_{k=0}^{n} F_i(n,k)$ for $n = 0,1,2,\dots$. Then

$$(m+2)^3\big(S_i(m+2) - F_i(m+2,m+2) - F_i(m+2,m+1)\big)$$
$$- 8(2m+3)\big(8m^2+24m+19\big)\big(S_i(m+1) - F_i(m+1,m+1)\big)$$
$$+ 1024(m+1)(2m+1)(2m+3)S_i(m)$$

$$= (m+2)^3\sum_{k=0}^m F_i(m+2,k) - 8(2m+3)\big(8m^2+24m+19\big)\sum_{k=0}^m F_i(m+1,k)$$
$$+ 1024(m+1)(2m+1)(2m+3)\sum_{k=0}^m F_i(m,k)$$
$$= \sum_{k=0}^m\big(G_i(m,k+1)-G_i(m,k)\big) = G_i(m,m+1)-G_i(m,0) = G_i(m,m+1).$$

From the above we deduce that for $i=1,2$ and $m=0,1,2,\dots,$

$$\begin{aligned}&(m+2)^3S_i(m+2)-8(2m+3)\big(8m^2+24m+19\big)S_i(m+1)\\&\quad+1024(m+1)(2m+1)(2m+3)S_i(m)\\&\quad= G_i(m,m+1)+(m+2)^3\big(F_i(m+2,m+2)+F_i(m+2,m+1)\big)\\&\quad-8(2m+3)\big(8m^2+24m+19\big)F_i(m+1,m+1)=0.\end{aligned} \tag{4.2}$$

Since $S_1(0)=1=S_2(0)$ and $S_1(1)=24=S_2(1)$, from (4.2) we deduce $S_1(n)=S_2(n)$ for all $n=0,1,2,\dots$. This completes the proof. □

Theorem 4.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\big(x(1-64x)\big)^k \equiv \left(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\big(x(1-64x)\big)^k &= \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}x^k\sum_{r=0}^{k}\binom{k}{r}(-64x)^r\\&=\sum_{m=0}^{2(p-1)}x^m\sum_{k=0}^{\min\{m,p-1\}}\binom{2k}{k}^2\binom{4k}{2k}\binom{k}{m-k}(-64)^{m-k}.\end{aligned}$$

Suppose $p\leqslant m\leqslant 2p-2$ and $0\leqslant k\leqslant p-1$. If $k>\frac{p}{2}$, then $p\mid\binom{2k}{k}$ and so $p^2\mid\binom{2k}{k}^2$. If $k<\frac{p}{2}$, then $m-k\geqslant p-k>k$ and so $\binom{k}{m-k}=0$. Thus, from the above and Lemma 4.1 we deduce that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\big(x(1-64x)\big)^k\\&\quad\equiv\sum_{m=0}^{p-1}x^m\sum_{k=0}^{m}\binom{2k}{k}^2\binom{4k}{2k}\binom{k}{m-k}(-64)^{m-k}\\&\quad=\sum_{m=0}^{p-1}x^m\sum_{k=0}^{m}\binom{2k}{k}\binom{4k}{2k}\binom{2(m-k)}{m-k}\binom{4(m-k)}{2(m-k)}\end{aligned}$$

$$=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\sum_{m=k}^{p-1}\binom{2(m-k)}{m-k}\binom{4(m-k)}{2(m-k)}x^{m-k}$$

$$=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\sum_{r=0}^{p-1-k}\binom{2r}{r}\binom{4r}{2r}x^r$$

$$=\left(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\right)^2-\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\sum_{r=p-k}^{p-1}\binom{2r}{r}\binom{4r}{2r}x^r \pmod{p^2}.$$

Now suppose $0\leqslant k\leqslant p-1$ and $p-k\leqslant r\leqslant p-1$. If $k\geqslant\frac{3p}{4}$, then $p^2\nmid(2k)!$, $p^3\mid(4k)!$ and so $\binom{2k}{k}\binom{4k}{2k}=\frac{(4k)!}{(2k)!k!^2}\equiv 0\pmod{p^2}$. If $k<\frac{p}{4}$, then $r\geqslant p-k>\frac{3p}{4}$ and so $\binom{2r}{r}\binom{4r}{2r}=\frac{(4r)!}{(2r)!r!^2}\equiv 0\pmod{p^2}$. If $\frac{p}{4}<k<\frac{p}{2}$, then $r\geqslant p-k>\frac{p}{2}$, $p\mid\binom{2r}{r}$ and $p\mid\binom{4k}{2k}$. If $\frac{p}{2}<k<\frac{3p}{4}$, then $r\geqslant p-k>\frac{p}{4}$, $p\mid\binom{2k}{k}$ and $p\mid\binom{2r}{r}\binom{4r}{2r}$. Hence we always have $\binom{2k}{k}\binom{4k}{2k}\binom{2r}{r}\binom{4r}{2r}\equiv 0\pmod{p^2}$ and so

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}x^k\sum_{r=p-k}^{p-1}\binom{2r}{r}\binom{4r}{2r}x^r\equiv 0\pmod{p^2}.$$

Now combining all the above we obtain the result. □

Corollary 4.1. *Let p be an odd prime and $m\in\mathbb{Z}_p$ with $m\not\equiv 0\pmod p$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\equiv\left(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}\left(\frac{1-\sqrt{1-256/m}}{128}\right)^k\right)^2 \pmod{p^2}.$$

Proof. Taking $x=\frac{1-\sqrt{1-256/m}}{128}$ in Theorem 4.1 we deduce the result. □

Theorem 4.2. *Let p be an odd prime, $m\in\mathbb{Z}_p$, $m\not\equiv 0\pmod p$ and $t=\sqrt{1-256/m}$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\equiv P_{[\frac{p}{4}]}(t)^2\equiv\left(\sum_{x=0}^{p-1}\left(x^3+4x^2+2(1-t)x\right)^{\frac{p-1}{2}}\right)^2 \pmod p.$$

Moreover, if $P_{[\frac{p}{4}]}(t)\equiv 0\pmod p$ or $\sum_{x=0}^{p-1}(x^3+4x^2+2(1-t)x)^{\frac{p-1}{2}}\equiv 0\pmod p$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\equiv 0\pmod{p^2}.$$

Proof. For $\frac{p}{2}<k<p$, $\binom{2k}{k}=\frac{(2k)!}{k!^2}\equiv 0\pmod p$. For $\frac{p}{4}<k<\frac{p}{2}$, $\binom{4k}{2k}=\frac{(4k)!}{(2k)!^2}\equiv 0\pmod p$. Thus, $p\mid\binom{2k}{k}\binom{4k}{2k}$ for $\frac{p}{4}<k<p$. Now combining Lemma 2.2, Theorem 2.1 with Corollary 4.1 gives the result. □

Theorem 4.3. *Let $p\equiv 1,3\pmod 8$ be a prime and so $p=c^2+2d^2$ with $c,d\in\mathbb{Z}$ and $c\equiv 1\pmod 4$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\equiv(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}\left(2c-\frac{p}{2c}\right) \pmod{p^2}.$$

Proof. By Lemma 2.2, Theorem 2.1 and [BE, Theorems 5.12 and 5.17],

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}&\equiv\sum_{k=0}^{[p/4]}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\equiv P_{[\frac{p}{4}]}(0)\equiv-\sum_{x=0}^{p-1}\left(\frac{x^3+4x^2+2x}{p}\right)\\&=-\sum_{x=0}^{p-1}\left(\frac{(-x)^3+4(-x)^2+2(-x)}{p}\right)=-\left(\frac{-1}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-4x^2+2x}{p}\right)\\&\equiv(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}2c\pmod p.\end{aligned}$$

Set $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}=(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}2c+qp$. Then

$$\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\right)^2=\left((-1)^{[\frac{p}{8}]+\frac{p-1}{2}}2c+qp\right)^2\equiv4c^2+(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}4cqp\pmod{p^2}.$$

Taking $x=\frac{1}{128}$ in Theorem 4.1 we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\right)^2\pmod{p^2}.$$

From [M] and [Su4] we have $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/256^k\equiv4c^2-2p\pmod{p^2}$. Thus

$$4c^2-2p\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\right)^2\equiv4c^2+(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}4cqp\pmod{p^2}$$

and hence $q\equiv-(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}\frac{1}{2c}\pmod p$. So the theorem is proved. □

We note that Theorem 4.3 was conjectured by Zhi-Wei Sun in [Su1].

5. Congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/m^k$

Theorem 5.1. *Let $p\neq2,3,7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k}\equiv\begin{cases}0\pmod{p^2}&\text{if } p\equiv3\pmod4,\\4a^2\pmod p&\text{if } p=a^2+4b^2\equiv1\pmod4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k}\equiv\begin{cases}0\pmod{p^2}&\text{if } p\equiv2\pmod3,\\4A^2\pmod p&\text{if } p=A^2+3B^2\equiv1\pmod3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k}\equiv\begin{cases}0\pmod{p^2}&\text{if } p\equiv3,5,6\pmod7,\\4C^2\pmod p&\text{if } p=C^2+7D^2\equiv1,2,4\pmod7.\end{cases}$$

Proof. Taking $m = 648, -144, -3969$ in Theorem 4.2 and then applying Theorems 2.2–2.4 and (1.6) we deduce the result. □

We mention that Theorem 5.1 was conjectured by the author in [S2].

Lemma 5.1. *(See [S3, Lemma 4.1].) Let p be an odd prime and let a, m, n be algebraic numbers which are integral for p. Then*

$$\sum_{x=0}^{p-1}(x^3+a^2mx+a^3n)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}}\sum_{x=0}^{p-1}(x^3+mx+n)^{\frac{p-1}{2}} \pmod p.$$

Moreover, if a, m, n are congruent to rational integers modulo p, then

$$\sum_{x=0}^{p-1}\left(\frac{x^3+a^2mx+a^3n}{p}\right) = \left(\frac{a}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3+mx+n}{p}\right).$$

Theorem 5.2. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$P_{[\frac{p}{4}]}\left(\frac{5\sqrt{-7}}{9}\right) \equiv \begin{cases} (\frac{3(7+\sqrt{-7})}{p})(\frac{C}{7})2C \pmod p & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = C^2+7D^2, \\ 0 \pmod p & \text{if } p \equiv 3, 5, 6 \pmod 7 \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod p & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = C^2+7D^2, \\ 0 \pmod {p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Proof. By Theorem 2.1,

$$P_{[\frac{p}{4}]}\left(\frac{5\sqrt{-7}}{9}\right) \equiv -\left(\frac{6}{p}\right)\sum_{x=0}^{p-1}\left(x^3-\frac{5}{2}(3+\sqrt{-7})x+7+5\sqrt{-7}\right)^{\frac{p-1}{2}} \pmod p.$$

Since

$$\frac{-\frac{5}{2}(3+\sqrt{-7})}{-35} = \left(\frac{1-\sqrt{-7}}{2\sqrt{-7}}\right)^2 \quad\text{and}\quad \frac{7+5\sqrt{-7}}{-98} = \left(\frac{1-\sqrt{-7}}{2\sqrt{-7}}\right)^3,$$

by the above and Lemma 5.1 we have

$$P_{[\frac{p}{4}]}\left(\frac{5\sqrt{-7}}{9}\right) \equiv -\left(\frac{6}{p}\right)\left(\frac{1-\sqrt{-7}}{2\sqrt{-7}}\right)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\left(\frac{x^3-35x-98}{p}\right) \pmod p.$$

As $x^3+21x^2+112x = (x+7)^3-35(x+7)-98$, by (2.4) we get

$$\sum_{x=0}^{p-1}\left(\frac{x^3-35x-98}{p}\right) = \begin{cases} -2C(\frac{C}{7}) & \text{if } p = C^2+7D^2 \equiv 1, 2, 4 \pmod 7, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases} \tag{5.1}$$

For $p\equiv 1,2,4 \pmod 7$ we see that

$$\left(\frac{6}{p}\right)\left(\frac{1-\sqrt{-7}}{2\sqrt{-7}}\right)^{\frac{p-1}{2}}=\left(\frac{6}{p}\right)\left(\frac{7+\sqrt{-7}}{2\cdot(-7)}\right)^{\frac{p-1}{2}}\equiv\left(\frac{3}{p}\right)\left(\frac{7+\sqrt{-7}}{p}\right) \pmod p.$$

Thus, from the above we deduce the congruence for $P_{[\frac{p}{4}]}(\frac{5\sqrt{-7}}{9}) \pmod p$. Applying Theorem 4.2 (with $m=81$) we obtain the remaining result. $\square$

Let $p>3$ be a prime and let $\mathbb{F}_p$ be the field of p elements. For $m,n\in\mathbb{F}_p$ let $\#E_p(x^3+mx+n)$ be the number of points on the curve E: $y^2=x^3+mx+n$ over the field $\mathbb{F}_p$. It is well known that (see for example [S1, pp. 221–222])

$$\#E_p\big(x^3+mx+n\big)=p+1+\sum_{x=0}^{p-1}\left(\frac{x^3+mx+n}{p}\right). \tag{5.2}$$

Let $K=\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and the curve $y^2=x^3+mx+n$ has complex multiplication by an order in K. By Deuring's theorem ([C, Theorem 14.16], [PV], [I]), we have

$$\#E_p\big(x^3+mx+n\big)=\begin{cases} p+1 & \text{if } p \text{ is inert in } K,\\ p+1-\pi-\bar{\pi} & \text{if } p=\pi\bar{\pi} \text{ in } K,\end{cases} \tag{5.3}$$

where π is in an order in K and $\bar{\pi}$ is the conjugate number of π. If $4p=u^2+dv^2$ with $u,v\in\mathbb{Z}$, we may take $\pi=\frac{1}{2}(u+v\sqrt{-d})$. Thus,

$$\sum_{x=0}^{p-1}\left(\frac{x^3+mx+n}{p}\right)=\begin{cases} \pm u & \text{if } 4p=u^2+dv^2 \text{ with } u,v\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{5.4}$$

In [JM] and [PV] the sign of u in (5.4) was determined for those imaginary quadratic fields K with class number 1. In [LM] and [I] the sign of u in (5.4) was determined for imaginary quadratic fields K with class number 2.

Theorem 5.3. *Let p be a prime such that $p\equiv\pm 1 \pmod{12}$. Then*

$$P_{[\frac{p}{4}]}\left(\frac{7}{12}\sqrt{3}\right)\equiv\begin{cases} (\frac{2+2\sqrt{3}}{p})2x \pmod p & \textit{if } p=x^2+9y^2\equiv 1 \pmod{12} \textit{ with } 3\mid x-1,\\ 0 \pmod p & \textit{if } p\equiv 11 \pmod{12}\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12\,288)^k}\equiv\begin{cases} 4x^2 \pmod p & \textit{if } p=x^2+9y^2\equiv 1 \pmod{12},\\ 0 \pmod{p^2} & \textit{if } p\equiv 11 \pmod{12}.\end{cases}$$

Proof. From [I, p. 133] we know that the elliptic curve defined by the equation $y^2=x^3-(120+42\sqrt{3})x+448+336\sqrt{3}$ has complex multiplication by the order of discriminant -36. Thus, by (5.4) and [I, Theorem 3.1] we have

$$\sum_{n=0}^{p-1}\left(\frac{n^3-(120+42\sqrt{3})n+448+336\sqrt{3}}{p}\right)$$
$$=\begin{cases}-2x(\frac{1+\sqrt{3}}{p}) & \text{if } p=x^2+9y^2\equiv 1 \pmod{12} \text{ with } 3\mid x-1,\\ 0 & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

By Theorem 2.1,

$$\begin{aligned}P_{[\frac p4]}\left(\frac{7}{12}\sqrt{3}\right)&\equiv -\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\left(n^3-\frac{60+21\sqrt{3}}{8}n+\frac{28+21\sqrt{3}}{4}\right)^{\frac{p-1}{2}}\\ &\equiv -\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\left(\left(\frac{n}{4}\right)^3-\frac{60+21\sqrt{3}}{8}\cdot\frac{n}{4}+\frac{28+21\sqrt{3}}{4}\right)^{\frac{p-1}{2}}\\ &\equiv -\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\left(\frac{n^3-(120+42\sqrt{3})n+448+336\sqrt{3}}{p}\right) \pmod p.\end{aligned}$$

Now combining all the above we obtain the congruence for $P_{[\frac p4]}(\frac{7}{12}\sqrt{3}) \pmod p$. Applying Theorem 4.2 (with $m=-12\,288$ and $t=\frac{7}{12}\sqrt{3}$) we deduce the remaining result. □

Remark 5.1. In [Su1, Conjecture A24], Z.W. Sun conjectured that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12\,288)^k}\equiv\begin{cases}(-1)^{[\frac x6]}(4x^2-2p) \pmod{p^2} & \text{if } p=x^2+y^2\equiv 1 \pmod{12} \text{ and } 4\mid x-1,\\ -4(\frac{xy}{3})xy \pmod{p^2} & \text{if } p=x^2+y^2\equiv 5 \pmod{12} \text{ and } 4\mid x-1,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Theorem 5.4. *Let p be an odd prime such that $p\equiv\pm1 \pmod 8$. Then*

$$P_{[\frac p4]}\left(\frac{2\sqrt{2}}{3}\right)\equiv\begin{cases}(-1)^{\frac{p-1}{2}}(\frac{\sqrt{2}}{p})(\frac{x}{3})2x \pmod p & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\equiv\begin{cases}4x^2 \pmod p & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ 0 \pmod{p^2} & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Proof. From [I, p. 133] we know that the elliptic curve defined by the equation $y^2=x^3+(-21+12\sqrt{2})x-28+22\sqrt{2}$ has complex multiplication by the order of discriminant -24. Thus, by (5.4) and [I, Theorem 3.1] we have

$$\sum_{n=0}^{p-1}\left(\frac{n^3+(-21+12\sqrt{2})n-28+22\sqrt{2}}{p}\right)$$
$$=\begin{cases}2x(\frac{2x}{3})(\frac{1+\sqrt{2}}{p}) & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2,\\ 0 & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

By Theorem 2.1,

$$P_{[\frac{p}{4}]}\Big(\frac{2\sqrt{2}}{3}\Big)\equiv -\Big(\frac{6}{p}\Big)\sum_{n=0}^{p-1}\Big(n^3-\frac{15+6\sqrt{2}}{2}n+7+6\sqrt{2}\Big)^{\frac{p-1}{2}} \pmod p.$$

Since

$$\frac{-(15+6\sqrt{2})/2}{-21+12\sqrt{2}}=\Big(\frac{\sqrt{2}+1}{\sqrt{2}}\Big)^2 \quad\text{and}\quad \frac{7+6\sqrt{2}}{-28+22\sqrt{2}}=\Big(\frac{\sqrt{2}+1}{\sqrt{2}}\Big)^3,$$

by Lemma 5.1 and the above we have

$$\begin{aligned}P_{[\frac{p}{4}]}\Big(\frac{2\sqrt{2}}{3}\Big)&\equiv -\Big(\frac{6}{p}\Big)\Big(\frac{\sqrt{2}(\sqrt{2}+1)}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3+(-21+12\sqrt{2})n-28+22\sqrt{2}}{p}\Big)\\&=\begin{cases}-(\frac{6}{p})(\frac{\sqrt{2}}{p})2x(\frac{2x}{3}) \pmod p & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}\end{aligned}$$

This yields the result for $P_{[\frac{p}{4}]}(\frac{2\sqrt{2}}{3}) \pmod p$. Taking $m=48^2$ and $t=\frac{2}{3}\sqrt{2}$ in Theorem 4.2 and applying the above we deduce the remaining result. $\square$

Let $b\in\{3,5,11,29\}$ and $f(b)=48^2,12^4,1584^2,396^4$ according as $b=3,5,11,29$. For any odd prime p with $p\nmid bf(b)$, Z.W. Sun conjectured that [Su1, Conjectures A14, A16, A18 and A21]

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{f(b)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2by^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } p=2x^2+by^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-2b}{p})=-1.\end{cases}\tag{5.5}$$

For $m\in\{5,13,37\}$ let

$$f(m)=\begin{cases}-1024 & \text{if } m=5,\\ -82\,944 & \text{if } m=13,\\ -2^{10}\cdot 21^4 & \text{if } m=37\end{cases}\quad\text{and}\quad t(m)=\sqrt{1-\frac{256}{f(m)}}=\begin{cases}\frac{\sqrt{5}}{2} & \text{if } m=5,\\ \frac{5}{18}\sqrt{13} & \text{if } m=13,\\ \frac{145}{882}\sqrt{37} & \text{if } m=37.\end{cases}\tag{5.6}$$

Suppose that p is an odd prime such that $p\nmid mf(m)$. In [Su1], Z.W. Sun conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{f(m)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{m}{p})=(\frac{-1}{p})=1 \text{ and so } p=x^2+my^2,\\ 2p-2x^2 \pmod{p^2} & \text{if } (\frac{m}{p})=(\frac{-1}{p})=-1 \text{ and so } 2p=x^2+my^2,\\ 0 \pmod{p^2} & \text{if } (\frac{m}{p})=-(\frac{-1}{p}).\end{cases}\tag{5.7}$$

Theorem 5.5. *Let $m\in\{5,13,37\}$ and let p be an odd prime such that $(\frac{m}{p})=1$ and $p\nmid f(m)$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{f(m)^k}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1 \pmod 4 \text{ and so } p=x^2+my^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Proof. Let $t(m)$ be given in (5.6). From [LM, Table II] we know that the elliptic curve defined by the equation $y^2=x^3+4x^2+(2-2t(m))x$ has complex multiplication by the order of discriminant $-4m$. Thus, by (5.4) we have

$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-2t(m))n}{p}\right)=\begin{cases}2x & \text{if } p\equiv 1 \pmod 4 \text{ and so } p=x^2+my^2,\\ 0 & \text{if } p\equiv 3 \pmod 4),\end{cases}$$

for $m=5$ see also [LM, Theorem 11]. Now applying the above and Theorem 4.2 we deduce the result. $\square$

Remark 5.2. Let $p\equiv 1 \pmod{20}$ be a prime and hence $p=a^2+4b^2=x^2+5y^2$ with $a,b,x,y\in\mathbb{Z}$. A result of Cauchy [BEW, p. 291] states that

$$\binom{\frac{p-1}{2}}{\frac{p-1}{20}}^2\equiv\begin{cases}4x^2 \pmod p & \text{if } 5\nmid a,\\ -4x^2 \pmod p & \text{if } 5\mid a.\end{cases}$$

Using the arguments similar to the proof of Theorem 5.5, [LM, Table II] (or [I]) and (5.4) we see that for any odd prime p,

$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-\frac{8}{9}\sqrt5)n}{p}\right)=\begin{cases}2x & \text{if } (\frac{5}{p})=(\frac{-2}{p})=1 \text{ and so } p=x^2+10y^2,\\ 0 & \text{if } (\frac{5}{p})=1 \text{ and } (\frac{-2}{p})=-1,\end{cases}$$
$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-\frac{140}{99}\sqrt2)n}{p}\right)=\begin{cases}2x & \text{if } (\frac{2}{p})=(\frac{-11}{p})=1 \text{ and so } p=x^2+22y^2,\\ 0 & \text{if } (\frac{2}{p})=1 \text{ and } (\frac{-11}{p})=-1,\end{cases}$$
$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-\frac{3640}{9801}\sqrt{29})n}{p}\right)=\begin{cases}2x & \text{if } (\frac{-2}{p})=(\frac{29}{p})=1 \text{ and so } p=x^2+58y^2,\\ 0 & \text{if } (\frac{-2}{p})=-1 \text{ and } (\frac{29}{p})=1,\end{cases}$$
$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-\frac{40}{49}\sqrt6)n}{p}\right)=\begin{cases}2x & \text{if } p\equiv 1,19 \pmod{24} \text{ and so } p=x^2+18y^2,\\ 0 & \text{if } p\equiv 5,23 \pmod{24},\end{cases}$$
$$\sum_{n=0}^{p-1}\left(\frac{n^3+4n^2+(2-\frac{161}{180}\sqrt5)n}{p}\right)=\begin{cases}2x & \text{if } p\equiv 1,9 \pmod{20} \text{ and so } p=x^2+25y^2,\\ 0 & \text{if } p\equiv 11,19 \pmod{20}.\end{cases}\tag{5.8}$$

From (5.8) and Theorem 4.2 we deduce the following results:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}\equiv\begin{cases}4x^2 \pmod p & \text{if } (\frac{5}{p})=(\frac{-2}{p})=1 \text{ and so } p=x^2+10y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{5}{p})=1 \text{ and } (\frac{-2}{p})=-1,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}}\equiv\begin{cases}4x^2 \pmod p & \text{if } (\frac{2}{p})=(\frac{-11}{p})=1 \text{ and so } p=x^2+22y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{2}{p})=1 \text{ and } (\frac{-11}{p})=-1,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}}\equiv\begin{cases}4x^2 \pmod p & \text{if } (\frac{-2}{p})=(\frac{29}{p})=1 \text{ and so } p=x^2+58y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p})=-1 \text{ and } (\frac{29}{p})=1,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1,19 \pmod{24} \text{ and so } p=x^2+18y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,23 \pmod{24},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{14}\cdot 3^4\cdot 5)^k}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1,9 \pmod{20} \text{ and so } p=x^2+25y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 11,19 \pmod{20}.\end{cases} \tag{5.9}$$

We remark that all the congruences in (5.9) were conjectured by Zhi-Wei Sun in [Su1].

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